EVOLUTIONARY STRUCTURAL OPTIMIZATION METHOD FOR SYSTEMS WITH STIFFNESS AND DISPLACEMENT CONSTRAINTS

by

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CERTIFICATE OF RESEARCH

This is to certify that except where specific reference to other investigation is made, the work described in this thesis is the result of the candidate's own investigations.

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DECLARATION

This is to certify that neither this thesis, nor any part of it, has been presented or is being concurrently submitted in candidature for any other degree at any other university.

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SUMMARY

This thesis presents the development of an evolutionary method for optimization of structures subject to displacement and stiffness constraints. Using an optimality criteria approach, the study provides a rigorous mathematical basis to the recently proposed evolutionary structural optimization (ESO) method. New types of sensitivity numbers for element removal have been formulated from the optimality conditions of the general weight minimization problem. The optimal shape and topology of a structure is obtained by repeated finite element analysis and element removal until the sensitivity numbers become uniform by which the optimality conditions are satisfied, or no further improvement in the objective can be achieved. It is shown that the method can be applied to other constraints on generalized displacements, stiffness, stress and frequency. Investigations on various aspects of the proposed method have been carried out to show its validity and efficiency for shape and topology optimization.

The method has been also further developed for sizing optimization problems. Similar types of sensitivity numbers for sizing elements have also been formulated which allow for solving the sizing optimization problems where sizing design variables are discrete. Solutions to problems of minimizing weight of a structure subject to displacement constraints, as well as problems of minimizing a specified displacement or maximizing the stiffness of a structure subject to a weight constraint, can be easily obtained by the proposed size selection techniques. A wide range of examples with results compared to existing solutions are included to demonstrate the capability of the proposed methods for topology, shape and discrete sizing problems.

CONTENTS

LIST OF ABBREVIATIONS

ECR	Element Changing Ratio
ERR	Element Removal Ratio
ESO	Evolutionary Structural Optimization
ESOTOPO	Program for topology optimization for truss/frame structures
ESOSHAPE	Program for shape and topology optimization for plate/shell structures
ESOSZPLT	Program for weight minimization for plate/shell structures
ESODOPLT	Program for displacement minimization and stiffness maximization for plate/shell structures
ESR	Element Shifting Ratio
ER	Evolutionary Rate
FE	Finite Element
FEA	Finite Element Analysis
FEM	Finite Element Method
MRR	Material Removal Ratio
MSR	Material Shifting Ratio
RR	Rejection Ratio
RS	Removal Strategy
2D	two-dimensional
3D	three-dimensional

NOTATIONS

a	design variable scaling parameter
$\{a^j\}$	eigenvector in jth mode
b	step control parameter
c_i, c_{ij}	element contribution or element strain energy
C, C_j	structural response
C_{ij}	ith term of structural responses
C^*, C_j^*	limiting values for structural responses
D_i	value relating to ith element
E	Young's modulus
e_i, e_{ij}	energy density within ith element
f	objective function
f_i	ith term of the objective function
$\{F^j\}$	virtual unit load vector
g_j	constraint functions
g_{ij}	ith term of the constraint function
i, j, k	indices
k_j	jth modal stiffness
$[K]$	stiffness matrix of a structure
$[K^*]$	stiffness matrix of a structure after removing an element
$[K^e]$, $[K^i]$	element stiffness matrix
l	number of real load cases
L	Lagrangian function
L_x	dimension along x axis
L_y	dimension along y axis
m	number of constraints or number of virtual unit loads
m_j	jth modal mass

$[M]$	mass matrix of a structure
$[M^e]$	element mass matrix
n	number of design variables or number of finite elements
p	index or step control parameter
P	point load
$\{P\},\{P^k\}$	real nodal load vectors
q	index or step control parameter
Q	point load
r	step control parameter
R	radius
s	number of generalised displacement constraints
s_i, s_{ik}	element strain energy
S, S_k	strain energy
S_k^*	limiting value for strain energy
t, t_s	thicknesses of plate elements
$\{u\}$, $\{u^k\}$	displacement vectors
$\{u^j\}$	virtual displacement vectors
$\{u^i\}$, $\{u^{ik}\}$	element displacement vectors
$\{u^{ij}\}$	element virtual displacement vectors
u_j	constrained displacement
u_j^*	limiting value for absolute value of constrained displacements
v_i	element volume
w_i	element weight
W	structure's weight or weight objective function
W_i	ith term of the weight objective function
X	design variable vector, $X=\{x_1, ..., x_n\}$
x_i	design variables
x_i^L	lower limits for design variables

x_i^U	upper limits for design variables
α_i, α_{ij}, α_{ijk}	element virtual energies (or contributions)
$\Delta\alpha_i^-$, $\Delta\alpha_{ij}^-$	change in element virtual energy due to size reduction
$\Delta\alpha_i^+$, $\Delta\alpha_{ij}^+$	change in element virtual energy due to size increase
$[\Delta K]$	change in stiffness matrix
$[\Delta K^i]$, $[\Delta K^e]$	change in element stiffness matrix
$[\Delta K^i]^-$	change in element stiffness matrix due to size reduction
$[\Delta K^i]^+$	change in element stiffness matrix due to size increase
Δk_i	change in modal stiffness
Δm_i	change in modal mass
Δt	change in thickness
$\{\Delta u\}$	change in displacement vector
Δu_j	change in displacement component
$\Delta\lambda_i$	change in eigenvalue
δ	tolerance
ρ_i	mass density
λ, λ_j	Lagrange multipliers
ν	Poisson's ratio
σ_e^{VM}	element von Mises stress
$\sigma_{\max}^{VM}$	maximum von Mises stress
φ	limit scaling parameter
φ_j	parameter characterising the activeness of a constraint.
γ_i, γ_{ij}	sensitivity number for single constraint
η_i, η_{ij}	sensitivity number for multiple constraints

Chapter 1

INTRODUCTION

1.1 Introduction

Structural optimization aims at finding the shape of a structure which will carry a given set of loads and has minimum weight or minimum manufacturing cost while satisfying constraints on various structural factors including stresses, displacements, natural frequencies and buckling loads.

Initially, the development of structural optimization was mainly based on analytical methods using variation calculus. Analytical methods give basic understanding of structural optimization, however they are not capable of dealing with practical problems. Only simple structures subjected to simple loading and boundary conditions can be solved by analytical methods.

More recently, structural optimization using numerical methods was developed based on the finite element method (FEM). With the introduction of high speed computers the finite element method offers a very powerful tool in solving various complex structural problems. A large number of finite element packages have been developed and used in design. Using the finite element method, two main approaches to structural optimization, mathematical programming and optimality criteria methods, have been developed and have become important tools in solving practical structural optimization problems. Both methods require repeated finite element analyses to carry out the design sensitivity analysis, where the derivatives of the objective and constraint functions with respect to all design variables are calculated. For large-scale problems, the time required for the sensitivity analysis may sometimes become prohibitive.

Due to the mathematical complexity, the application of structural optimization has gained less popularity compared with the finite element method itself. The development of commercial software for practical structural optimization has been held back by the lack of a really robust and efficient optimization method suitable for solving general engineering design problems.

The research described in this thesis is to explore an effective structural optimization method for systems with stiffness and displacement constraints. It is based on the simple concept that by systematically removing or reducing unwanted material, the residual shape of a structure evolves towards an optimum. The results of a finite element analysis are used to identify the best location for material removal/reduction. In doing so, shape and topology optimization can be achieved very easily by repeating the process of analysis and material removal. Initial investigations have shown that the proposed technique can be applied to a wide range of engineering problems.

1.2 Aims and scope of investigation

The aim of the study is to develop a generalised method, based on using the finite element method and the concept of material removal, for optimization of structures subject to displacement and stiffness constraints. Shape, topology and sizing optimization problems will be considered. Computer programs are developed and linked to finite element software to perform the optimization tasks.

The study focuses on the following points:

1. To develop an efficient scheme to evaluate the effect of removing or sizing finite elements on a specified displacement or the stiffness of a structure under a given set of loads. This is referred to as sensitivity analysis.
2. To develop a structural optimization procedure based on the idea of systematically removing or sizing elements to obtain an optimal design of the structure while keeping specified displacements or the stiffness within the given limits, i.e. satisfying displacement or stiffness constraints.
3. To develop computer programs for calculating the effect of removing or sizing elements on specified displacements or stiffness and link these programs to a finite element analysis package, to carry out the structural optimization process automatically, so that it can be used as a design tool for structural design.
4. To investigate various aspects of the proposed method to prove the validity and efficiency.

The initial design domain represented by a finite element model is chosen so that it is big enough to cover the total allowable space. The finite element analysis will reveal that not all material is effectively utilised. Using a selected rejection criterion, the ineffective material will be eliminated resulting in a more efficient structure.

Generally the optimum structure cannot be achieved in one step. The optimization process has to be evolutionary, where only a small amount of material is removed at each iteration. The cycle of analysis and material removal continues until a desired optimum is reached, for example, when a specified displacement reaches its given limit.

The finite element analysis package STRAND6, developed by G+D Computing Pty. Ltd. in Australia, is used as the tool for the analysis phase. Computer programs for the sensitivity analysis and element removal or sizing are developed by the candidate and linked to the FEM package STRAND6. The same computer programs may be linked to other commercial finite element codes with few modifications.

Depending on the type of structures to be optimised, the type of elements can be beams, plates or bricks. However, this study concentrates on two-dimensional continuum structures modelled using plate/shell elements and skeletal structures (trusses, frames) modelled using beam/bar elements. It is assumed that all materials behave within the linear elastic range.

1.3 Outline of the thesis

The thesis consists of eight chapters, a list of references and appendices. A literature review on structural optimization, with a particular emphasis on shape and topology optimization and on a background of the evolutionary structural optimization method is provided in Chapter 2.

Chapter 3 presents the theoretical basis of the evolutionary structural optimization method for shape and topology problems with displacement constraints, which include the problem formulation, sensitivity analysis and evolutionary optimization procedures for a single constraint, multiple constraints and multiple load cases.

Chapter 4 presents the application of the proposed method to truss topology optimization. The problem of singularity of the stiffness matrix is encountered. A technique is proposed for avoiding singularity of the stiffness matrix and allowing continuity of the optimization process. A number of examples are provided to show the effectiveness of the proposed method.

Chapter 5 presents the application of the proposed method to shape and topology optimization of two-dimensional continuum structures. A measure for avoiding singularity of the stiffness matrix is provided. The evolutionary method is able to perform shape and topology optimization simultaneously or pure shape optimization. Examples are provided to show the capability of the proposed method.

Chapter 6 provides a comprehensive study on validity and reliability of the proposed method for shape and topology problems. Various aspects of the proposed method including influences of element removal ratio, ground structure, mesh size and element type are discussed. Additional steps for obtaining solutions without checker-board patterns for two-dimensional continuum structures are also suggested.

Chapter 7 presents the evolutionary structural optimization method for discrete sizing problems, which includes problems of minimum weight design of structures subject to displacement constraints and minimization of a specified displacement or strain energy of structures subject to a constraint on the weight. For these problems, suitable types of sensitivity numbers and evolutionary optimization procedures are formulated. The proposed method is capable of dealing with discrete design variables directly. Examples of optimum design of plates with discrete variable thicknesses are provided.

Chapter 8 summarises the findings of this study. Some recommendations for further investigation are also given.

A list of references includes only works which are closely related to the study and directly referred to in the thesis.

Appendix I includes the analytical solution for the problem of the optimal configuration of two bar truss. Appendices II, III and IV include copies of the published papers, where the initial results of the study are first reported.

Chapter 2

BACKGROUND AND LITERATURE REVIEW

2.1 Background

2.1.1 Classification of structural optimization problems

In general, three major criteria need to be satisfied in the design of all structures: strength, rigidity and stability. For a particular type of structure one or more of these requirements governs the design. For example, the strength requirement is the dominant factor in the design of low-height buildings. However, for high-rise buildings, the rigidity and stability requirements are more important and are often the dominant factors in the design. Furthermore, lateral deflections and the peak acceleration at the top floor of high-rise buildings should be limited to acceptable levels to satisfy the standpoints of serviceability and comfort (Taranath 1988). Optimal designs in terms of least weight (cost) while maintaining the strength, rigidity, stability and serviceability of structures within the required limits, have been continuously sought.

Structural optimization aims to reduce the weight (cost) or improve the behaviour of the structure while satisfying certain requirements. The basic concepts and solution methods in structural optimization can be found in Gallagher and Zienkiewicz (1973), Morris (1982) and Haftka *et al.* (1990). The type of structural optimization problem depends on the *objective,* the *constraints* and the nature of the *design variables.*

In structural optimization, the following criteria can be treated either as the objective or as constraint functions:

(1) weight (volume);

(2) displacement at a given point;

(3) maximum displacement in the structure;

(4) stress (several types) at a given point;

(5) maximum stress (several types) in the structure;

(6) buckling load factor;

(7) frequencies;

(8) overall stiffness.

Different combinations of these entries from the above list lead to different optimization problems. The most common structural optimization problem is usually stated as

Minimize *the weight (volume) of a structure*

subject to *constraints imposed on some structural responses (one or more entries from (2)-(8)).*

Another kind of structural optimization problem is often stated as

Optimize *structural response (one of entries (2)-(8))*

subject to *a weight (volume) constraint.*

A mathematical statement for a general optimization problem can be given as

Find the set X, that will

$$\begin{array}{lll} \textit{minimize} & f(X) & \\ \textit{subject to} & g_j(X) \leq 0 & (j = 1, m) \\ & x_i^L \leq x_i \leq x_i^U & (i = 1, n) \end{array} \qquad (2.1)$$

where $f(X)$ - *objective function;*

$g_j(X)$ - *constraint functions;*

$X = \{x_1, \ldots, x_n\}$ - *design variables;*

x_i^L *and* x_i^U - *lower and upper limits for* x_i.

The constraints may be given in the form of equality or inequality conditions. The additional constraints imposed on the design variables are called side constraints. There are also equality constraints in the form of equilibrium and compatibility conditions which are usually satisfied as a sub-problem via finite element analysis and, therefore, are often not included in the general statement of optimization problems.

The objective and constraint functions can be either linear or non-linear functions of design variables. When all the objective and constraint functions are linear functions, the problem is a *linear optimization problem.* When at least one of them is a non-linear function of design variables the problem becomes a *non-linear optimization problem.* Most structural optimization problems are highly non-linear where the objective and/or constraint functions are implicit functions of design variables. When the objective and constraint functions are expressed as a sum of functions, and each is a function of only one design variable, i.e.

$$f(X) = \sum_{i=1}^{n} f_i(x_i) \tag{2.2}$$

$$g_j(X) = \sum_{i=1}^{n} g_{ij}(x_i) \tag{2.3}$$

they are separable functions and the problem is referred to as a *separable optimization problem.* Separability is an important feature in derivation of the optimality criteria for optimization problems.

When all design variables are continuous, i.e. they can have any value between the given lower and upper limits, the problem is called as a *continuous variable problem*. In many cases, where one or more design variables can have only a certain value from a set of discrete values, the problem is referred to as a *discrete variable problem*.

Design variables can be any quantity relating to description of the geometry of structure (nodal coordinates), member cross-sectional dimensions, member cross-sectional properties, member material properties and topology (the pattern of connection to nodes by members or member connectivities), which are required to be determined in the solution process. Each type of design variables requires different mathematical approach. Depending on the nature of the design variables, structural optimization problems are commonly classified in three types: *sizing optimization*, *shape optimization* and *topology optimization problems*.

Sizing optimization problems: In sizing optimization problems the geometry (nodal coordinates) and topology (member connectivities) of structures are fixed. Sizing design variables may include bar cross-sectional areas, beam element cross-sectional dimensions, cross-sectional properties (cross-sectional area, moments of inertia) and plate/shell element thicknesses. They may also include the material properties such as Young's modulus, Poisson's ratio as in optimization of composite structures.

Shape optimization problems: In contrast to sizing problems, the shape of structures is to be determined in shape optimization problems. The exterior and interior boundary shapes of the structures are changed. In such a shape optimization process, no new holes (cavities) can be created, i.e. topology of the structure remains the same. Design variables in shape optimization problems can be nodal coordinates relating to the finite element method.

Often the term shape optimization is used in a narrow sense referring only to the optimal design of the shape of the boundary of two- and three-dimensional structural components. In the broad sense, shape optimization can be any problem where it is needed to change the position of nodes of the finite-element model. In this sense it also includes *geometry* (or *configuration*) optimization of skeletal structures (frames and trusses) where joint locations are design variables. In more general case, besides nodal coordinates, member cross-sectional dimensions (sizing variables) are also included into design variables.

The above definition for shape optimization is related to more traditional methods where shape optimization is achieved by changing nodal coordinates. Other methods using the ground structure approach can also perform shape optimization by removing elements only from the existing boundaries. Although nodal coordinates have never changed in these methods, a new shape of the structure is formed by those external nodes, to which remaining elements are connected.

Topology optimization problems: With reference to the finite element method, topology optimization is any problem where the pattern of element connectivity to the nodes needs to be determined. Topology optimization of continuum structures involves creation of internal holes. Topology optimization of skeletal structures seeks the number and spatial sequence of elements, joints and supports. During topology optimization elements can be removed from the structure. Topology design variables are usually the variables which are able to describe the presence or absence of each element. When the position of nodes are allowed to change, the nodal coordinates are also design variables. In this case simultaneous shape and topology optimization is performing. The most general optimization problems can include topology, shape and sizing design variables.

2.1.2 Main approaches to structural optimization

The main approaches to structural optimization are *mathematical programming* and *optimality criteria methods*. Recent developments of these approaches to structural optimization can be found in Kamat (1993).

2.1.2.1 Mathematical programming methods

Mathematical programming offers a general tool for structural optimization. It includes linear programming, penalty function method, feasible direction method, sequential linear programming and sequential non-linear approximate optimization (Vanderplaats 1993). These methods require calculation of derivatives of objective and constraint functions with respect to all design variables, which is referred to as sensitivity analysis (Arora and Haug 1979; Adelman and Haftka 1986). Often, repeated finite element analyses are performed to carry out the sensitivity analysis, which is very costly for large problems. The computational time for sensitivity analysis sometimes becomes prohibitive. Many approximation methods and techniques have been developed and employed, to improve the efficiency of the sensitivity analysis and optimization algorithms (Vanderplaats *et al.* 1991).

2.1.2.2 Optimality criteria methods

Fully stressed design: Fully stressed design is the earliest intuitive optimality criteria approach for strength optimization of structural systems. In the simplest procedure, the design variables, for example member cross-sectional areas, are scaled by the ratio of the element stress to the allowable stress using the formula

$$x_i^{new} = x_i^{old}\left(\frac{\sigma_i}{\sigma^a}\right) \tag{2.4}$$

where σ_i is the stress of the ith element and σ^a is the allowable stress. The ratio σ_i/σ^a is called the *stress ratio*. An iterative process of analysis and design rationing can result in a structure where all members, except those which are at the minimum gauges, are fully stressed, i.e. their stresses are at allowable level.

Fully stressed design procedures are very attractive and efficient in the sense that the iterative process usually converges in few iterations. The shortcomings of the method is that it can not necessarily give minimum weight designs due to the fact that no objective is involved. It has been pointed out later that the fully stressed design method can give minimum weight designs only for statically determinate structures under a single loading condition with equal allowable stresses on tension and compression. However, because of its effectiveness, this method is still used to get a starting point for more rigorous optimization procedures (Gallagher and Zienkiewicz 1973; Morris 1982).

Optimality criteria methods: Rigorous optimality criteria methods try to satisfy the Kuhn-Tucker necessary conditions for optima. A common criterion is that the strain energy density in each member of the structure should be uniform. A recursive formula is derived which leads to the desired solution using iterative process. Is was pointed out by Fleury (1979, 1980) that mathematical programming and optimality criteria methods have a common basis in the duality of the original problem statement. Optimality criteria is valid for a mathematically separable problem. The optimality criteria approach is shown to be quite effective as a design tool. Its principal attractiveness is that the method can be easily programmed for the computer, is relatively independent of problem size, and usually provides a near optimum design with relatively smaller number of detailed structural analyses compared with mathematical programming methods (Vanderplaats 1981).

The common approach to optimality criteria methods is the Lagrange multiplier method. The separability of the objective and constraint functions with respect to the design variables, i.e. equations (2.2) and (2.3) are satisfied, is the most important condition for derivation of optimality criteria (Berke and Khot 1988). Following the standard procedure, the constrained optimization problem (2.1) is replaced by the unconstrained problem of minimizing the so-called Lagrangian which is formed as a combination of the initial objective and constraint functions. For the ease of reference and simplicity in discussion, the optimality criteria method is summarised below. The summary is mainly based on the work by Berke and Khot (1988). The result can also be found elsewhere in the works by Venkayya *et al.* (1973) and Venkayya (1993), and in the books by Morris (1982) and Haftka *et al.* (1990).

The separability of the problem of weight minimization subject to a displacement constraint can be easily pointed out, if there is only one design variable for each element, i.e. n (the number of design variables) is also the number of elements. The separability condition (2.2) is satisfied for the weight objective

$$W(X) = \sum_{i=1}^{n} W_i(x_i) \tag{2.5}$$

where $W_i(x_i)$ is the weight of the ith element. The separability condition (2.3) is also satisfied for a displacement constraint. By using the virtual unit load method the specified constrained displacement, denoted by C, can be expressed as the sum of element virtual strain energies

$$C(X) = \sum_{i=1}^{n} C_i(x_i) \tag{2.6}$$

where $C_i(x_i)$ is the virtual strain energy of the ith element. It is assumed that the term C_i is an explicitly function of only x_i. This is justified in statically determinate structures. In statically indeterminate structures the terms C_i are also implicit functions of all the design variables representing members cross section properties. The constraint in the problem statement (2.1) will have the form

$$g(X) = C(X) - C^* = \sum_{i=1}^{n} C_i(x_i) - C^* \le 0 \tag{2.7}$$

where C^* is the given limit for the specified displacement C.

The Lagrangian for the problem of weight minimization subject to a single constraint is defined as

$$L(X,\lambda) = W(X) + \lambda g(X) = \sum_{i=1}^{n} W_i(x_i) + \lambda\left(\sum_{i=1}^{n} C_i(x_i) - C^*\right) \tag{2.8}$$

where λ is a Lagrange multiplier. The necessary Kuhn-Tucker condition for optima of the problem is obtained by differentiating the Lagrangian with respect to the design variables. This gives

$$\frac{\partial L(X,\lambda)}{\partial x_i} = \frac{\partial W(X)}{\partial x_i} + \lambda \frac{\partial C(X)}{\partial x_i} = \frac{dW_i(x_i)}{dx_i} + \lambda \frac{dC_i(x_i)}{dx_i} = 0 \quad (i = 1,\ n) \tag{2.9}$$

where separability is utilized. Further derivation of the optimality criteria needs explicit forms of $W_i(x_i)$ and $C_i(x_i)$ in term of the design variable x_i. Berke and Khot (1988) proposed the following expressions

$$W_i(x_i) = w_i x_i \qquad (i = 1,\ n) \tag{2.10}$$

$$C_i(x_i) = \frac{c_i}{x_i} \qquad (i = 1,\ n) \tag{2.11}$$

where w_i is the specific weight. Further specialization, for example, for truss and displacement constraints results in $w_i = \rho_i l_i$ and $c_i = T_i^v T_i l_i / E_i$, where ρ_i is the bar mass density, l_i is the bar length, E_i material Young's modulus, T_i is the bar force due to the actual load and T_i^v is the bar virtual force due to the virtual unit load applied to the location of and in the direction of the constrained displacement.

The optimality condition (2.9) now becomes

$$w_i - \lambda \frac{c_i}{x_i^2} = 0 \qquad (i = 1,\ n) \qquad (2.12)$$

or

$$D_i = \lambda \frac{c_i}{w_i x_i^2} = \lambda \frac{C_i}{W_i} = 1 \qquad (i = 1,\ n) \qquad (2.13)$$

or

$$\frac{c_i}{w_i x_i^2} = \frac{C_i}{W_i} = \frac{C_i}{\rho_i v_i} = \frac{e_i}{\rho_i} = \frac{1}{\lambda} = constant \qquad (i = 1,\ n) \qquad (2.14)$$

where

$$e_i = \frac{C_i}{v_i} \qquad (i = 1,\ n) \qquad (2.15)$$

is the element virtual strain energy density, v_i is the element volume and ρ_i is the element mass density.

Equation (2.14) states an optimality criterion that at the optimum the ratio of the element virtual strain energy density and mass density is equal for all elements. In the case where all elements are made from the same material, ρ_i is the same for all elements, the optimality criterion is that the virtual energy density is uniform for all elements (Venkayya *et al* 1973).

There are three essential recurrence formulas that are possible to obtain from equation (2.13). By multiplying equation (2.13) by $(x_i)^q$ and taking qth root on both sides, the exponential recurrence formula is obtained in the form as

$$x_i^{new} = (x_i D_i^{\frac{1}{q}})^{old} \qquad (i = 1,\ n) \qquad (2.16)$$

The linearized form for formula (2.16) is as

$$x_i^{new} = [x_i(1+\tfrac{1}{q}(D_i-1))]^{old} \qquad (i=1,\ n) \tag{2.17}$$

If the above steps are performed using reciprocal variables and then reconverting to the original variables, the linearized reciprocal form is obtained as

$$x_i^{new} = \left[\frac{x_i}{(1-\tfrac{1}{q}(D_i-1))}\right]^{old} \qquad (i=1,\ n) \tag{2.18}$$

The parameter q is called a step size parameter. The Lagrange multiplier is determined from the condition that the constraint is active. In the case of a single constraint, this condition can be easily satisfied by scaling all design variables without the necessity of determination of the Lagrange multiplier. It should be pointed out that in statically determinate structures, the term c_i in (2.12) is constant. Formula (2.12) can be solved for x_i and gives the correct optimum value to x_i in a single sizing step.

In problems with multiple constraints, the constrained displacements are expressed as

$$C_j(X) = \sum_{i=1}^{n} C_{ij}(x_i) \qquad (j=1,\ m) \tag{2.19}$$

where $C_{ij}(x_i)$ is the virtual strain energy of the ith element with respect to the jth constrained displacement and m is the number of constraints. The constraints will have the form

$$g_j(X) = C_j(X) - C_j^* = \sum_{i=1}^{n} C_{ij}(x_i) - C_j^* \le 0 \qquad (j=1,\ m) \tag{2.20}$$

where C_j^* is the given limit for the jth constraint. The Lagrangian will have the form as

$$L(X,\lambda)=\sum_{i=1}^{n}W_i(x_i)+\sum_{j=1}^{m}\lambda_j(\sum_{i=1}^{n}C_{ij}(x_i)-C_j^*) \tag{2.21}$$

where λ_j (j = 1, m) are m Lagrange multipliers. The necessary Kuhn-Tucker conditions are

$$\frac{\partial L}{\partial x_i}=\frac{dW_i(x_i)}{dx_i}+\sum_{j=1}^{m}\lambda_j\frac{dC_{ij}(x_i)}{dx_i}=0 \qquad (i=1,\ n) \tag{2.22}$$

The explicit forms of $W_i(x_i)$ is the same as in (2.9). Similar explicit forms for $C_{ij}(x_i)$ are given as

$$C_{ij}(x_i)=\frac{c_{ij}}{x_i} \qquad (i=1,\ n;j=1,\ m) \tag{2.23}$$

For a bar member $c_{ij}=T_{ij}^{\ v}T_i\, l_i/E_i$ where $T_{ij}^{\ v}$ is the bar virtual force due to the virtual unit load applied to the location of and in the direction of the jth constrained displacement.

The optimality condition (2.22) now becomes

$$w_i-\sum_{j=1}^{m}\lambda_j\frac{c_{ij}}{x_i^2}=0 \qquad (i=1,\ n) \tag{2.24}$$

or

$$D_i=\sum_{j=1}^{m}\lambda_j\frac{c_{ij}}{w_i x_i^2}=\sum_{j=1}^{m}\lambda_j\frac{C_{ij}}{W_i}=1 \qquad (i=1,\ n) \tag{2.25}$$

or

$$D_i=\sum_{j=1}^{m}\lambda_j\frac{e_{ij}}{\rho_i}=1 \qquad (i=1,\ n) \tag{2.26}$$

where

$$e_{ij}=\frac{C_{ij}}{v_i} \qquad (i=1,\ n;j=1,\ m) \tag{2.27}$$

is the element virtual strain energy density corresponding to the jth constrained displacement.

The equation (2.26) expresses the optimal criterion for multiple constraints that at the optimum the weighted sum of ratios of element virtual strain energy density and mass density is equal to unity for all elements, where the weighting parameters are the Lagrange multipliers (Venkayya *et al.* 1973)

The evaluation of the Lagrange multipliers λ_j is important for multiple constraints. They are determined from conditions that constraints are active. Assuming all constraints in (2.19) are equality constraints, we have

$$C_j = C_j^* \qquad (j = 1,\ m) \tag{2.28}$$

By multiplying by $(\lambda_i)^b$ and taking bth root on both sides of (2.28), the exponential recurrence formulae for the Lagrange multipliers are obtained in the form as

$$\lambda_j^{new} = \left[\lambda_j \left(\frac{C_j}{C_j^*} \right)^{\frac{1}{b}} \right]^{old} \qquad (j = 1,\ m) \tag{2.29}$$

The Lagrange multipliers can also be determined by linear equations which are obtained by combining the optimality criterion (2.25) and the active constraint equations (2.28), where C_j is determined by (2.18). The linear equations have the form

$$\sum_{p=1}^{m} \lambda_k \left(\sum_{i=1}^{n} \frac{c_{ij} c_{ik}}{w_i x_i^3} \right) = C_j^* \qquad (j = 1,\ m) \tag{2.30}$$

The incremental procedure based on (2.30) to update λ_j is given as follows

$$\sum_{p=1}^{m} \lambda_k \left(\sum_{i=1}^{n} \frac{c_{ij} c_{ik}}{w_i x_i^3} \right) = (1+p)C_j - pC_j^* \qquad (j = 1,\ m) \tag{2.31}$$

where p is a step size parameter.

The formulae (2.29), (2.30) and (2.31) are valid under the assumption that constraints are active. There is a need of a mechanism for keeping constraints active. This can be easily done by scaling design variables. Thus scaling design variables should precede the evaluation of the Lagrange multipliers.

The recurrence formulae to update design variables in the case of multiple constraints have the same forms as in equations (2.16), (2.17) and (2.18), where D_i is given in equation (2.26).

2.2 Review on recent developments in structural optimization

2.2.1 Sizing optimization

Both mathematical programming and optimality criteria methods were initially developed for sizing optimization problems where design variables are continuous. A review of basic developments of these methods has been presented by Vanderplaats (1981). It is seen that substantial efforts have been devoted to the associated mathematical and computational backgrounds to improve the accuracy of analysis, especially the accuracy of sensitivity analysis, to facilitate higher convergence rates and to ensure the reliability and efficiency of solution algorithms. In this direction, various approximation concepts and methods have been proposed. A good review on approximation concepts for structural synthesis has been given by Vanderplaats *et al.* (1991). It is pointed out that constraint deletion, by which only critical or near-critical constraints remain under consideration in each step, can help to reduce the computation time significantly. The use of intermediate variables and responses, for example reciprocals of cross-sectional properties and member forces, can change highly nonlinear dependence of responses with respect to design variables to more linear dependence. Therefore a linear approximation can be of a higher quality, which increases the accuracy of sensitivity analysis. The approximate problem is then formulated and solved without the necessity of full finite element analysis. The result of the approximate problem is used to update the design and the next design step commences. Using approximation concepts will reduce the number of costly full finite element analyses. The current status of this trend in structural optimization can also be found in Kamat (1993). All these attempts aim to apply the optimization methods to large scale and more complex structures.

In many practical applications, the design variables must be selected from a list of discrete values. For example, structural members may have to be selected from standard sections or thicknesses commercially available from manufacturers. Relating to practical applications, more attention has been given to dealing with discrete sizing variables optimization. Huang and Arora (1995) gave an overview of methods for solving such problems. It is pointed out that the solution of the mixed (discrete-continuous) variable optimization problems usually requires considerably more computation effort compared to the continuous variable optimization problems. Often the problem is solved for continuous optimal solution by using traditional optimization techniques, such as mathematical programming or optimality criteria methods, assuming all designs variables be continuous. Then, one of the methods, such as rounding-off, branch and bound methods, simulated annealing, genetic algorithm, Lagrangian relaxation methods, is used to get the discrete solution (Huang and Arora 1995; Ringerts 1988; Sandgren 1990; Schmit and Fleury 1980). The values given for each discrete design variable are usually required to be close to each other for validity of converting a continuous optimal solution to a discrete one (Ringerts 1988).

Olsen and Vanderplaats (1989) presented a sequential linear discrete programming method for nonlinear optimization with discrete design variable. The continuous optimum solution is rounded in a direction away from constraint violation to get a starting point for discrete optimization. The approximation techniques are used to create sub-problems suitable for linear mixed-integer programming method. The use of a truncated set of discrete possibilities reduces the size of the approximate problem and hence improves the efficiency. Each discrete design variable is allowed three possibilities: its present value, the adjacent higher value, and the adjacent lower value. Examples show that the method gives results with lower weights compared with the continuous solution and pointed out the potential non-unique solution to a discrete problem.

The rounding up method is the simplest way to get a discrete solution from the continuous one. This may result in a large disturbance from the continuous variable solution. It is recognised that while many design variables are assigned the closest higher discrete values, it is possible to reduce some of them to the closest lower discrete values to obtain a feasible design close to the continuous variable solution. Chan (1994) proposed a pseudo-discrete section selection technique to achieve a smooth progressive transition from the continuous variable design to the optimum final design using discrete standard steel sections (see also Chan *et al.* 1994). Once the continuous variable solution is obtained, the strategy starts selecting sections for member or group of members involving the least change in material weight. These members then are treated as inactive by being fixed at the assigned discrete sections. Other active members may possibly reduce in size by re-evaluating the Lagrange multipliers and reusing the recursive relation. This section selection process is progressively applied until each member is assigned a final discrete standard section.

2.2.2 Shape and topology optimization of continuum structures

In comparison to sizing optimization, shape optimization is more complex because the shapes are continuously changing in the design process. The first serious problem associated with boundary shape optimization of two- or three-dimensional bodies is mesh deformation. As the shape of the structure changes, the finite element mesh needs to be updated to maintain an adequate element geometry. Highly deformed finite elements due to change of boundary will result in loss of accuracy in calculations of sensitivity derivatives. This problem is addressed by manually remeshing during the optimization process or by employing sophisticated automatic mesh generators. Another problem is that of existence or creation of internal boundaries or holes. In many problems the optimal design will have internal cavities. It is impossible to produce these cavities with a standard optimization approach without prior knowledge of their existence. One approach to deal with this problem is to assume that the material is not homogeneous, but instead has an microstructure with microcavities in the material (Haftka and Grandhi 1986; Haftka *et al.* 1990, p. 197).

Ding (1986) gave a review on numerical and analytical methods for shape optimization of structures. Several steps in the shape optimization process, such as model description, selection of the objective function and shape variables, representation of boundary shape, finite element mesh generation and refinement, sensitivity analysis and solution methods, are reviewed in detail. Examples of shape optimization of two- or three-dimensional structures are given to show the state of the art in the mid 80's in shape optimization. Ding (1986) pointed out that careful consideration has to be given in order to describe the changing shape, to maintain an adequate finite element mesh, to enhance the accuracy of the sensitivity analysis, to impose proper constraints and to utilise existing optimization methods to solve the shape optimization problems.

Homogenization method

An important recent advance in the shape optimization is the homogenization method proposed by Bendsøe and Kikuchi (1988). In this method a structure was represented by a model of finite elements with microvoids (a micro rectangular hole is included in each element). By changing the sizes of the rectangular hole the element can become a complete void or solid, as well as generalized porous medium. The orientation of such holes is also important. Thus, the sizes and the orientation of the microscale rectangular holes are the design variables which characterize the porosity of the porous medium. The objective is to minimize the mean compliance of the structure subject to equilibrium equations and a volume constraint. An optimality criteria method is used which gives the optimal porosity of such a porous medium. Many interesting results of the homogenization method can be found in Bendsøe and Kikuchi (1988, 1993), Bendsøe (1988, 1989), Suzuki and Kikuchi (1991), Díaz and Bendsøe (1992).

Tenek and Hagiwara (1994) presented a study on optimization of rectangular plate and shallow shell using the homogenization method. The objective was to minimise the strain energy function under a volume constraint. An optimum distribution of thickness or microstructural density is sought under the hypothesis that the design variables can only be assigned their extreme allowable bounds, or values very near them, so that the material can be removed from low density areas, thus an optimum topology can be determined. The finite element method was used to obtain the structural response and the feasible direction method was used for the optimization process.

Another microstructure-based design domain method, similar to homogenization method, has been proposed by Gea (1996). New microstructures in the form of spherical inclusions are introduced in finite elements. A closed form of the problem is achieved which allows the use of linear programming algorithms for optimization process. The method significantly reduces the number of design variables per element and can give similar results to that obtained by the homogenization method.

Density function approach

A method proposed by Yang and Chuang (1994) based on an empirical relationship between the density and the Young's modulus and linear programming techniques, gave a similar result to that obtained by the homogenization method. The drawback of the method is in the intuitive relationship between the density and the Young's modulus.

Changing element connectivity

One possibility to overcome difficulties of changing shape description, remeshing and creation of cavities is to rely on the entire finite element mesh chosen for the initial design domain. The shape of the structure will be obtained by *removing unnecessary elements* from the mesh (e.g. by changing the element connectivities). The key point in this method is to work out an effective measure to evaluate the importance of each element to the whole structure and an appropriate criterion for element removal. The idea of element removal has been tried by other researchers such as Maier (1973), Rodriguez-Velazquez and Seireg (1985), but such studies have not resulted in a generalized method. A systematic investigation is under way at Victoria University of Technology and The University of Sydney on the full potential of such a simple but attractive idea of removing unwanted material.

Atrek (1989) reported an optimization program called SHAPE for shape optimization of continuum structures. The structure is represented by a finite element model and the optimal shape of the structure is obtained by removing elements. The program can optimize the shapes of solid, shell, or plane-stress systems for multiple load cases, with multiple constraints related to stresses, displacement, and stiffness. Material may be removed from inside the domain as well as from the boundaries. A linear maximization sub-problem for Lagrange multipliers has to be solved in order to identify the optimum locations for element removal.

Evolutionary structural optimization

Recently, a simple approach for shape and layout optimization, called *evolutionary structural optimization* (ESO), has been proposed by Xie and Steven (1993). With this procedure, structural shape and layout optimization can be easily achieved using standard finite element analysis codes. The original idea of evolutionary structural optimization involves obtaining an optimal shape of a structure by systematically removing lowly stressed elements from the structure (Xie and Steven, 1993, 1994a). This method has been extended for frequency optimization problems (Xie and Steven, 1994b, 1996), for which by removing appropriate elements from the structure, the frequency of the resulting structure will be changed in a desired direction. A wide range of examples has been presented by Xie and Steven (1993, 1994a, 1994b, 1996) to demonstrate the capacities of such an evolutionary procedure for solving structural shape and layout optimization of continuum structures.

A recent application of the proposed ESO method is shape optimization of metallic insert in composite bolted joints (Rispler and Steven 1995). The optimal shape for the metallic insert is obtained by gradually changing material of highly stressed elements from carbon fibre composite to aluminium. The obtained shape significantly reduces the high stress concentration and increases the bearing strength of pin loaded and bolted joints.

2.2.3 Truss topology optimization

In truss topology optimization problem, one seeks the optimal pattern of connection of members or the number and spatial sequence of elements, joints, and supports. Compared with the sizing problem, the topology optimization is much more complex due to the changes in both the finite element model and the set of design variables. Despite the significant difficulties involved in the solution process, it is recognised that substantial savings can be achieved in topology optimization compared with sizing optimization (Kirsch 1989, Topping 1993).

Due to the difficulties involved in the topology optimization, various simplifications and approximations are often used. A common approach to the topology optimization of truss structures is based on an initial *ground structure* or *structural universe*, which contains many members connecting to the chosen set of joints. Members are being removed during the solution process. An optimal solution is obtained as a sub-set of such a ground structure. This approach is referred to as the *ground structure method.* Dorn *et al.* (1964) were the first to use the ground structure approach. Both force and displacement methods in structural analysis are employed in truss topology problems.

By using the force method and ignoring the compatibility conditions, the truss topology optimization problem for a minimum weight can be formulated as a linear programming problem in member areas and forces. The resulting topology may represent a statically determinate, statically indeterminate or unstable structure. However, the obtained design is not the final optimum because it may not satisfy the compatibility conditions. Some modification to account for the elastic compatibility or stability might be required (Kirsch 1989, 1993).

In the displacement formulations of truss topology problems, non zero lower bounds on the cross-sectional areas have been imposed to avoid singularity of the stiffness matrix. One way of tackling this problem is to use special methods for compliance minimization (Bendsøe and Kikuchi 1993; Bendsøe and Ben-Tal 1993). The problem is to find the minimum compliance truss subject to the equilibrium condition and a given volume of material. In this formulation, the design variables (bar volumes) and the displacements appear as independent variables. The presence of the displacements in the problem formulation does not require a positive definition of the stiffness matrix. This allows to set zero to the lower bound on the design variables and therefore the bars of the ground structure can be removed, and the problem statement covers topology design. The limitation of the compliance formulation is that only a constraint on the volume of the truss is considered.

A more general way for avoiding the problem of singularity of the stiffness matrix is to use the simultaneous analysis and design (SAND) approach which treats displacements as additional design variables and equilibrium conditions as equality constraints (Haftka 1985). This method does not require inversion or factorisation of the stiffness matrix so it can be applied to truss topology optimization where the zero lower bounds for design variables are allowed (Sankaranarayanan *et al.* 1993, 1994). To reduce the computational time, an additional member elimination strategy is incorporated into SAND. After every five optimization circles, elements with small cross-sectional areas are eliminated. An element is removed if its cross-sectional area is less than 1% of the maximum area in the current design and if simultaneously the elemental stress is less than 75% of the maximum stress. Although the SAND approach avoids the problem of singularity of the stiffness matrix, it generally increases the number of design variables substantially.

A simpler method for the design of structures with variable topology, based on theorems of structural variations, is proposed by Majid (1974). The method includes both stress and displacement constraints. The benefit to weight saving due to removal of every element is assessed. The member, which gives the largest weight saving without violating constraints, is removed. However, repeated analyses for a pair of unit loads acting at the ends of each member are required, which increases the computation time substantially. Thus, the method is limited only to small size structures due to efficiency considerations

Tabatabaei and Marsh (1993) demonstrated the effect of diagonal removal on space structures. Removing appropriate diagonals creates a more uniform distribution of forces between chords of equal size, and consequently increases the overall load capacity of the structure. Their examples illustrated that more efficient designs can be obtained by removing select elements from the given structure. However, the proposed procedure is very time-consuming because repeated finite element analysis is required for identifying the best element to be removed at each step.

It has been pointed out by other researchers that fully stressed design methods can also be employed to eliminate members from the ground structure. By applying the stress ratio procedure to a highly connected ground structure, many of the members will reduce to zero. For a structure subject to a single loading condition with stress constraints for members in tension and compression at the same level, the resulting structural layout will be the same as that obtained by solving the linear programming problem and will generally be statically determinate (Kirch 1989).

It is worth noting that the selection of a solution method for a particular optimal design problem greatly depends on the FEA packages and optimization programs available to the designer. At present, finite element packages are popular and accessible to all engineers while optimization programs are less popular because of the mathematical complexity involved in structural optimization problems and lack of general-purpose optimization programs. The evolutionary structural optimization method offers a simple way to solve optimization problems by using standard finite element packages. This method is still at the early stage in development. The current status of this simple method is given in the next section.

2.3 State of the art of evolutionary structural optimization

2.3.1 Original evolutionary procedure for stress optimization

In the original procedure of evolutionary structural optimization, proposed by Xie and Steven (1993), the shape and layout of a structure is obtained by gradually removing lowly stressed elements. A design domain is chosen large enough to cover the final design and is divided into a fine mesh of finite elements. Loads and boundary conditions are imposed and a stress analysis is carried out by using a standard finite element code. It is often revealed that some parts of the material are lowly stressed and they can be removed from the structure. The Rejection Criterion (*RC*) is based on the von Mises stress in elements. For a structure under a single loading condition, elements, in which the von Mises stress is less than a Rejection Ratio (*RR*) times the maximum von Mises stress over the structure, will be removed from the structure. In other words, elements are removed if the following condition is satisfied

$$\frac{\sigma_e^{VM}}{\sigma_{\max}^{VM}} \leq RR \tag{2.32}$$

The cycle of finite element analysis and element elimination is repeated for the same value of RR until a steady state is reached when no more elements are deleted. At this stage an Evolutionary Rate (ER) is introduced and added to the RR, i.e.

$$RR_{i+1} = RR_i + ER \qquad (i = 0,\ 1,\ 2,\ 3,\ 4,...) \tag{2.33}$$

where i refers to iteration number. The iterative process takes place again until a new steady state is reached. This evolutionary process continues until a desired optimum is reached, for example, when all stress levels are within 25% of the maximum. The absolute best value of the final rejection ratio to terminate the iteration process might not exist, however the evolutionary procedure provides the possibility of knowing every stage of shape and layout path towards the final design. The final design is the optimum in the sense that more uniform stress distribution is achieved.

In this method two parameters, the initial rejection ratio RR_o and the evolutionary rate ER, are introduced. The typical values RR_o = 1% and ER = 1% are small enough to give satisfactory results. For some specific problems where stress levels do not vary much over the whole design domain, an initial Rejection Ratio as high as 10% and an evolutionary rate as large as 5% have been suggested and used by these authors.

The previously described evolutionary procedure has been extended to optimal design of structures subjected to multiple load cases (Xie and Steven 1994a). After a finite element analysis, the ratio of the element stress over the maximum stress is checked for each load case and an element will be eliminated from the structure only if the ratio is less than current RR_i for all load cases, i.e.

$$\frac{\sigma_e^k}{\sigma_{\max}^k} \leq RR_i \qquad (k = 1,\ l) \tag{2.34}$$

where σ_e^k is element von Mises stress and σ^k_{max} is the maximum von Mises stress over the current design of the structure corresponding to k^{th} load case and l is number of load cases. The process of finite element analysis and element elimination is repeated with the same value of RR_i until a steady state is reached. By then, the current rejection ratio is increased by the Evolutionary Rate following the formula (2.33). The optimization process continues until the final value for the Rejection Ratio is reached. The final design is optimum in the sense that more uniform stress distribution results and every element has its own role to play for at least one load case and possibly for all load cases. Examples provided by Xie and Steven (1994a) show that the proposed method can reproduce many existing optimal solutions obtained by other methods.

2.4.2 Evolutionary procedure for frequency optimization

The evolutionary procedure for frequency optimization has been also proposed by Xie and Steven (1994b, 1996). The distinction of this method from the previously described one is the achievement in evaluation of the effect of element removal on the chosen frequency. Having these effects, it is easy to identify the best elements to be removed in order to shift the specified frequency in a desired direction.

Following Xie and Steven (1994b, 1996), a structure is divided into a fine mesh of finite elements. The dynamic behaviour of the structure in the finite element method is represented by the following general eigenvalue problem

$$([K]-\omega_j^2[M])\{a^j\}=0 \qquad (2.35)$$

where $[K]$ is the global stiffness matrix, $[M]$ is the global mass matrix, ω_j^2 is the jth eigenvalue and $\{a^j\}$ is the eigenvector corresponding to ω_j^2. Note that different notations from those given in the original works are used to keep a consistency in notations throughout the thesis.

The eigenvalue ω_j^2 and the eigenvector $\{a^j\}$ are related to each other by the Rayleigh quotient

$$\omega_j^2 = \frac{k_j}{m_j} \tag{2.36}$$

where the modal stiffness k_j and the modal mass m_j are defined as

$$k_j = \{a^j\}^{\mathrm{T}}[K]\{a^j\} \tag{2.37}$$

$$m_i = \{a^j\}^{\mathrm{T}}[M]\{a^j\} \tag{2.38}$$

From equation (2.36) we can find the change of the eigenvalue as

$$\Delta(\omega_j^2) \approx \frac{\Delta k_j}{m_j} - \frac{k_j \Delta m_j}{m_j^2} = \frac{1}{m_j}(\Delta k_j - \omega_j^2 \Delta m_j) \tag{2.39}$$

In order to obtain the value of $\Delta(\omega_j^2)$ from the information of the previous eigenvalue analysis, we assume that the eigenvector $\{a^j\}$ is approximately the same before and after the removal of that element. The assumption that the mode shape does not change significantly during the design cycles has been commonly used in frequency optimization. With this assumption, the changes in the modal stiffness and the modal mass due to removal of the *i*th element from the structure can be approximated as

$$\Delta k_j \approx \{a^{ij}\}^{\mathrm{T}}[K^i]\{a^{ij}\} \tag{2.40}$$

$$\Delta m_j \approx \{a^{ij}\}^{\mathrm{T}}[M^i]\{a^{ij}\} \tag{2.41}$$

in which $[K^i]$ and $[M^i]$ are the stiffness and mass matrices of the *i*th element. The element eigenvector $\{a^{ij}\}$ contains the entries of $\{a^j\}$ which are related to the removed *i*th element.

Substituting equations (2.40) and (2.41) into (2.39) we have the approximation of the change of the eigenvalue due to the removal of an element, i.e.

$$\Delta(\omega_j^2) \approx \frac{1}{m_j}\{a^{ij}\}^{\mathrm{T}}(\omega_j^2[M^i]-[K^i])\{a^{ij}\} \tag{2.42}$$

To decide which elements should be removed from the structure so that the eigenvalue will increase or reduce, we calculate for each element the following *sensitivity number*

$$\varsigma_{ij} \approx \frac{1}{m_j}\{a^{ij}\}^{\mathrm{T}}(\omega_j^2[M^i]-[K^i])\{a^{ij}\} \tag{2.43}$$

which indicates the change in the eigenvalue ω_j^2 due to the removal of the *i*th element. When only one frequency is considered, the modal mass m_j in formula (2.43) can be omitted since it is the same for all elements.

It has been pointed out that the values of ζ_{ij} for all elements are ranging from a minimum to a maximum. The minimum is negative and the maximum is positive. Thus, removing the elements with highest (positive) values of ζ_{ij} will make a maximum increase in the frequency of the structure and removing the elements with lowest (negative) values of ζ_{ij} will make a largest reduction in the frequency. It is also possible to reduce the structural weight while making the least change in the frequency by removing elements with ζ_{ij} close to zero.

The optimal structure, however, cannot be achieved in one step. The optimization process has to be evolutionary. Only a small number of elements (i.e. small amount of material), should be eliminated from the structure at each iteration, for example 1% of elements (e.g. 1% of material). The evolutionary procedure for frequency optimization is summarized as follows:

Step 1: Discretize the structure using a fine mesh of finite elements.

Step 2: Solve the eigenvalue problem.

Step 3: Calculate the value of ζ_{ij} for each element.

Step 4: To increase the eigenvalue, remove a small number of elements whose value of ζ_{ij} are highest; To reduce the eigenvalue, remove a small number of elements whose value of ζ_{ij} are lowest; To keep frequency unchanged, remove elements with ζ_{ij} close to zero.

Step 5: Repeat Step 2 to Step 4 until an optimum is reached.

Examples given by Xie and Steven (1994b, 1996) show that the frequency of the structure is significantly increased or decreased by removing a number of elements with highest or lowest values of ζ_{ij}. An extension of this procedure has been given to increase the gap between any two frequencies by removing the elements with the largest difference $\zeta_{ik} - \zeta_{ij}$ $(k > j)$ in the concerning modes.

2.4 Discussion on existing ESO procedures and further developments

It has been shown that the evolutionary procedures proposed by Xie and Steven (1993, 1994a, 1994b, 1996) for shape and layout problems for stress and frequency optimization are simple and can easily implemented into any finite element analysis package. These methods are capable of reproducing many existing solutions obtained by other mathematically more complex methods. However, both the procedures have limitations due to lack of rigorous mathematical backgrounds to ensure the true minimum of the objective. In both procedures, there is no clear statement of the objective and constraints in the problem formulations.

In the original evolutionary procedure, the element stress is used as an indicator of the element performance. The rejection criterion is based on the ratio of the element stress and the maximum stress over the structure. The procedure can create lighter designs with more uniform stress distributions. However, neither an objective nor a constraint has been involved in the problem statement. It is uncertain that the procedure could give minimum weight topology or could produce the design satisfying certain requirement on stresses or weight. It is seen that the proposed procedure can be referred to as a modification of the well known fully stressed design method, where a hard-kill option is used to remove elements. In fact, the Rejection Criterion in the evolutionary procedure based on (2.32) can be re-stated in terms of a design variable x_i relating to each element as

$$
\begin{aligned}
x_i^{new} = x_i^{old} \times R_i \quad \text{where } R_i &= 0 \ \text{ if } \left(\frac{\sigma_i^{VM}}{\sigma_{\max}^{VM}}\right) \leq RR, \\
R_i &= 1 \ \text{ if } \left(\frac{\sigma_i^{VM}}{\sigma_{\max}^{VM}}\right) > RR
\end{aligned}
\qquad (2.44)
$$

When $x_i = 0$, the ith element is totally removed from the structure. In fully stressed design method, the size of an element is rationed by the formula (2.4) which is referred to as soft-kill option. Both the evolutionary procedure and the fully stressed design method are based on element stress ratios. Also no objective weight is involved in the fully stressed design approach and in the evolutionary procedure. It has been pointed out by many researchers that the fully stressed design can only give minimum weight for statically determinate structures under a single loading condition with equal allowable requirements on tension and compression. Therefore, the proposed evolutionary procedure may share the same conclusion.

Progress has been made in the evolutionary method for frequency optimization, where the effects of removal of elements on the change in the eigenvalue, called sensitivity numbers, are determined using results from solution of the eigenproblem. The method provides procedures to make a maximum increase or decrease in the frequency by removing elements with highest or lowest sensitivity numbers. These procedures offer a very effective way to shift the frequency to a desired direction as far as possible with minor modifications of the existing structure. In the way that these procedures have been presented, the objective has aimed to change the frequency as much as possible by small changes in the weight. Therefore, it can not be expected that these procedures could give a minimum weight design for the prescribed frequency or the frequency is extremized for the specified weight. These procedures should be accompanied by scaling the design to keep the frequency at the prescribed value and track the change in the weight, or to keep the weight at the specified value and see how the frequency varies. The method also offers a procedure to keep a minimum change in the frequency by removing a number of elements whose absolute values of sensitivity numbers are smallest. It seems that this procedure might be a rational way to minimize the weight of the structure for the prescribed frequency or to optimize (minimize or maximize) the frequency for the specified weight. Unfortunately, little attention has been paid to this problem by the proposed method. Further investigations in this direction are needed.

It has been seen that in the proposed evolutionary methods several parameters, such as the initial Rejection Ratio (RR_o), and Evolutionary Rate (ER) for stress consideration, and the number of elements to be removed in each iteration for frequency optimization, have to be specified before starting the optimization process. Although the proposed values for these parameters provide good results, there is a need to investigate the effect of these parameters on the final designs in term of structural weight and topology. Other factors such as the effect of mesh size and element type on the final designs also need further investigation. The most important is to improve these procedures to obtain the extremum of the objective for prescribed limits of the constraints.

As mentioned in the introduction, to satisfy serviceability limit state, displacements of the structure have to be limited to an acceptable magnitude. It is natural and desirable to develop an evolutionary procedure for optimization of structures with displacement constraints, by which a minimum weight can be achieved for the prescribed displacements. This procedure is important for the evolutionary structural optimization in relation to the other existing procedures proposed by Xie and Steven (1993, 1994a, 1994b, 1996). The success of this procedure will make a significant contribution to the recently proposed evolutionary structural optimization method, and an additional contribution to structural optimization in general. The candidate has initialised the study to develop an evolutionary structural optimization method for systems with displacement constraints. The results of the study are presented in details in the following chapters of this thesis.

Chapter 3

EVOLUTIONARY STRUCTURAL OPTIMIZATION METHOD FOR TOPOLOGY PROBLEMS - THEORETICAL BASIS

3.1 Introduction

With reference to the finite element method, topology optimization seeks the optimal pattern of connection by elements to the nodes. Although topology optimization is more complex than sizing optimization due to continual changing of both the finite element model and the set of design variables, it is well recognised that greater savings can be achieved in topology optimization. Due to the difficulties involved in topology optimization, various simplifications and approximations are often used. A common approach to the topology optimization is the ground structure method (or the design domain method). The optimal solution is then found as a subset of the initially chosen set of elements in the ground structure. Elements are successively removed during the solution process. Compared with other existing methods, the ESO method proposed by Xie and Steven (1993, 1994a, 1994b, 1996) for stress and frequency optimization offers a simple and effective tool for solving topology problems. The ESO method is still at an early stage of development. Recent achievements in the method have initialized a wide range of studies to apply to problems with other types of objectives and constraints, as well as to place more rigorous mathematical background to the method.

This chapter presents the theoretical basis of the ESO method for topology optimization of structures subject to displacement constraints. Some of these results have been recently reported by Chu *et al.* (1995, 1996a-b, 1997). The Lagrange multiplier approach is used to derive the optimality criterion for the problem. A more general type of sensitivity numbers for element removal is then formulated. An evolutionary procedure is proposed to drive the solution to the optimum where the optimality criterion is satisfied. It is seen that the proposed method can be easily applied to other constraints on generalized displacements, stiffness, stress and frequency. Therefore, the method given here can be referred to as *an optimality criteria based approach to the ESO method.*

3.2 Problem formulation

Consider a structure with a given boundary subjected to a loading condition. A number of constraints are imposed on displacement components at given points of the structure. Following the analysis phase, it is generally recognised that some parts of the structure are not effectively used and thus can be removed from the structure while satisfying the imposed constraints. By doing so, the resulting structure will have a lower weight and a modified shape. The main idea of the ESO method is to find an optimal shape and topology for the structure by gradually removing unnecessary material from the structure. The key point in this method is to work out an appropriate criterion to determine which parts of the structure are unnecessary, that is, which elements are unnecessary.

The shape and topology optimization problem with displacement constraints can be stated as follows:

By means of element removal,

minimize *the weight of the structure*

subject to *constraints on the specified displacements.*

By removing elements, new topologies are created during the solution process. It is clear that this problem is discrete in nature in the sense that elements are either totally removed or remain for the next design cycle. It belongs to the class of integer programming problems, or more precisely, zero-one programming problems. For each element an integer variable with only two values, zero and one, is needed, where zero corresponds to absence and one to presence of this element. The branch and bound method is often employed for this type of discrete problems. However, a simpler procedure can be used to solve the above problem. With the presence of all concerning elements, the current analysis provides the values of the objective and constraint functions. The effect of removal (absence) of each element on the corresponding objective and constraint needs to be assessed. This can be done either by using additional exact analyses or by employing some approximation procedures to reduce the number of costly exact analyses. By comparing these effects, the best element to be removed can be identified. Removal of such an element will result in a more efficient design in terms of a lighter weight without violating the given constraints.

The study of the effects of element removal on specified displacements is referred to as *the sensitivity analysis* and the indicator for effective element removal is called *the sensitivity number*. It will be shown that the effect of element removal can be derived from the information available following a finite element analysis and the sensitivity number is formulated from optimality conditions for the general weight minimization problem.

3.3 Sensitivity analysis

3.3.1 Effect of element removal on displacements

Removal of any element, in general, will change the displacements of the structure. The change in displacements can be found by using finite element analysis. In the finite element method the static behaviour of a structure is represented by the following general equation

$$[K]\{u\} = \{P\} \tag{3.1}$$

where $[K]$ is the global stiffness matrix, $\{u\}$ is the global nodal displacement vector and $\{P\}$ is the nodal load vector (Zienkiewicz 1971; Rockey *et al.* 1983; Cook *et al.* 1989).

Suppose that the structure is modelled with a mesh of n finite elements and the ith element is to be removed from the structure. The stiffness matrix of the structure will change by

$$[\Delta K] = [K^*] - [K] = -[K^i] \tag{3.2}$$

where $[K^*]$ is the stiffness matrix of the resulting structure after removal of the ith element and $[K^i]$ is the stiffness matrix of the ith element. For simplicity, assume that the removal of the element has no effect on the nodal load vector $\{P\}$, i.e. $\{\Delta P\} = 0$. As a consequence, the displacement vector will have a finite change $\{\Delta u\}$ which needs to be determined. The equilibrium condition for this new state of the structure is given by

$$[K + \Delta K]\{u + \Delta u\} = \{P\} \tag{3.3}$$

Subtracting equation (3.1) from (3.3) and ignoring the higher order term $[\Delta K]\{\Delta u\}$, we can find the change of the displacement vector as

$$\{\Delta u\} = -[K]^{-1}[\Delta K]\{u\} \tag{3.4}$$

To find the change in the jth displacement component u_j due to removal of an element, we introduce a virtual unit load vector $\{F^j\}$, in which only the corresponding jth component is equal to unity and all the others are equal to zero. By multiplying equation (3.4) by $\{F^j\}^{\mathrm{T}}$, the change in u_j is obtained as

$$\Delta u_j = \{F^j\}^{\mathrm{T}}\{\Delta u\} = -\{F^j\}^{\mathrm{T}}[K]^{-1}[\Delta K]\{u\} = -\{u^j\}^{\mathrm{T}}[\Delta K]\{u\} \tag{3.5}$$

where $\{u^j\}$ denoted the virtual nodal displacement vector, which is the solution of equation (3.1) for the virtual unit load vector $\{F^j\}$.

By using equation (3.2), formula (3.5) becomes

$$\Delta u_j = \{u^{ij}\}^{\mathrm{T}}[K^i]\{u^i\} \qquad (i = 1,\ n) \tag{3.6}$$

where $\{u^i\}$ and $\{u^{ij}\}$ are the element displacement vectors containing the entries of $\{u\}$ and $\{u^j\}$, respectively, which are related to the ith element. We denote for each element the value defined by (3.6) as

$$\alpha_{ij} = \{u^{ij}\}^{\mathrm{T}}[K^i]\{u^i\} = \Delta u_j \qquad (i = 1,\ n) \tag{3.7}$$

which indicates the change in the specified displacement u_j due to the removal of the ith element. The value α_{ij} in (3.7) is known as the element virtual strain energy. So, it can be stated that *the change in the specified displacement due to removal of an element is equal to the element virtual strain energy*. It should be noted that α_{ij} can be either positive or negative, which implies that u_j may change in opposite directions.

The most important feature of the formula (3.7) is that the change in any displacement due to removal of any element can be easily calculated using the displacements due to the real load and the corresponding virtual unit load. Finite element packages usually provide solutions for several load cases, so that one analysis can provide displacements for the real load and several virtual unit loads. Matrix multiplications for equation (3.7) are performed only at element level, so the time required for this calculation is small compared to the time required for a full analysis.

It is worth pointing out that the sum of α_{ij} for i from 1 to n, where n is the total number of elements in the structure, is equal to the displacement u_j. That is

$$\begin{aligned}\sum_{i=1}^{n}\alpha_{ij} &= \sum_{i=1}^{n}\{u^{ij}\}^{\mathrm{T}}[K^i]\{u^i\} = \sum_{i=1}^{n}\{u^j\}^{\mathrm{T}}[K^i]\{u\} \\ &= \{u^j\}^{\mathrm{T}}(\sum_{i=1}^{n}[K^i])\{u\} = \{u^j\}^{\mathrm{T}}[K]\{u\} = \{F^j\}^{\mathrm{T}}\{u\} = u_j \end{aligned} \tag{3.8}$$

Due to the relationship (3.8), the value α_{ij} is also considered as the element contribution to the specified displacement u_j. Therefore, it can be also stated that *the change in a specified displacement due to removal of an element is equal to the element contribution to that displacement.*

The values $|\alpha_{ij}|$ have been used as the sensitivity numbers for element removal for the problems with a constraint imposed on the absolute value of a displacement (Chu *et al.* 1995, 1996a). Removal of the element with the lowest value of $|\alpha_{ij}|$ will make a minimum change in the absolute value of the specified displacement. When all elements have equal weights (this happens, for instance, when all elements are identical as in the case of regular meshes), removal of the element with the lowest $|\alpha_{ij}|$ is always the best choice because the weight of structure is reduced by the same amount while the least change in the specified displacement is resulted. This means that we will have the design with the minimum displacement for the prescribed weight, or conversely, with the minimum weight for the prescribed displacement. However, when element weights differ from each other, which are typical in irregular meshes, the removal of an element, depending on its own weight w_i, will have different effects on the weight of the resulting design. Comparing two elements with the same $|\alpha_{ij}|$, it is obvious that removal of the element with a heavier weight results in a lighter design for the same displacement. In this case, removal of the element with a lower ratio $|\alpha_{ij}|/w_i$ is more efficient. Unfortunately, the proposed removal strategy based on the element contributions, corresponding to the values of $|\alpha_{ij}|$, would fail to distinguish these two elements. This infers that for the prescribed displacement, the weight of the resulting design may not necessarily be the minimum, suggesting the need for a more general type of sensitivity numbers for element removal.

It will be shown that the sensitivity numbers for element removal, which will guarantee a minimum weight topology in the general case, can be formulated from optimality conditions of the general weight minimization problems using the Lagrange multiplier approach.

3.3.2 Sensitivity number for element removal

3.3.2.1 Single displacement constraint

The objective of the problem is to minimize the weight of the structure

$$W = \sum_{i=1}^{n} w_i \tag{3.9}$$

subject to a single constraint imposed on the absolute value of a specified displacement u_j, given as

$$|u_j| - u_j^* \le 0 \tag{3.10}$$

where u_j^* is the prescribed limit for $|u_j|$ and w_i is the weight of the ith element. It should be noted that the formula (3.10) expresses, in fact, two separate constraints, namely $u_j \leq u_j^*$ for $u_j \geq 0$ and $-u_j \leq u_j^*$ for $u_j \leq 0$, and the discussion should be given for each case. However, the results of these two cases are combined together and are given below.

Using the Lagrange multiplier approach, the Lagrangian for the problem is given by

$$L = W - \lambda(|u_j| - u_j^*) \tag{3.11}$$

where λ is a Lagrange multiplier. Referring to w_i (i = 1, n) as design variables and taking into account (3.9), the optimality conditions for the problem, which are known as Kuhn-Tucker conditions, are

$$\frac{\partial L}{\partial w_i} = \frac{\partial W}{\partial w_i} - \lambda\left|\frac{\partial u_j}{\partial w_i}\right| = 1 - \lambda\left|\frac{\partial u_j}{\partial w_i}\right| = 0 \qquad (i = 1,\ n) \tag{3.12}$$

Further discussion usually needs an explicit expression of u_j in terms of design variables w_i. However, equation (3.12) can be approximated by

$$1 - \lambda\left|\frac{\Delta u_j}{\Delta w_i}\right| = 0 \qquad (i = 1,\ n) \tag{3.13}$$

Taking into account that when the ith element is removed, the change in the weight of the element is given by

$$\Delta w_i = -w_i \qquad (i = 1,\ n) \tag{3.14}$$

and the change in the specified displacement will be determined by (3.7). Therefore, the optimality conditions (3.13) become

$$1 - \lambda\frac{|\alpha_{ij}|}{w_i} = 0 \qquad (i = 1,\ n) \tag{3.15}$$

or at the optimum

$$\gamma_i = \frac{|\alpha_{ij}|}{w_i} = \frac{|\alpha_{ij}|}{\rho_i v_i} = \frac{e_{ij}}{\rho_i} = \frac{1}{\lambda} = constant \qquad (i = 1,\ n) \tag{3.16}$$

where ρ_i is the mass density, v_i is the volume, $w_i = \rho_i v_i$, and

$$e_{ij} = \frac{|\alpha_{ij}|}{v_i} \qquad (i = 1,\ n) \tag{3.17}$$

is the virtual strain energy density within the *i*th element.

Equation (3.16) represents the well known principle that "*at the optimum the ratio of virtual energy density to mass density is equal for all elements*" (Venkayya *et al.* 1973; Morris 1982). A minimum weight structure is achieved when γ_i is equal for all elements. In non-optimal design, elements have different γ_i, which reflect the different levels of effectiveness of material in each element. This means that γ_i can be viewed as an essential measure of efficiency of the material within the *i*th element. It is worth noting that, by using (3.7), an explicit expression u_j in terms of design variables w_i is not necessary in derivation of optimality criterion (3.16). Furthermore, the use of element weights w_i as intermediate design variables, one per each element, provides the separability of the problem which is an important feature for the derivation of the optimality criterion.

Optimality criterion (3.16) has been derived by other researchers using the Lagrange multiplier approach where the separability was employed and an explicit form of the constrained displacement in terms of design variables was proposed (Venkayya *et al.* 1973; Morris 1982; Berke and Khot 1988; Haftka *et al.* 1990). For sizing problems where the topology of the structure is fixed and design variables are continuous, various recurrence formulas, based on the optimality conditions, have been suggested to find the optimum where γ_i are equal for all elements and the weight of the structure is indirectly minimized. It appears that by using the similar recurrence formula, the elements with very small γ_i may have their volumes reduced to zero. However, these traditional techniques cannot be directly used for topology optimization due to the fact that non-zero lower bounds must be imposed on design variables. To overcome this limitation, the ESO technique is a choice, by which a direct element removal strategy is employed.

Investigations have shown that by removing elements with the lowest γ_i, a more uniform distribution of γ_i in the resulting structure can be achieved while making larger reduction in the weight and less change in the constrained displacement. An iterative process is used to remove a small number of elements with the lowest γ_i at each iteration, whilst the specified response is increasing slowly and approaching the prescribed limit. It is possible that the proposed technique can create an uniform state of γ_i for all elements, by which the optimal topology is achieved and the minimum weight is reached at the same time. It should be noted that by means of removing elements, where design variables are discrete in nature, it is not always possible to reach such a uniform state. However, a more uniform state of γ_i is always achieved, which means a more efficient design is obtained. A typical trend observed through examples of a single constraint is that removing elements with the lowest sensitivity numbers γ_i does result in increases in the sensitivity numbers of the remaining elements. This reflects the fact that the remaining elements will have a greater contribution after each iteration. However, the maximum sensitivity number over the resulting structure is increases slowly while the minimum sensitivity number increases rapidly and may approach the maximum if a uniform state becomes possible later.

Thus, *the sensitivity numbers for element removal for single displacement constraint problems* are defined as

$$\gamma_i = \gamma_{ij} = \frac{|\alpha_{ij}|}{w_i} = \frac{|\{u^{ij}\}^T[K^i]\{u^i\}|}{w_i} \qquad (i = 1,\ n) \tag{3.18}$$

This type of sensitivity numbers is simply defined by the ratio of the effect on the constraint to the effect on the weight due to removal of an element. It is obvious from the definition that the most effective way to minimize the weight of the structure is to remove the element which has the largest effect on the weight and the least effect on the constraint, i.e. the element with the lowest sensitivity number.

3.3.2.2 Multiple displacement constraints

Consider the problem of minimizing the weight of a structure subject to multiple displacement constraints given as

$$|u_j| - u_j^* \leq 0 \qquad (j = 1,\ m) \tag{3.19}$$

where m is number of constraints, u_j^* is the given limit for $|u_j|$. The Lagrangian for the problem is

$$L = W - \sum_{j=1}^{m} \lambda_j (|u_j| - u_j^*) \tag{3.20}$$

where λ_j is the Lagrange multiplier for the jth constraint, $j = 1,\ m$. The optimality conditions are

$$\frac{\partial L}{\partial w_i} = \frac{\partial W}{\partial w_i} - \sum_{j=1}^{m} \lambda_j \left| \frac{\partial u_j}{\partial w_i} \right| = 1 - \sum_{j=1}^{m} \lambda_j \left| \frac{\partial u_j}{\partial w_i} \right| = 0 \qquad (i = 1,\ n) \tag{3.21}$$

where $\lambda_j(|u_j| - u_j^*) = 0$. For the active constraints, $\lambda_j > 0$ and $|u_j| - u_j^* = 0$, and for the passive constraints $\lambda_j = 0$ and $|u_j| - u_j^* < 0$. The optimality conditions (3.21) can be approximated by

$$1-\sum_{j=1}^{m}\lambda_j\left|\frac{\Delta u_j}{\Delta w_i}\right|=0 \qquad (i=1,\ n) \tag{3.22}$$

Taking into account (3.7) and (3.14), similar to the single constraint problem, we have at the optimum

$$1-\sum_{j=1}^{m}\lambda_j\frac{|\alpha_{ij}|}{w_i}=0 \qquad (i=1,\ n) \tag{3.23}$$

or

$$\eta_i=\sum_{j=1}^{m}\lambda_j\frac{|\alpha_{ij}|}{w_i}=\sum_{j=1}^{m}\lambda_j\frac{e_{ij}}{\rho_i}=1 \qquad (i=1,\ n) \tag{3.24}$$

where e_{ij} is the element virtual strain energy density, defined by (3.17), and ρ_i is the element mass density.

Equation (3.24) represents the well known optimality criterion for multiple constraints that "*at the optimum the weighted sum of the ratio of virtual strain energy density to mass density is equal to unity for all elements, where the weighting parameters are the Lagrange multipliers*" (Venkayya *et al.* 1973; Morris 1982; Berke and Khot 1988; Haftka *et al.* 1990). The value η_i can be viewed as a measure of the efficiency of an element in multiple constraint problems so it is used as the sensitivity number for element removal. Removing the less efficient elements, i.e. elements with the lowest values of η_i, results in a more efficient design in the sense of a more uniform state of η_i being obtained.

Thus, *the sensitivity numbers for element removal for multiple displacement constraint problems* are defined as

$$\eta_i=\sum_{j=1}^{m}\lambda_j\frac{|\alpha_{ij}|}{w_i} \qquad (i=1,\ n) \tag{3.25}$$

where α_{ij} is determined by (3.7). Using (3.18), formula (3.25) can be rewritten as

$$\eta_i = \sum_{j=1}^{m} \lambda_j \gamma_{ij} \qquad (i = 1,\ n) \tag{3.26}$$

which implies that the sensitivity number of an element for multiple constraints is the weighted sum of its sensitivity numbers corresponding to each constraint.

3.3.2.3 Method of determination of the Lagrange multipliers

To calculate the sensitivity numbers for the case of multiple displacement constraints by (3.25) or (3.26), the values for Lagrange multipliers are needed. The method for evaluation of the Lagrange multipliers plays an important role in any optimization procedure using an optimality criterion approach. Various formulae with their advantages and disadvantages for determining Lagrange multipliers have been given in literature (Morris 1982; Berke and Khot 1988; Haftka *et al.* 1990).

Following the common approach, a simple way is to assume all the constraints in (3.19) are equality constraints, which gives

$$|u_j| = u_j^* \qquad (j = 1,\ m) \tag{3.27}$$

Multiplying both sides of equation (3.27) by $\lambda_j^{\,b}$ and taking the bth root, the recurrence formulae for the Lagrange multipliers can be written as

$$\lambda_j^{new} = \lambda_j^{old}\left(\frac{|u_j|}{u_j^*}\right)^{\frac{1}{b}} \qquad (j = 1,\ m) \tag{3.28}$$

where b is a step control parameter.

The advantage of the recurrence formulae (3.28) is that an equivalent single virtual load vector in the form of $\{F\} = \sum\lambda_j\{F^j\}$ can be used to determine the sensitivity numbers η_i of the elements without the necessity of determining individual values α_{ij}. This reduces the computation time substantially when the number of constraints increases. The use of the recurrence formula (3.28) requires that the initial values of the Lagrange multipliers have to be assumed. The value of a Lagrange multiplier corresponding to a less potentially active constraint is reduced, which scales down the contribution from the corresponding energy density of elements to the weighted sensitivity number defined by (3.25). It has been found that because the recurrence formula (3.28) successively reduces the Lagrange multiplier corresponding to a less potentially active constraint, it may not be able to pick up a change when the constraint becomes the most potentially active in a later iteration. The Lagrange multiplier corresponding to this constraint may still remain smaller than the others.

To avoid the drawback of the recurrence formula (3.28), we proposed a simpler formula for Lagrange multipliers given in the form

$$\lambda_j = \left(\frac{|u_j|}{u_j^*} \right)^{\frac{1}{b}} \qquad (j = 1, m) \tag{3.29}$$

where Lagrange multipliers are determined only by the ratios of the current values of the constrained displacements to their given limits. A special case of the formula (3.29) where $b = 1$ has been used by Chu *et al.* (1995, 1996a, 1997).

Similar to (3.28), formula (3.29) also reduces the contribution from a potentially less active constraint to the sensitivity numbers defined by (3.25). The advantage of the formula (3.29) is its simplicity and ability to pick up any change in the relative order of a constraint. The Lagrange multiplier corresponding to the most current potentially active constraint always has the largest value, so that the contribution from this constraint to the weighted sensitivity number is dominant. The formula (3.29) shares the same advantage with (3.28) that an equivalent single virtual load vector in the form as $\{F\} = \Sigma\lambda_j\{F^j\}$ can be used to reduce computation time. A comparison between (3.28) and (3.29) will be given through the examples in later chapters.

The Lagrange multipliers can also be determined by linear equations which are obtained by combining the optimality criterion (3.24) and the constraint equations (3.27), where u_j is determined by (3.8). Taking into account that $|u_j| = u_j$ signu_j, where signu_j denotes the sign of u_j (i.e. signu_j = 1 if $u_j \geq 0$ and signu_j = −1 if $u_j < 0$), the optimality conditions (3.24) can be rewritten using different index notation as

$$\sum_{j=1}^{m} \lambda_p \frac{\alpha_{ip} \operatorname{sign} u_p}{w_i} = 1 \qquad (i = 1,\ n) \qquad (3.30)$$

By substituting (3.8) into (3.27), the equality constraints now become

$$\sum_{i=1}^{n} \alpha_{ij} \operatorname{sign} u_j = u_j^* \qquad (j = 1,\ m) \qquad (3.31)$$

Multiplying the ith term in the left side of (3.31) by the ith equation in (3.30) and rearranging the terms, the constraint equations (3.31) will have the form

$$\sum_{p=1}^{m} \lambda_p \left(\sum_{i=1}^{n} \frac{(\alpha_{ij} \operatorname{sign} u_j)(\alpha_{ip} \operatorname{sign} u_p)}{w_i} \right) = u_j^* \qquad (j = 1,\ m) \qquad (3.32)$$

The equations (3.32) form a system of linear equations in terms of the Lagrange multipliers. It should be noted that although equation (3.32) is valid only at the optimum, it can be used to estimate the Lagrange multiplier for each constraint.

The incremental procedure based on (3.32) to update λ_j is given as follows

$$\sum_{p=1}^{m} \lambda_p \left(\sum_{i=1}^{n} \frac{(\alpha_{ij} \operatorname{sign} u_j)(\alpha_{ip} \operatorname{sign} u_p)}{w_i} \right) = (1 + r)|u_j| - r u_j^* \qquad (j = 1,\ m) \qquad (3.33)$$

where r is a step size control parameter.

The Lagrange multipliers determined by equations (3.32) or (3.33) take into account the interdependence of different constraints by including coupling terms. However, the computation effort to determine every α_{ij}, to assemble the coefficients for the Lagrange multipliers and to solve the set of simultaneous equations increases substantially with increasing number of constraints. It should be noted that equations (3.32) or (3.33) are solved only for the set of active constraints. It may give negative values for some of the Lagrange multipliers. This means the corresponding constraint is passive and should be excluded from the set of active constraints. Several revisions in the set of active constraints may be required to get positive values for Lagrange multipliers (Haftka *et al.* 1992). Some researchers temporarily set any negative Lagrange multiplier obtained from solution of these linear equations to zero (e.g. Chan 1994).

3.3.2.4 Constraint limit scaling versus design variable scaling

Equations (3.28), (3.29), (3.32) and (3.33) for the Lagrange multipliers are valid under the assumption that one or more constraints are active. The use of these equations to determine Lagrange multipliers requires a procedure to make one or more constraints active. A simple procedure widely used in sizing optimization problem, is scaling design variables by a scalar or parameter. The scaling parameter is defined for the most critical constraint. By scaling, a feasible design can be obtained easily after each iteration. This helps to keep track of the reduction in the weight of the structure after each iteration and also assists to identify the most active constraints. Scaling the design variables also keeps the most critical constraint at the limit (Morris 1982).

In the proposed ESO method for topology optimization, it is desirable to employ the element removal without involving member sizing, i.e. no changes in the size of the remaining elements are allowed. To this end, an alternative procedure to make constraints active is to scale all given displacement limits, so that the most potentially active constraint becomes active. In order to compare the active level of each constraint, all constraints should be normalised. The constraints given in (3.19) are converted into the following form

$$\frac{|u_j|}{u_j^*} \leq 1 \qquad (j = 1,\ m) \tag{3.34}$$

It is obvious from (3.34) that the activeness of a constraint is characterised by the ratio of the actual and limiting values of the corresponding constrained displacement, i.e. the value

$$\varphi_j = \frac{|u_j|}{u_j^*} \qquad (j = 1,\ m) \tag{3.35}$$

The closer the value of φ_j to unity, the more potentially active is the constraint. The scaling factor for the displacement limits is determined as

$$\varphi = \max_{j=1,\, m} \varphi_j = \max_{j=1,\, m} \left(\frac{|u_j|}{u_j^*} \right) \tag{3.36}$$

Thus, the limits u_j^* in equations (3.28), (3.29), (3.32) and (3.33) are scaled to φu_j^*. It is easy to check that the most potentially active constraint has the current value of the corresponding displacement as its new limit, hence becomes active. Other less potentially active constraints remain less active or passive. As displacements increase due to removal of elements, φ increases to unity and the scaled limits approach the given limits. By using the scaled limits, we try to minimize the weight of the structure while keeping displacements to levels governed by the most potentially active constraints.

Thus, by using the constraint limit scaling procedure, the Lagrange multipliers are determined from one of the following schemes, namely:

the recurrence formulae

$$\lambda_j^{new} = \lambda_j^{old}\left(\frac{|u_j|}{\varphi u_j^*}\right)^{\frac{1}{b}} \qquad (j = 1,\ m) \tag{3.37}$$

the ratio formulae

$$\lambda_j = \left(\frac{|u_j|}{\varphi u_j^*}\right)^{\frac{1}{b}} \qquad (j = 1,\ m) \tag{3.38}$$

or the linear equations

$$\sum_{p=1}^{m}\lambda_p\left(\sum_{i=1}^{n}\frac{|\alpha_{ij}||\alpha_{ip}|}{w_i}\right) = (1+r)|u_j| \ - \ r\varphi u_j^* \qquad (j = 1,\ m) \tag{3.39}$$

where the new form of the linear equations (3.32) is not given here since it is considered as a special case of (3.39) when $r = -1$. It is noted that the sign of α_{ij} also depends on the direction of the virtual unit load. Based on numerical experiences, to make the linear equations work equally for both positive and negative directions of the virtual unit loads, values (α_{ij} signu_j) and (α_{ip} signu_p) are replaced by $|\alpha_{ij}|$ and $|\alpha_{ip}|$.

It can be shown that these two procedures, the design variable scaling and the constraint limit scaling, are equivalent. Suppose all design variables w_i are scaled by a parameter a to the new value aw_i. When the stiffness matrix is a linear function of the design variable, as in the case of trusses or membrane structures, the stiffness matrices of the elements and of the structure become $a[K^i]$ and $a[K]$. From the equilibrium equation (3.1), the actual and virtual displacement vectors will change to $a^{-1}\{u\}$ and $a^{-1}\{u^j\}$. The design scaling parameter a is determined from the following condition $a^{-1}|u_j| = u_j^*$ or $a = |u_j|/u_j^*$ for the most critical constraint. The most critical constraint has the maximum value of the ratio $|u_j|/u_j^*$, therefore, taking into account equation (3.36) we have

$$a = \max_{j=1,m}\left(\frac{|u_j|}{u_j^*}\right) = \varphi \tag{3.40}$$

This means the scaling parameter for design variable is the same as the scaling factor for the limits. In other cases, when the stiffness matrix is a non-linear function of the design variable, a $\neq \varphi$, an iterative procedure should be employed in order to obtain an acceptable design scaling parameter a corresponding to the value of φ for the current design.

In general, due to scaling the design variables to aw_i, the element displacements will change to $\varphi^{-1}\{u^i\}$ and $\varphi^{-1}\{u^{ij}\}$. For linear static, the element stiffness matrix must be scalled to $\varphi[K^i]$. From (3.7), the element virtual energies have their new values given by $\varphi^{-1}\alpha_{ij}$. Therefore, by using the design variable scaling procedure, the Lagrange multipliers denoted by $(\lambda_j)'$ are determined from one of the following equations

$$(\lambda_j)'_{new} = (\lambda_j)'_{old}\left(\frac{\varphi^{-1}|u_j|}{u_j^*}\right)^{\frac{1}{b}} \qquad (j = 1,\ m) \tag{3.41}$$

$$(\lambda_j)' = \left(\frac{\varphi^{-1}|u_j|}{u_j^*}\right)^{\frac{1}{b}} \qquad (j = 1,\ m) \tag{3.42}$$

$$\sum_{p=1}^{m}(\lambda_p)'\frac{1}{a\varphi^2}\left(\sum_{i=1}^{n}\frac{|\alpha_{ij}||\alpha_{ip}|}{w_i}\right) = (1+r)\varphi^{-1}|u_j| - ru_j^* \qquad (j = 1,\ m) \tag{3.43}$$

It is obvious that the formulas (3.37) and (3.41), (3.38) and (3.42) give the same values for the Lagrange multipliers. Equations (3.39) and (3.43), show that the values for Lagrange multipliers in the constraint limit scaling procedure can be obtained from the values in the design variable scaling procedure by multiplying by a constant $(a\varphi)^{-1}$ or φ^{-2} if $a = \varphi$. As it will be shown in section 3.4, this does not affect the solution by the proposed ESO method, so either procedure can be employed for evaluation of the Lagrange multipliers.

The advantage of the limit scaling procedure is that no change in the size of the remaining elements is involved. It makes use of all results of the current analysis and is easy to implement, independent to whether the stiffness matrix is a linear function of the design variables or not. Because no sizing design variable is involved, the optimization process must be started from a feasible design where all displacements are smaller than the given limits. The optimization process will be terminated when the most potentially active constraint reaches the originally given limit. When the stiffness matrix is a linear function of the design variables, a simple scaling of the actual weight by the parameter $\varphi = a$ defined by formula (3.36) or (3.40) will give the objective (scaled) weight of the structure at each iteration for the initially prescribed displacement limits.

Atrek (1989) has used a slightly different procedure where the originally given limits for the constrained displacements are replaced by the current values of the displacements. However, this procedure will make all the constraints in the active set become equally active, which does not reflect the fact that some constraints are more critical than the others. In fact, it alters the constraint conditions for the problem.

3.3.2.5 Multiple displacement constraints and multiple load cases

Consider the case where a structure is subject to multiple displacement constraints under multiple load cases. The displacement constraints are given as

$$|u_{jk}| - u_j^* \le 0 \qquad (j = 1,\ m;\ k = 1,\ l) \tag{3.44}$$

where u_j^* is the given limit for the jth constrained displacement for all load cases, u_{jk} is the value of the jth constrained displacement under kth load case, m is number of constraints and l is number of load cases. The change in the jth constrained displacement under the kth load case due to removal of the ith element, similar to (3.7), is as follows

$$\Delta u_{jk} = \alpha_{ijk} = \{u^{ij}\}^{T}[K^{i}]\{u^{ik}\} \qquad (i = 1,\ n;\ j = 1,\ m;\ k = 1,\ l) \qquad (3.45)$$

where $\{u^{ik}\}$ denotes displacement vector of the ith element due to the kth load case. There are $m \times l$ constraints in (3.44), so $m \times l$ Lagrange multipliers λ_{jk} are needed to form the Lagrangian for the problem. By using again the single index j taking values from 1 to $m \times l$ to represent both j and k, the problem of multiple displacement constraints and multiple loading conditions has the exact form as the problem of multiple constraints.

3.4 Evolutionary procedures for topology optimization

Using the sensitivity numbers derived in the previous sections one can produce better topologies by gradually removing elements with the lowest sensitivity numbers. It should be noted that only a small number of elements should be removed from the structure at each step. This is because the sensitivity number is derived by neglecting the higher order term $[\Delta K]\{\Delta u\}$, i.e. it is assumed that both $[\Delta K]$ and $\{\Delta u\}$ are small. When a large number of elements are removed from the structure, $[\Delta K]$ and consequently $\{\Delta u\}$ will be large. Thus, their product can become significant and cannot be ignored. The optimum solution cannot be achieved in one step. An iterative process of analysis and element removal, where only a small number of elements are removed from the structure at each step, has to be adopted for the solution process to find the optimum.

Another important issue, which requires attention, is the problem of singularity of the stiffness matrix. Singularity causes interruption of the optimization process. The initial design can always be chosen to get the non-singular stiffness matrix. The problem often arises when a large number of elements have been removed from the structure. Removal of elements will change the connection of elements to the nodes. In certain configurations the remaining elements, which are connected to a node, cannot provide restraints on some degrees of freedom of the node. The node can move freely in those directions and the stiffness becomes singular. Therefore, to facilitate the continuity of the evolutionary optimization process, certain measures are needed to maintain the non-singularity of the stiffness matrix. These measures are closely related to each type of finite elements and will be discussed in the applications of the method to each type of structures in Chapter 4 and Chapter 5 of this thesis.

The main steps of the solution procedure of the proposed method are analysis, calculation of sensitivity numbers and element removal. However, depending on what scaling procedure is used to keep constraints active, two different evolutionary procedures can be used to obtain the solution. Different procedures require different ways of implementation.

The evolutionary procedure for topology optimization of a structure subject to displacement constraints using design variable scaling is as follows:

Step 1: Choose a ground structure (initial FEA model) for the structure.

Step 2: Analyse the structure for the given loads and virtual unit loads.

Step 3: Scale the design variables to make the most active constraints at the limits.

Step 4: Check reduction of the objective weight, if the objective weight increases in several successive iterations, go to step 9. Otherwise, go to step 5.

Step 5: Calculate the sensitivity numbers γ_{ij} for each element corresponding to each constraint. For multiple constraints, scale the responses, select the active set of constraints, calculate the Lagrange multipliers for the active set and the weighted sensitivity numbers η_i for each element.

Step 6: If the sensitivity numbers are uniform, go to step 9. Otherwise, go to step 7.

Step 7: Remove a number of elements which have the lowest sensitivity numbers.

Step 8: Maintain the non-singularity of the stiffness matrix and go back to step 2.

Step 9: Stop and select the best topology from the obtained topologies.

It should be noted that the weight of the structure obtained after scaling the design variables is referred to as *the objective weight* or *the scaled weight*. The use of design variable scaling, which keeps the critical constraints at the limits, allows one to keep track of the reduction of the objective weight. Due to the discrete nature resulting from element removal, smooth changes in the objective weight cannot be always expected. Smooth convergence of the objective weight to the minimum value cannot be guaranteed either. Therefore, the prescribed tolerance for the relative change in the objective cannot be used alone as the only termination criterion for the iterative process. It is observed in many cases that the objective weight reduces in the overall iterative process, it may increase at a particular iteration and then decreases in subsequent iterations. So the iterative process will be terminated when the objective weight continues to increase in a prescribed number of successive iterations. When unity is assigned to this number, the optimization process is interrupted at the step when the objective weight is first increased. It is possible to repeat this step to have the objective weight reduced by removing a smaller number of elements. When the uniform state in sensitivity numbers is reached, the necessary optimality conditions are satisfied and the optimization process is also terminated. When the uniform sensitivity numbers cannot be reached and no further reduction in the objective weight can be achieved, the optimum topology is the one with the lowest weight among the evolved topologies.

The evolutionary procedure for topology optimization of a structure subject to displacement constraints using constraint limit scaling is as follows:

Step 1: Choose a ground structure (initial FEA model) for the structure.

Step 2: Analyse the structure for the given loads and virtual unit loads.

Step 3: If any constraint is violated, go to step 8. Otherwise, go to step 4.

Step 4: Calculate the sensitivity numbers γ_{ij} for each element corresponding to each constraint. For multiple constraints, scale the constraint limits, select the active set of constraints, calculate the Lagrange multipliers for the active set and the weighted sensitivity numbers η_i for each element.

Step 5: If the sensitivity numbers are uniform, go to step 8. Otherwise, go to step 6.

Step 6: Remove a number of elements which have the lowest sensitivity numbers.

Step 7: Maintain the non-singularity of the stiffness matrix and go back to step 2

Step 8: Stop. Scaling actual weights to get the objective weights and select the best topology from the available topologies.

It is noted that, by using the constraint limit scaling, elements are simply removed from the structure and no changes in the sizes of the remaining element are involved. The weight of the remaining elements in each iteration is referred to as *the actual weight* of the structure The objective weight (or scaled weight) for the specified limits is not available during the iterative process. Therefore a different termination criterion for the iterative process is employed. Based on the fact that the actual value of constrained displacements usually increases very slowly due to removing elements with the lowest sensitivity numbers, the optimization process is allowed to continue until one of the constraints is violated. The uniform state of the sensitivity numbers is also used to terminate the iterative process. Although the obtained topologies by the

ESO method are much more efficient compared to the initial topology, they cannot be equally efficient. Step 8 is necessary to identify the optimum topology. The actual weight and the value of the most potentially active displacement are provided for each ESO topology obtained at each iteration. After completion of the iterative process, the objective weight for each topology is determined by scaling the design to make the most potentially active displacement at its originally prescribed limit. The optimum topology is the one with the minimum objective weight.

The principal difference between these two evolutionary procedures is whether the design scaling is included inside or outside of the loop. In the former procedure, inclusion of design scaling inside the loop requires more efforts for implementation when the stiffness matrix is non-linear function of the design variables, for which an inner loop is needed to find the design scaling scalar. In the latter, the constraint limit scaling is very simple to implement and works for any kind of stiffness matrices. Furthermore, no change in the sizes of elements is involved. So, the evolutionary procedure with constraint limit scaling is chosen for implementation to carry out the optimization process. Fig. 3.1 presents a flow chart of this procedure for topology optimization.

It is seen that in the proposed method, the relative order of the sensitivity number of an element is more important than its actual value. Scaling sensitivity numbers by a constant does not affect the solution. When all elements are identical and made from the same material, i.e. $w_i = \rho_i v_i = constant$, the element virtual energies (contributions) $|\alpha_{ij}|$ can be used as sensitivity numbers (Chu *et al.* 1995, 1996a). When elements are different but made from the same material, i.e. $\rho_i = constant$, the element energy densities $e_{ij} = |\alpha_{ij}|/v_i$ can also be used as sensitivity numbers for element removal. For the same reason, scaling all Lagrange multipliers by a constant

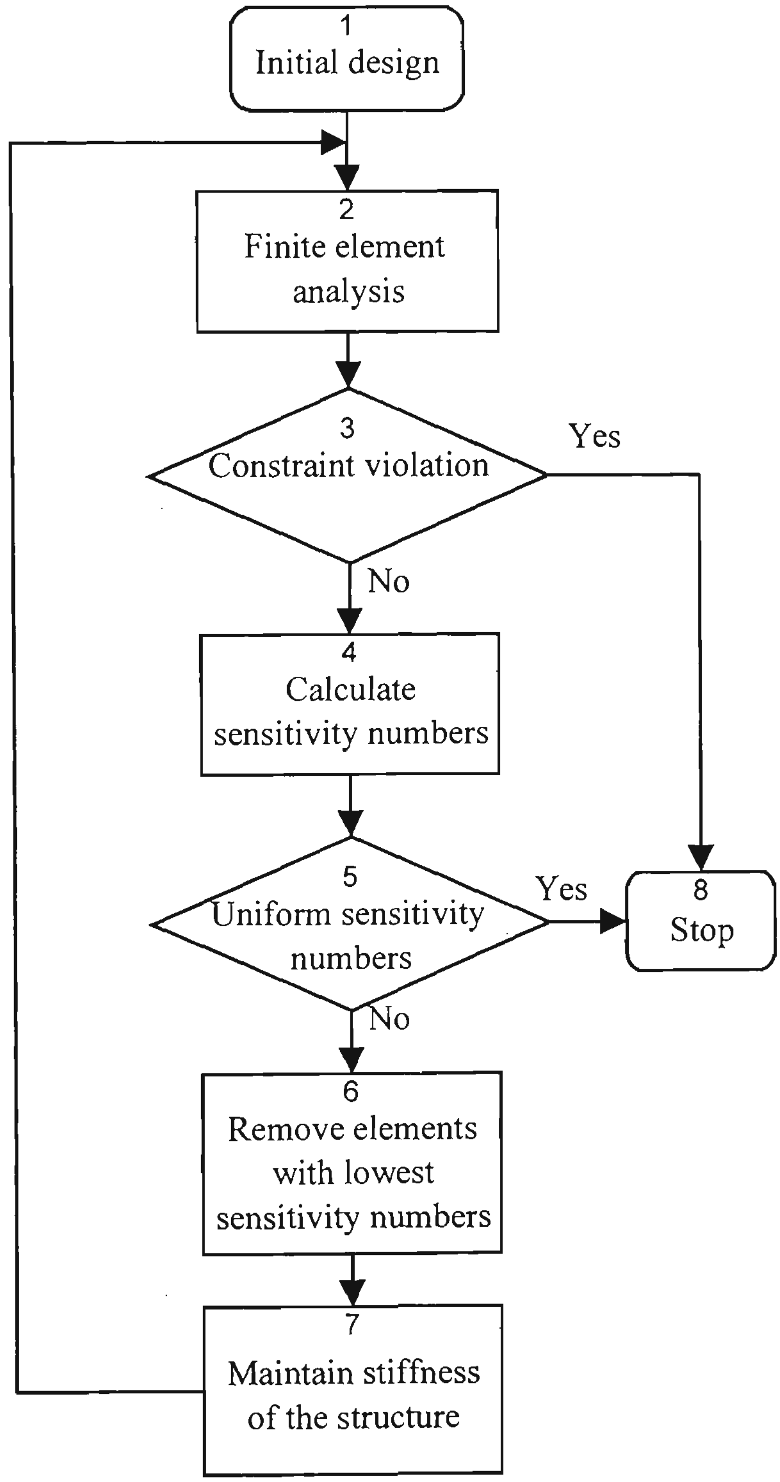


Fig. 3.1 Flow chart of the ESO procedure for topology optimization with constraint limit scaling.

does not affect the solution. One can use either the limit scaling procedure, equations (3.37)-(3.39), or the design variable scaling procedure, equations (3.41)-(3.43), to determine the Lagrange multipliers. Furthermore, even equations (3.28), (3.29) and

(3.32) can be used without any scaling procedure to obtain the Lagrange multipliers for calculation of sensitivity numbers.

It is noted that in the simplest case when a loading condition consists of only one point load Q acting on a structure and the objective is to minimize the weight of structure subject to a displacement constraint at the same location and in the direction of the load, the virtual displacement due to the virtual unit load can be obtained from the displacement due to the real load as $\{u^j\} = (1/Q)\{u\}$. So, there is no need to analyse the structure for the virtual unit load corresponding to that constraint.

The most important parameter in the proposed procedure is the number of elements to be removed at each iteration. This number controls the magnitude of the change in each design step. The number of elements to be removed at each iteration can be prescribed by one of the following options:

(1) by an integer number; or
(2) by the ratio of the number of elements to be removed to the total number of elements, called the *element removal ratio* (ERR); or
(3) by the ratio of the weight (material) to be removed in each iteration to the total weight (material), called the *material removal ratio* (MRR).

Option (2) gives the number of elements to be removed at each iteration equal to ERR times the total number of elements, rounded off to the nearest integer. This number is constant when the total number is the initial number of elements. In this case option (2) is equivalent to option (1). This number is decreasing when the total number is the current number of elements as the current number of elements is reducing. Options (2) and (3) are equivalent when all elements have equal weights. All three options are equivalent and give constant number only when all elements have equal weights, the total number is the initial number of elements and the total weight is the initial weight.

When the total weight is considered as the current weight and all elements have equal weights, the number of elements to be removed also decreases gradually. In general, when elements are different and the material removal ratio (MRR) is prescribed, the number of elements to be removed varies from iteration to iteration.

To get an accurate solution, the number of elements to be removed, the value of ERR or the value of MRR should be sufficiently small to ensure a smooth change in stiffness of the structure and consequently a smooth change in displacements. For ERR, the values from 1% to 4% for two-dimensional continuum structures (Chu *et al.* 1995, 1996a, 1997) and up to 10% for truss structures (Chu *et al.* 1996b) have been adopted. Two removal strategies can be used. When the number of elements to be removed is determined by ERR with respect to the total number of elements in the current design, it decreases as the number of elements in the resulting design reduces. The minimum number of elements to be removed is set to be one. Care must be taken when the number of elements to be removed is constant (i.e. it is determined by ERR with respect to the total number of elements in the initial design). The solution process is terminated whenever the number of remaining elements of the resulting design becomes equal to or less than the number of elements to be removed. In that case, the optimization process may be continued by using a smaller ERR. Discussions on practical aspects of application of the proposed method to different types of structures will be given in the next chapters. Furthermore, the influence of the element removal ratio on the solutions will be studied.

3.5 Generalization to other constraints

The proposed evolutionary method for topology optimization of structures subject to displacement constraint can be easily generalized to other constraints imposed on the generalized displacements, stiffness, stress and frequency. The sensitivity numbers for these constraints are derived in exactly the same manner as for displacement constraints. A brief discussion is provided below.

3.5.1 Generalized displacement constraints

In some cases, it is required to limit the value of a function relating to several displacement components. This function is called the generalized displacement. Generalized displacements in the form of linear combination of several displacement components are often considered in practice. These can be given in the following common form

$$G_q = \{z^q\}^{\mathrm{T}}\{u\} \qquad (q = 1,\ s) \tag{3.46}$$

where $\{z^q\}$ is the given constant vector which contains the coefficients for all displacement components, s is the number of generalised displacements. The vector $\{z^q\}$ in (3.46) plays a very similar role to the virtual load vector $\{F^j\}$ in (3.5). Therefore, the vector $\{z^q\}$ is usually referred to as a virtual or dummy load. When the dummy load is a virtual unit load $\{F^j\}$, the generalised displacement becomes a single displacement, by which obviously $u_j = \{F^j\}^{\mathrm{T}}\{u\}$.

A consideration of the generalised displacement is needed when the relative displacement between a pair of points is taken into account in the design process. The inter-storey drift is a typical example of the relative displacement. In the design of high-rise buildings, in addition to the limit on the deflection on the top floor, the inter-storey drifts should also be limited to an acceptable level. The inter-storey drift between two floors can be expressed by the formula (3.46), where the vector $\{z^q\}$ has non-zero components corresponding to the horizontal displacements of these two floors, with other components being equal to zero. In this case, it is said that the dummy load consists of a pair of opposite horizontal unit forces acting on two floors.

Another example of the generalized displacements is the stress in a truss member, which is determined from the relative longitudinal displacements of its nodes. The dummy load is composed of a pair of equal and opposite forces acting in the longitudinal direction on the two ends of the member.

It is easy to show that the change in a generalised displacement due to removal of an element is determined, similarly to the case of a 'pure' displacement, as

$$g_{iq} = \Delta G_q = \{z^q\}^{\mathrm{T}}\{\Delta u\} = \{u^{iq}\}^{\mathrm{T}}[K^i]\{u^i\} \qquad (q = 1,\ s) \tag{3.47}$$

where g_{iq} denotes the change in the jth generalised displacement due to removal of the ith element, $\{u^{iq}\}$ is the element displacement vector obtained from the displacement vector $\{u^q\}$ which is the solution of the static problem (3.1) for the dummy load $\{z^q\}$, and s is the number of generalized displacements. Furthermore, it is also easy to show that

$$G_q = \sum_{i=1}^{n} g_{iq} \qquad (q = 1,\ s) \tag{3.48}$$

The constraints on the generalised displacements are given in the form as

$$|G_q| \leq G_q^* \qquad (q = 1,\ s) \tag{3.49}$$

where G_q^* is the given limit for absolute value of the generalized displacement G_q.

By following the same procedure, the sensitivity numbers for element removal for a single generalised displacement constraint will be determined as

$$\gamma_i = \gamma_{iq} = \frac{|g_{iq}|}{w_i} = \frac{|\{u^{iq}\}^{\mathrm{T}}[K^i]\{u^i\}|}{w_i} \qquad (i = 1,\ n) \tag{3.50}$$

and for multiple generalised displacement constraints as

$$\eta_i = \sum_{q=1}^{s} \lambda_q \gamma_{iq} = \sum_{q=1}^{s} \lambda_q \frac{|g_{iq}|}{w_i} \qquad (i = 1,\ n) \tag{3.51}$$

The same method is used to determine the Lagrange multipliers. Removal of elements with lowest values of this type of sensitivity numbers will result in the minimum weight topology for the prescribed limits on the generalised displacements.

3.5.2 Stiffness constraints

In many cases, structures are required to be stiff enough to carry the given loads. The strain energy is commonly considered as an inverse measure for the overall stiffness of a structure. The lesser the strain energy stored in the structure, the stiffer the structure is said to be. Maximizing the overall stiffness of a structure is equivalent to minimizing its strain energy.

The strain energy of a structure under the *k*th load case is defined as

$$S_k = \tfrac{1}{2}\{P^k\}^{\mathrm{T}}\{u^k\} \tag{3.52}$$

where S_k denotes the strain energy and $\{u^k\}$ is the displacement vector corresponding to the *k*th load case $\{P^k\}$. Comparing (3.52) to (3.46), it is noted that the strain energy, hence the stiffness, can be viewed as a special case of the generalised displacement, when the real load is used instead of the dummy load. Imposing a lower limit on the overall stiffness is done by setting an equivalent upper limit on the strain energy. The stiffness constraints are usually given in the form

$$S_k \le S_k^* \qquad (k = 1,\ l) \tag{3.53}$$

where S_k^* is the given limit for S_k, l is the total number of load cases.

Removal of any element will, in general, reduce the stiffness of the structure, or inversely increase its strain energy. Similar to the case of displacement constraints, the change in the strain energy due to removal of the ith element can be found (Chu *et al.* 1996a) as

$$s_{ik} = \Delta S_k = \tfrac{1}{2}\{u^{ik}\}^{T}[K^{i}]\{u^{ik}\} \qquad (i = 1,\ n;\ k = 1,\ l) \tag{3.54}$$

The value s_{ik} is referred to as the element strain energy for the kth load case. It is worth pointing out that by taking the sum of s_{ik} in (3.54) over all elements, we have

$$S_k = \sum_{i=1}^{n} s_{ik} \qquad (k = 1,\ l) \tag{3.55}$$

where s_{ik} is also referred to as element contribution to the total strain energy of the structure.

Using a similar approach to the case of displacement constraints, by which s_{ik} in (3.54) is considered instead of α_{ij} in (3.7), the sensitivity numbers for element removal are defined for a single stiffness constraint as

$$\gamma_i = \gamma_{ik} = \frac{s_{ik}}{w_i} = \frac{\frac{1}{2}\{u^{ik}\}^{T}[K^{i}]\{u^{ik}\}}{w_i} \qquad (i = 1,\ n) \tag{3.56}$$

and for multiple stiffness constraints as

$$\eta_i = \sum_{k=1}^{l} \lambda_k \gamma_{ik} = \sum_{k=1}^{l} \lambda_k \frac{s_{ik}}{w_i} \qquad (i = 1,\ n) \tag{3.57}$$

The same procedure is also used to calculate the Lagrange multipliers. By using this type of sensitivity numbers, a minimum weight topology can be obtained for the prescribed limits on stiffness by removing elements with the lowest sensitivity numbers. For multiple load cases, a minimum weight topology with the strain energy less than a prescribed value is obtained.

It is noted that for the simplest case when the loading consists of only one point load Q acting on a structure and the objective is to minimize the weight of structure subject to a single displacement constraint at the same location and in the direction of the load, we have $\{u^j\}= (1/Q)\{u\}$. The sensitivity numbers for elements for the problem with the displacement constraint are exactly in the same order as for the problem with the stiffness constraint (scaled by a factor $2/Q$). This means that solutions for displacement and stiffness constraints will be identical.

It has been shown that constraints imposed on displacements, generalised displacement and stiffness are treated in exactly the same manner. They are commonly referred to as displacement constraints. The general optimization problem with multiple displacement constraints and multiple load cases can include constraints on some displacement components, on some generalised displacements and on stiffness for different load cases.

3.5.3 Stress constraints

In the original ESO method for topology optimization proposed by Xie and Steven (1993), the element stress has been used as a measure for element removal. It has been shown that removing the less stressed elements will result in a more efficient design in the sense that more uniform stress distribution over the resulting design is achieved. However, due to the fact that no objective and no constraint functions have been employed and no optimality criterion has been used, the proposed procedure may not lead to a minimum weight design.

Stress constraint can be treated directly by specifying the constraint as a function of the stress in an element or indirectly by converting it into a generalized displacement constraint. To employ the method given in the previous section for stress constraints, evaluation of the change in the stresses in an element due to removal of another element is needed. This can be done easily by using the indirect approach. It has been well known that in the finite element method the stress in an element can be expressed in terms of a linear combination of the element nodal displacements. Therefore, a stress constraint can be viewed as a generalised displacement constraint. Thus, the same procedure as given in section 3.5.1 can be used to calculate the sensitivity numbers for element removal, provided that further specialisation for $\{z^q\}$ is needed for each stress constraint.

According to this procedure, each stress constraint requires the analysis result for the corresponding dummy load. For a bar structure the number of stress constraints, hence the number of dummy loads, will be equal to the number of elements. For other type of structures the number of dummy loads can be greater than the number of elements. For large scale structures the time required for analysis for additional dummy loads may become prohibitive. An alternative way of tackling this problem is to consider only critical constraints. In some cases, only a constraint on the maximum stress needs to be considered. Removal of elements with the lowest sensitivity numbers will result in a large weight saving and a small increase in the maximum stress. It is worth pointing out that, unlike the case of displacements where for a certain type of structures the location of the maximum displacement is known before analysis, the element with maximum stress often cannot be identified before analysis. Additional analysis for a dummy load corresponding to the maximum stress is needed. This dummy load may also be different at different steps due to the fact that the maximum stress may shift from element to element over the structure. This makes the solution process for the constraint on the maximum stress more complicated.

3.5.4 Frequency constraints

The ESO method for frequency optimization has been presented by Xie and Steven (1994b, 1996). According to these authors, the change in the eigenvalue in the jth mode due to removal of the ith element can be approximated by

$$\zeta_{ij} = \Delta(\omega_j^2) = \frac{1}{m_j}\{a^{ij}\}^{\mathrm{T}}(\omega_j^2[M^i]-[K^i])\{a^{ij}\} \qquad (i = 1,\ n) \qquad (3.58)$$

where m_j is the modal mass, $[K^i]$ and $[M^i]$ are stiffness and mass matrices of the ith element, the element eigenvector $\{a^{ij}\}$contains the related entries of mode shape $\{a^j\}$ [see also formula (2.43) in section 2.4.2].

Following the newly proposed method as for the displacement constraints, to get minimum weight topology for the prescribed frequency, the sensitivity number for element removal should be determined for a single mode as

$$\gamma_i = \gamma_{ij} = \frac{\zeta_{ij}}{w_i} = \frac{\{a^{ij}\}^{\mathrm{T}}(\omega_j^2[M^i]-[K^i])\{a^{ij}\}}{m_j w_i} \qquad (i = 1,\ n) \qquad (3.59)$$

and for multiple modes as

$$\eta_i = \sum_{j=1}^{m}\lambda_j\gamma_{ij} = \sum_{j=1}^{m}\lambda_j\frac{\zeta_{ij}}{w_i} \qquad (i = 1,\ n) \qquad (3.60)$$

where m is number of modes.

It should be noted the only formulas similar to (3.37) and (3.38) can be used for Lagrange multipliers. Similar linear equations cannot be derived for this case due to the fact that $\omega_j^2 \neq \sum\zeta_{ij}$ over all elements. To obtain the minimum weight design for the prescribed frequency, the element with lowest sensitivity number defined by (3.59) or (3.60) should be removed, as it will result in the largest weight reduction and a least change in the frequency. Only in the case where all elements have equal weights, the weights of elements in (3.59) and (3.60) may be omitted and the removal strategy based on the value ζ_{ij} can also give a minimum weight topology.

3.6 Concluding remarks

It is shown in this chapter that the optimality criteria method has been used to formulate the sensitivity numbers for element removal. An iterative process of analysis followed by element removal is proposed to drive the solution to the optimum. Unlike the traditional approaches where the explicit forms of the objective and constraint functions of the design variables are needed, the use of the virtual unit load method for assessment of the effect of element removal on the change in the constraints in the proposed method avoids the necessity of explicit forms for objective and constraint functions. Furthermore, the use of element weights as the intermediate design variables provides separability of the problem which allows derivation of the same optimality criteria as obtained by other researchers using traditional approaches. It is noted that the proposed method for displacement constraints can also be applied to other constraints such as constraints on generalized displacements, stiffness, stress and frequency. So the method given here can be referred to as *an optimality criteria approach to the ESO method.*

The next two chapters will deal with practical aspects of the application of the proposed method to topology optimization of skeletal and two-dimensional continuum structures. The performance of the method will be investigated through considering examples. Most examples are selected from existing literature to show the validity and efficiency of the method. Parametric studies to investigate the influence of factors including the element removal ratio and the ground structure on the solution will be carried out in Chapter 6.

Chapter 4

TOPOLOGY OPTIMIZATION OF SKELETAL STRUCTURES - APPLICATION OF THE ESO METHOD

4.1 Introduction

The problem of optimal truss topology has drawn great attention from many researchers. The earliest work dealing with this problem was done by Michell in the beginning of this century (Michell 1904; Hemp 1973). However, Michell's theory based on variation calculus only gave solutions for structures subjected to simple loads and boundary conditions. Later, most researches have been devoted to developments of the solution methods based on the finite element analysis. It has been recognized that topology optimization can provide significant savings compared with sizing optimization. As reviewed in Chapter 2, topology optimization faces the significant complexities due to the continuing change in the finite element model, the changing set of design variables, and the limitations of the displacement based approach of the finite element analysis. To overcome these difficulties, various simplifications and approximation methods have been proposed. Some are more general such as the SAND approach, others are more specialized such as the force methods or compliance minimization. These methods include the displacements or member forces as additional design variables and treat equilibrium equations as equality constraints. Therefore, they are not easy to implement into finite element analysis packages because most of these packages are developed using the displacement methods. It can be seen that the evolutionary method given in Chapter 3 is simple and straightforward. By extracting information from a finite element analysis, the sensitivity numbers for element removal can be calculated. Once the sensitivity numbers are available, by adopting a suitable criterion, a number of elements can be removed to create more efficient topologies for the structure.

This chapter presents the application of the ESO method to topology optimization of skeletal structures with particular emphasis on truss topology optimization. The problem of singularity of the stiffness matrix is discussed. Measures for preventing the stiffness matrix from being singular are introduced to facilitate the continuity of the solution process. A wide range of examples are included to show the capability of the method. Some of the results have been reported in Chu *et al.* (1996b).

4.2 Singularity of the stiffness matrix and measures for prevention

Theoretically, the ESO method given in Chapter 3 may be applied to various types of structures. Practical application of the method raises an issue relating to the singularity of the stiffness matrix. As more elements are removed, the stiffness matrix is more likely to become singular. In the finite element method, a structure is modelled by a set of finite elements connecting a chosen set of nodes. Removal of elements may cause the singularity of the stiffness matrix in the following cases:

(1) Whole structure may have experience rigid body movements;

(2) Element or part of the structure may have experience rigid body movements;

(3) Node can freely move in a certain direction or freely rotate about a certain axis;

(4) Node becomes isolated (i.e. it does not connect to any element).

The initial design must have sufficient supports. During the optimization process, the resulting structure as a whole may become a "global" mechanism, loosing load carrying capability. In this instance no further solution can be obtained. Most finite element analysis programs can detect the singularity of the stiffness matrix upon which the solution process is automatically terminated. During the iterative process, a node may become isolated when all elements connected to it in the initial configuration have been removed. Fortunately, most finite element analysis programs have the ability of ignoring isolated nodes, in which case there is no concern. Otherwise, an additional subroutine is required to check and remove the isolated nodes from the structural data of the model before proceeding to the next design step.

The cases requiring the greatest attention are cases (2) and (3), when the structure still maintains the overall load carrying capability. A close look reveals that when an element or a part of the structure experiences rigid body movements, some of nodes can move freely, causing singularity of the stiffness matrix due to insufficient number of restraints provided by the remaining elements at the subject node. This causes interruption of the optimization process. By consideration of the performance of the finite element model of the structure, singularity of the stiffness matrix needs to be avoided to maintain the continuity of the solution process. This can be done by modifying the finite element model of the structure.

Trusses and frames are referred to as skeletal structures which are modelled using bar and beam elements connecting to a grid of joints (nodes), respectively. Beam elements used to model frames have all six stiffness components corresponding to six degrees of freedom of a joint. In three-dimensional frames where all six stiffness components of a beam element are non-zero, the beam element will put restraints on all six degrees of freedom of a node when it is connected to the node. The joint (node) is sufficiently restrained when it is connected to this beam element. Singularity is not a problem in the application of the ESO method to three-dimensional frame structures where all six stiffness components of all beam elements are non-zero. In two-dimensional frame structures, although non-zero values are provided only for three stiffness components of beam elements, singularity of the stiffness does not rise due to additional restraints on other three degrees of freedom of joints corresponding to zero components.

Singularity of the stiffness matrix becomes a serious issue in the application to truss structures due to the fact that a bar element only possesses a single axial stiffness along its longitudinal axis. Consequently a bar element only offers a single restraint to the joint it connects to, increasing the possibilities of singularity. Joints used in modelling truss structures have three translational degrees of freedom. Removal of elements often results in the case where the remaining elements cannot offer sufficient restraints on these degrees of freedom of a joint. For a particular configuration of the bars connecting to the joint, the joint can move freely to a certain direction. For example, if there is only one bar connected to a joint, this joint is obviously not sufficiently restrained because it can move freely in any direction normal to the bar. If this joint is free of load or not externally restrained, the bar will have zero stress by considering equilibrium. The element with zero stress obviously does not contribute to the structure, hence can be removed without affecting the load carrying capacity of the structure. Once this element is removed, the corresponding joint becomes isolated and is automatically removed. Another type of insufficiently restrained joint occurs when the joint is connected by two bar elements which are aligned along a straight line. The joint between these bars has no stiffness in the direction normal to that line. Such a joint is called an inner nodal point and is ignored by simply replacing these two bar elements by a single bar element. In certain cases, bars connecting to some insufficiently restrained joints can form a "local" mechanism. Depending on the configuration and the number of bars connecting to each joint, the equilibrium conditions for the joint will reveal the members with zero stresses. Removing these zero stressed elements will ignore the insufficiently restrained joints. In other cases where the equilibrium conditions cannot help to ignore the insufficiently restrained joint, additional restraints should be imposed on the corresponding degrees of freedom of this joint.

The following cases with alternative modifications to the finite element model of truss structures to prevent the stiffness matrix from being singular are taken into account:

a) One bar connects to a free joint: The bar has zero stress and is removed.
b) Two bars extend from a free joint and form a straight line: The joint becomes an inner nodal point and is ignored by replacing these two bars with a single bar.
c) Two bars extend from a free joint and do not form a straight line: These two bars may be a part of a local mechanism. By considering equilibrium conditions for this free joint, these two bars will have zero stresses and they are removed.
d) Three bars connect to a free joint and two of which are on a straight line: By considering equilibrium conditions for this joint, the third bar has zero stress and is removed. The joint between the two remaining bars becomes an inner nodal point and also is ignored.
e) Three bars connect to a free joint and form a straight line: This may happen when bars connecting a node to all the others nodes in the ground structure. Similar to case b, this inner nodal is ignore by simply removing one bar and extend the remaining bars to the other end of the removed bar.
f) Three bars connect to a free joint and do not lie on one plane: From the equilibrium conditions for the free joint, these bars will have zero stresses and they are removed.
g) Three bars connect to a free joint and lie on a plane: The joint will have no stiffness in the direction normal to the plane. These bars may also have stresses and cannot be removed. Hence additional restraint on this degree of freedom needs to be put on the joint to avoid singularity of the stiffness matrix.
h) Four or more bars connect to a free joint and form a straight line: Similar to case e.

Therefore, features for removing zero stressed members and ignoring inner nodal points are included in Step 7 of the evolutionary procedure using constraint limit scaling (see section 3.4) to maintain stiffness of the resulting structure before starting analysis for the next iteration. Thus, the number of removed elements at some iterations may be greater than that determined by the specified value of the element removal ratio. It is noted that ignoring inner nodal points reduces the number of elements. In addition, if two connecting bars have the same cross sectional areas and are made of the same material, ignoring the inner nodal point does not reduce the weight of the current design. Furthermore, removing zero stressed members does not alter the load carrying capacity of the resulting designs.

4.3 Computer implementation for topology optimization of skeletal structures

As discussed above, for the ease and simplicity in the implementation of the proposed method, the existing finite element software should have the following two features:

1) can ignore any isolated node;
2) can remove any element by simply "switch off" the element without altering the format of the data for the whole structure. The element that has been switched off will not be assembled in future finite element analyses.

If the above features are not available, additional efforts are needed to develop appropriate subroutines to incorporate these features in order for the method to be implemented.

The FEA package STRAND6 (G+D Computing, 1991) has these abilities built-in without altering the format of the data for the model. The isolated nodes and the elements, which are designated to be removed, are logically ignored by not being assembled into the following designs, but physically they still remain in data files for the structure. This helps to keep track of the changes in the design.

A computer program named ESOTOPO has been developed by the candidate for topology optimization of skeletal structures (trusses and frame) subject to multiple displacement constraints and multiple loading conditions. This is linked to the finite element analysis software STRAND6 to carry out the evolutionary optimization process. The program ESOTOPO consists of four parts. The first part is designed to check the constrained displacements. If one of the constrained displacements reaches its given limit, the optimization process is terminated. The second part is used to calculate the element sensitivity numbers. If an uniform state of sensitivity numbers is taken place, the optimization process is terminated as the optimum solution is reached. The third part is assigned the task of removing those elements, which have the lowest sensitivity numbers. The final part is checks and removes zero stressed elements, and to ignore inner nodal points to maintain stiffness of the structure before starting the next iteration. For frame structures, the final part is not needed. The program ESOTOPO is a post-processor to the finite element analysis. A batch file is set up to create a loop for the iterative process of analysis and element removal.

For the calculation of sensitivity numbers, displacements and element stiffness matrices are needed. The displacements due to real and virtual unit loads are available following the finite element analysis. The element stiffness matrices are also available in the finite element analysis and can easily be extracted by minor modification to the source codes. When access to the source codes is not allowed, a separate subroutine to calculate the element stiffness matrices is needed. Calculation of sensitivity numbers involves only matrix multiplication at the element level.

The program makes use of the built-in features of STRAND6 (G+D Computing, 1991). It removes an element by assigning zero to the material property number of the element. In many cases, an even number of elements are removed from the structure to maintain symmetry of the structure. It is possible to keep a number of elements unchanged during the optimization process by specifying the property of the corresponding element as a non-design property.

To implement features for maintaining non-singularity of the stiffness matrix, after the element removal phase, the number of bars connecting to each node is calculated and stored in an array INTER(INODE) where INODE is nodal identification number. For the simplicity of programming, only cases a to f in section 4.2 are incorporated into the program. The program checks the free nodes with INTER(INODE) less than four for cases a to f to remove zero stressed elements or ignore inner nodal points. This simple implementation can completely avoid singularity of the stiffness matrix for 2D trusses with elements connecting a node to the immediate neighbouring nodes. For other cases, appropriate modification to the model of the structure is needed additionally when the singularity of the stiffness matrix is encountered.

The input data file for ESOTOPO provides the value of the optimization parameters, which control the optimization process. The data include the number of real load cases, the number of constraints, the element removal ratio, structure's type, symmetry control parameter, non-design properties and the displacement limits. Data for real load cases have to be input first. The number of real load cases is provided to distinguish real load data from virtual load data. To improve the numerical accuracy, virtual unit loads are replaced by virtual loads with magnitudes of the order as the real loads. These values are then corrected by appropriate virtual load factors.

Output files of ESOTOPO provide information such as the actual weight of the structure, the values of constrained displacements, the maximum and minimum sensitivity numbers at each step. At the end of each iteration the beam/bar element connection file is copied to a separate file for recovering structure's topologies later.

The FEA package STRAND6 gives solutions only for ten load cases, limiting the combined number of real load and virtual load cases to 10. However, in the case where each load case consists of only one point load and one constraint is imposed on the displacement at the loaded point in the direction of load for each load case, there is no need to provide solutions for virtual load cases. Therefore, for this particular case there is no need to input the virtual loads, and the maximum number of real load cases can be 10 with 10 displacement constraints at the loaded points. Another way to overcome this limitation is to apply STRAND6 repeatedly until all load cases are analysed.

4.4 Examples of truss topology optimization

This section will show the capability of the proposed method for truss topology optimization. All examples are solved using a 486DX2/66MHz personal computer.

4.4.1 Two bar truss

Consider a problem of two bar truss transferring a vertical load to a vertical line of support, for which an analytical solution exists. The optimal configuration for a two bar truss, is the one where the distance between two vertical supports is twice as much as the given horizontal distance from the force to the line of support (see Fig. 4.1 and Appendix I). Different ground structures containing a sub-set of elements, which forms the configuration of the analytical solution, are considered to see whether we can achieve the same solution by the proposed method.

1) Ground structure I: This ground structure consists of 25 bars connecting the loaded point to 25 fixed points equally placed with the step of 0.1 m along a vertical line. The loaded point is in the distance of 1 m horizontally from the middle fixed

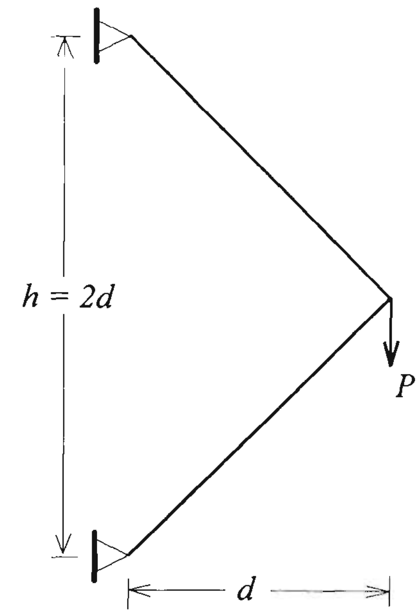
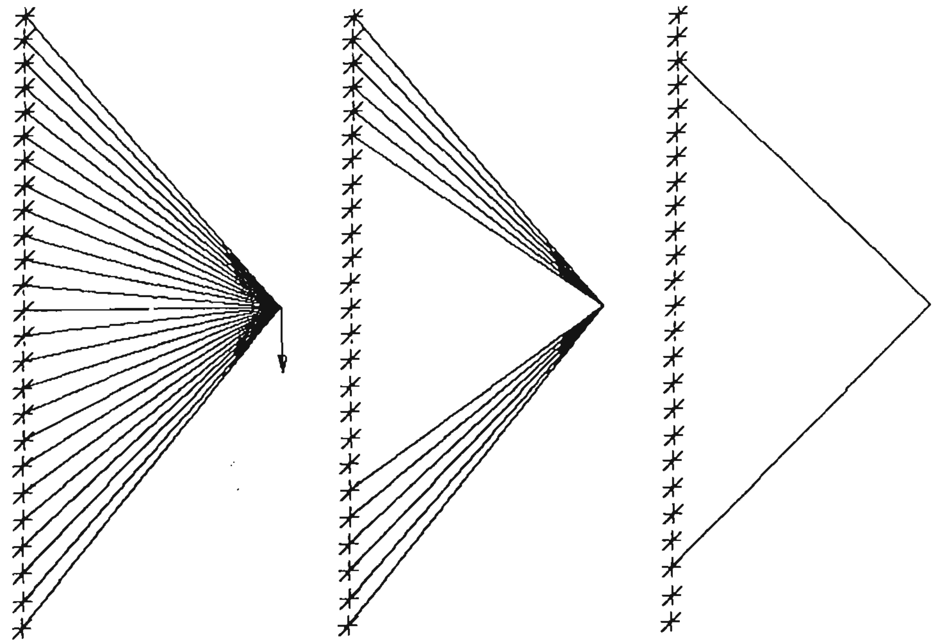


Fig. 4.1 Optimal configuration of two bar truss.

point (Fig. 4.2a). All bars have the same cross-sectional area A = 1 cm^2 and Young modulus $E = 2\times10^{11}$ Pa. Under the vertical point load P = 200 kN, the initial vertical displacement at the loaded point is 1.87 mm. The optimal configuration consists of two bars of the same cross-sectional area, as shown in Fig. 4.1, when subjected to the same load has a vertical displacement of 14.14 mm at the loaded point. So the limit on this displacement is set at 15 mm to see whether we can evolve to this solution.

The ESO procedure is used where two bars are removed each time (ERR = 8% of the initial number of elements). The symmetry of the structure is also reflected by the element sensitivity numbers. Bars in symmetrical positions have equal sensitivity numbers. To maintain symmetry of the resulting design, at any stage an additional third bar is removed automatically if its sensitivity number equals to the sensitivity number of the last element which has been removed by the specified value of ERR. In the first iteration three bars in the centre are actually removed due to symmetry in sensitivity numbers.



(a) (b) (c)

Fig. 4.2 ESO topologies for the ground structure I:

(a) initial topology; (b) at iteration 6; (c) optimum topology.

The optimization process is terminated when a uniform state in sensitivity numbers is reached for the remaining elements Under this criterion, two bars remained in the final topology as illustrated in Fig. 4.2c. This final solution is the same as the optimal configuration of the two bar truss. Fig. 4.2b shows an intermediate topology at iteration 6. It is noted that sensitivity numbers are significantly different in the initial design, with the ratio of the minimum and the maximum sensitivity numbers of about 10^{-30}. They become more uniform in the following designs and are equal in the final design. This example shows that the proposed method has derived an exact optimum solution.

During the solution process, the actual displacement and the actual weight of the ESO topologies are recorded. The change in the constrained displacement is presented in Fig. 4.3a. The constrained displacement has negligible change in the first several iterations despite elements being successively removed. This confirms that the

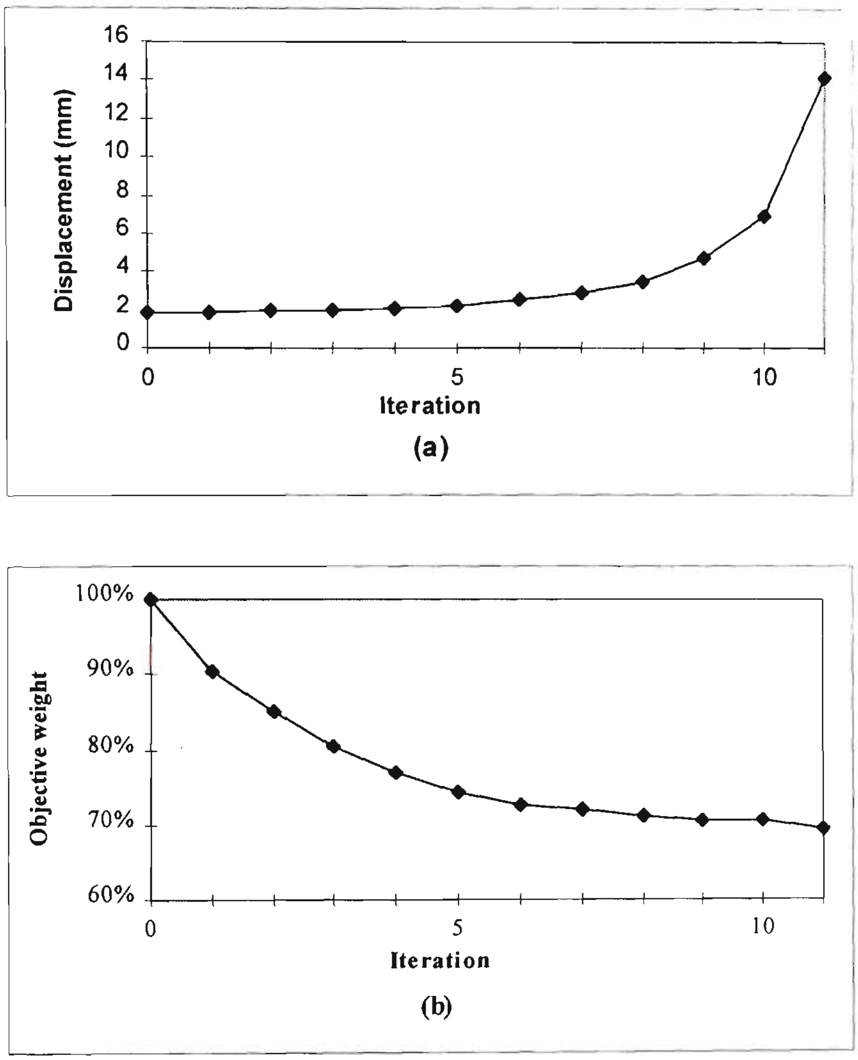


Fig. 4.3 Evolutionary history for the ground structure I:

(a) constrained displacement; (b) objective weight.

proposed ESO method removes inefficient elements to create a more efficient design than the preceding iteration. The same trend has been observed in all following examples subjected to a single load and a single constraint.

The efficiency of the ESO method can be assessed by observing the changes in the objective weight during the optimization process. For truss structures, the objective weight of any ESO topology is obtained by simply scaling its actual weight by the ratio of its actual displacement and the specified limit. This is due to the fact that the stiffness matrix of truss structures is a linear function of the cross sectional area. Suppose the current design (the ESO topology) has an actual weight W^{act} and a displacement u^{act}. To achieve the displacement limit u^*, the actual displacement is scaled by the factor (u^*/u^{act}). Therefore, the objective weight W^{obj} of the design is

$$W^{obj} = W^{act}\left(\frac{u^{act}}{u^*}\right) \tag{4.1}$$

Assuming that the initial design has an initial weight W_0 and initial displacement u_0, the initial objective weight is

$$W_0^{obj} = W_0\left(\frac{u_0}{u^*}\right) \tag{4.2}$$

The relative objective weight of any design with respect to the initial objective weight is calculated as

$$W_{rel}^{obj} = \frac{W^{obj}}{W_0^{obj}} = \left(\frac{W^{act}}{W_0}\right)\times\left(\frac{u^{act}}{u_0}\right) = W_{rel}^{act}\times\left(\frac{u^{act}}{u_0}\right) \tag{4.3}$$

where

$$W_{rel}^{act} = \frac{W^{act}}{W_0} \tag{4.4}$$

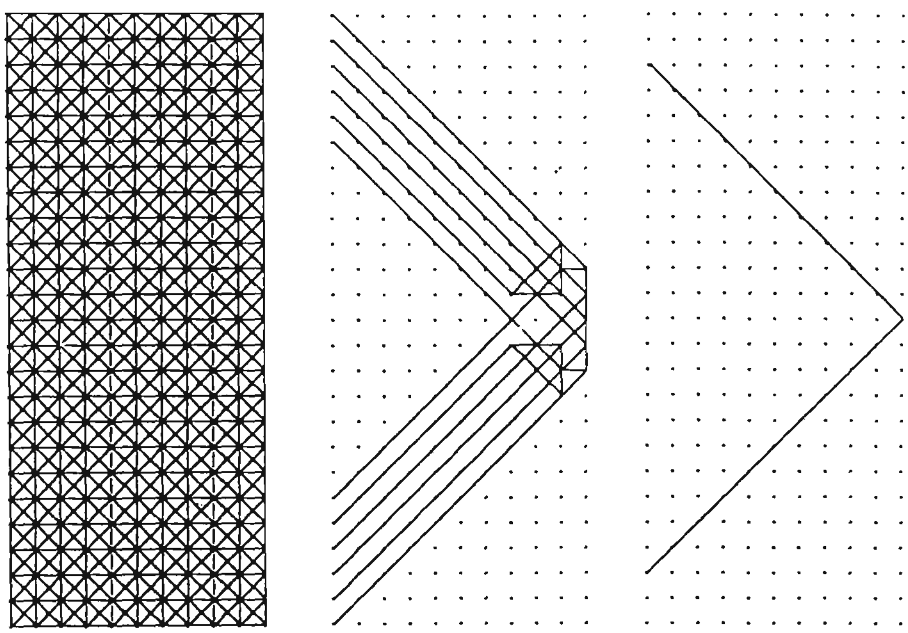
is the relative actual weight of the current design with respect to the initial actual weight. It is seen that the limit u^* is eliminated in the formula (4.3) indicating that the relative change in the objective weight is the same for any value of the limit. Thus, using (4.3) the relative objective weight is obtained simply by scaling the relative actual weight by the ratio of the actual and the initial values of the displacement.

For this example, the initial displacement is 1.87 mm. The ESO topologies illustrated in Figs. 4.2b and 4.2c have actual displacements of 2.49 mm and 14.14 mm with actual weights of 54.4% and 9.3% of the initial weight, respectively. By using (4.3), their objective weights are 54.4% × (2.49/1.87) = 72.47% and 9.3% × (14.14/1.87) =

69.42% of the objective weight of the initial design. These ESO topologies have the weight savings of 27.53% and 30.58% compared with the initial topology. The history of the objective weight, which is the same for any value of the displacement limit, is presented in Fig. 4.3b, where the objective weight substantially decreases in the first several iterations and then asymptotically approaches to the minimum. The final ESO topology in Fig. 4.3c is the absolute optimum for this type of structure.

During the solution process, both element energy and sensitivity number (ratio of element energy and element weight) are recorded for each element. Two bars, which remain in the final topology, always have the maximum sensitivity numbers. However, the two outer bars always have the maximum energies and they would remain in the final design if the element energies are used as an indicator for element removal. Removal of elements with lower energies results in minimum displacement topologies, but not necessarily minimum weight. The design with two outer bars is always heavier for the same displacement. This particular example shows that a removal decision based on element energies may be misleading for the case where elements have different weights. The ratio between the element energy and weight provides a more meaningful measure with regard to efficient use of material within each element.

2) Ground structure II: The ground structure shown in Fig. 4.4a is in a domain of $1\text{m} \times 2.4\text{m}$ with a grid of 11×25 nodes and 994 bars connecting only neighbouring nodes. All bars have the same cross-sectional area and the same Young's modulus as adopted for the ground structure I and the vertical force $P = 200$ kN acts in the middle of the right side. All nodes on the left side are fixed in location. The initial displacement at the loaded point is 1.52 mm. The same limit of 15 mm on the vertical displacement at loaded point is imposed.



(a) (b) (c)

Fig. 4.4 ESO topologies for the ground structure II:

(a) initial topology; (b) at iteration 55; (c) optimum topology.

Using ERR = 2% of the current number of elements (the number of elements to be removed decreases from 20 to 2 during the solution process), the ESO procedure produces the final topology at iteration 60, shown in Fig. 4.4c, which is the same as the analytical solution. Fig. 4.4b is the ESO topology at iteration 55.

The history of the change in the displacement is presented in Fig. 4.5a. It is seen that the displacement has only small changes up to iteration 53 despite the fact that elements are progressively removed. A sharp increase is observed only in the last iteration. Fig. 4.5b shows the change in the objective weight. The objective weight of the ESO topologies reduces progressively and approaches the minimum value when the optimal configuration of two bars is reached, as shown in Fig. 4.5b. It is worth pointing out that the method has produced a series of topologies (iterations 48-53) with almost equal structural weight efficiency.

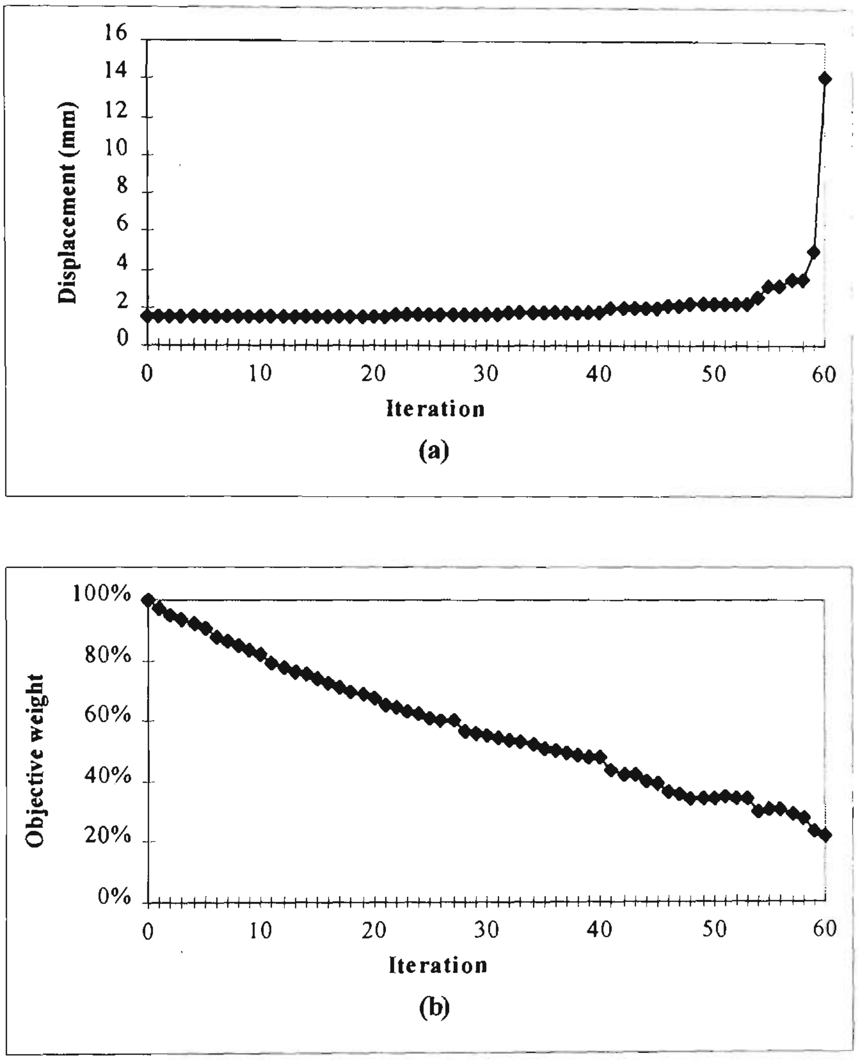
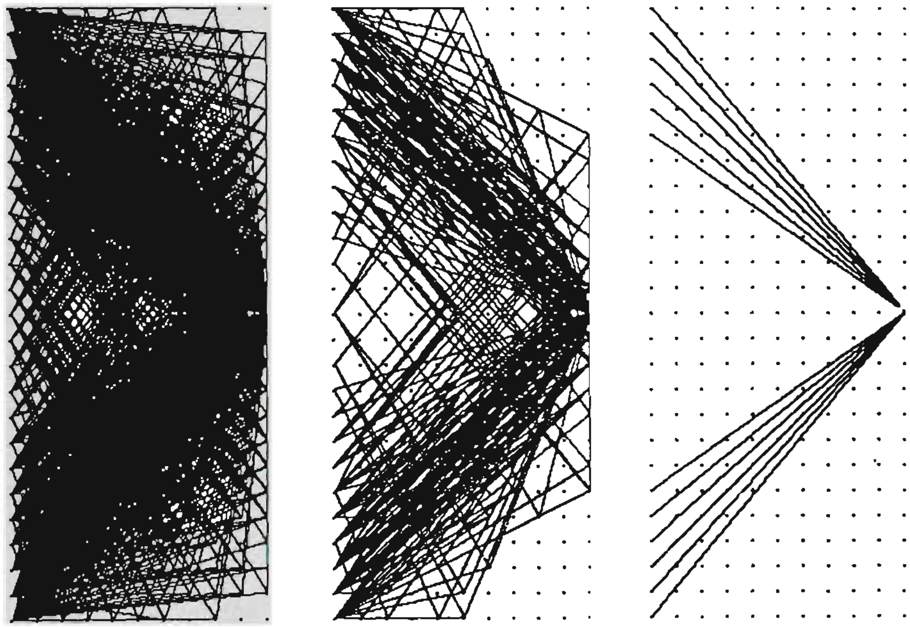


Fig. 4.5 Evolutionary history for the ground structure II:

(a) constrained displacement; (b) objective weight.

The optimum topology shown in Figs. 4.4c has the actual displacement of 14.14 mm and the actual weight 2.4% of the initial weight. So the minimum objective weight is only of 21.96% of the initial objective weight which results in a weight saving of 78.04% compared to the initial design. The intermediate topology obtained at iteration 55 as shown in Fig. 4.4b has the actual displacement of 3.11 mm and the actual weight of 14.9% of the initial. This topology with the objective weights of 30.44% gives a weight saving of 69.56% with respect to the initial topology.



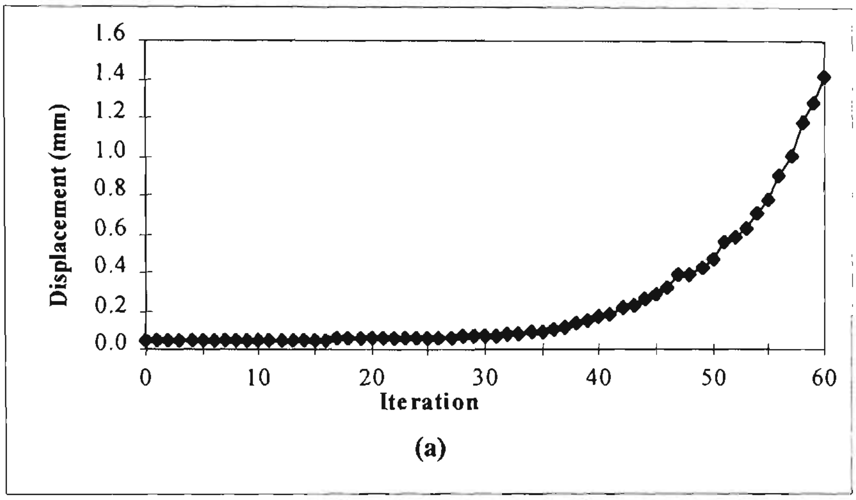
(a) (b) (c)

Fig. 4.6 ESO topologies for the ground structure III:

(a) iteration 30; (b) iteration 40; (c) iteration 56.

3) Ground structure III: With the same domain and grid given in the ground structure II, the new ground structure consists of 37,675 bars connecting each node to all other nodes. The initial displacement at the loaded point is 0.05 mm. One may wonder how other existing methods could easily deal with such a large structure. For this structure, the ESO method has no more difficulties than for the ground structure I comprising only 25 bars.

Using ERR = 10% of the current number of elements, the ESO method evolves an optimal configuration corresponding to a two bar truss at iteration 60, where each side contains ten equal overlapping bars. This results in a structure ten times heavier subjected to a tenth of the displacement compared with the two bar configuration, inferring that there is no change from a topological point of view. Fig. 4.6 presents some steps toward the optimal configuration. The topologies clearly converge to the optimum two bar configuration.



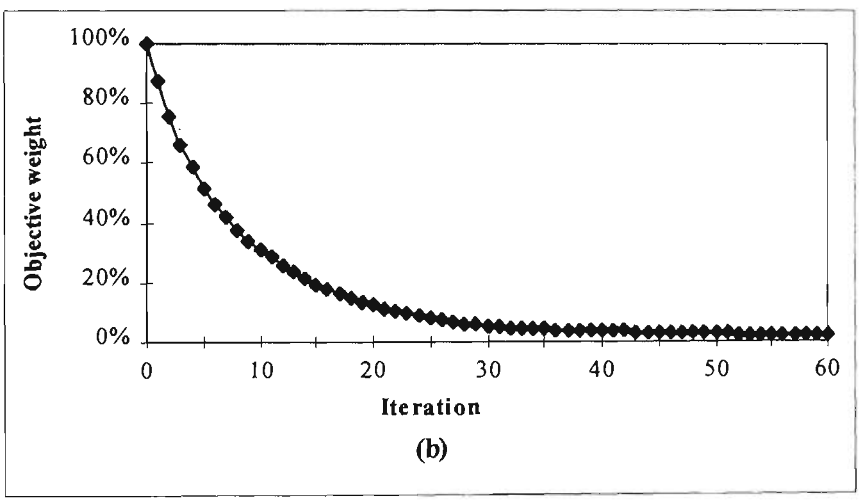


Fig. 4.7 Optimization history for the ground structure III:

(a) constrained displacement; (b) objective weight.

Fig. 4.7a clearly illustrates that the constrained displacement changes insignificantly over the first 30 iterations and then rises rapidly. At the same time as shown in Fig. 4.7b, the objective weight sharply decreases to 51.78%, 31.17%, 12.24% and 5.37% at iterations 5, 10, 20 and 30 respectively, and then slowly reaches the minimum, which is only 2.19% of the initial value.

It is worth pointing out that ground structure III contains ground structure I as a subset and the topology obtained from ground structure III at iteration 56 is the same as obtained from ground structure I at iteration 6 (Figs. 4.2b and 4.6c). Their difference is that overlapping bars exist in the topology obtained from ground structure III.

4.4.2 Three dimensional truss

Consider the topology optimization problem of a space (3D) truss. The initial design of the truss is in a domain of $4\ \mathrm{m} \times 4\ \mathrm{m} \times 4\ \mathrm{m}$ with a grid of $5 \times 5 \times 5$ nodes and bar elements connect each node to its immediate neighbouring nodes. Four bottom corner nodes are fixed in location. A vertical point load $P =$ 400 kN acts at the centre of the top surface. The Young's modulus $E = 2 \times 10^{11}$ Pa is assumed.

Due to symmetry, only a quarter of the structure is considered, which consists of 45 nodes and 296 bars, shown in Fig. 4.8. All bars in the initial design for a quarter are assumed to have the same cross-sectional area A = 1 cm^2. Therefore, in the whole structure, the elements along the axis of symmetry will have the cross-sectional area of $4A$ and elements lying on the planes of symmetry will have the area of $2A$.

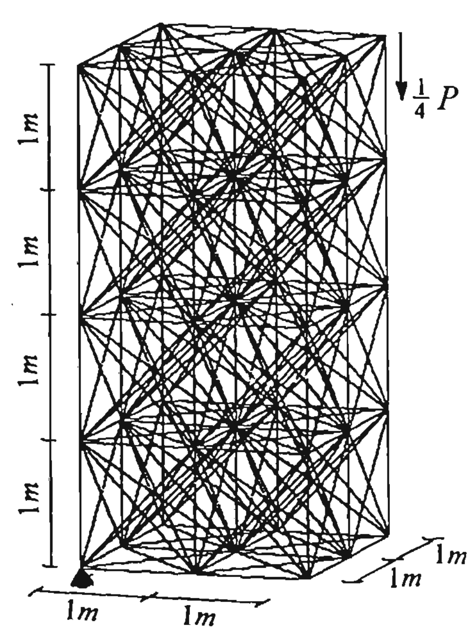
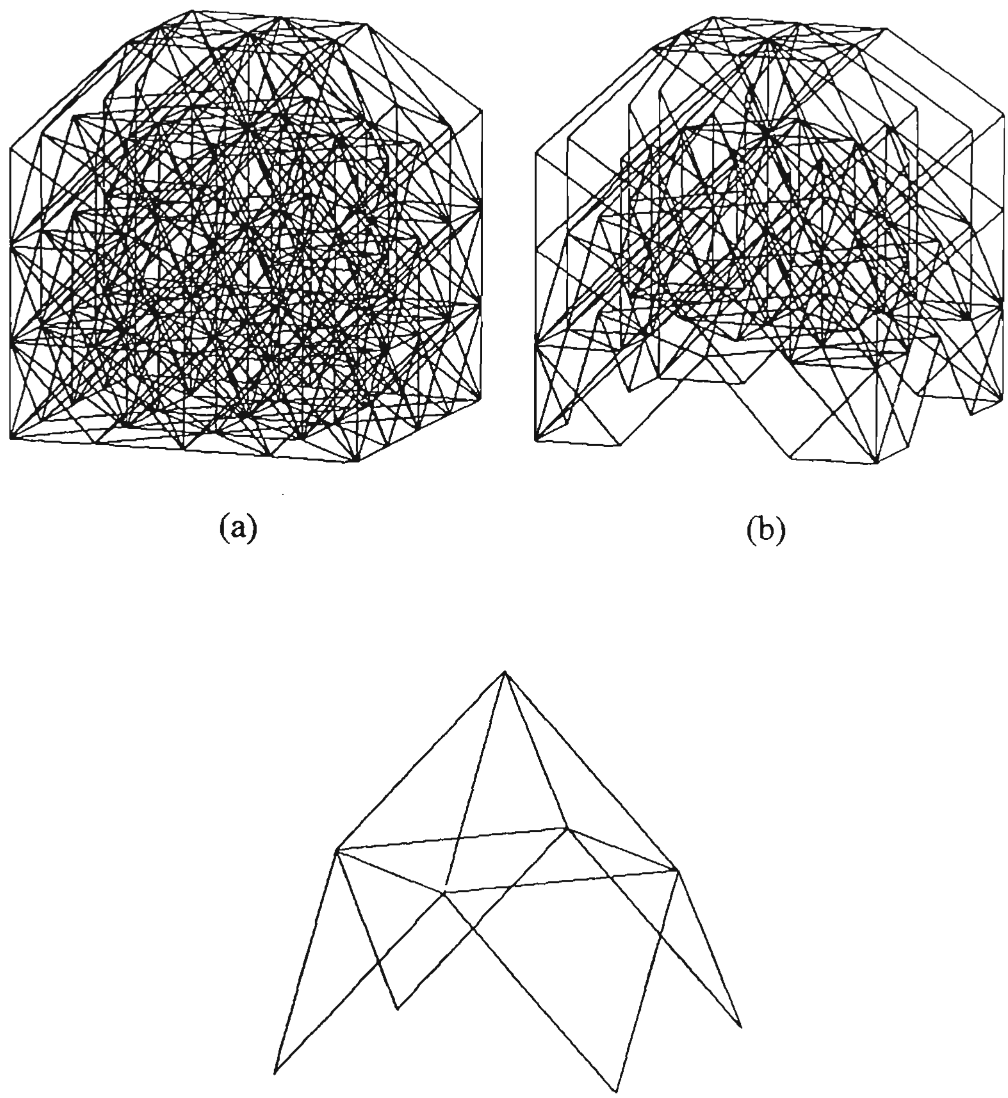


Fig. 4.8 Ground structure for a quarter of a 3D truss.



(a) (b) (c)

Fig 4.9 ESO topologies for the 3D truss:

(a) iteration 10; (b) iteration 15; (c) optimum topology.

The initial vertical displacement at the loaded point is 7.3 mm. A limit of 50 mm is imposed on this displacement. Using ERR = 4% of the initial number of elements, the optimization process continues up to iteration 18. Further solutions are obtained by reducing ERR to 1% and the final topology shown in Fig. 4.9c is reached at iteration 20, where sensitivity numbers are equal for all remaining elements. Fig. 4.9a-b shows the topologies at iterations 10 and 15.

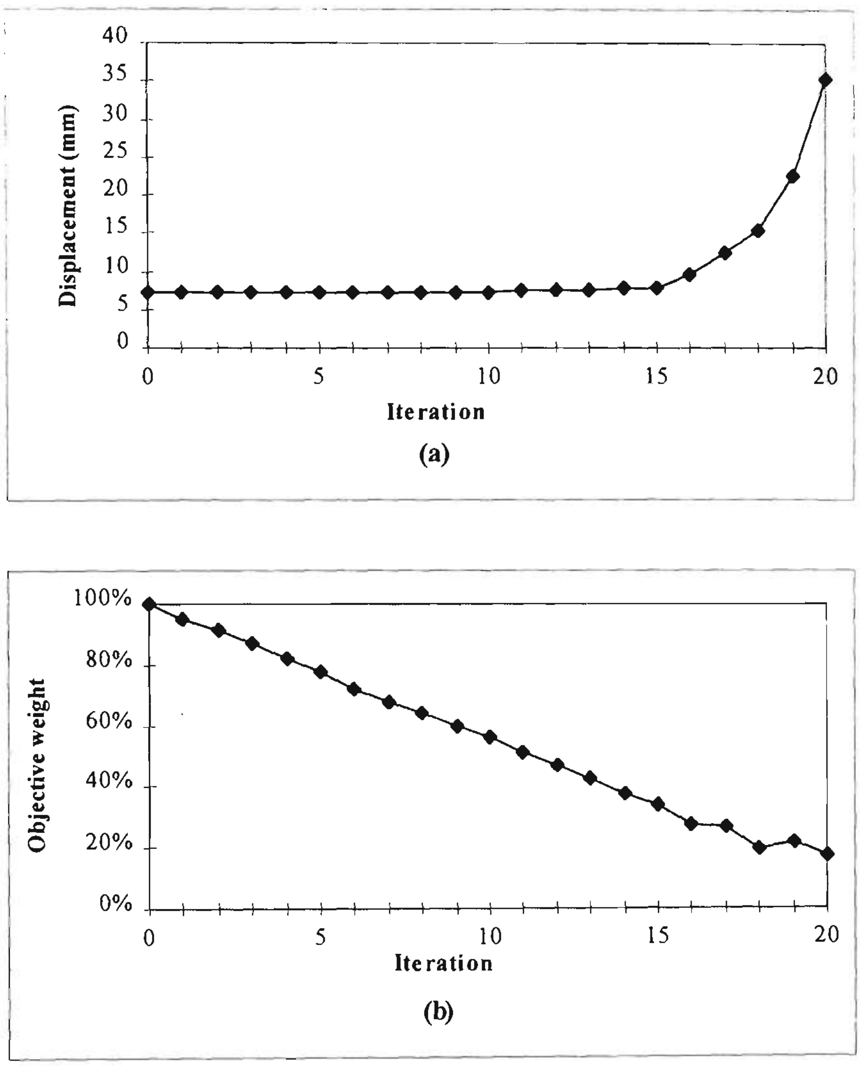


Fig. 4.10 Evolutionary history for the 3D truss:

(a) constrained displacement; (b) objective weight.

This example shows that the ESO method can drive solution to the optimum with an impressive topology (Fig. 4.9c), where the optimality criteria are satisfied for all remaining elements. In the initial design the difference between the maximum and minimum sensitivity numbers is in the order of 10^7. In the final topology all elements have equal sensitivity numbers. This final topology represents a statically determinate structure, which consists of elements with equal lengths. Elements on the top have double cross-sectional areas. This final topology has the actual weight of 3.5% of the initial weight and a displacement of 35.36 mm at the loaded point.

The change in the displacement is shown in Fig. 4.10a with the similar trend as observed in previous examples. It is seen from Fig. 4.10b that the objective weight steadily reduces to 19.28% at iteration 18, followed by a slight rise to 21.87% at iteration 19 before approaching a minimum value of 17.16% of the initial value. The final topology is the optimum with a weight saving of 82.84% with respect to the initial topology.

4.4.3. Arch bridge under multiple displacement constraints

Consider an example where a structure is subjected to multiple displacement constraints. The initial topology of an arch bridge and its dimensions are given in Fig. 4.11, which is a modification of an example given in Majid (1974). The initial design has 39 members, where no bar is allowed to bypass any loaded node on the bottom line, i.e. no overlapping bars. The bridge is simply supported. Three vertical point loads $P = 10$ kN act simultaneously at points A, B & C. The same cross-sectional area and material property as given in the previous example are adopted. Initially, the maximum vertical displacement is 6.3 mm which occurs at point B. The limit is set at 20 mm for the vertical displacements at all loaded points.

The ESO procedure is used by removing two members at each iteration (ERR = 5% of initial elements). To maintain symmetry, a additional third bar is removed if its sensitivity number equals to the sensitivity number of the second bar which has been currently removed.

Three different types, namely Type (1): the recurrence formulae (3.37), Type (2): the ratio formulae (3.38) and Type (3): linear equations (3.39), are used to calculate the Lagrange multipliers. The initial values of all Lagrange multipliers are set to unity in

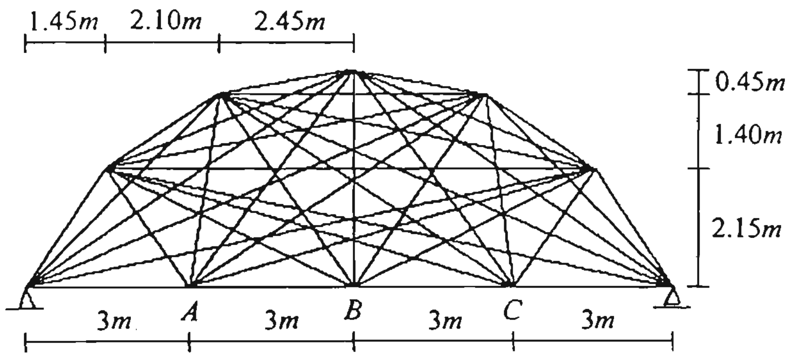


Fig. 4.11 Ground structure for an arch bridge.

the recurrence formulae and for both the recurrence and ratio formulae $b = 1$. For linear equations $r = -1$ is used. Scaled displacement limits are used throughout the study to compare the performance of different formulae for the Lagrange multipliers.

The solution processes continues until the resulting design becomes a mechanism or the maximum displacement reaches the given limit. In all cases, more uniform sensitivity numbers among the remaining elements are achieved. The minimum sensitivity number, which is in the order of 10^{-3} of the maximum in the initial ground structure, becomes in the order of 10^{-1} or in the same order as the maximum in the final topologies. Unlike the case of a single constraint where the maximum sensitivity number always increases, decreases in the maximum sensitivity number are observed during the solution process for this structure subject to multiple constraints. This is due to the re-distribution in contributions among the remaining elements. Histories of the change in the maximum displacement and in the objective weight for different types of formulae for the Lagrange multipliers are provided in Fig. 4.12.

For the structure under multiple constraints, the objective weight is determined from the actual weight and the actual displacement corresponding to the most active constraint. In the case of trusses where the stiffness matrix is a linear function of the design variable, the design scaling parameter a is equal to the limit scaling parameter φ defined by equation (3.36) as

$$\varphi = \max_{j=1,\,m}\left(\frac{|u_j|}{u_j^*}\right) \tag{4.4}$$

To keep the constraint at the limit, the current design should be scaled by the parameter φ. Therefore, the objective weight W^{obj} of the current design is also obtained by scaling its actual weight W^{act} by the same parameter , i.e.

$$W^{obj} = W^{act}\varphi \tag{4.5}$$

Assume that the initial design has the initial weight W_0 and the initial value of φ is φ_0, where

$$\varphi_0 = \max_{k=1,\,m}\left(\frac{|u_{k0}|}{u_k^*}\right) \tag{4.6}$$

where the index k is used instead of j clarifying the fact that at a different iteration a different constraint becomes the most active. The initial objective weight is determined as

$$W_0^{obj} = W_0\varphi_0 \tag{4.7}$$

So the relative objective weight of any design with respect to the initial objective weight is calculated as

$$W_{rel}^{obj} = \frac{W^{obj}}{W_0^{obj}} = \left(\frac{W^{act}}{W_0}\right)\times\left(\frac{\varphi}{\varphi_0}\right) = W_{rel}^{act}\times\left(\frac{\varphi}{\varphi_0}\right) \tag{4.8}$$

where W_{rel}^{act} is the relative actual weight of the current design with respect to the initial weight defined by (4.4).

In the case where a common limit is specified for all constraints, i.e. $u_j^* = u^*$ for j from 1 to m, equation (4.8) becomes

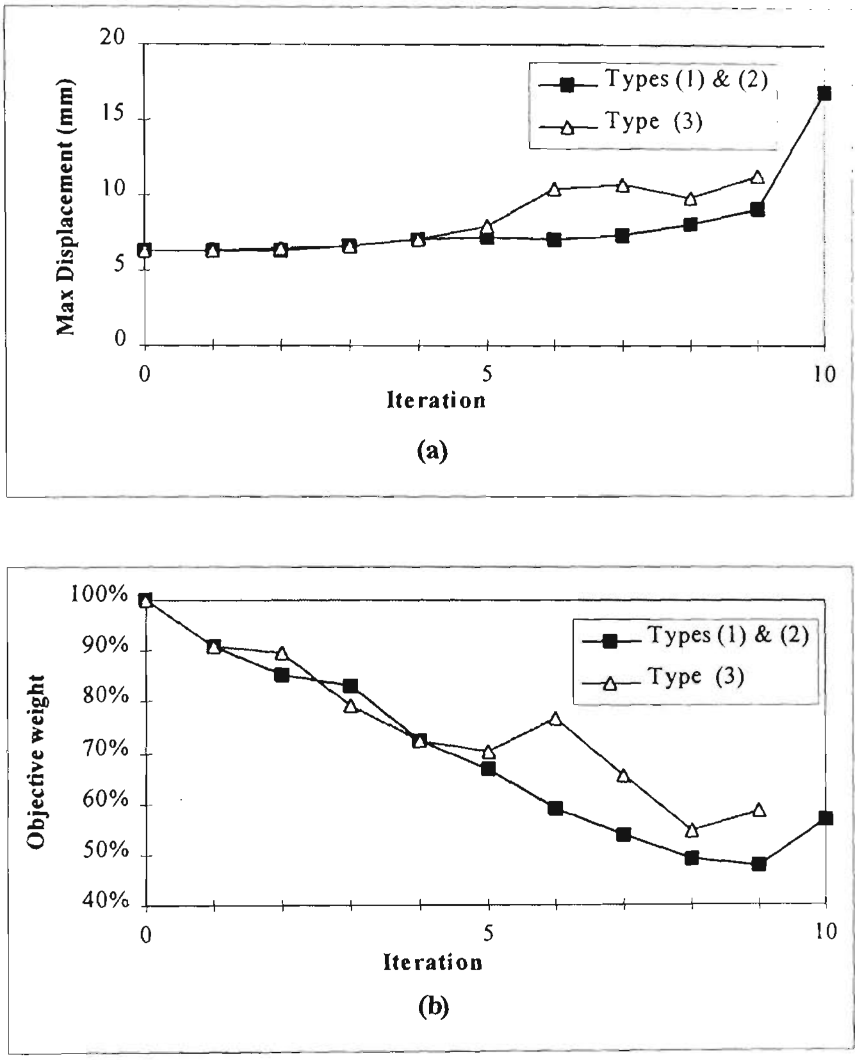
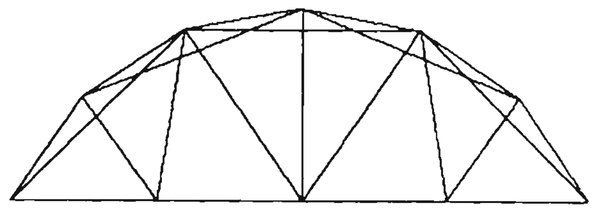


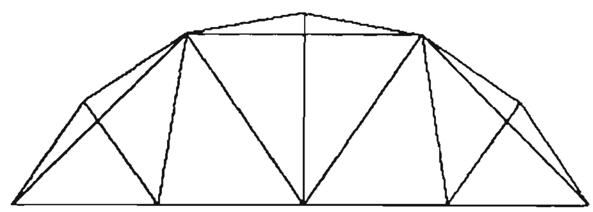
Fig. 4.12 Evolutionary history for the arch bridge under multiple constraints: (a) maximum displacement; (b) objective weight

$$W_{rel}^{obj} = W_{rel}^{act} \times \frac{\max_{j=1,\,m} |u_j|}{\max_{k=1,\,m} |u_{k0}|} \tag{4.9}$$

Thus, the relative objective weight is obtained by simply scaling the relative actual weight by the ratio of the maximum actual displacement and the maximum initial displacement. It is seen that the common limit u^* is eliminated in equation (4.9) indicating that the relative change in the objective weight is the same for any value of



(a)



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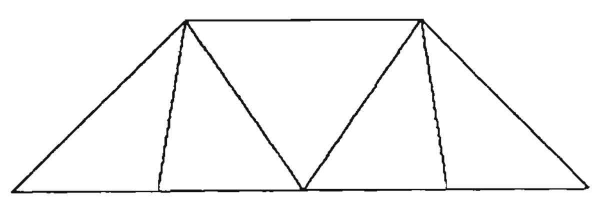
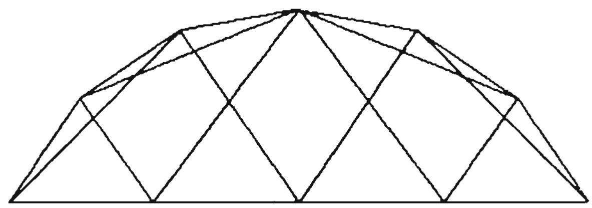


(c)

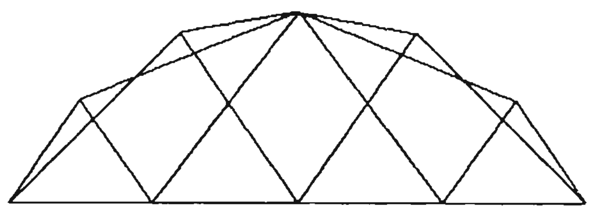
Fig. 4.13 ESO topologies for the arch bridge under multiple constraints with Lagrange multipliers determined by the recurrence and ratio formulae: (a) near-optimum; (b) optimum; (c) final stable topology.

the common limit. This infers that the optimum topology is the same for any value of the common limit. In other cases, the optimum topology is different for different sets of limits.

In this example, both the recurrence and ratio formulae, types (1) and (2), produce identical topologies during the solution process, despite the fact that they give different values for the Lagrange multipliers. Transitions from a statically indeterminate to a determinate structure and then to a mechanism are also observed for both types. The final stable topology is obtained at iteration 10, which is a



(a)



(b)

Fig. 4.14 ESO topologies for the arch bridge under multiple constraints with Lagrange multipliers determined by linear equations: (a) optimum topology; (b) final stable topology.

statically determinate structure (Fig. 4.13c). The maximum displacement slowly increases in the first five iterations and then jumps up sharply in the last iteration (Fig. 4.12a). At the same time the objective weight, shown in Fig. 4.12b, sharply reduces to the minimum value at iteration 9 and then goes up in the last iteration. Fig. 4.13 presents the near-optimum topology at iteration 8, the optimum topology at iteration 9 and the final stable design at iteration 10. These topologies have the objective weight of 49.12%, 47.84% and 56.69% of the initial value, respectively. The minimum weight topology, shown in Fig. 4.13b, is a statically indeterminate structure. The final stable topology, Fig. 4.13.c, is a statically determinate structure. In this example, the statically indeterminate topologies in Figs. 4.13a and 4.13b have lower objective weights than the statically determinate topology in Fig. 4.13c. These results suggest that more material should be put along the upper chord to obtain more efficient designs.

The use of linear equations leads to different topologies. The final stable topology is obtained at iteration 9, which is statically indeterminate as shown in Fig. 4.14b. No further stable and statically determinate topology can be obtained in this case. A similar trend in the change in the maximum displacement is also observed as seen in Fig. 4.12a. The objective weight changes in a different pattern. As seen in Fig. 4.12b, it decreases from iteration 1 to iteration 5, and increases in iteration 6, then reduces to the minimum at iteration 8 before it increases in the final iteration. The minimum weight topology (Fig. 4.14a) is a statically indeterminate which is of 54.56% of the initial objective weight. The final topology (Fig. 4.14b) has an objective weight of 58.38% of the initial.

This example illustrates that the proposed simple ratio formulae for Lagrange multipliers can produce the same topologies as the recurrence formulae and lower minimum weight topology in comparison with linear equations.

It should be noted that as seen in the previous examples of structures under a single load and a single constraint, the minimum weight topologies are the final stable designs which are also statically determinate structures. With multiple constraints, the method can also produce significantly more efficient topologies compared with the initial chosen topology. However, the minimum weight topology is not the final stable topology. To find the best topology, the evolutionary process should be continued until no further stable structure can be achieved. The best topology is the one with the minimum objective weight.

4.4.4. Arch bridge under multiple load cases

Consider again the problem given in Example 4.3.3, however with each point load acting at a different time and limits are imposed on displacements at loaded points, i.e.

the bridge is under multiple displacement constraints and multiple loading conditions. For all load cases, the maximum initial vertical displacement at loaded points is 2.8 mm. The structure is solved by the ESO method using a similar removal strategy. Only the recurrence and ratio formulae are used for the Lagrange multipliers with the initial values equal to unity and $b = 1$.

Unlike the previous example, the use of recurrence and ratio formulae produces different topologies for this problem. Only the topologies obtained at iterations 1, 4, 6 and 7 are identical. As shown in Fig. 4.15a, the patterns of change in the maximum displacement in both cases are similar. The change in the objective weight is different. The objective weight in both cases reaches the near-minimum at early stage, iteration 6, then slightly increases as shown in Fig. 4.15b.

The use of the recurrence formulae gives the minimum weight topology at iteration 8, as shown in Fig. 4.16b, with the objective weight of 56.50% of the initial objective weight. This optimum topology is a statically indeterminate structure. Other topologies obtained at iteration 6 and 10 by using the recurrence formula are shown in Figs. 4.16a and 4.16c. The topology shown in Fig. 4.16a is a near-optimum topology with the objective weights of 56.77% of the initial. The topology in Fig. 4.16c has the objective weight of 65.25% with respect to the initial design.

The use of the ratio formulae produces the optimum topology at the final iteration as shown in Fig 4.17c, which is a statically determinate structure. The optimum topology has the minimum objective weight of 55.59% of the initial value. A near-optimum topology is obtained in this case at iteration 6, which is the same near-optimum obtained in previous case shown in Fig. 4.16a. Figs. 4.17a and 4.17b present the topologies obtained at iterations 7 and 9 by using the ratio formulae.

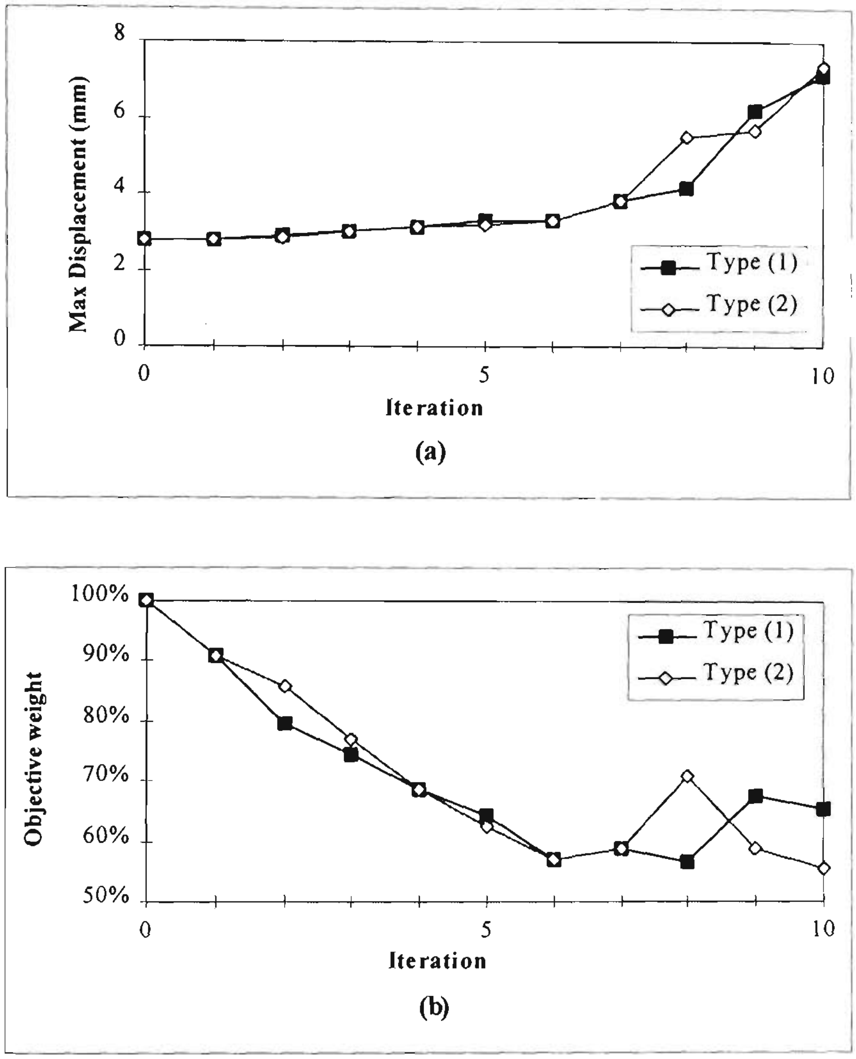
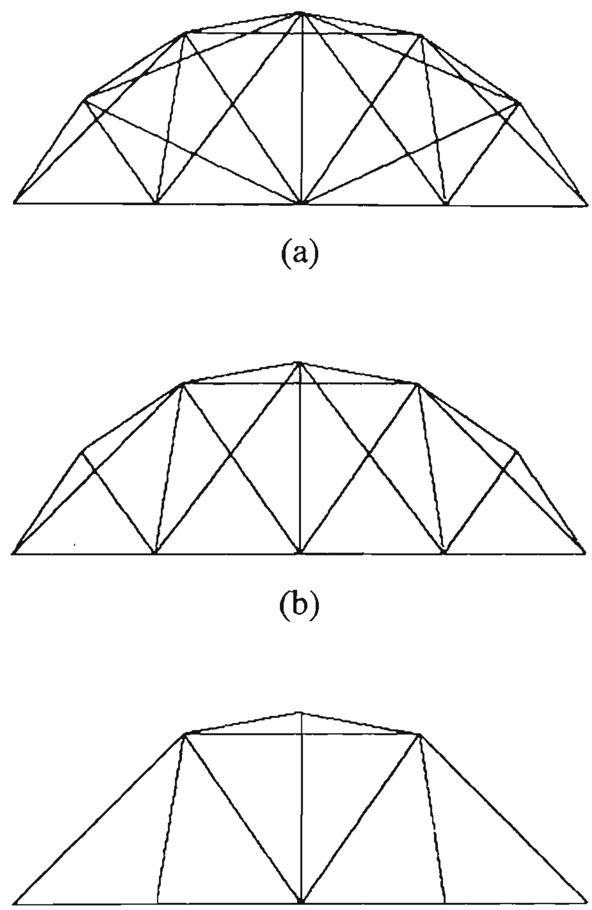


Fig. 4.15 Evolutionary history for the arch bridge under multiple load cases: (a) maximum displacement; (b) objective weight

These topologies have the objective weights of 58.87% and 58.64% of the initial objective weight. For this example, the statically determinate topology in Fig. 17c is the best.

It is seen that for the same value of *b* and assuming unity for initial values of Lagrange multipliers, both the recurrence and ratio formulae will give the same answer in the first step only. With the use of recurrence formulae, a change in the set of active constraints has taken place after iteration 9 when displacements at points *A*



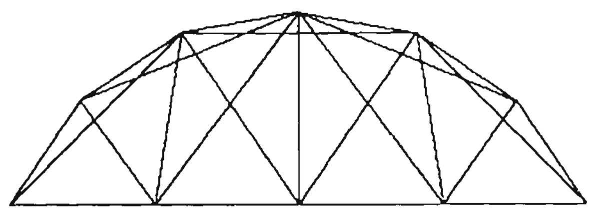
(a)

(b)

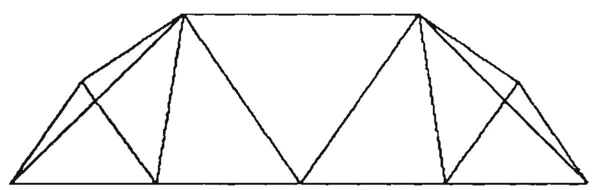
(c)

Fig. 4.16 ESO topologies for the arch bridge under multiple load cases with Lagrange multipliers determined by recurrence formulae: (a) near-optimum; (b) optimum; (c) final stable topology.

& *C* become larger than at *B*. However, the values given by the recurrence formulae for the Lagrange multipliers corresponding to these active constraints at *A* & *C* are approximately half smaller than the Lagrange multiplier corresponding to the passive constraint at *B*, which contradicts the initial physical meaning of Lagrange multipliers. Fortunately, this does not happen to Lagrange multipliers determined by the ratio formulae. The set of active constraints changes twice after iterations 5 and 10. The Lagrange multiplier determined by the ratio formulae corresponding to the most active constraint is always the maximum.



(a)



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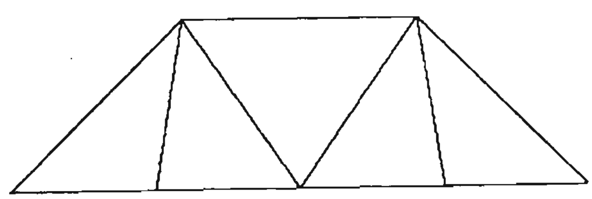


(c)

Fig. 4.17 ESO topologies for the arch bridge under multiple load cases with Lagrange multipliers determined by the ratio formulae: (a) iteration 7; (b) iteration 9; (c) optimum topology.

4.5 Concluding remarks

This chapter has presented the application of the proposed ESO method to truss topology optimization problems. The proposed method is simple and can overcome difficulties in topology optimization faced by other methods. The method can produce significantly improved topologies compared with the initial ground structure. Although examples presented in this chapter are truss structures, there are no difficulties in applying this method to frame structures.

Truss topology obtained by the proposed method, may represent a statically indeterminate structure, a statically determinate structure or a mechanism (unstable structure). In many cases, transitions from statically indeterminate structure to statically determinate and then to mechanism are observed. When a statically indeterminate structure is obtained, further solutions may be produced by using a smaller element removal ratio. When a statically determinate structure is reached, further removal of elements will either destroy the structure completely or create a mechanism. The solution process is terminated at this step.

In the given examples of structures under a single point load subject to a single constraint on the displacement at the location and in the direction of the load, the optimum has been statically determinate structures. As pointed out above that the pattern of change in the objective weight is the same for any value of the limit, it is suggested that one should run the process until no further stable structure can be achieved by setting a large value for the displacement limit. The optimum topology is the solution with the minimum objective weight. For multiple constraints, the optimum often is a statically indeterminate structure. When a common limit is specified for all constraints the pattern of change in the objective weight does not depend on the limit value, and the optimum is the same for any limit. In general, the optimum topology is different for different sets of limits.

It is seen that the element removal ratio (ERR) is an important parameter in the proposed method, and values of up to 10% have been used for the given examples. Further study on its influence on the solution should be carried out. For problems with multiple constraints, Lagrange multipliers play an important role. Methods of determining Lagrange multipliers significantly influence the solutions and require further investigation. Based on examples given in this chapter and in the previous works (Chu *et al.* 1995, 1996a, 1997), the proposed simple ratio formulae with $b = 1$ can give as good solutions as and sometimes better solutions than those obtained by the recurrence formulae and linear equations ($r = -1$). Therefore, the ratio formulae (3.38) are recommended.

Chapter 5

TOPOLOGY OPTIMIZATION OF CONTINUUM STRUCTURES - APPLICATION OF THE ESO METHOD

5.1 Introduction

It has been shown in Chapter 4 that the proposed ESO method is an effective tool for topology optimization of skeletal structures such as trusses or frames. This chapter presents the application of the method to topology optimization of two-dimensional continuum structures. Removal of elements from the design domain of two-dimensional continuum structures can create new internal holes in addition to changing in the shape of the existing external and internal boundaries. This means that the ESO method can perform shape and topology optimization simultaneously when elements are allowed to be removed from any part of the structure. The proposed procedure can also be switched to "pure" shape optimization by allowing elements to be removed from the existing boundaries only. Similar to any other method using the design domain (ground structure) approach, such as the homogenization (Bendsøe and Kikuchi 1988) and the density function method (Yang and Chuang 1994), the ESO method also produces solutions with non-smooth boundaries. However, these solutions can be used as good starting points for other boundary variation methods to obtain necessary smooth boundaries.

Similar to truss topology problems, singularity of the stiffness matrix is also encountered for plate/shell structures during the optimization process. This causes interruption of the solution process. A measure for avoiding singularity of the stiffness matrix is proposed to provide continuity of the solution process.

5.2 Singularity of the stiffness matrix and a measure for prevention

During the process of removing elements, singularity of the stiffness matrix may result due to the fact that one or more remaining elements do not have sufficient connectivities to other elements. The most common case is where a triangular and quadrilateral plate element is connected to the remaining structure at only one node. Due to the fact that triangular and quadrilateral plate elements adopted in the FEM typically do not have in-plane rotational stiffness corresponding to drilling freedom, they can experience free in-plane rotation about the node, causing singularity of the stiffness matrix. If the remaining nodes of these elements are free of load or are not externally restrained, these elements obviously do not contribute to the stiffness of the structure and hence, can be removed without affecting the overall load carrying capacity of the structure. A feature, for checking the element connectivities and for removing insufficiently connected elements, is included in the proposed evolutionary procedure to prevent the stiffness matrix of the structure from being singular. This may result in the number of elements that are removed at each iteration to exceed the number specified based on the element removal ratio (ERR).

5.3 Computer implementation for shape and topology optimization of continuum structures

A computer program named ESOSHAPE has been developed by the candidate for shape and topology optimization of the plate/shell structures subject to multiple displacement constraints and multiple load cases. The program is linked to the finite element analysis software STRAND6 to carry out the proposed optimization process.

Similar to the program for truss topology optimization, the program ESOSHAPE also consists of four parts. The first part is designed to check constraint violation. If one of the specified displacements reaches its given limit, the optimization process will be terminated. The second part is used to calculate the element sensitivity numbers, and where the optimization process is terminated once a uniform state of the sensitivity numbers is reached. The third part is assigned the task of removing elements, which have the lowest sensitivity numbers. The final part involves checking element connectivities and removing any elements which have insufficient connections to maintain non-singularity of the stiffness matrix. This program is developed as a post-processor to the finite element analysis package where a batch file is set up to create a loop for the iterative process of analysis and element removal.

The program makes use of the features of STRAND6 (G+D Computing, 1991). Removal of a plate element is done by assigning zero to the property number of this element. In many cases, an even number of elements are removed from the structure in each iteration to maintain symmetry of the structure. It is possible to keep a number of elements untouched (i.e. not removed) during the optimization process. For this purpose their properties are specified as non-design properties. Elements with non-design properties, called non-design elements, will not be removed during the optimization process.

The program can be switched to either *topology optimization* (simultaneous topology and shape) or to "pure" *shape optimization* by assigning 0 or 1 to the hole control parameter. For topology optimization, elements are allowed to be removed from any part of the structure (holes are created). For shape optimization, elements are allowed to be removed only from the existing boundaries (no internal holes are created). In the shape optimization mode, before the element removal phase, the number of elements connecting to each node in the current design is calculated and stored in an array INTER(INODE) where INODE is nodal identification number. Depending on the type of elements used, these numbers are used to distinguish the internal nodes from the boundary nodes and then the internal elements from the boundary ones. For instance, for 4-node plate element meshes, the maximum number of elements connecting to a node is 4. The nodes, to which 4 elements are connected, i.e. INTER(INODE) = 4, are the internal nodes. Other nodes are boundary nodes. The elements connecting to 4 internal nodes are internal elements. If shape optimization is switched on, the internal elements are not removed. Only the boundary elements are removed during the element removal phase. Similar strategies can be used for other types of elements.

In both topology and shape optimization modes, after the element removal phase, the number of remaining elements connected to each node is calculated and stored in the array INTER(INODE). These values are used to identify the insufficiently connected elements. For a plate element, if there is only one node which has the corresponding value of INTER(INODE) greater than unity, then the element has insufficient connectivities (connection to other elements by only one node). All insufficiently connected elements are removed to prevent the stiffness matrix from being singular.

The input file for the program ESOSHAPE provides the values of optimization parameters, which control the optimization process. Data in the input file for ESOSHAPE include the number of real load cases, the number of constraints, the element removal ratio, the values for the hole control parameter and for the symmetry control parameter, non-design properties and the displacement limits. Data for real load cases have to be input first. The number of real load cases is provided to distinguish real load data from virtual unit load data. To improve the numerical accuracy, virtual unit loads are replaced by virtual loads with magnitudes of the order as the real loads, and an associated virtual load factor (magnitude) is specified.

Output files of the program ESOSHAPE provide information including the actual weight, the values of constrained displacements, the maximum and minimum sensitivity numbers at each step. At the end of each iteration, the plate element connection file is copied and stored in a separate file for later recovery of the structure's topology.

5.4 Examples of topology optimization of continuum structures

This section illustrates the capability of the proposed method for solving shape and topology optimization of two-dimensional continuum structures. Examples include structures subjected to multiple displacement constraints and multiple load cases. Most examples are chosen from existing literature to show the validity of the proposed method. Some initial results of these examples have been reported in Chu *et al.* (1995, 1996a) where the element virtual energies (contributions) have been used as a basis for element removal criteria. Because regular meshes are used in all examples by which all elements have equal weights, these results will remain the same under the newly proposed method based on the optimality criteria approach. They are revised and presented in this section with an additional step to obtain the objective weight for identifying the absolute optimum topology for each example.

5.4.1 *Short cantilever*

The design domain for a cantilever beam is shown in Fig. 5.1. This short cantilever beam is under plane stress conditions. The left hand side of the beam is fixed and a vertical load $P = 3$ kN is applied at the middle of the free end. The dimensions of the beam are $L_x = 0.16$ m, $L_y = 0.10$ m and a thickness $t = 0.001$ m. The Young's modulus $E = 207$ GPa and the Poisson's ratio $\nu = 0.3$ are assumed. The initial vertical displacement at the middle of the free end is 0.33 mm. This example has been solved by Suzuki and Kikuchi (1991) using the homogenization method.

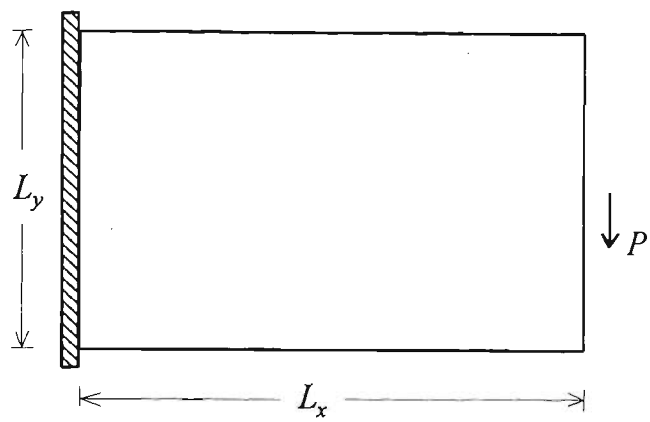


Fig. 5.1 Design domain for a short cantilever.

The design domain is divided into 32 × 20 four-node quadrilateral elements. The optimization process is carried out using an element removal ratio of 2% of initial elements, and symmetry is maintained by removing an even number of elements (12 elements in each iteration). The limit of 1 mm on the vertical displacement is imposed.

The effectiveness of the proposed sensitivity number for element removal can be seen from the change in the specified displacement during the optimization process as shown in Fig. 5.2a. The figure clearly shows that the removal of elements has negligible effects on the specified displacement in the first ten iterations. From iteration 11 to 25 the specified displacement increases slowly. However significant changes in the displacement occurs between iterations 30-31 and 33-34, which correspond to the large changes in the topology of the design.

It is worth noting that the sensitivity numbers of elements become more uniform during the optimization process. The minimum sensitivity number, which is in the

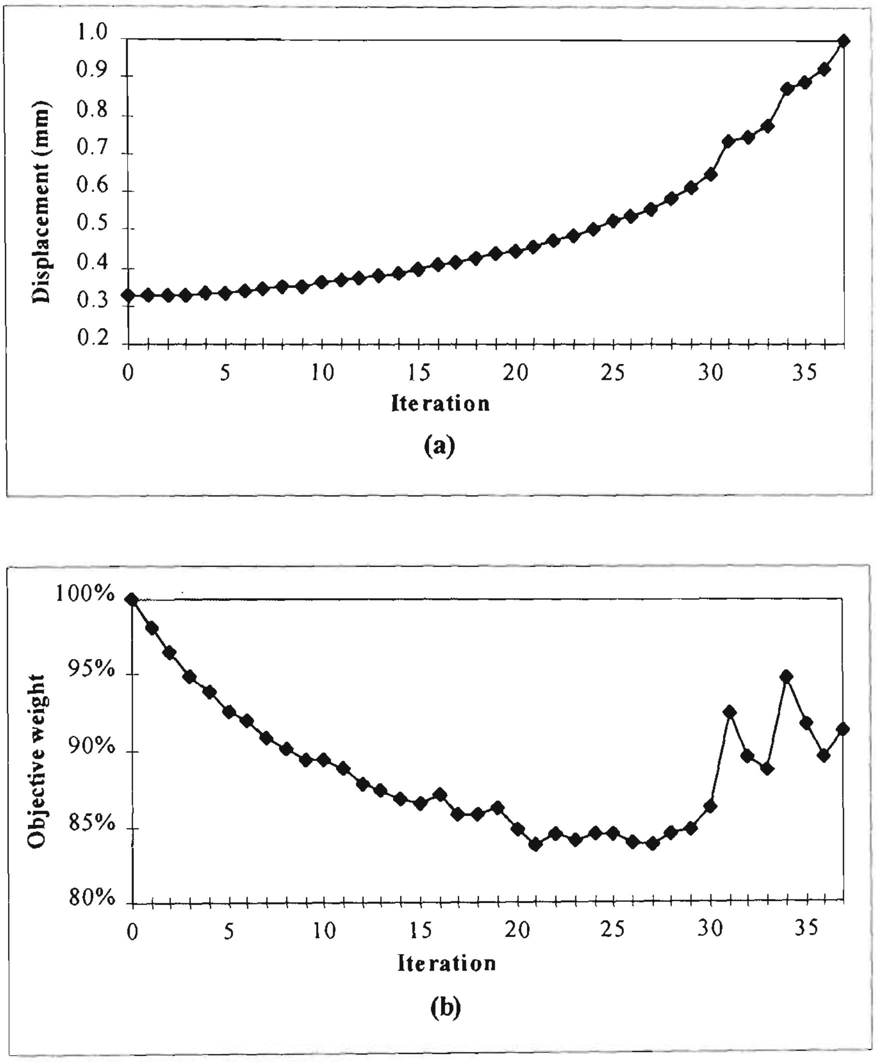
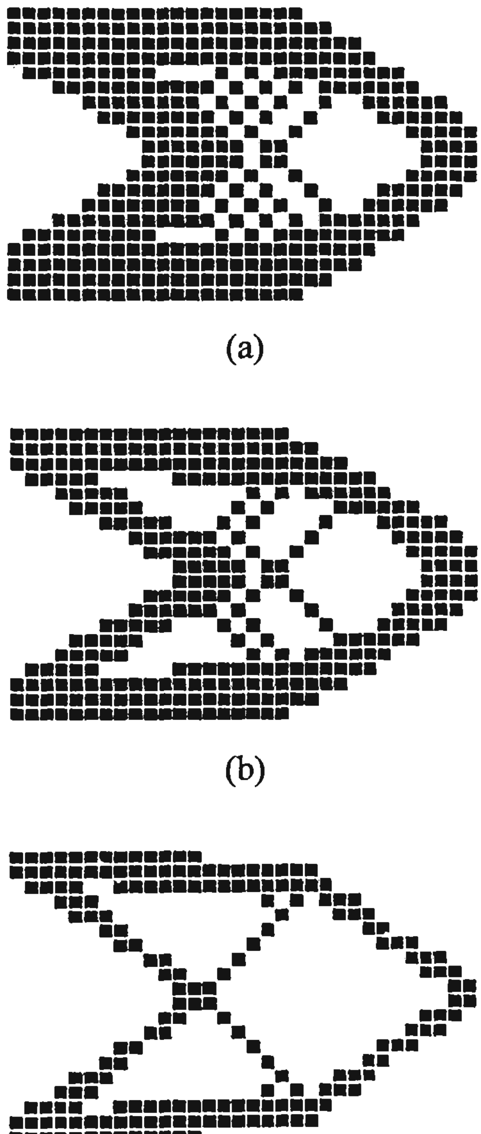


Fig. 5.2 Evolutionary history for the short cantilever:
(a) constrained displacement; (b) objective weight.

order of 10^{-5} of the maximum over the initial domain, converges to the order of 10^{-2} of the maximum, when the displacement reaches the limit of 1 mm at iteration 37.

Similar to truss structures, in the plane stress conditions, the stiffness matrices of plate elements and of the plate structure are linear functions of the thicknesses. The relative objective weight in this case can be also calculated by equation (4.3). The pattern of change in the relative objective weight is shown in Fig. 5.2b, which is also independent of the limit. It is seen that the objective weight steadily reduces in the



(a)

(b)

(c)

Fig. 5.3 ESO topologies for the short cantilever:
(a) iteration 21; (b) optimum at iteration 27; (c) at iteration 37.

first 15 iterations, with minor increases at iterations 16 and 19, and reaches the near-minimum at iteration 21 with the value 83.73% of the initial. The minimum of 83.70% is obtained at iteration 27. The topologies obtained at iterations from 20 to 29 have the objective weights less than 85%, indicating they are almost equally efficient. Sharp increases are observed in the last few iterations. The final topology has the objective weight of 91.28% of the initial. Fig. 5.3 shows the near-optimum design (iteration 21), the optimum topology (iteration 27) and the final topology (iteration 37) obtained for the problem. These results are very similar to solutions obtained by Suzuki and Kikuchi (1991) using the homogenization method (see Fig. 2 in Appendix II for comparison).

The probable reason for sharp increases in the objective weight at some iterations may due to significant changes in the topology, which are followed by redistribution of contributions among the remaining elements. The large increases in the objective weight in the last few iterations may be due to the fact that the plate element does not work efficiently in tension or compression along one of its diagonals, which consequently reduces the overall efficiency of the design. Another reason for moving away from the optimum is that the specified number of elements to be removed is too large for the last few iterations. This problem is also observed in other optimization methods when the chosen move limit or step size is not small enough.

5.4.2 *The Michell type structure*

In this example a test to check whether the proposed simple method can reproduce the classical Michell type structure (Michell 1904; Hemp 1973) is carried out. An optimum structure is to be designed to transfer a vertical force P to the circular fixed support.

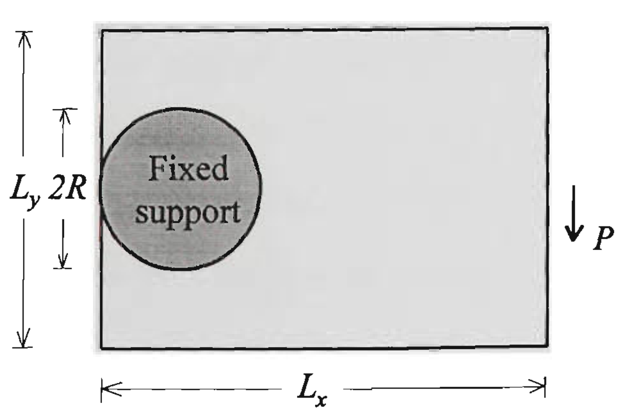
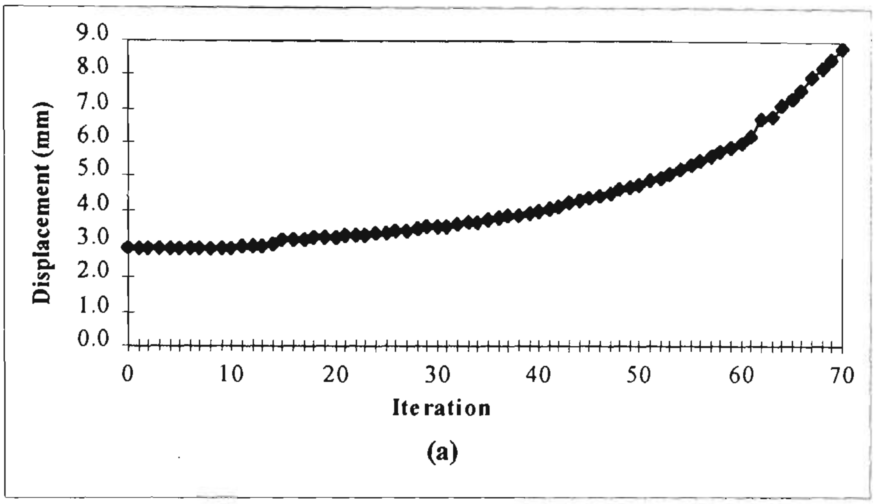


Fig. 5.4 Design domain for the Michell type structure.



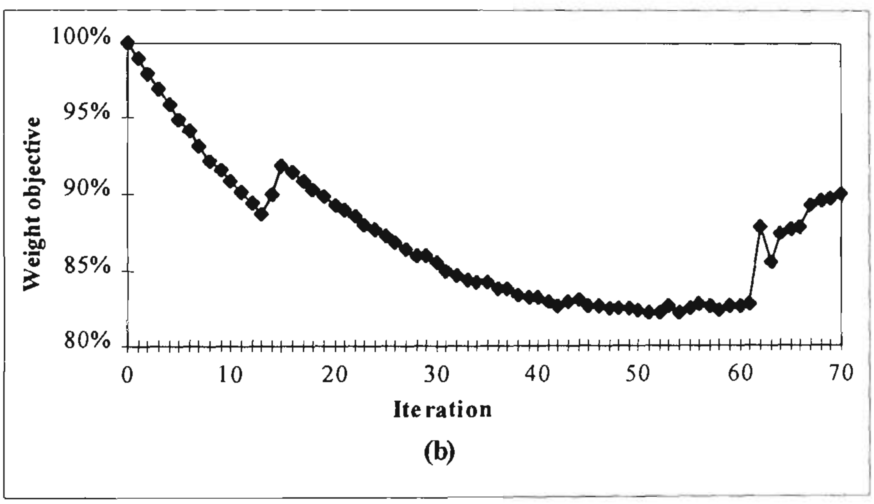
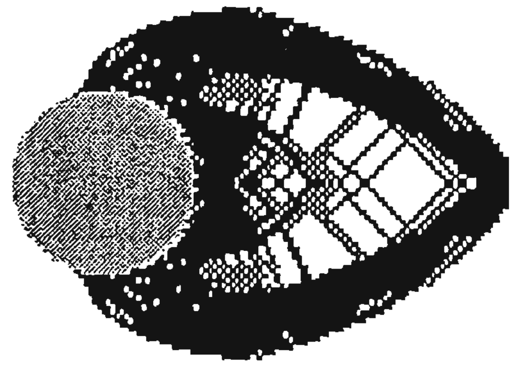


Fig. 5.5 Evolutionary history for the Michell type structure:

(a) constrained displacement; (b) objective weight.

A rectangular design domain L_x = 0.55 m and L_y = 0.4 m similar to that used by Suzuki and Kikuchi (1991) is adopted in this study as illustrated in Fig. 5.4. The radius of the circular fixed support is R = 0.1 m. The Young's modulus E = 205 GPa and the Poisson's ratio ν = 0.3 are assumed. For a thickness t = 0.001 m and a force P = 50 kN, the initial vertical displacement at the loaded point is 2.87 mm.

The whole design domain (including the fixed support area) is divided into 110 × 80



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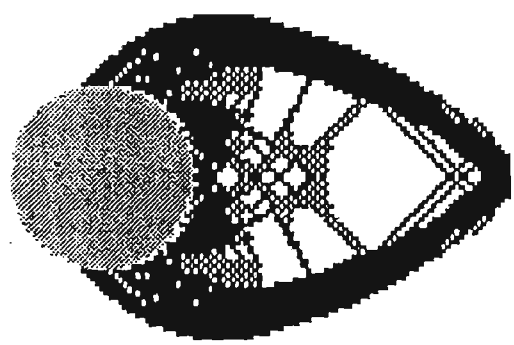

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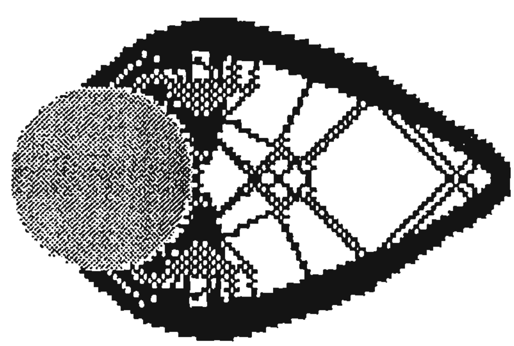


(c)

Fig. 5.6 ESO topologies for the Michell type structure:

(a) optimum at iteration 54; (b) iteration 63; (c) iteration 70.

four-node quadrilateral elements. The circular fixed support is approximated by fixing nodes close to the circle. The elements inside the support area are removed. There are 7520 elements in the design area. A constraint on the vertical displacement at the loaded point is imposed.

Due to symmetry, only a half-model is analysed. The element removal ratio of 1% of the initial elements is used. The solution process using the proposed evolutionary optimization method continues until the vertical displacement at the loaded point reaches the limit of 9 mm.

It can be seen in Fig. 5.5a that the displacement is almost unchanged in the first 13 iterations and then slowly approaches the specified value. At the same time, despite increases at iterations 14 and 15, the objective weight reduces to the minimum value of 82.15% of the initial objective weight at iteration 54. The topologies generated by the method at iterations from 41 to 61 are near-optimum with the relative objective weights less than 83%. Similar to the previous example, less efficient topologies are obtained in the following steps with increases in the objective weight. It is seen that the method can create savings up to 17.85% of the initial objective weight.

The optimal topology obtained at iteration 54 is shown in Fig. 5.6a. Figs. 5.6b and 5.6c present topologies at iterations 63 and 70 with relative objective weights of 85.57% and 90.01%, respectively. All ESO shapes in Fig. 5.6 are similar to the results obtained by Suzuki and Kikuchi (1991) (see Fig. 6 in Appendix II).

5.4.3 *The MBB beam*

The beam is designed to carry the floor in the fuselage of an Airbus passenger carrier. This support beam is produced by MBB in Germany. The initial geometry with loads and boundary conditions of the MBB beam is shown in Fig. 5.7. This problem has previously been solved by several researchers as one of the most challenging problems in topology optimization, e.g. Olhoff *et al.* (1991), Rozvany and Zhou (1991) and Zhou and Rozvany (1991). It will be shown that similar results for this beam can be easily obtained using the proposed ESO method.

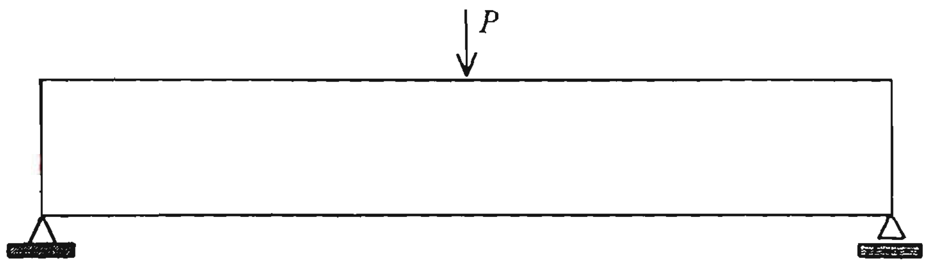


Fig. 5.7 Design domain for the MBB beam.

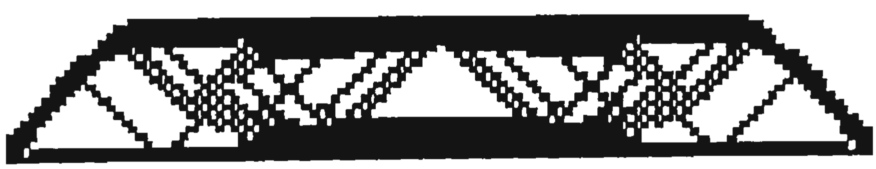


Fig. 5.8 Optimal topology for the MBB beam.

The beam is 2400 mm long and 400 mm deep with a point load of 20 kN acting at the middle. The Young's modulus E = 200 GPa and the Poisson's ratio ν = 0.3 are assumed. The initial volume is 1.07×10^6 mm^3. The initial displacement at the loaded point is 6.3 mm.

Due to symmetry, only a half of the structure is modelled with 60 $\times$ 20 four-node quadrilateral elements. The element removal ratio of 1% of the current number of elements is used. So the number of elements to be removed at each iteration is decreasing as the current number of elements reduces. The optimal shape for the MBB beam by the ESO method is shown in Fig 5.8. The trend of the change in the displacement in this example as shown in Fig. 5.9a is similar to other examples. The objective weight reaches the minimum value of 74.28% at iteration 67 and slightly increases to 74.29% at the final iteration as seen in Fig. 5.9b. All ESO topologies generated from iteration 63 to the final iteration have the relative objective weights

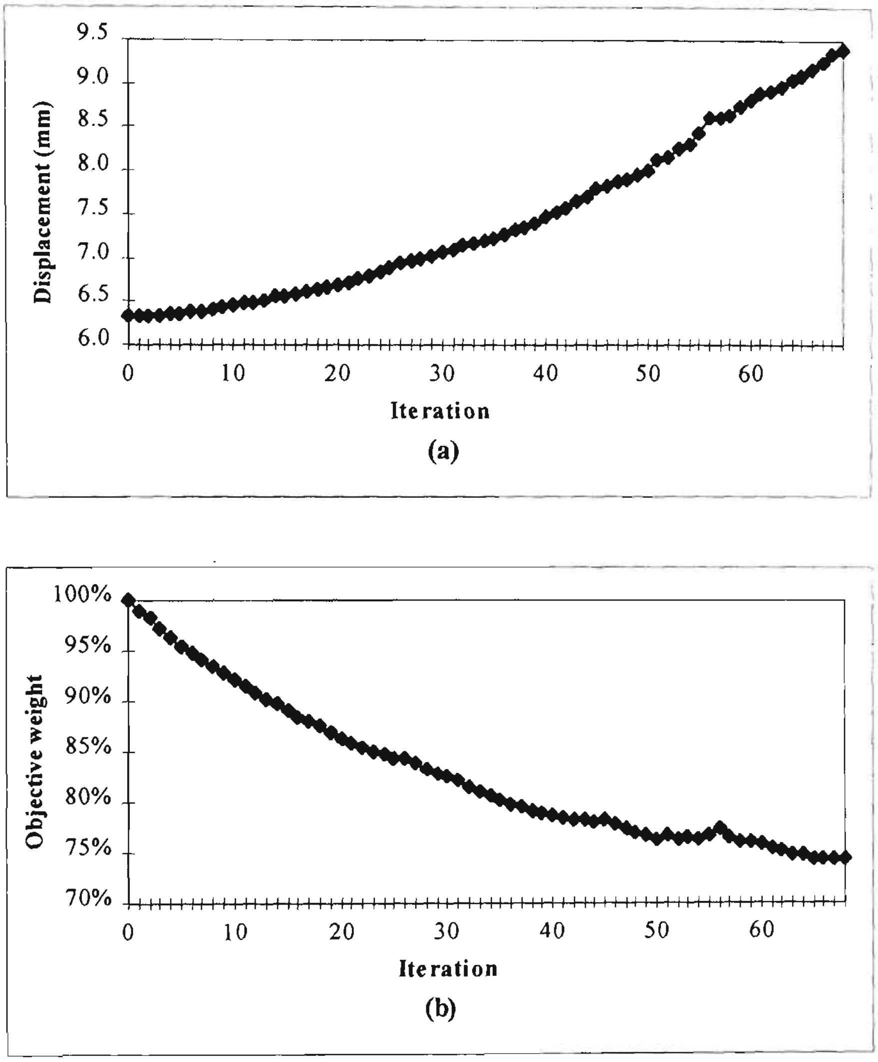


Fig. 5.9 Evolutionary history for the MBB beam:

(a) constrained displacement; (b) objective weight.

less than 75%. The largest weight saving is 25.72% of the initial objective weight.

5.4.4 *Plate in bending*

A simply supported square plate (0.20 m × 0.20 m × 0.0001 m) is loaded at the centre by a point load P = 0.04 N normal to its plane, which is shown in Fig. 5.10. The Young's modulus E = 174.7 GPa and Poisson's ratio ν = 0.3 are assumed. The initial out-of-plane displacement at the centre is 1.16 mm.

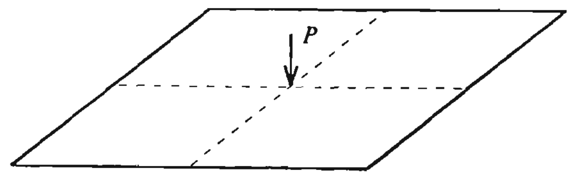


Fig. 5.10 Initial design for a plate in bending.

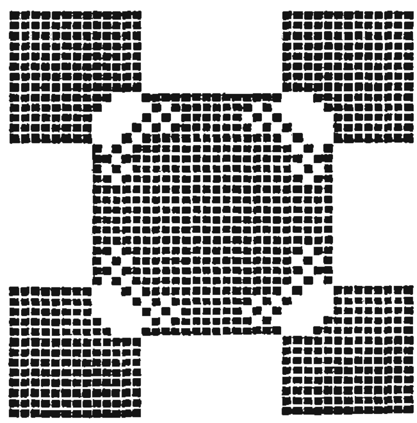
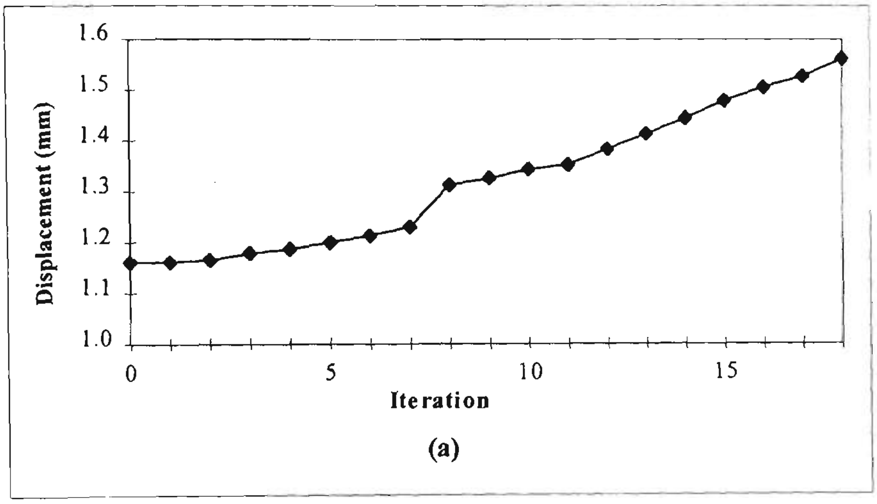


Fig. 5.11 Optimal design for the plate in bending.

Due to symmetry, one quarter of the plate is modelled with 20 × 20 four-node quadrilateral plate elements. In order to maintain symmetry, an even number of elements is removed at each step. The limit of 1.6 mm is imposed on the out-of-plane displacement at the centre.

Using an element removal ratio of 2% of the initial number of elements, the optimum design is obtained as shown in Fig. 5.11. This result is very close to the solution obtained by Atrek (1989) (see Fig. 7 in Appendix II for comparison). Hinges appear in the optimal design for the plate by the proposed ESO method. Such hinge lines have been previously reported in Tenek & Hagiwara (1994).

The optimization history for the plate in bending is shown in Fig. 5.12. It is seen that the displacement at the centre increases very slowly in the first 6 iterations. A large



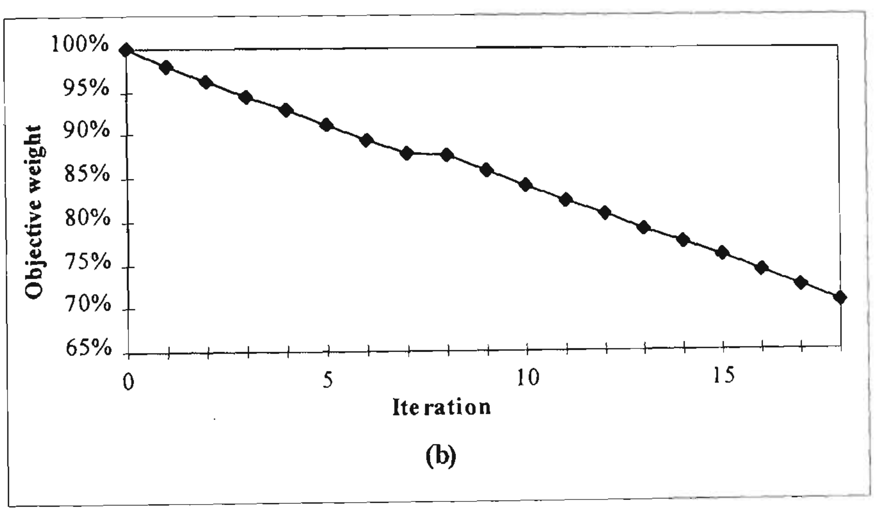


Fig. 5.12. Evolutionary history for a plate in bending:

(a) constrained displacement; (b) objective weight.

increase is observed in the following iteration and it then gradually increases to the specified limit.

The objective weight for this example can be obtained by the following procedure. It is known that the stiffness matrix of the simply supported plate in bending is a function of the cube of the thickness. When the thickness is scaled by a factor a, the stiffness matrix will be scaled by a^3, and the displacement will be reduced by $1/(a^3)$. Conversely, to get to the limit u^*, the actual displacement u^{act} should be scaled by the ratio u^*/u^{act}, which can be done by scaling the thickness by the cubic root of the reciprocal u^{act}/u^*. Therefore, the objective weight for any ESO topology is obtained as

$$W^{obj} = W^{act} \times \left(\frac{u^{act}}{u^*} \right)^{\frac{1}{3}} \tag{5.1}$$

Similarly the initial objective weight is calculated from the initial actual weight W_0^{act} and the initial displacement u_0^{act} as

$$W_0^{obj} = W_0^{act} \times \left(\frac{u_0^{act}}{u^*} \right)^{\frac{1}{3}} \tag{5.2}$$

Thus, the relative objective weight is

$$W_{rel}^{obj} = \frac{W^{obj}}{W_0^{obj}} = \frac{W^{act}}{W_0^{act}} \times \left(\frac{u^{act}}{u_0^{act}} \right)^{\frac{1}{3}} = W_{rel}^{act} \times \left(\frac{u^{act}}{u_0^{act}} \right)^{\frac{1}{3}} \tag{5.3}$$

where W_{rel}^{act} is the relative actual weight. It is seen in equation (5.3) that the limit u^* is eliminated, which means that the pattern of the relative change in the objective weight does not depend on the specified value for the limit in this case.

The change in the objective weight in this example is shown in Fig. 5.12b. The objective weight reduces successively and reaches the minimum value of 70.68% at iteration 18. The connection between the central part and corners is broken up in the next step which means no further solution can be obtained for this structure.

5.4.5 *Structure under multiple displacement constraints*

A structure is to be designed to support three loads, each at 10 kN, under the given boundary conditions shown in Fig. 5.13. The dimensions for the design domain are $L_x = 0.20$ m, $L_y = 0.10$ m and thickness $t = 0.005$ m. The Young's modulus $E = 207$

GPa and the Poisson's ratio $\nu = 0.3$ are assumed. The maximum initial vertical displacement is 0.22 mm.

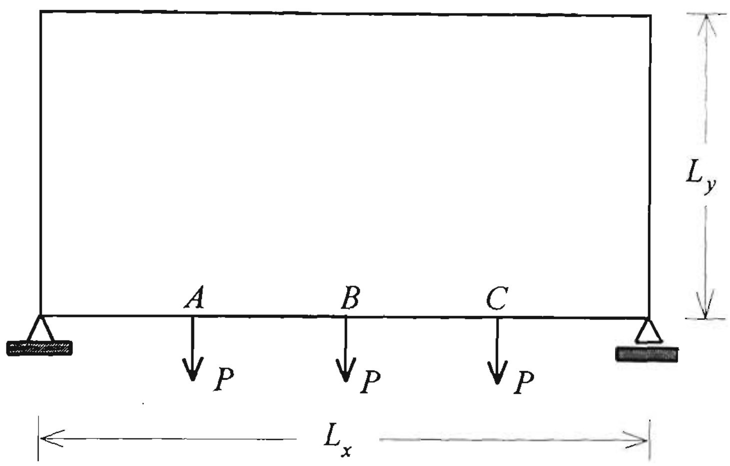


Fig. 5.13 Design domain for a structure under multiple displacement constraints.

Due to symmetry, only a half of the structure is analysed using a mesh of 40×40 four-node quadrilateral elements. The element removal ratio of 1% of the initial elements is used and the limits are set at 0.45 mm for all vertical displacements at the loaded points *A*, *B* and *C*. The ratio formulae (3.38) are used to determined the Lagrange multipliers. The evolutionary optimization process continues until the maximum displacement becomes greater than the limits. No elements were removed on the basis of having insufficient connections to other elements.

The changes in the displacements at the loaded points are shown in Fig. 5.14a, where the constrained displacements increase slowly during the optimization process. The displacement at *B* is always the maximum. The objective weight for this type of structure is calculated by the same formula (4.9) given for truss structures under multiple constraints. As seen in Fig. 5.14b the objective weight continuously reduces, except in iteration 36, to the minimum value of 59.49% at iteration 68 and then

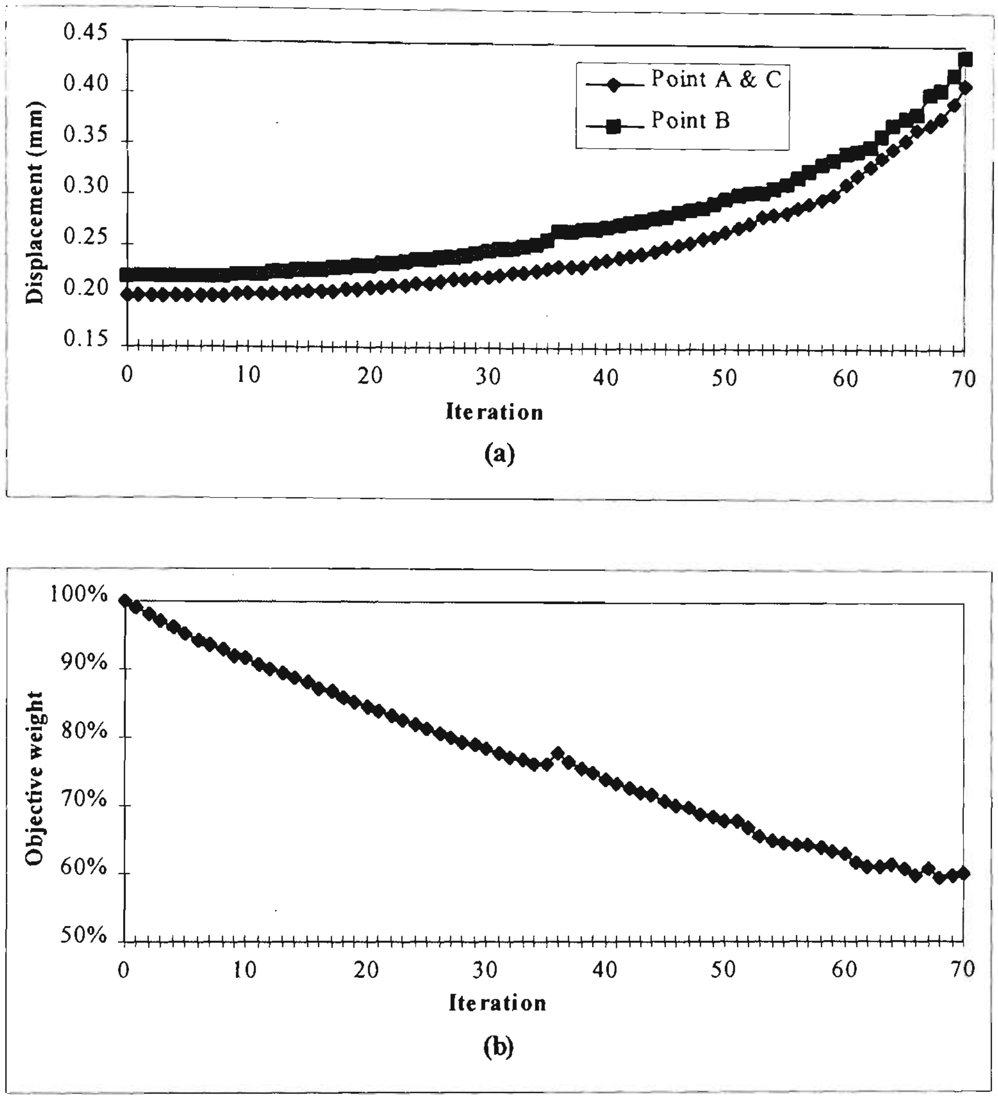
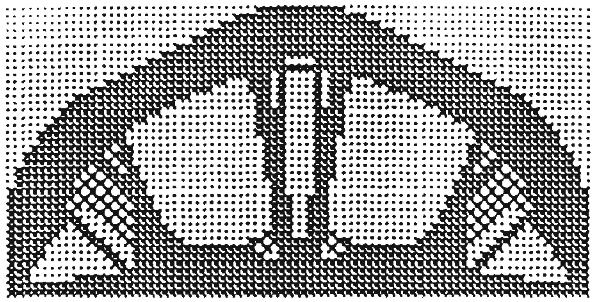
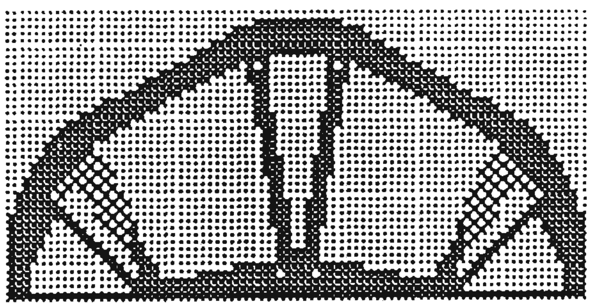


Fig. 5.14 Evolutionary history for structure under multiple displacement constraints: (a) constrained displacements; (b) objective weight.

increases slightly to the value of 60.12% at iteration 70. From iteration 61, all obtained topologies have the objective weights less than 62%. Weight savings greater than 40% have been achieved by the proposed method. It should be noted that to ensure the absolute minimum, the limits should be set at larger values to follow further changes in the objective weight. Fig. 5.15 present the optimum topology at iteration 68 and the other topologies at iterations 50 and 70. In this figure, the black areas represent the remaining elements. The light grey areas represent the nodes of the initial FEA model



(a)



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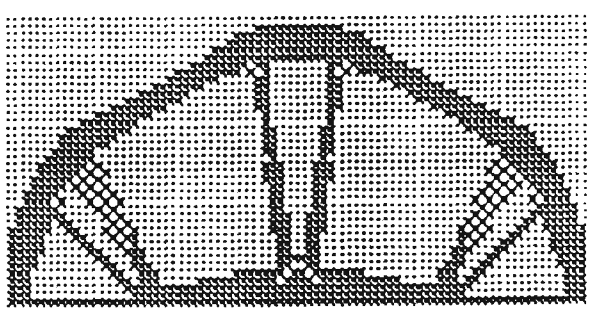


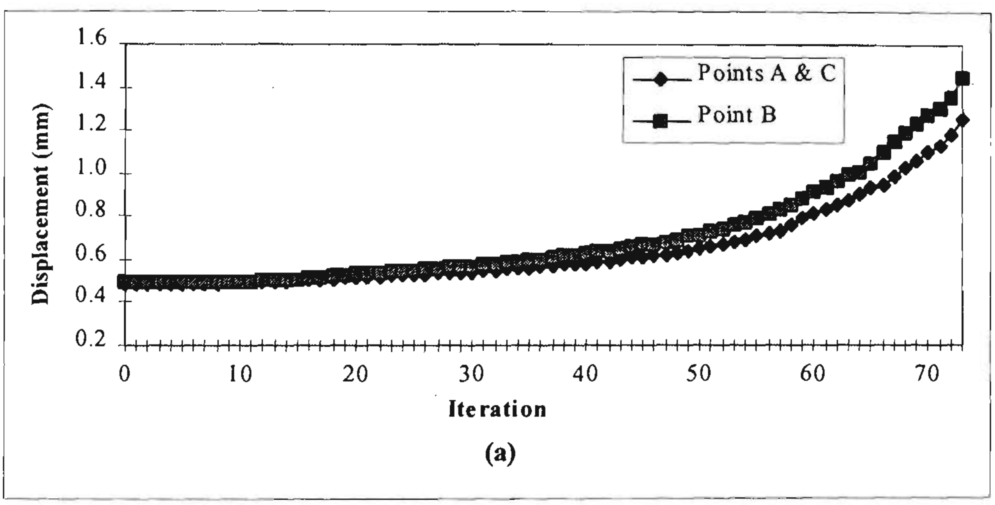
(c)

Fig. 5.15 ESO topologies for the structure under multiple displacement constraints: (a) iteration 50; (b) optimum at iteration 68; (c) iteration 70.

The optimum topology (Fig. 5.15b) has the minimum objective weight of 59.49% of the initial. The topologies in Figs. 5.15a and 5.15c have the objective weights of 67.89% and 60.12% of the initial. Similar solutions to this example have been obtained by Díaz and Bendsøe (1992) using the homogenization method and by Yang and Chuang (1994) using the density function approach (see Fig. 4 in Appendix II for comparison).

5.4.6 *Structure under multiple load cases*

Consider the same structure as in the previous example in Fig. 5.13, but with each of the three given loads acting at different times, i.e. the structure is subjected to three load cases, each load case consists of a single point load. For the thickness $t = 0.001$ m, the maximum of initial vertical displacements for the three load cases is 0.49 mm. Due to asymmetry of the loads, the whole structure has to be analysed. A mesh of 80×40 four-node quadrilateral elements is used.



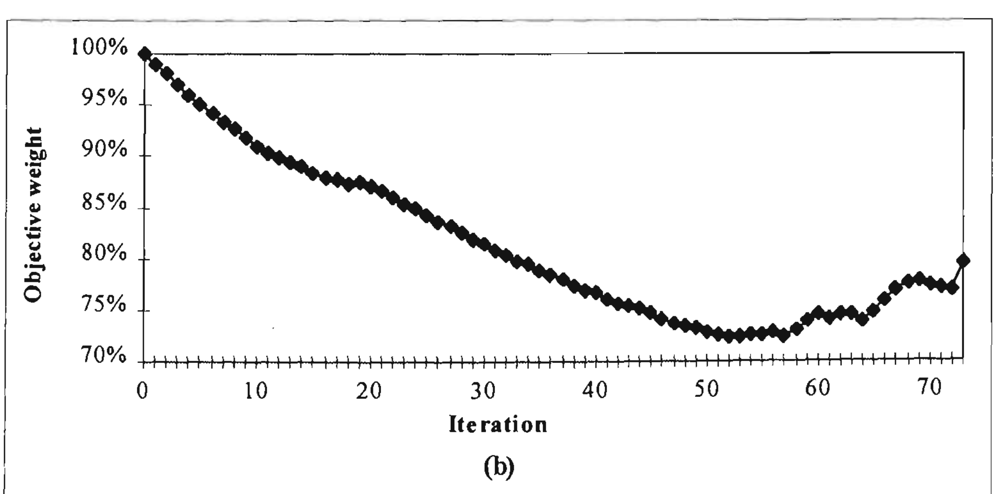
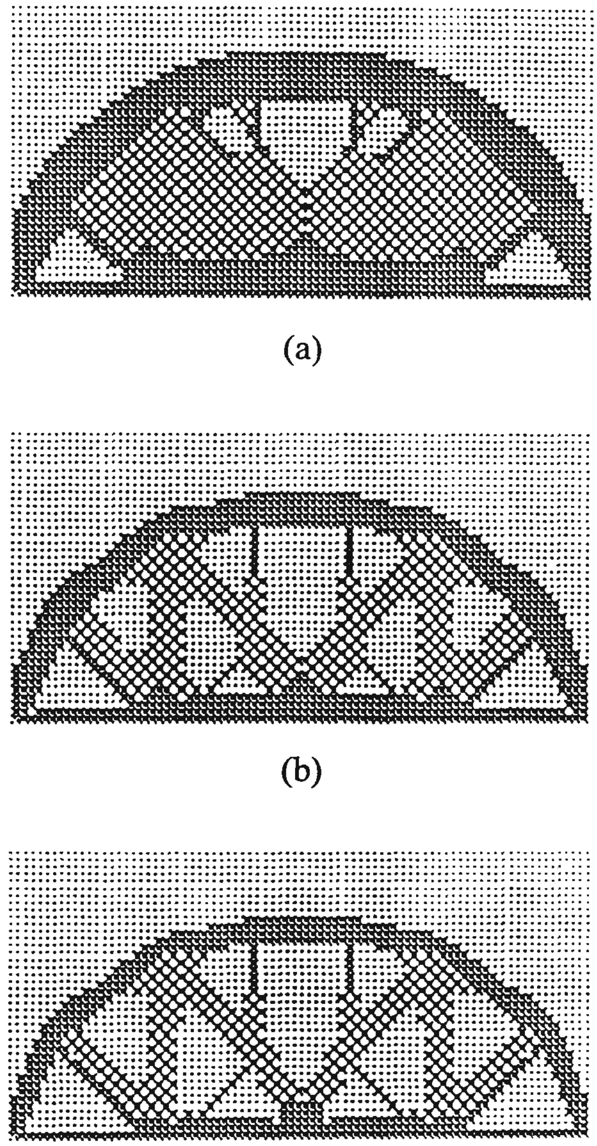


Fig. 5.16 Evolutionary history for structure under multiple load cases: (a) constrained displacements; (b) objective weight.



(a)

(b)

(c)

Fig. 5.17 Optimal designs for the structure under multiple load cases:
(a) optimum at iteration 52; (b) iteration 66; (c) iteration 73.

The element removal ratio of 1% of the initial elements is used which allows an even number of elements to be removed at each step to maintain the symmetry of the structure. The limits of 1.5 mm are set for all displacements at the loaded points A, B and C. For the given three load cases, there are a total of nine constraints. It is obvious that for each load case the displacement at the loaded point is always the maximum. Only one constraint corresponding to this maximum displacement in each load case is considered as the active constraint. Thus, for this problem only three constraints, one for each load case, are included in the active set. The ratio formulae (3.38) are used to calculate the Lagrange multipliers for the active constraints.

The changes in the active displacements are shown in Fig 5.16a, which are similar to the case where the structure is under one load case in the previous example. The change in the objective weight is different as seen clearly in Fig. 5.16b. The objective weight gradually reduces to the minimum value of 72.29% at iteration 52 and then increases until the limit is reached. The topologies obtained from iteration 50 to 58 have the objective weights of less than 73% which are near the minimum value. The savings of up to 27.71% can be obtained in this case. The optimum topology is shown in Fig 5.17a. The ESO topologies obtained at iterations 66 and 73 are shown in Figs. 5.17b and 5.17c with the corresponding relative objective weights of 75.81% and 79.46%. The meaning of black and light grey areas is the same as given in the previous example. These results are similar to the solutions obtained by Díaz and Bendsøe (1992) using the homogenization method.

5.4.7 *Bridge with a moving load*

The initial design for the bridge is shown in Fig. 5.18. The body of the bridge, with dimensions of 16 m × 5 m × 0.1 m, is supported by four solid piers underneath. A point load P = 1000 kN, travelling from the left to the right of the bridge on the top surface, is approximated by 9 load cases with an equal distance of 1.75 m between each other. The Young's modulus E = 30 GPa and the Poisson's ratio ν = 0.2 are assumed. The maximum initial vertical displacement for all load cases is 1.39 mm.

The whole structure is modelled by a mesh of 64 x 40 four-node quadrilateral elements and each solid pier is approximately represented by four fixed nodes.

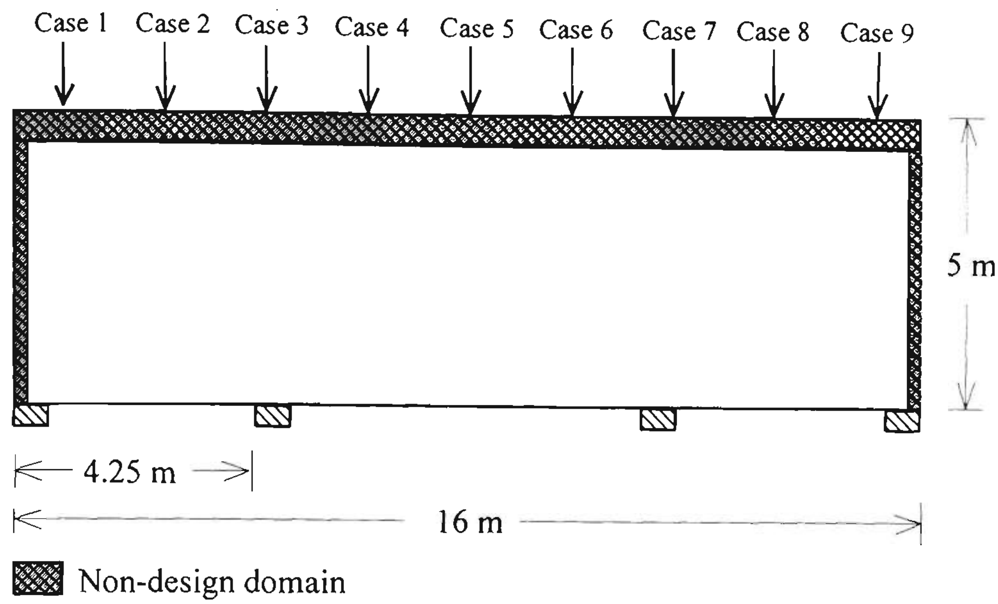


Fig. 5.18 Initial design for a bridge with a moving load.

Minimum thicknesses of 0.5 m and 0.25 m are required at the top and along two sides of the bridge, respectively. This can be done by specifying properties of the elements on the top and along the sides as non-design properties. The limits of 2 mm are imposed on the vertical displacements at the loaded points and the element removal ratio of 1% of the initial number of elements is used. Similar to example 5.4.6 only one constraint on the displacement at the loaded point in each load case is included in the active set. The ratio formulae (3.38) are used to calculate the Lagrange multipliers for these nine active constraints.

The ESO method is applied to the structure for the following cases:

(I) shape optimization: elements are removed from the bottom boundary (no internal holes are allowed).

(II) topology optimization: element are removed from any part of the design domain except the above specified non-design areas (internal holes are allowed).

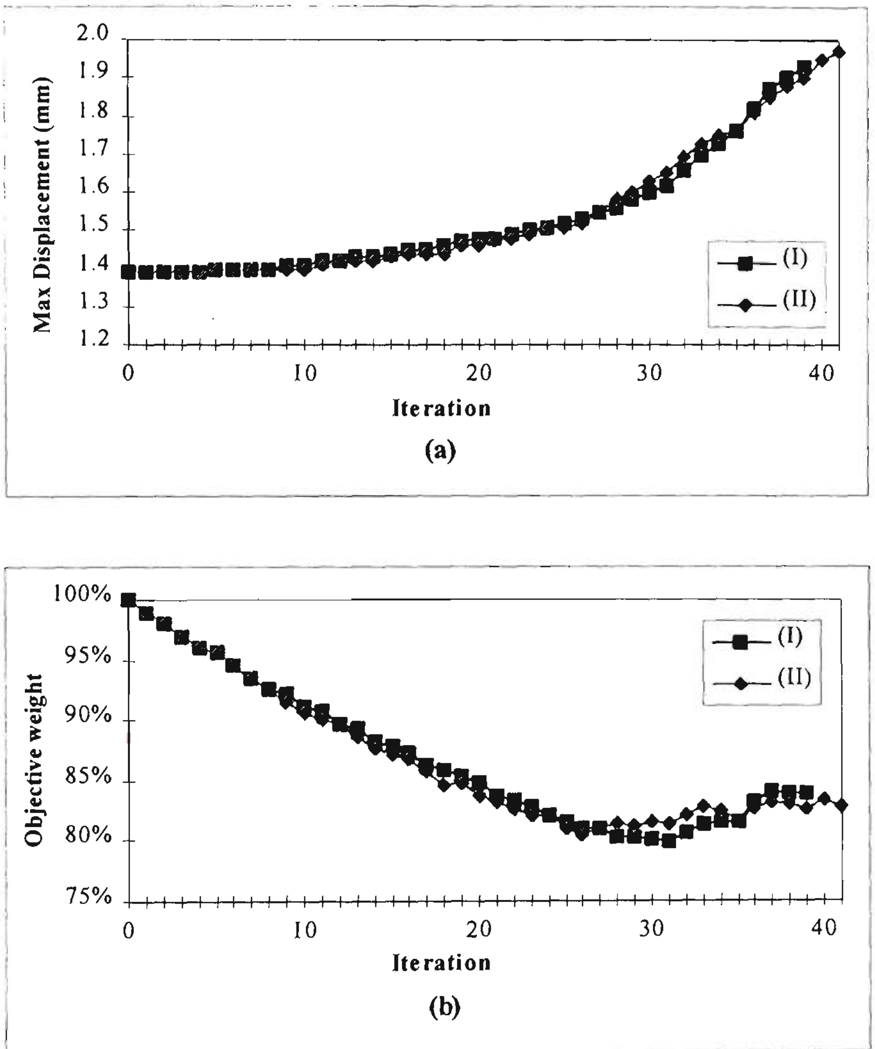
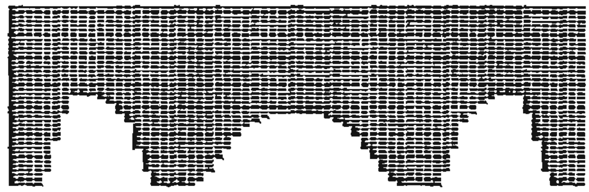


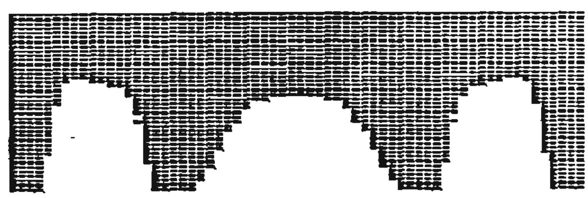
Fig. 5.19 Evolutionary history for the bridge with a moving load:

(a) maximum displacement; (b) objective weight.

The changes in the maximum displacements and objective weight in both cases are very similar as seen in Fig. 5.19. In shape optimization, case (I), where no internal holes are allowed, the objective weight reaches the minimum of 79.85% at iteration 31 and all topologies at iterations from 26 to 35 have the objective weights less than 81% of the initial value. The optimum topology is shown in Fig. 5.20b. Other ESO topologies obtained at iterations 22 and 39 for case (I) are shown in Figs. 5.20a and 5.20c. These topologies have the objective weights of 83.24% and 85.43% of the initial value, respectively.



(a)



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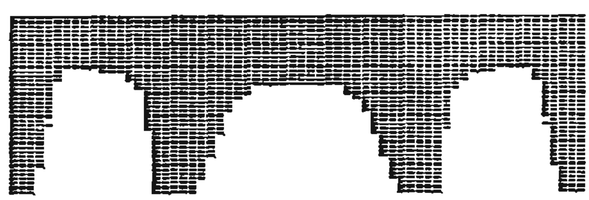


(c)

Fig. 5.20 ESO shapes for the bridge with a moving load by shape optimization: (a) iteration 22; (b) optimum at iteration 31; (c) iteration 39.

In topology optimization, case (II), where internal holes are allowed, the minimum objective weight of 80.48% of the initial is achieved at iteration 26. The topologies derived at three iterations from 25 to 27 have objective weights less than 81% of the initial objective weight. Fig. 5.21a presents the optimum topology. Other topologies obtained at iterations 32 and 41, as shown in Figs. 5.21b and 5.21c, have the objective weights 82.07% and 82.71% of the initial objective weight, respectively.

A similar example was considered by Xie and Steven (1994a) and a slightly different optimal design was obtained by removing the lowly stressed elements.

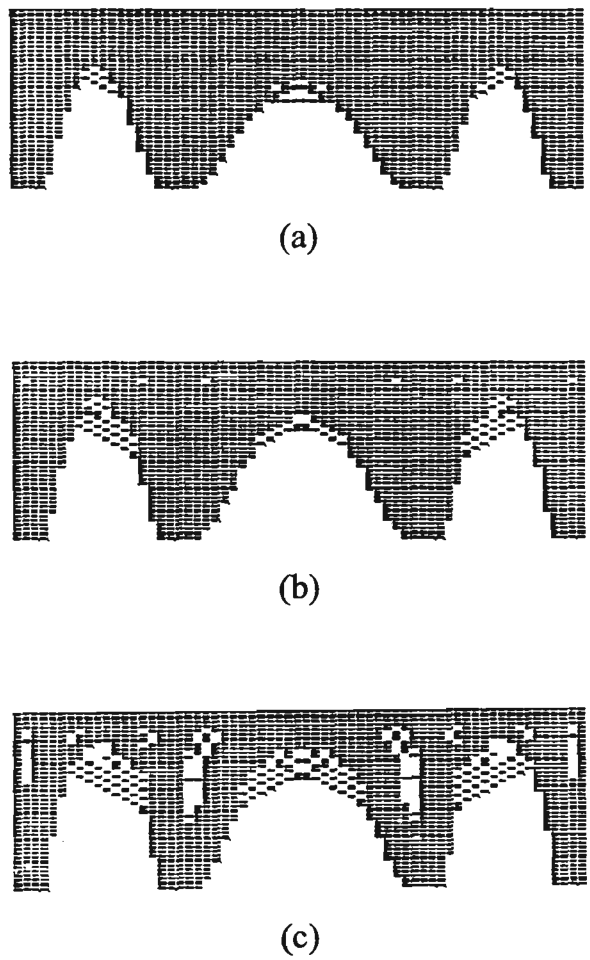


Fig. 5.21. ESO topologies for the bridge with a moving load by topology optimization: (a) optimum at iteration 26; (b) iteration 32; (c) iteration 41.

It is interesting to note that despite more freedom for element removal in topology optimization (case (II)), it does not result in more efficient designs as in shape optimization (case (I)). This may be due to the fact that plate elements are more efficiently utilized when elements are solidly stuck together as seen in Fig. 5.20 compared to the situation where many of them are working in the simple tension or compression along one of their diagonals as seen in Fig. 5.21. As more elements are removed from the design domain, more plate elements in the resulting topologies work in an inefficient manner (simple tension or compression along one diagonal). This reduces the overall efficiency of the design. The same trend has been observed in other previously provided examples.

5.5 Concluding remarks

It has been seen in this chapter that the proposed method can be easily applied to solve the shape and topology optimization problem for two-dimensional continuum structures. The shapes and topologies generated by the method are much more efficient compared with the initial design. For some examples, the minimum objective weight as low as 60% of the initial value has been achieved. The results compare well with existing solutions obtained by other methods such as the homogenization method and the density function method.

It has also been shown that, the pattern of change in the objective weight for plane stress, for plane strain and for pure bending problems does not depend on the specified limits. This is true for structures with a single constraint as well as for structures subject to multiple constraints with equal limits. It has been seen that for continuum structures, uniform sensitivity numbers cannot be reached. The constraint violation is used as termination condition for the iterative process. For most examples the topology obtained when the displacements reach the limits is not the optimum. It is suggested that the optimization process should be continued by setting larger values for the limits until no better results can be obtained.

It should be noted that the element removal ratio is an important parameter which controls the magnitude of the change between two consecutive iterations. The value chosen for this parameter will influence the optimization process. Although the values of 1% and 2% for the element removal ratio have been used, which gave desirable results for the selected examples, further investigation of its influence on the solution is clearly needed. This will be considered in Chapter 6.

Chapter 6

INVESTIGATION ON VARIOUS ASPECTS OF THE ESO METHOD FOR TOPOLOGY OPTIMIZATION

6.1 Introduction

In iterative procedures for structural optimization, parameters are introduced to control the change of the design in each step. In the proposed ESO method, the element removal ratio (ERR) is one of these parameters. It controls the magnitude of the change in each design step. The ERR plays a similar role as the move limit adopted in mathematical programming methods or the step size parameter used in existing optimality criteria methods. The value for ERR should be chosen sufficiently small enough to ensure smooth changes between successive designs. The performance of the proposed method will depends on the value chosen for this parameter. The best value of ERR for all types of structures does not exist. Therefore, investigations on the influence of the ERR on the solutions are needed to determine appropriate values for the ERR for each type of structures.

Similar to other methods based on the ground structure approach, the optimal topology derived by the ESO method can only be a subset of the initial chosen set of elements. Dorn *et al.* (1964) were the first to use the ground structure approach and pointed out that the ground structure grid had a significant effect on both the weight and layout of the optimum structure (Kirsch 1989). Therefore, the influence of the ground structure including the influences of the initial grid of joints, element connection, mesh size, and element type, on the solution by the proposed method also need to be examined.

This chapter illustrates how the weight and topology of the optimal solution by the ESO method will change by varying one of the above mentioned factors while keeping others unchanged. A detailed study on the computation time, which greatly depends on the above factors, is also carried out.

6.2 Aspects of the ESO method in truss topology optimization

6.2.1 Influence of the element removal ratio on optimal topology for trusses

It is generally recognised that, the smaller the value of the element removal ratio (ERR) is used, the more accurate is the solution, but at the expense of larger computation time. The use of larger element removal ratio will reduce the number of the elements in the resulting design more rapidly, leading to a sharp decrease in the computation time for the subsequent iteration. By studying the influence of the element removal ratio, appropriate values are recommended. In the examples given in Chapter 4, different values up to 10% of the total elements for ERR have been used to optimize different truss structures. In this section, different values of the ERR are applied to an initially chosen ground structure to show how they affect the solution.

It is obvious that for each value of ERR, two different removal strategies (RS) can be employed depending on whether the initial or the current number of elements is considered as the total number of elements. They are termed as follows:

A) Removal strategy A (RS = A): The total number of elements is the current number of elements. The ERR is defined with respect to the current number of elements. The number of elements to be removed decreases as iterations proceed.

B) Removal strategy B (RS = B): The total number of element is the initial number of elements. The ERR is defined with respect to the initial number of elements. The number of elements to be removed at each iteration is constant.

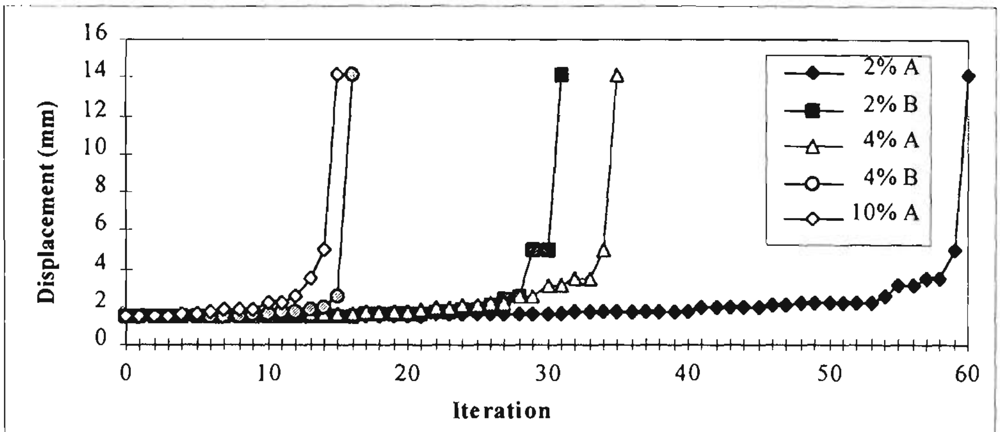
For simplicity in identification of which removal strategy to be used, a character A or B is introduced following the value of ERR. For example the expression ERR = 2%A and ERR = 2%B indicates that the element removal ratio is 2% of the current and 2% of the initial number of elements, respectively.

6.2.1.1 Truss structure with a grid of 11 × 25 nodes in a domain of 1 m × 2.4 m

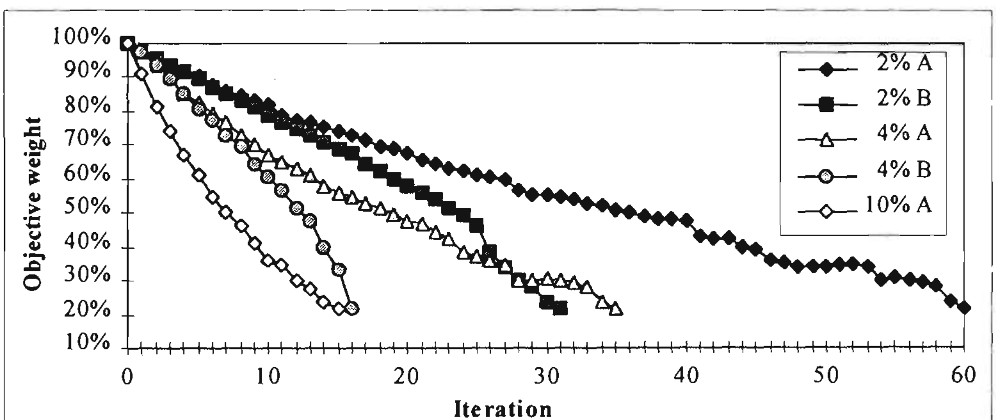
The ground structure, which is in a domain of $1 \text{ m} \times 2.4 \text{ m}$ with a grid of 11×25 nodes and 994 potential bars connecting only neighbouring nodes given in Fig. 4.4a, is used to examine the influence of the element removal ratio on the solution. The same cross-sectional areas, material properties, load and support conditions as in Chapter 4 are assumed. The initial vertical displacement at the loaded point is 1.52 mm and the limit of 15 mm on this displacement is prescribed.

The structure is optimized using the values of 2%, 4% and 10% for ERR. Only even numbers of elements are removed to maintain symmetry in the resulting designs. In combination with the two removal strategies, the following five cases are considered:

1) ERR = 2%A : The number of elements to be removed decreases from 20 to 2 during the solution process. The solution by the ESO procedure converges to the two bars in the optimal configuration as obtained by the analytical method. This result has been presented in Fig. 4. 4c.
2) ERR = 2%B : 20 bars are removed in each iteration. The solution process is terminated after iteration 29 because the number of the remaining elements is less then 20. The number of elements to be removed is then reduced to 2 and the analytical optimal topology is derived at iteration 31.



(a)



(b)

Fig. 6.1 Influence of the ERR on solution for a truss structure with a grid of 11 × 25 in 1 m × 2.4 m domain: (a) displacement; b) objective weight.

3) ERR = 4%A : The number of elements removed decreases from 40 to 2 during the solution process. The analytical optimal topology is reached at iteration 35.

4) ERR = 4%B : 40 elements are removed each time. The analytical optimal topology is obtained at iteration 16.

5) ERR = 10%A : The number of elements to be removed decreases from 100 to 2. The analytical optimal topology is obtained at iteration 15.

It is seen that the same solution is obtained by using different values for the element removal ratio. The use of smaller ERR requires more iterations and hence more computation time is needed. As the number of elements in the resulting designs decreases, the time required for each following iteration decreases sharply. The histories of the change in the displacement and objective weight are shown in Fig. 6.1 for all cases. As seen in Fig 6.1a, the specified displacement only has significant changes in the last few iterations. This proves the effectiveness of the proposed removal criterion based on the formulated sensitivity numbers. Fig. 6.1b shows that the objective weight in all cases continuously reduces to the minimum of 21.96% of the initial value.

It is observed during the optimization process that, at early stages different element removal ratios and strategies produce slightly different topologies for the same value of the constrained displacement. However, the main patterns and orientations of the remaining bars in the resulting designs are very similar. In the last few iterations the topologies obtained in all cases are identical and finally converge to the analytical solution. For this particular example, different values of the ERR has not influenced the optimal solution.

6.2.1.2 Cantilever truss with a grid of 7 × 5 nodes in a domain of 2 m × 1 m

The optimal topology design of a cantilever truss in a domain of 2 m × 1 m to transfer a vertical load to a vertical line of support is considered to examine the influence of the ERR. A ground structure with a grid of 7×5 nodes and 106 bars connecting only neighbouring nodes is chosen as shown in Fig 6.2. All nodes on the left hand side are fixed. A vertical load P = 20 kN is acting in the middle of the right hand side. The cross-sectional area A = 1 cm^2 and the Young's modulus $E = 2 \times 10^{11}$ Pa are assumed.

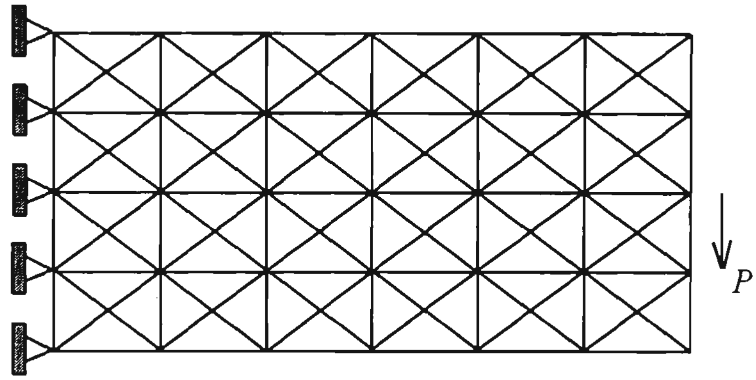
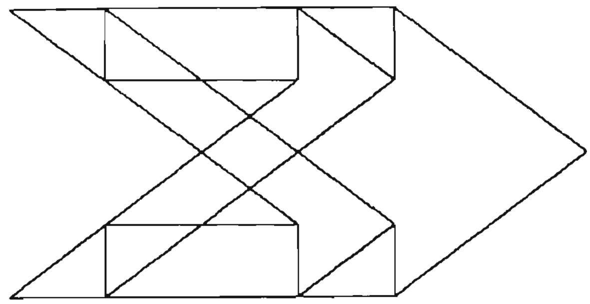


Fig. 6.2 Ground structure of a cantilever truss.

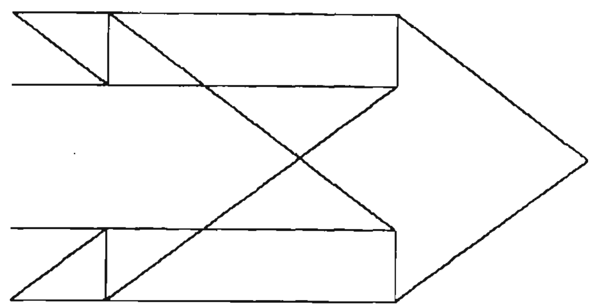
To maintain symmetry of the resulting design, only even numbers of elements are removed at each step. Thus, the minimum number of elements to be removed is 2. The following cases are considered:

1) ERR = 2% (A or B) : 2 elements are removed at each iteration. The final stable topology is obtained at iteration 20 as shown in Fig 6.3a.
2) ERR = 4%A : The number of elements to be removed decreases from 4 to 2. The final stable topology is obtained at iteration 15 which is exactly the same as in the case ERR = 2% shown in Fig. 6.3a.
3) ERR = 4%B : Four elements are removed at each iteration. The final stable topology is obtained at iteration 11 and is shown in Fig. 6.3b.
4) ERR = 10%A : The number of elements to be removed decreases from 10 to 2. The final stable design is reached at iteration 8 as shown in Fig. 6.3c.

All the final stable topologies in Fig. 6.3 are statically determinate structures. Following the change in the objective weights given in Fig. 6.4b, these topologies are optimum solutions for each case. It is seen that for the same initial ground structure different values of the ERR lead to different optimal topologies. The topology in the case ERR = 4%B is slightly different from that obtained by ERR = 2% and 4%A. The topology for ERR = 10%A is completely different from the others.



(a)



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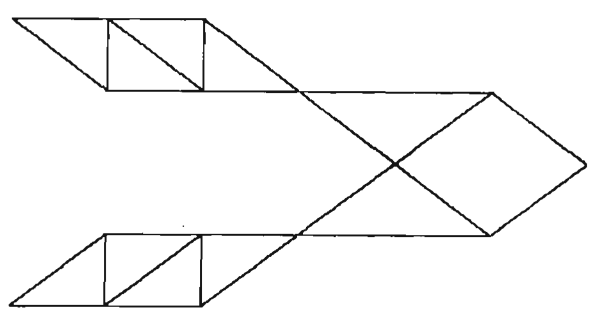
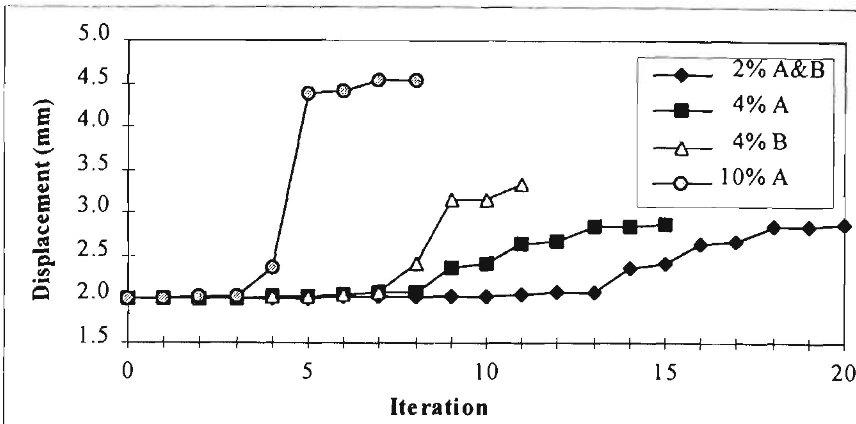


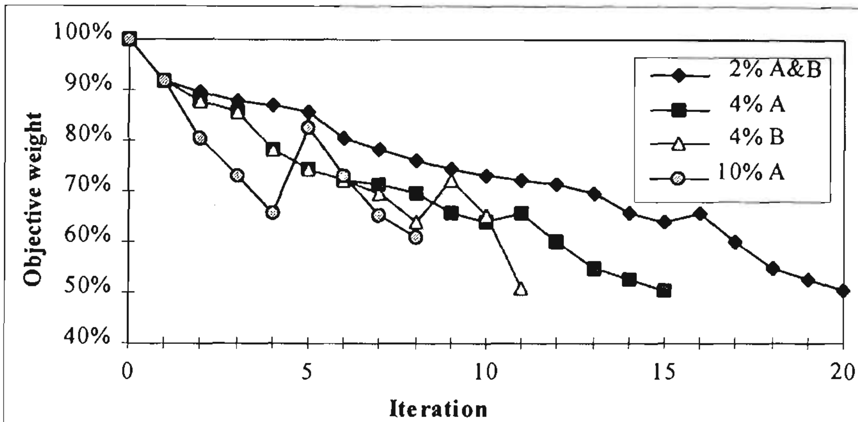
(c)

Fig. 6.3 Optimal topologies for the cantilever truss:
(a) ERR = 2% and 4%A; (b) ERR = 4%B; (c) ERR = 10%A.

The change in the displacements in all cases is given in the Fig. 6.4a. In the first few iterations, the change in the displacement is not affected by the ERR. However, as iterations progress, larger changes in the displacement are observed with larger ERR values. The changes in the objective weight with respect to the initial value during the optimization processes are shown in Fig. 6.4b. In all cases, the objective weight significantly reduces. The minimum values are 50.37% for ERR = 2% and 4%A, 51.01% for ERR = 4%B and 60.98% for ERR = 10%A. It is observed that lower



(a)



(b)

Fig. 6.4 Influence of the ERR on solution for a cantilever truss:

(a) displacement; (b) objective weight.

minimum objective weights are evolved with smaller values of ERR. Comparing Figs. 6.4a and 6.4b, it is observed that increases in the objective weight often correspond to the large jumps in the displacement. Due to large oscillations observed in the objective weight, the value of 10% for the ERR is considered too large for this ground structure. In this example, where the absolute minimum weight topology is not known prior, from an initially chosen ground structure the proposed method can generate topologies with substantial weight savings. Although the topology obtained by ERR = 10%A is heavier, it is still much more efficient than the initial design.

6.2.2 Influence of the ground structure on optimal topology for trusses

6.2.2.1 Influence of the ground structure on the solution for a truss structure in a domain of 1 m × 2.4 m

To examine the influence of the ground structure, consider again the problem of optimal topology design for the truss to transfer a vertical load to a vertical line of support in the domain of 1 m × 2.4 m as shown in Fig 4.4a. Four different grids with decreasing aspect ratio are considered:

1. Grid of 6 × 25 with aspect ratio of 1/0.5;
2. Grid of 11 × 25 with aspect ratio of 1/1;
3. Grid of 11 × 21 with aspect ratio of 1/1.2;
4. Grid of 11 × 13 with aspect ratio of 1/2.

Firstly, three ground structures where elements are connecting only to neighbouring nodes are chosen. The same cross sectional area, load and material property as in the previous example are assumed. These ground structures have different initial weights and vertical displacements at the loaded point. Only the ground structure for the grid of 11 × 25 contains a sub-set, which forms the optimal configuration for two bar truss. For this grid, the method has derived to the optimal configuration as shown in Fig. 4.4c. We will show how the ground structures affect the solutions and whether we can get the results close to this optimal configuration from other ground structures.

Using the value for ERR as high as 10% of the current number of elements, the solution is obtained within 15 iterations as given in Fig. 6.5. The solution for the grid of 11 × 25 has been given in Fig. 4.4c. For the first three grids, the corresponding ground structures contain subsets of elements which form a pair of straight lines connecting the load to the supporting points. The configuration of these lines in each

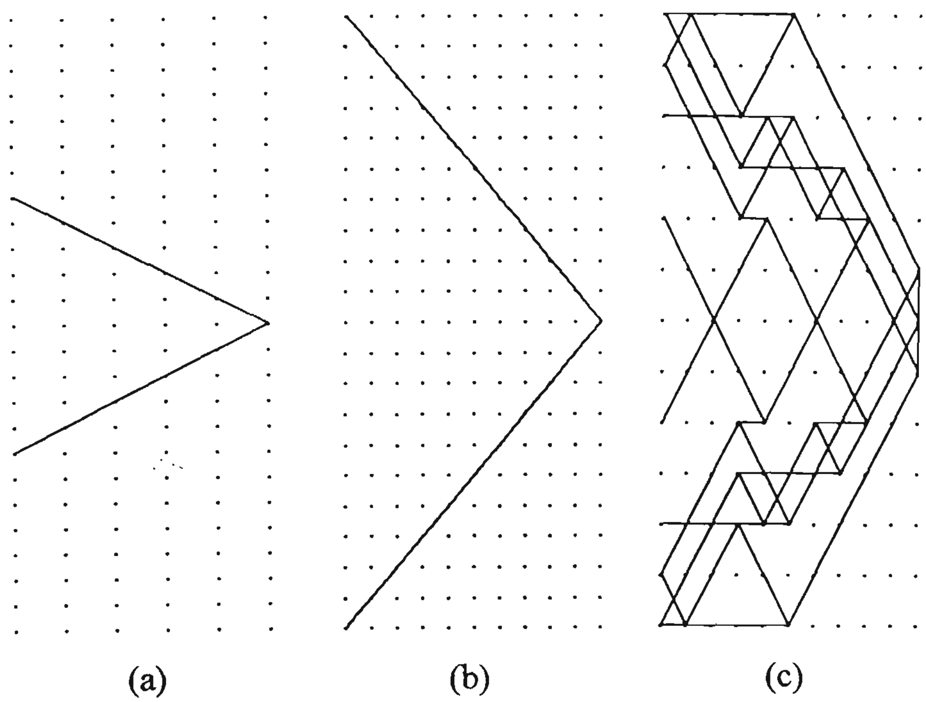


Fig. 6.5 Optimal topologies in 1 m × 2.4 m domain for different ground structures: (a) grid of 6 × 25; (b) grid of 11 × 21; (c) grid of 11 × 13.

case is the optimal topology for each ground structure (Fig. 4.4c, Figs. 6.5a and 6.5b). The grid of 11 × 13 does not have a similar subset and as a result, forms a different topology at the optimum (Fig. 6.5c). More elements are needed in this case to create a better load path. It is noted that this topology is also statically determinate. Among the remaining elements, 24 bars have the minimum sensitivity number. Removing any of them will destroy the design.

It can be seen in Fig. 6.6a that the patterns of changes in the displacement are similar in all cases except for the last few iterations. To compare the results, the objective weights for all grids are scaled by the initial objective weight of the grid of 11 × 25. Fig 6.6b presents the relative changes in the objective weights for all grids. It is seen that the three ground structures are not equally efficient at the beginning. The objective weights in all cases reduce to values less than half of their initial value.

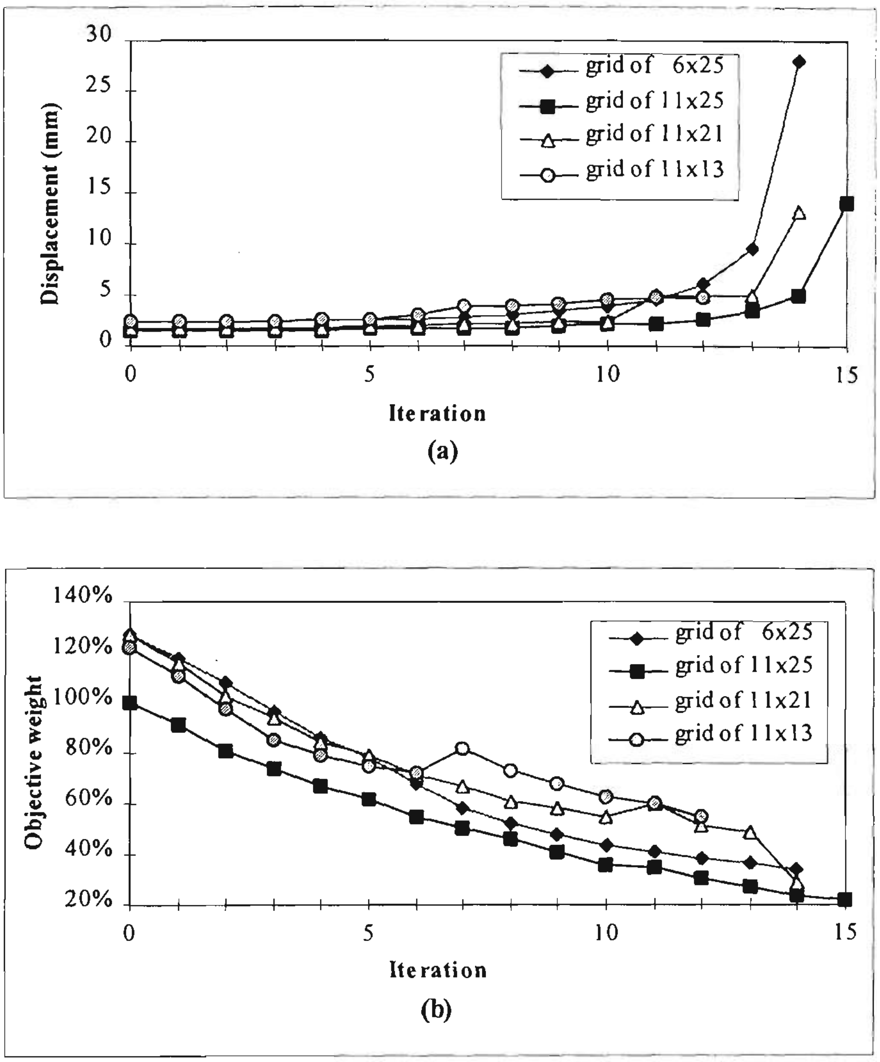


Fig. 6.6 Influence of the ground structure on solution for a truss structure in 1 m × 2.4 m domain: (a) displacement; (b) objective weights (with reference to grid of 11 × 25)

The grid of 11 × 25 always has the lowest objective weights during the optimization process. Following the change in the objective weight, the final stable topology for each case is also the optimum (minimum weight) topology. The grid with aspect ratio of 1/1 gives the lowest minimum weight, which is in agreement with the analytical solution. This example shows that the ground structure grid has significant effect on both weight and topology of the optimum solution derived by the proposed ESO method. This was also pointed out by Dorn *et al.* (1964) by using another method.

Table 6.1 presents the initial and the minimum objective weights for each grid with reference to the initial objective weight of the grid of 11 × 25. At the optimum, the closer the obtained topology to the analytical optimal configuration is, the lower is the minimum weight. The grid of 11 × 13 results in a minimum objective of 2.5 times as much as the optimal two bar truss configuration. However, compared to its initial objective weight, the proposed method can produce an optimum topology for this grid with a saving almost of a half of the material needed in the initial design.

Table 6.1 Objective weight for different ground structures in a domain of 1 m × 2.4 m.

	Grid of 6×25	Grid of 11×25	Grid of 11×21	Grid of 11×13
Initial objective weight*	126.80%	100.00%	127.29%	121.85%
Minimum objective weight*	34.31%	21.95%	29.08%	54.92%

(* with reference to the initial objective weight of the grid of 11×25)

For a given grid of joints, there are different ways to connect elements to the nodes. The most common option is to connect one node to either neighbouring nodes only or to all other nodes. For the grid of 11 × 25 these two different ground structures (ground structures II and III) have been already considered in section 4.4.1. The solutions for both these ground structures are identical because they contain the optimal analytical solution as a subset. Now, for other above chosen grids (6 × 25, 11 × 21 and 11 × 13 grids) the ground structures with elements connecting one node to all the other nodes are used to examine the influence of the element connection on the solution by the proposed method. It is obvious that all these ground structures contain at least a set of bars connecting the loaded node to all the supporting nodes. Based on the evolutionary solution process obtained for the ground structures I and III in section 4.4.1 and others ground structures lately considered in the same domain, it is obvious that the optimal topology for these ground structures is a pair of straight lines connecting the loaded node to the supporting nodes, whose configuration is closest to the analytical optimal configuration for two bars. For this type of element connections, the grid of 11×13 gives the analytical solution, although the earlier type of element connection gives the topology with highest minimum. From this example, the use of the latter type of element connection can give solution closer to the absolute minimum even for a grid with a relatively small number of nodes.

6.2.2.2 Influence of ground structures on the solution for a cantilever truss in 2 m × 1 m domain

Consider the topology design problem for a cantilever truss in a domain of 2 m × 1 m which transfers a vertical load to a vertical line of support, similar to the example given in section 6.2.1.2. Three different grids with decreasing aspect ratio are considered:

1. Grid of 5×5 with aspect ratio of 1/0.5;
2. Grid of 9×5 with aspect ratio of 1/1;
3. Grid of 17×5 with aspect ratio of 1/2.

Three ground structures where elements are connecting only to immediate neighbouring nodes are first investigated. The same cross sectional areas, load, material properties as in the previous example are assumed. These ground structures, as shown in Fig. 6.7, have different weights and displacements at the loaded points.

The value of 4% of the current elements is used for ERR. Only even numbers of elements are removed at each step to maintain the symmetry of the resulting designs. The prescribed limit on the vertical displacement at the loaded point as large as 50 mm is prescribed to continue the optimization process until no further stable design can be achieved.

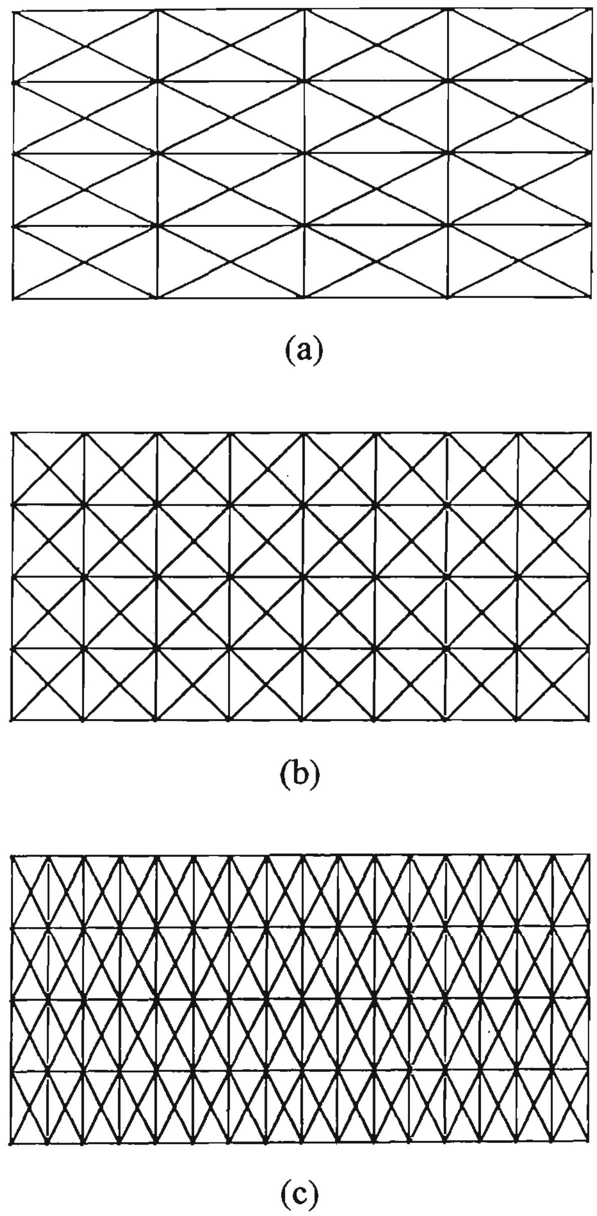
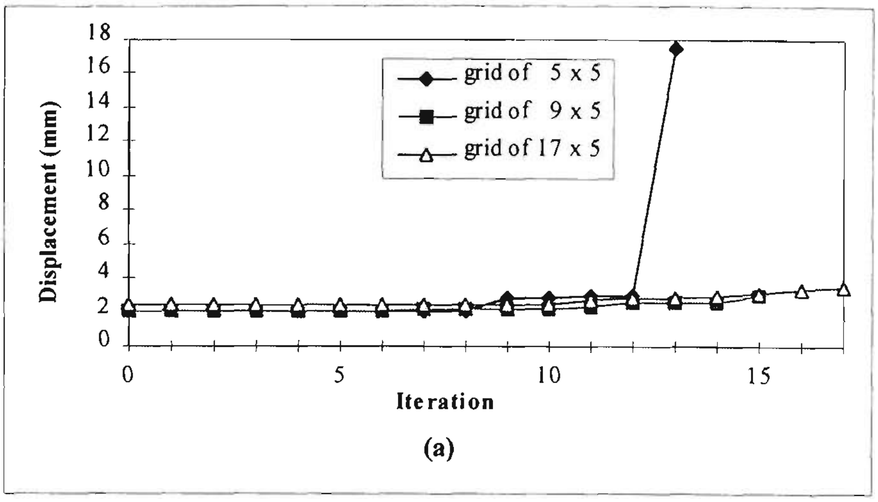


Fig. 6.7 Ground structures for a cantilever truss in 2 m × 1 m domain:

(a) grid of 5 × 5; (b) grid of 9 × 5; (c) grid of 17 × 5.

The evolutionary optimization process for the grid of 9 × 5 is terminated at iteration 13 because the loaded node does not have sufficient restraint in the horizontal direction. An additional restraint on this degree of freedom has been imposed to continue the process. The optimization histories for these ground structures are shown in Fig 6.8. It can be seen in Fig. 6.8a that the displacements in all cases have negligible changes in the first eight iterations. The design created for the grid of 5 × 5



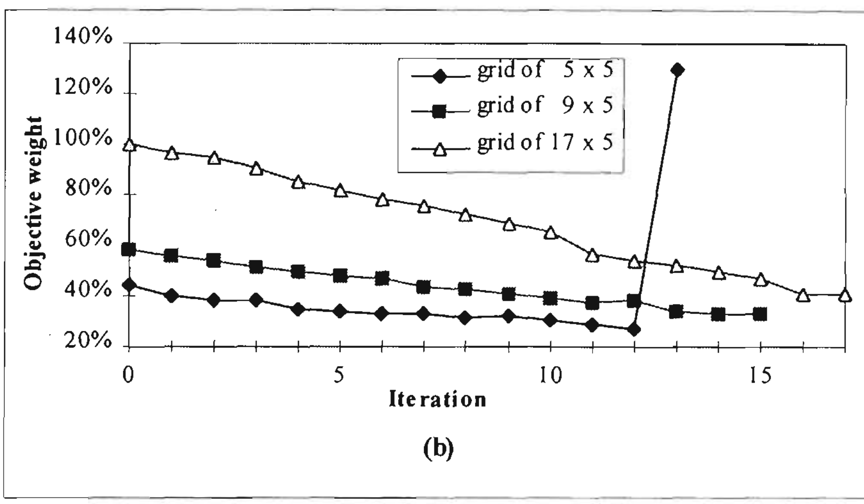
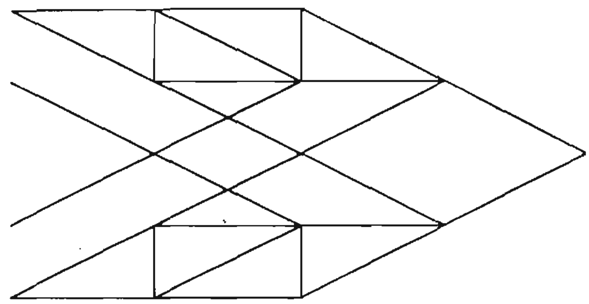
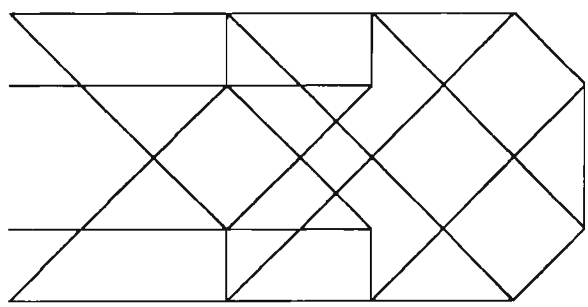


Fig. 6.8 Influence of the ground structure on solution for a cantilever truss in 2 m × 1 m domain: (a) displacement; (b) objective weight (with reference to the grid of 17 × 5).

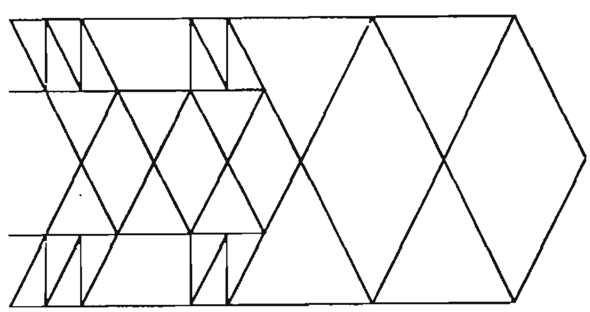
in the last iteration experiences a very large increase the displacement, although it is still a stable structure. The objective weights for all ground structures are given in reference to the initial objective weight of the grid of 17 × 5 and presented in Fig. 6.8b. For the grid of 5 × 5 the optimum topology is obtained at iteration 12. The optimum for the grid of 9 × 5 is reached in iteration 15. The minimum weight design for the grid of 17 × 5 is created at iteration 16 and the topology at iteration 17 is the very close to the minimum. The optimum topologies for these ground structures are



(a)



(b)



(c)

Fig. 6.9 Optimal topologies for a cantilever truss in 2 m × 1 m domain:

(a) grid of 5 × 5; (a) grid of 9 × 5; (a) grid of 17 × 5.

given in Fig. 6.9. It is seen that different grids lead to different optimal topologies. The elements in the diagonal directions are dominant in these optimal designs. The optimal topologies in Figs. 6.9a and 6.9c are not the final stable designs. The minimum objective weight for each case can be seen in Table 6.2. Significant reductions in the objective weight have been achieved for all grids compared to their initial values. The largest reduction results for the grid of 17 × 5. However, due to its inefficiency at the beginning, this grid does not result in the lowest minimum objective weight. It is suggested that one should start from the best among the chosen ground structures. For the above three grids, the larger the grid aspect ratio is, the better is the solution.

Table 6.2 Objective weight of the cantilever truss using different ground structures.

	Grid of 5×5	Grid of 9×5	Grid of 17×5
Initial objective weight*	44.14%	57.88%	100.00%
Minimum objective weight*	27.17%	32.85 %	40.85%

(* with reference to the initial objective weight of the grid of 17×5)

For the above chosen grids, three other ground structures where elements are connecting one node to all other nodes are also considered. These ground structures contain sets of bars connecting the loaded node to the supporting nodes. Similar to the previous example for the structure in the domain of 1 m × 2.4 m, even without solving the problem, it can be concluded that the optimum topology for all these ground structures is a pair of straight lines connecting the loaded node to the top and bottom supporting nodes and this solution is the absolute optimum for the structure in the given domain. This is due to the fact that for all the chosen grids in the given domain, these lines form the topology closest to the optimal configuration of the two bar truss. Overlapping bars may exist in this optimal design similar to the solution for ground structure III in the domain of 1 m × 2.4 m (section 4.4.1).

6.3 Aspects of the ESO method in topology optimization of continuum structures

This section examines the influence of various aspects on the optimal solutions by the proposed method for two-dimensional continuum structures. Unlike truss topology problems where material in the ground structure cover only a fraction of the total allowable design domain, the material in the ground structure for continuum structure covers the entire design domain. The choice of the ground structure depends on the mesh size and element type of finite element models. We will show how the solutions for continuum structures are affected by the ERR, mesh size and element type. In addition, the problem of checker-board patterns in solutions for continuum structures is discussed and a procedure for suppression of checker-board pattern is suggested.

The following results have been initially obtained by using the element contributions as the sensitivity numbers for element removal and reported in Chu *et al.* (1977). Because regular meshes have been used, these results are also valid for the newly proposed method based on the optimality criteria. They are revised and presented with an additional discussion on the change in the objective (scaled) weight. This helps to identify the minimum weight topology among the available ESO topologies obtained during solution process. The solution time required for continuum structures (plates and shells) is usually much longer than for truss structures with a similar number of elements. The solution time is an important indicator of the efficiency of the proposed method. Therefore, the influences of the ERR, the mesh size and element type on the solution time are also studied. It should be noted the all examples given in this section are solved using a 486DX2/66MHz personal computer.

6.3.1 Influence of the element removal ratio on topology solution for two-dimensional continuum structures

This section examines the influence of the element removal ratio (ERR) using several examples including structures subject to multiple displacement constraints under multiple load cases. As discussed before, for each value of the element removal ratio, two removal strategies (RS) can be used:

A) Removal strategy A (RS = A): The element removal ratio is defined with respect to the current number of elements. The number of elements to be removed decreases as iterations proceed.

B) Removal strategy B (RS = B): The element removal ratio is defined with respect to the initial number of elements. The number of elements to be removed at each iteration is constant.

The values of 1%, 2% and 4% for the element removal ratio are used and its influence on the solution is examined by considering the following cases:

Case 1A: ERR = 1%, RS = A;

Case 1B: ERR = 1%, RS = B;

Case 2A: ERR = 2%, RS = A;

Case 2B: ERR = 2%, RS = B;

Case 3A: ERR = 4%, RS = A;

Case 3B: ERR = 4%, RS = B.

6.3.1.1 The MBB beam

The problem of the MBB beam given in section 5.4.3 is used to examine the influence of the element removal ratio on the optimal solution of a continuum structure subject to a single displacement constraint. The same design domain, material properties, loading and boundary conditions are assumed (see Fig. 5.7). Using the same finite element mesh, the structure is optimized again for six above-mentioned cases until the displacement at the loaded point reaches the limit of 9.4 mm.

The changes in the constrained displacement in all cases are shown in Fig. 6.10a. When a larger element removal ratio is used, more material is removed from the design domain, and larger change in the displacement is observed in each iteration.

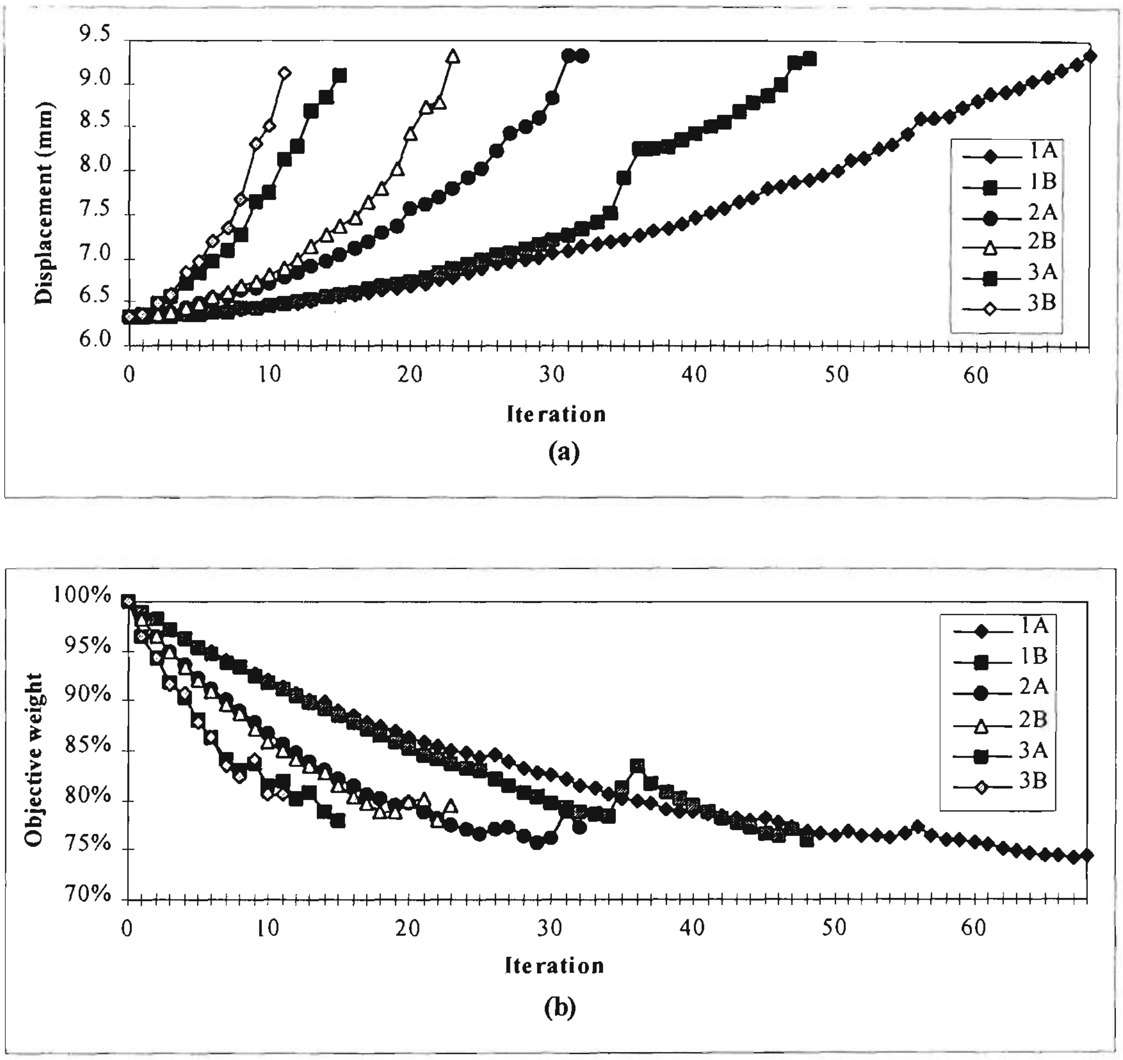


Fig. 6.10 Influence of the element removal ratio on the solution for the MBB beam: (a) displacement; (b) objective weight.

The change in the objective (scaled) weights with respect to the initial objective weight during the optimization process for all cases are given in Fig. 6.10b. Although the actual weight is successively reduced, some increases in the objective weight are observed in all cases. The largest increases occur in the cases 1B and 2A. It should be reminded that the pattern of the change in the objective weight for this problem does not depend on the specified value for the limit. It largely depends on the value for ERR. Despite some increases, the objective weight in all cases are significantly reduced overall as seen in Fig. 6.10b. In this example, the optimal (minimum weight) topologies are not always the final ones. Table 6.3 presents the minimum objective weight and the iteration identification number where the optimum is reached in each case. Case 1A gives the lowest minimum objective weight. Case 3B gives the highest minimum objective weight. Using the larger element removal ratio with removal strategy A (Case 2A) can produce a lower minimum objective weight than using the smaller ERR with removal strategy B (Case 1B).

Table 6.3 Objective weight for the MBB beam using different ERR.

	Case 1A	Case 1B	Case 2A	Case 2B	Case 3A	Case 3B
Minimum objective*	74.28%	76.03%	75.63%	77.85%	77.90%	80.59%
At iteration	67	48	29	22	15	11

(* with reference to the initial objective weight)

It should be noted that based on the trend of the objective weight where it continuously reduces in the last few iterations for some cases, it is possible to obtain a lower objective weight by continuing the optimization process with a larger limit on the displacement and/or smaller value for the ERR until no further reduction in the objective weight or no further stable design can be achieved.

Table 6.4 Total number of iterations and computational time for the MBB beam with the limit of 9.4 mm.

	Case 1A	Case 1B	Case 2A	Case 2B	Case 3A	Case 3B
Number of iterations	69	49	33	24	16	12
Time (hours)	4.3	3.1	1.5	1.1	0.9	0.6

The number of iterations and the solution time required for the limit of 9.4 mm are given in Table 6.4. This table includes the last iteration that creates the design whose

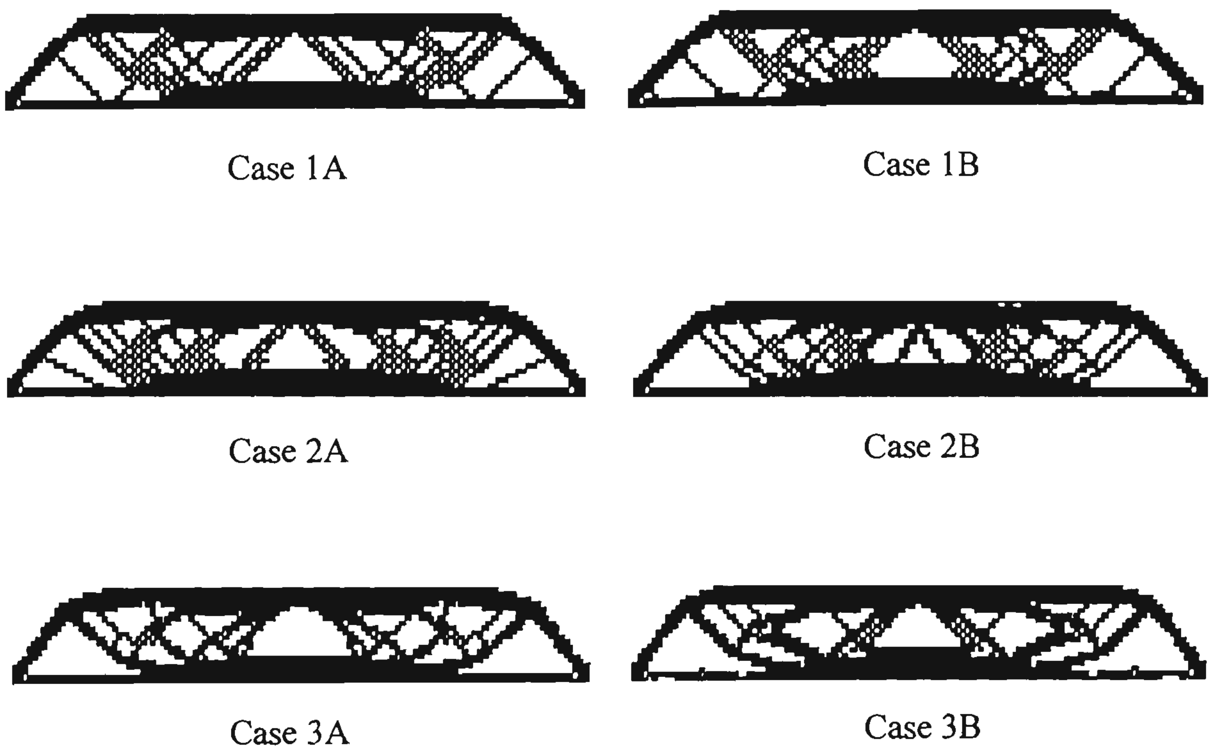

Case 1A Case 1B
Case 2A Case 2B
Case 3A Case 3B

Fig. 6.11 Optimal topologies for the MBB beam
(influence of the element removal ratio)

displacement first becomes greater than the specified limit. It is observed that by using larger element removal ratio the number of iterations and the computation time are dramatically reduced. For the same value of the element removal ratio, strategy B requires less time than strategy A.

Fig. 6.11 presents the optimal topologies obtained for each case during the optimization process for the displacement limit of 9.4 mm (see Table 6.3 for their objective weights and iteration numbers). These topologies provide an idea about the influence of the ERR on the optimum topology for this problem. It is seen that the outer shapes of these optimal designs in all cases are very similar. The element removal ratio has greater influences on the details of the internal parts, however, the main pattern and the orientation of these details are similar.

6.3.1.2 Structure under multiple displacement constraints

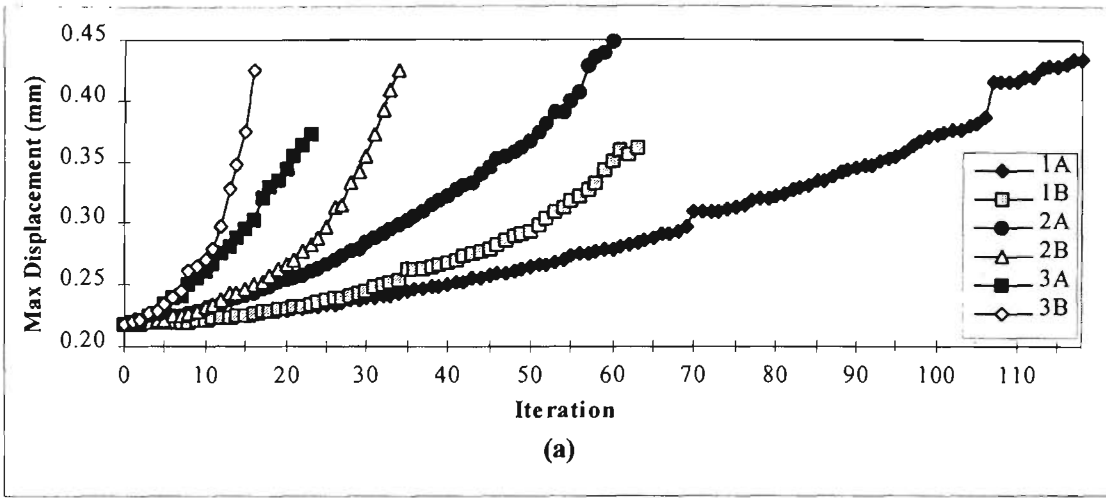
Consider the two-dimensional structure under multiple displacement constraints given in section 5.4.5, as shown in Fig 5.13. The same dimensions, material properties, loading and boundary conditions are assumed. The maximum initial vertical displacement is 0.22 mm. Constraints are imposed on the vertical displacements at the three loaded points. Due to symmetry only half of the structure is analysed using a mesh of 40 × 40 four-node quadrilateral elements. The structure is optimized using different element removal ratios and removal strategies as mentioned above until the maximum displacement reaches the limit of 0.45 mm. Elements are removed additionally on the basis of having insufficiently connectivities to avoid singularity of the stiffness matrix. The ratio formulae (3.38) are used for the Lagrange multipliers.

The changes in the maximum displacements for all six cases are presented in Fig. 6.12a. In all cases, the maximum displacements increase slightly in the first few iterations and then gradually approach the specified value. Except for few large changes, the level of changes in the maximum displacement between two successive iterations largely depends on the value of the ERR. However, as observed in this figure, the maximum displacement has no change or even reduces at some iterations. This can only happen in the case of multiple constraints due to the re-distribution in element contributions. It also proves the efficiency of the method.

Table 6.5 Objective weight of optimal topologies for the structure under multiple constraints.

	Case 1A	Case 1B	Case 2A	Case 2B	Case 3A	Case 3B
Minimum objective*	60.51%	61.75%	59.08%	61.39%	65.77%	67.62%
At iteration	105	62	60	34	23	15

(* with reference to the initial objective weight)



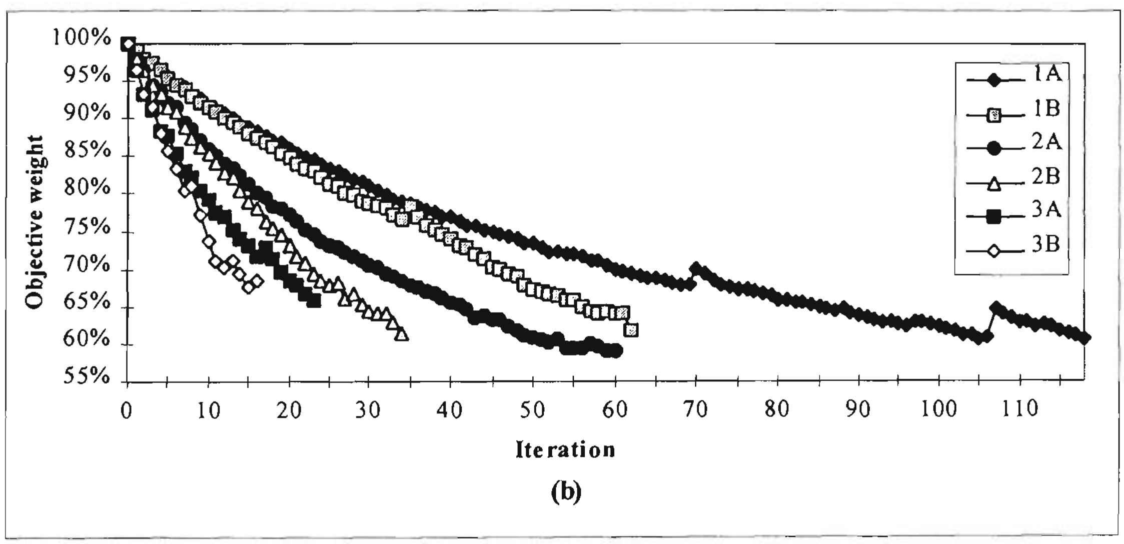
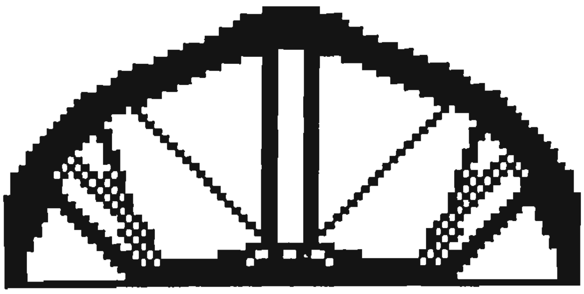
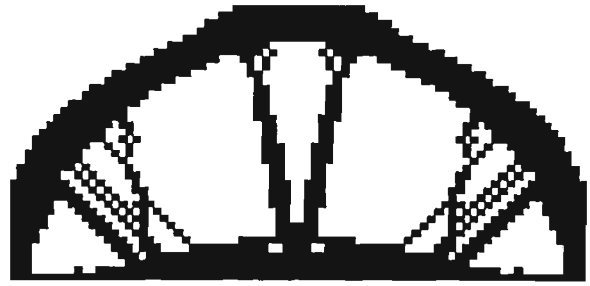


Fig. 6.12 Influence of the ERR for the structure under multiple constraints: (a) maximum displacement; (b) objective weight.

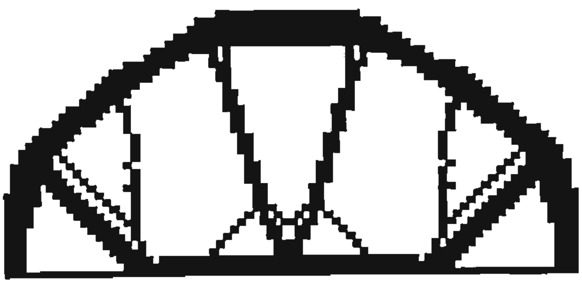
The changes in the objective weight during optimization process are given in Fig. 6.12b. In all cases the objective weight reduces to the values less than 68% of the initial value. Table 6.5 provides the minimum weight and the iteration number where the minimum is reached in each case. The lowest minimum of 59.08% of the initial is obtained in Case 2A. Cases 1A, 1B and 2B give the minimum weights close to this lowest minimum. For all cases lower minimum objective weight may be obtained by setting a larger value for the limit. For this example ERR = 2% of the current elements works better than all other ERR values (even 1%).



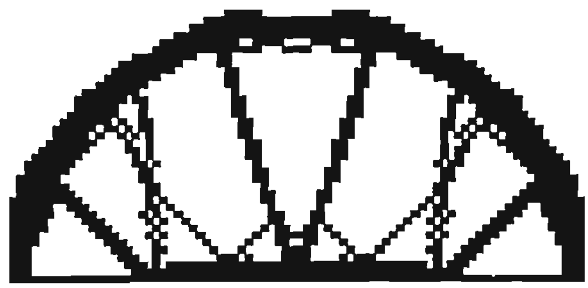
Case 1A



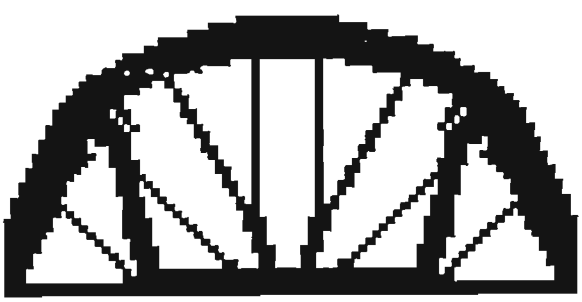
Case 1B



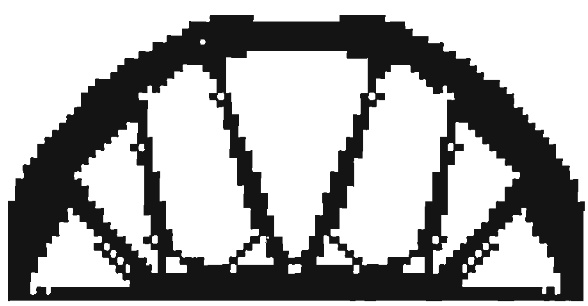
Case 2A



Case 2B



Case 3A



Case 3B

Fig. 6.13. Optimal topologies for the structure under multiple constraints (influence of the element removal ratio).

The optimal topologies obtained for different cases are shown in Fig. 6.13. It is seen that the outer shapes of all these optimal designs are similar. There are differences between the inner parts. The optimal designs for the Case 1B, 2A, 2B and 3B have very similar topologies. The number of iterations and the solution time required for the displacement limit of 0.45 mm are given in Table 6.6. It is seen that the number of iterations and computational time are dramatically reduced by using a larger value for the element removal ratio.

Table 6.6 Number of iterations and computational time for the structure under multiple constraints.

	Case 1A	Case 1B	Case 2A	Case 2B	Case 3A	Case 3B
Number of iterations	119	63	61	35	24	17
Time (hours)	14.7	9.6	5.7	4.8	3.5	1.3

6.3.1.3 Bridge with a moving load

Consider again the design of the bridge with a moving load as given in section 5.4.7, for which the initial design domain and boundary conditions are shown in Fig. 5.18. The same loads, material properties and constraints are assumed. There are nine load cases. The maximum initial vertical displacement for all load cases is 1.39 mm. Nine constraints, one per each load case, are included in the active set for this problem. The ratio formulae (3.38) are used for the Lagrange multipliers. This serves as an example of a structure subject to multiple constraints under multiple load cases. The structure is optimized using different ERR as given in Cases 1B, 2B and 3B.

The change in the maximum displacement for the bridge in three cases (1B, 2B and 3B) is given in Fig 6.14a. This trend of change in the maximum displacement in this example is similar to that observed in the previous example. Fig. 6.14b presents the objective weight during the optimization process. The objective weight reaches the minimum at earlier stages and rises in later iterations which is different compared to the previous example. Table 6.7 provides the minimum objective and the iteration number where the minimum is reached for each case. It is seen that different ERR values lead to very similar minimum objective weights. The minimum for Case 2B is the lowest. The differences between these minimum values are less than 2%. Also in this example, ERR = 2% works slightly better than ERR = 1%. The element removal ratio has little effect on the minimum objective weight in this example.

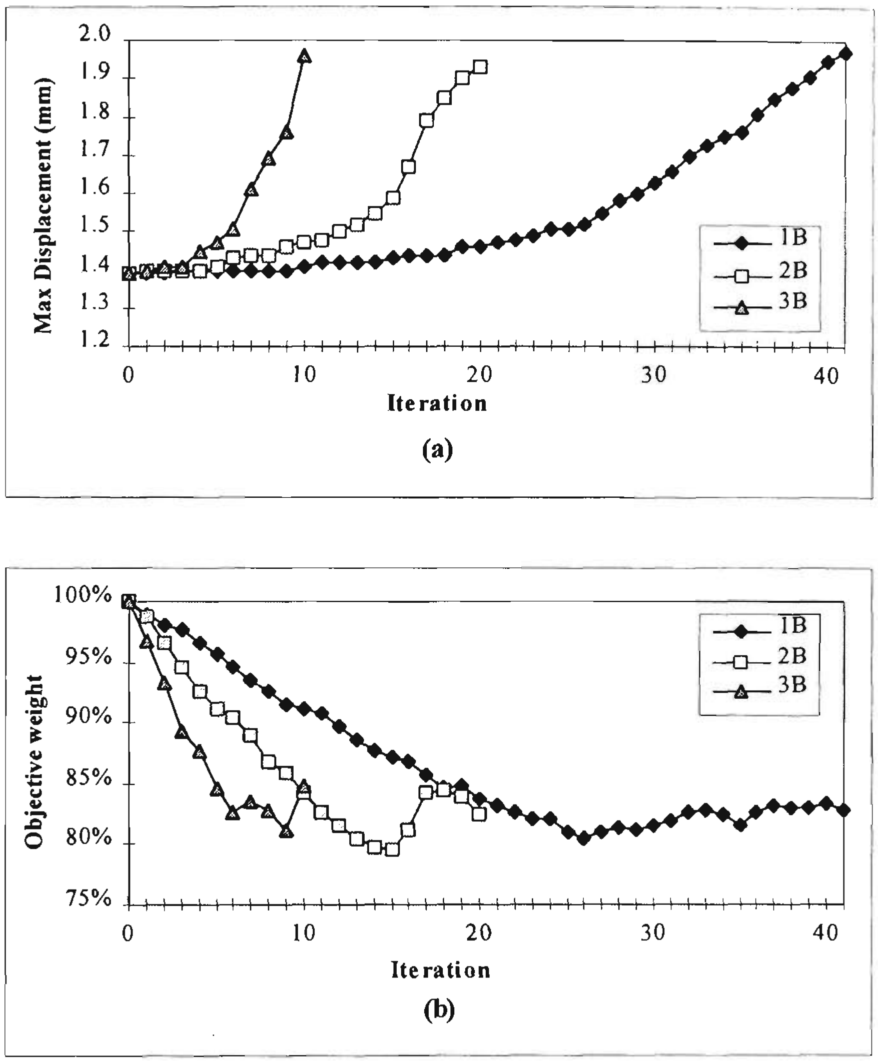
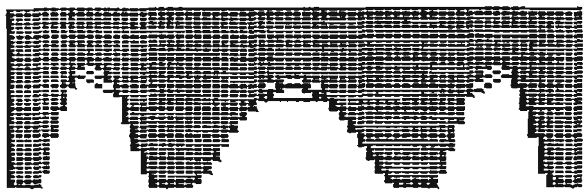


Fig. 6.14 Influence of ERR on solution for bridge with a moving load: (a) maximum displacement; (b) objective weight.

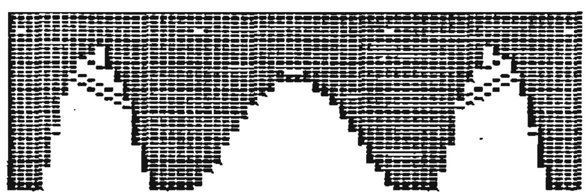
Table 6.7 Objective weight of optimal designs for the bridge with a moving load

	Case 1B	Case 2B	Case 3B
Minimum objective*	80.48%	79.54%	81.21%
At iteration	26	15	9

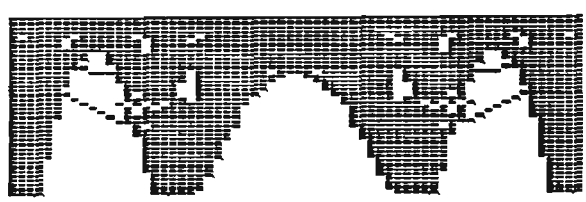
(* with reference to the initial objective weight)



Case 1B



Case 2B



Case 3B

Fig. 6.15 Optimal designs for the bridge with a moving load (influence of the element removal ratio).

The optimal topologies obtained for each case in the optimization processes before the maximum displacement exceeds the limit of 2.0 mm are shown in Fig. 6.15. The overall shapes of these solutions are similar. Although these optimum designs differ in details, their objective weights are close to each other as seen in Table 6.7.

The number of iterations and the solution time for the limit of 2.0 mm are given in Table 6.8. It is observed that the time for solution is dramatically reduced when larger element removal ratio is used. The use of ERR = 2% in Case 2B results in the lowest minimum objective weight.

Table 6.8 Number of iterations and time for the bridge with limit of 2.0 mm.

	Case 1B	Case 2B	Case 3B
Number of iterations	42	21	11
Time (hours)	124	60	27.2

6.3.2 Influence of mesh size

This section examines the influence of mesh size on the optimal solution. Obviously when a finer mesh is used, the number of elements in the structure's FEA model increases, thus more computation time is required for the problem. The study on the influence of mesh size on the solution is carried out by using different meshes for a fixed value of the element removal ratio.

The example of topology design for the short cantilever given in section 5.4.1 is used to study the influence of the mesh size on the solution by the proposed ESO method. This example was also used by Suzuki and Kikuchi (1991) to show the convergence property of the homogenization method. The initial design domain, the loading and boundary conditions are shown in Fig. 5.1. The same dimensions, load and material properties as in section 5.4.1 are assumed.

The design domain for the short cantilever is modelled by meshes of 32×20, 48×30 and 64×40 four-node quadrilateral elements. The element removal ratio of 2% of initial elements is used. The optimization process for each mesh continues until the constrained displacement becomes greater the limit of 1 mm. The optimization history for the short cantilever using different mesh size is shown in Fig. 6.16.

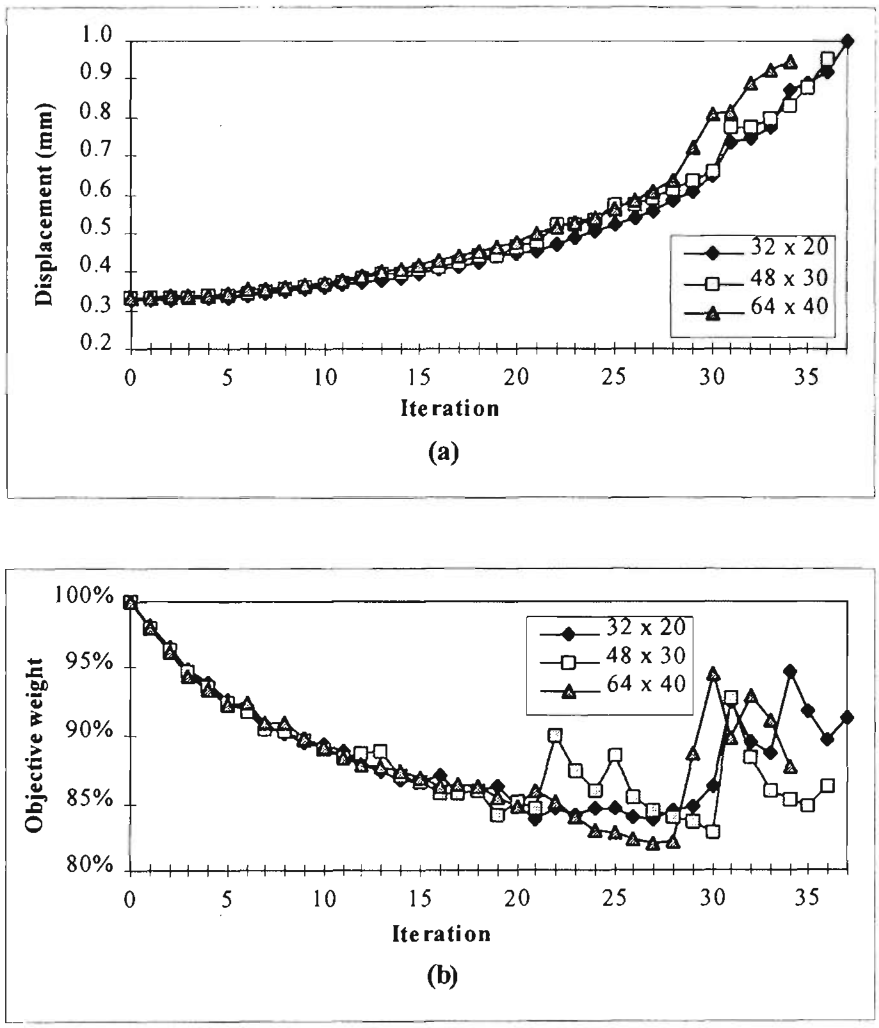
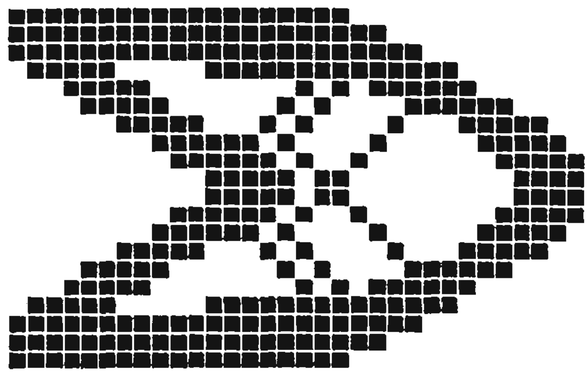


Fig. 6.16 Influence of mesh size on the solution for the short cantilever: (a) displacement; (b) objective weight.

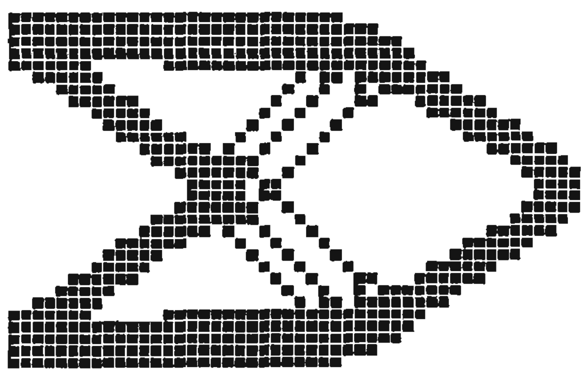
Table 6.9 Objective weight of optimal designs for the short cantilever.

	Mesh of 32 × 20	Mesh of 48 × 30	Mesh of 64 × 40
Minimum objective*	83.70%	82.71%	81.96%
At iteration	27	30	27

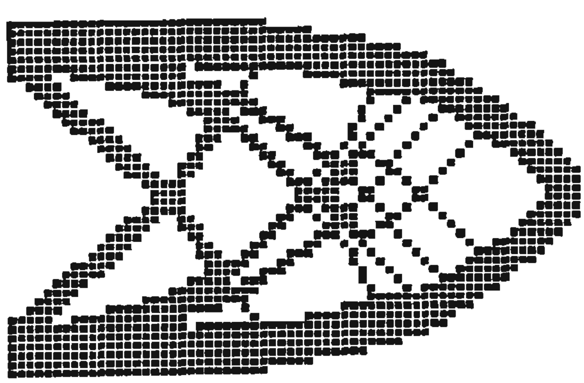
(* with reference to initial objective weight)



mesh 32 × 20



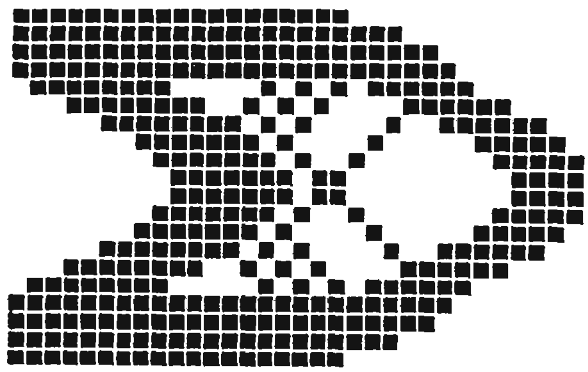
mesh 48 × 30



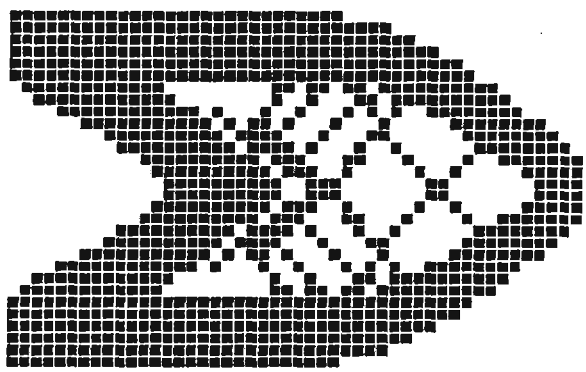
mesh 64 × 40

Fig. 6.17 Optimal designs for the short cantilever
(influence of the mesh size)

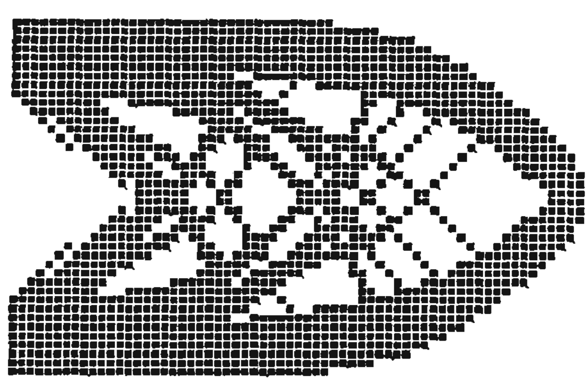
It is seen in Fig. 6.16a that, for the same value of ERR, the mesh size does not have much influence on the change in the displacement. The changes in the objective weights are given in Fig. 6.16b, where similar changes in the objective weight are observed for different meshes up to iteration 21. After reaching the minimum values,



mesh 32 × 20



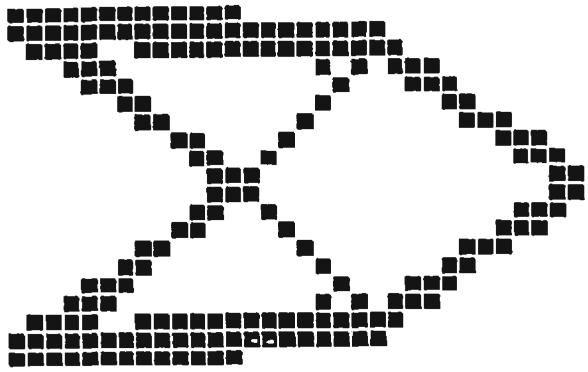
mesh 48 × 30



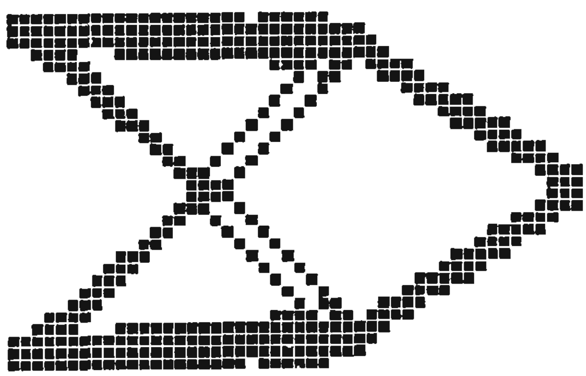
mesh 64 × 40

Fig. 6.18 ESO designs of the short cantilever for displacement of 0.5 mm (influence of the mesh size).

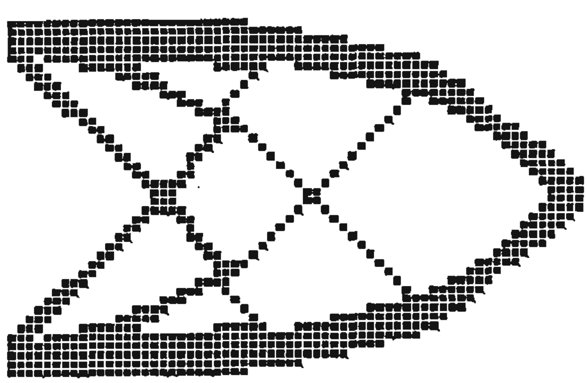
the objective weight starts increasing with large oscillations. The minimum objective weight for each mesh is given in Table 6.9. As expected, the finer the mesh is, the lower is the minimum objective weight. However, the differences between these minimum values are less than 2% of the initial value. This suggests that the mesh size



mesh 32 × 20



mesh 48 × 30



mesh 64 × 40

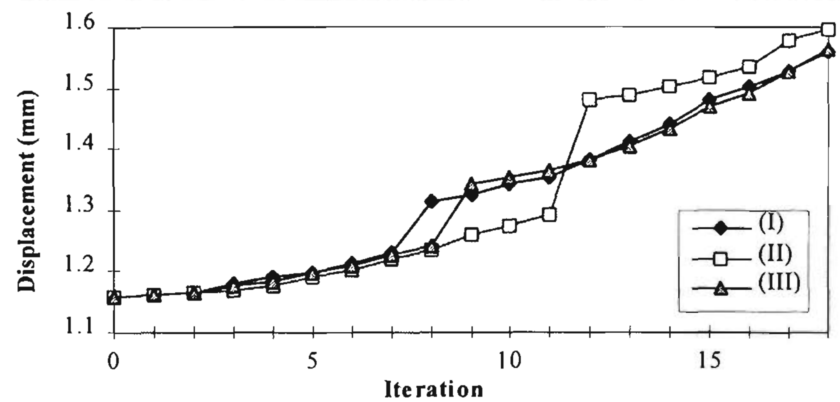
Fig. 6.19 ESO designs of the short cantilever for displacement of 1.0 mm (influence of the mesh size).

has little influence on the minimum objective weight. The optimal designs for the three meshes are shown in Figs. 6.17. It is seen that the size of elements has a little effect on the outer shape but it considerably affects on the inner parts of the optimal designs. Figs. 6.18 and 6.19 provide the topologies obtained when the displacement reaches the value of 0.50 mm and 1.0 mm. An interesting observation from the obtained results is that truss-like structures are formed if the displacement limit is sufficiently large. For the limit closer to the initial displacement, curved frames are generated, and more continuum-like shapes are created. As is seen in Fig. 6.16b the truss-like structures in the last few iterations are less efficient than the curved frames obtained in earlier iterations. This is due to the fact that in the truss-like structures more quadrilateral plate elements work in the inefficient manner as they can only take simple tension or compression along one of their diagonals. It would be more efficient to model such structures using beam elements.

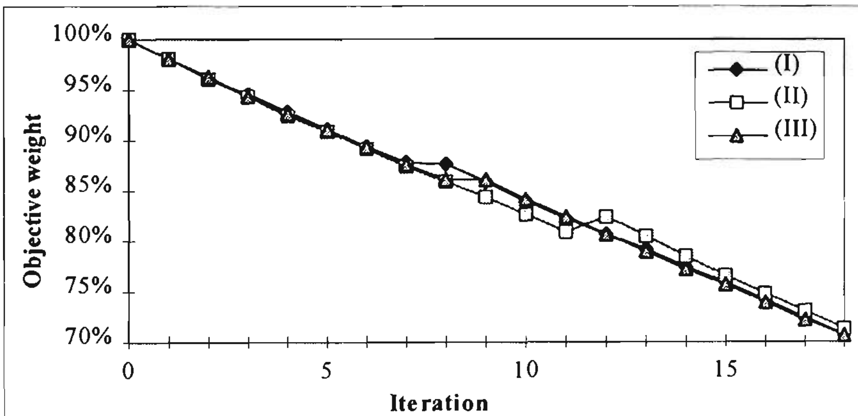
6.3.3 Influence of element type

The simply supported square plate in bending as given in section 5.4.4 is used to investigate the influence of the element type on the solution by the proposed method. For this purpose, elements with similar sizes but different types are used. The same dimensions, load, material properties are assumed. Three models for a quarter of the plate are considered using 400 four-node quadrilateral plate elements, 800 one-way and 1600 two-way constant stress triangular plate elements. Using an element removal ratio of 2% of the initial elements, the optimization process continues until the displacement at the centre reaches the limit of 1.6 mm.

The optimization history for the plate using these element types is given in Fig. 6.20. As seen in Fig. 6.20a, the changes in the displacement at the centre are almost the same in the first 6 iterations for all cases. The change in the displacement using one-ways triangular elements largely differs from other types of elements from iteration 7 to 12. Similar changes can be seen in the last iterations in all cases. The quadrilateral



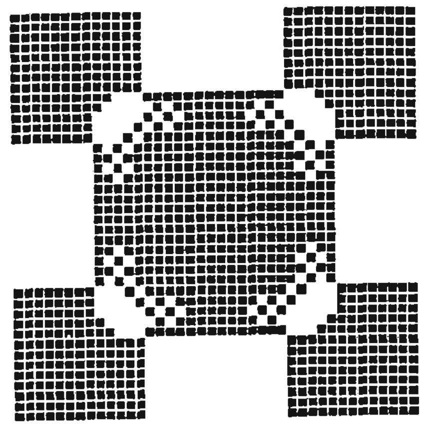
(a)



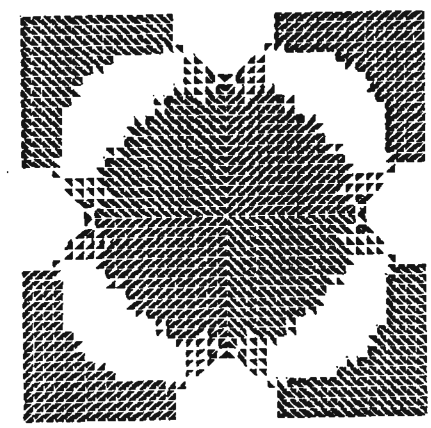
(b)

Fig. 6.20 Influence of element type on the solution for the plate in bending:
(a): displacement; (b): objective weight.

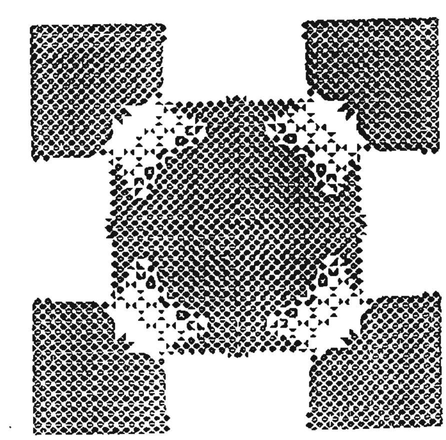
and two-way triangular element types perform very similar. It is seen in Fig. 6.20b that using different element types does not effect much on the objective weight. The minimum weight for each case is obtained in the last iteration before the displacement exceeds the limit. As seen in the Table 6.10 the minimum objective weight is almost the same. No further solution can be obtain for the model with quadrilateral elements. Based on the trend that the objective weight reduces successively in the last iterations for models with triangular elements, it is possible to obtained a lower minimum objective weight by continuing the solution process for a larger limit.



quadrilateral elements



one-way triangular elements



two-way triangular elements

Fig. 6.21 Optimal designs for the plate in bending

(influence of the element type).

Table 6.10 Objective weight of optimal designs for the plate in bending

	Quad elements	One-way triangles	Two-way triangles
Minimum objective*	70.68%	71.20%	70.60%
At iteration	18	18	18

(* with reference to the initial objective weight)

Three corresponding optimal designs are given in Fig. 6.21. The use of quadrilateral and two-way triangular elements gives almost identical shapes. Similar to the results reported in Tenek and Hagiwara (1994), hinge lines are formed between the central part and the four corners in all cases. The optimal solutions in Fig. 6.21 are similar to each other although the hinges appear at different locations along the hinge lines. These three optimal solutions have almost the same weight and displacement. This means that the locations of hinges on hinge lines are not important in this example.

6.3.4 Problem of checker-board patterns

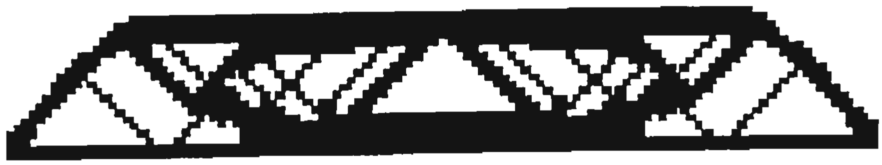
It is observed through examples given in the previous chapter that areas of checker-board patterns often appear in solutions derived by the proposed method for two-dimensional continuum structures. With checker-board patterns the obtained shapes may become practically unacceptable. There is a need of some ways to control the formation of checker-board patterns in order to make the solutions more practical.

Patches of checker-board patterns also appear often in solutions obtained by homogenization methods. The origin of checker-board patterns is still not fully understood but it is likely to be related to the finite element approximation as a numerical phenomenon (Bendsøe *et al.* 1993). There are different ways for suppression of the formation of checker-board patterns. One way is to use higher-order elements. Rodrigues and Fernandes (1993) showed that the checker-board patterns which appear by using 4-node elements, can be avoided by using 9-node elements. Another way is to use a special algorithm to control formation of checker board patterns as proposed by Bendsøe *et al.* (1993). A simpler algorithm for density re-distribution was suggested by Youn and Park (1995) to suppress formation of checker board patterns.

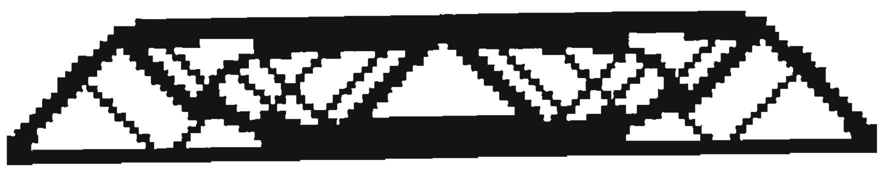
In the proposed ESO method, where element removal technique is employed and no intermediate thicknesses are allowed, the only way to control formation of checker-board pattern during the optimization process is to use higher order elements. However, this requires much higher computation time. To avoid the use of higher order elements and to make the existing ESO solutions more practical, we suggest the following additional steps:

1. Remove checker-board patterns from the optimum topology design by restoring appropriate elements to get a starting design (this can be very easy to program).
2. Switch the program ESOSHAPE to the pure shape optimization mode (elements are removed only from the existing boundaries) and continue the optimization process to obtain the optimum design without checker-board patterns.

The design of the MBB beam in section 5.4.3 is used to illustrate the suggested procedure. The optimum topology design of the MBB, shown in Fig. 5.8, is used to obtain a solution without checker-board patterns. The checker-board patterns are removed to create a starting design as shown in Fig. 6.22a. The program is then switched to pure shape optimization mode. The elements are removed only from existing boundaries including internal boundaries using the same value for the ERR (1% of the current number of elements). The optimal design without checker-board patterns for the limit of 9.4 mm is obtained after 8 additional iterations as shown in Fig. 6.22b.



(a)



(b)

Fig. 6.22 Solution for the MBB beam without checker-board patterns:

(a) the starting design; (b) the optimal solution.

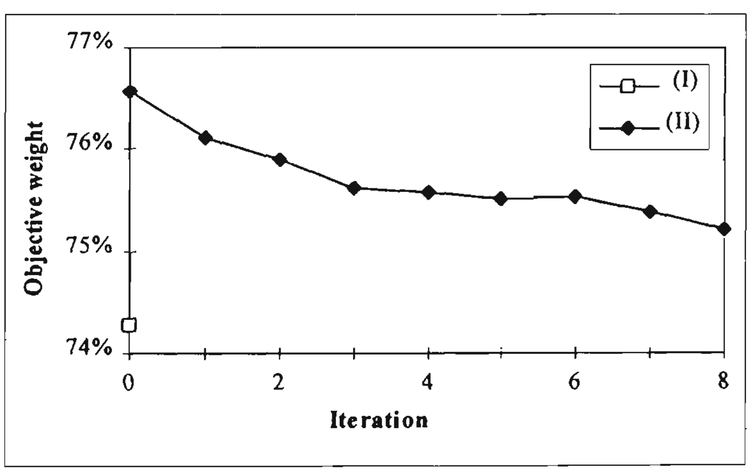


Fig. 6.23 Optimization history for the MBB beam:

(I): the minimum objective weight with checker-board patterns;

(II): objective weight of solutions without checker-board patterns.

To show the efficiency of the obtained designs, the weights are scaled for the displacement at the limit and then is given relative to the objective weight in the initial rectangular design domain (Fig. 5.7). The change in the objective weight is shown in

Fig. 6.23. The optimum design with checker board patterns is of 74.28% of the initial value. Removing checker-board patterns from the optimum design (Fig. 5.8) produces a starting design (Fig. 6.22a) with the objective weight increased to 76.58%. After 8 iterations, the objective weight reaches the minimum value of 75.21% when the displacement reaches the limit of 9.4 mm. The weight difference between the optimum design with checker-board patterns (Fig. 5.8) and the optimum design without checker-board patterns (Fig. 6.22) is less than 1% of the initial value. In this example, the efficiency of the solution without checker-board patterns is slightly lower than the solution with checker-board patterns. It is seen from Fig. 5.8 and Fig. 6.22b that, except for the checker-board areas, the overall topologies of these two optimum designs are very similar. Further investigation on the suggested procedure is needed.

6.4 Concluding remarks

This chapter has investigated various aspects of the proposed ESO method such as the influence of the element removal ratio and that of the ground structure (the initial grid, element connection, mesh size and element type) on the solution for topology problems of truss and two-dimensional continuum structures. An alternative way to obtain an optimum solution without checker-board patterns is also suggested, which allows to create more practical solutions using the proposed method.

It has been shown that the element removal ratio is an important parameter in the proposed method. It controls the magnitude of the change in each design step. The element removal ratio plays a very similar role as the move limit in the mathematical programming methods or the step size parameter in the traditional optimality criteria methods. As expected, the smaller the value for the element ratio is, the more accurate will be the solution, but at a higher computational cost. Based on provided examples, suitable values of the element removal ratio for each type of structures has been suggested to reduce the solution time while keeping the accuracy of the solution at acceptable levels. It is worth noting that, one can use the material removal ratio (MRR) instead of the element removal ratio (ERR) to control the change in each design step as described in section 3.4. When all elements have the same weight, the element removal ratio and the material removal ratio are identical. In other cases, the material removal ratio controls the number of elements to be removed at each iteration indirectly, by which this number varies from iteration to iteration.

Similar to other methods based on the ground structure approach, the solution by the proposed ESO method is also affected by the initial ground structure chosen for the design domain. The investigation has shown the extend of the influence of the ground structure on the solution by considering various examples.

The following conclusions can be drawn from the study in this chapter for truss structures:

1. The element removal ratio has considerable effect on the optimal topology. From a chosen ground structure, different values for the element removal ratio can lead to different optimal topologies with different values of the objective weight. The overall shapes of these solutions are similar but their internal details are considerably different. The smaller the ERR is, the lower the minimum objective can be obtained. Only for the case where a simple topology can be achieved, the element removal ratio has no effect on the weight and topology of the optimal solution. In many cases the value as high as 10% for the element removal ratio can give an efficient solution and this value is suggested as the first choice to get a rough idea on the optimal topology for the structure.

2. The ground structure has significant influence on the optimal solutions. In general, different optimal solutions (topologies and minimum objective weights) will be expected for different ground structures. When elements are connecting only neighbouring nodes, different optimal topologies are obtained for different ground structure grids. When element are connecting any node to all other nodes, different grids may lead to the same answer if these ground structures share a subset which form the optimum design. For the same grid, the ground structure where element are connecting one node to all other nodes gives the better solution compared to other ground structures with reduced set of elements. This type of element connection with a grid of a relatively small number of nodes is suggested as the first choice if no other requirement is imposed on the ground structure.

The following conclusions can be drawn from the study in this chapter for two-dimensional continuum structures:

(1) When the element removal ratio varies from 1% to 4%, it has little effect on the minimum objective weight and the outer shape of the optimal design. The removal ratio does affect the details of the inner parts. However, the main pattern and orientation of these inner parts are similar. In some cases the value of 2% for ERR works better than the value of 1%. It is suggested that one could use an element removal ratio as high as 4% to obtain optimal shape and topology with sufficient accuracy and significant time saving.

(2) Mesh size has little effect on the minimum objective weight, even though it affects the details of the final design. However, even coarse mesh can provide a rough idea of the shape and topology of the optimal design.

(3) The type of elements with similar sizes has almost no effect on the minimum objective weight but affects slightly the shape and topology of the optimal designs.

(4) The solution without checker-board patterns can be obtained by using the suggested procedure. This solution is more practically acceptable with the efficiency compared to the solution with checker-board patterns.

Despite the influences of the element removal ratio and the ground structure on the optimal solution, the proposed ESO method offers a very simple but effective tool for shape and topology optimization. Even non-optimum solutions generated by the method during the optimization process are much more efficient than the initial design. The solutions by the proposed method compare well to results obtained by other mathematically more complex methods.

Chapter 7

EVOLUTIONARY STRUCTURAL OPTIMIZATION METHOD FOR DISCRETE SIZING PROBLEMS

7.1 Introduction

In many practical applications of structural optimization, the design variables have to be selected from a given set of discrete values. For example, structural members may have to be selected from standard sections or thicknesses commercially available from manufacturers. In the literature, a wide range of sizing optimization problems have been solved by using mathematical programming and optimality criteria methods. However, these methods mostly handle problems where sizing design variables are required to be continuous in order to calculate derivatives of the objective and constraint functions with respect to design variables. They usually need special mathematical treatment when dealing with discrete design variables. Often the problem is solved for continuous optimal solution assuming all designs variables to be continuous and then a rounding-off or branch and bound method is used to obtain the discrete solution (Huang and Arora 1995; Sandgren 1990; Olsen and Vanderplaats 1989; Ringertz 1988; Schmit and Fleury 1980). Usually the values given for each discrete design variable are required to be close to each other for validity of converting the continuous optimal solution to a discrete solution (Ringertz 1988).

It appears that the ESO method, which has initially been developed for topology and shape optimization, can also be applied to sizing optimization problems. Using a very similar approach as developed for frequency optimization by Xie and Steven (1994b, 1996), Manickarajah *et al.* (1995) proposed an evolutionary procedure for sizing members to increase the buckling load factor while keeping the weight unchanged.

This chapter presents the applications of the ESO method developed in Chapter 3 to sizing optimization problems where design variables are discrete. In sizing problems the geometry of structures (nodal locations and element connectivities) is fixed, i.e. elements are not to be removed from the structure. To reduce the weight of a structure for a prescribed displacement limit, inefficient material is gradually removed from the structure. To improve the response of a structure for a prescribed weight, the inefficient material is systematically shifted to places where it can be more efficiently utilised. The effect of sizing elements on the specified displacement can be evaluated by using the virtual unit load method, which allows for dealing with discrete design variables directly. Sensitivity numbers for element reduction are derived using optimality criteria methods. An optimal design is obtained by gradually changing sizes of elements according to their sensitivity numbers until one of the constrained displacements reaches its given limit or no further improvement in the objective can be achieved. The initial results in this topic have been reported in Chu *et al.* (1996c, 1996d).

The method presented in this chapter is equally applicable to continuum structures (plates and shells) and to skeletal structures (trusses and frames). For simplicity, the discussion will be concentrated on continuum structures. Examples are provided to demonstrate the capability of the proposed method for discrete sizing optimization. Three types of sizing optimization problems will be considered including

1. minimum weight design subject to displacement constraints;
2. minimum displacement design subject to a weight constraint;
3. maximum stiffness (minimum strain energy) design subject to a weight constraint.

7.2 Minimum weight design subject to displacement constraints

7.2.1 Problem formulation

Consider the problem of minimizing the weight of structures subject to displacement constraints, where thicknesses (or cross-sectional dimensions) of elements must be assigned the values from the given sets of discrete values. Each element or each group of elements can have its own set of discrete values. The values in each set are often put in ascending or descending order for convenience. In sizing optimization problems elements cannot be removed. The only way to reduce the weight of a structure is to remove under-utilised material from the elements by reducing the element sizes (thicknesses or other cross-sectional dimensions). This can be done by simply assigning to their sizes the next lower values available from the given sets of discrete sizes.

The problem of minimum weight design of a structure subject to displacement constraints, where design variables are discrete, can be stated as follows:

By selecting the sizes for elements,

minimize *the weight of a structure*

subject to *constraints on the specified displacements.*

It is obvious that the problem belongs to the class of discrete variable problems. Besides the constraints on displacements, there are also constraints on the design variables, i.e. the design variables can only have values from given sets of discrete values. Because these constraints are being treated in a special way, they are not directly included into the constraint set in the problem formulation. Similar to the ESO method for topology and shape optimization problems given in Chapter 3, the effect of element reduction on a specified displacement can be assessed using information available from a finite element analysis and then sensitivity numbers for element reduction can be derived from optimality criteria for the general weight minimization problems.

7.2.2 Effect of element sizing on displacements

Suppose a structure is modelled by n finite elements and the ith element is to be sized to the next lower or higher available dimension. This results in the change in the element weight by

$$\Delta w_i = w_i^{new} - w_i \tag{7.1}$$

and the changes in the stiffness matrix of the element and of the structure by

$$[\Delta K] = [\Delta K^i] = [K^i]^{new} - [K^i] \tag{7.2}$$

By considering equilibrium conditions before and after changing the size of the ith element, i.e. equations (3.1) and (3.3) in Chapter 3, and assuming no change in the nodal load vector, the change in the displacement vector will be determined again by equation (3.4). By introducing a virtual unit load vector $\{F^j\}$, in which only the component corresponding jth degree of freedom is equal to unity and all other components are equal to zero, the change in the displacement component u_j due to changing the size of the ith element can be determined as

$$\begin{aligned} \Delta u_j &= \{F^j\}^T\{\Delta u\} = -\{F^j\}^T[K]^{-1}[\Delta K]\{u\} \\ &= -\{u^j\}^T[\Delta K]\{u\} = -\{u^{ij}\}^T[\Delta K^i]\{u^i\} \\ &= \{u^{ij}\}^T[K^i]\{u^i\} - \{u^{ij}\}^T[K^i]^{new}\{u^i\} \end{aligned} \tag{7.3}$$

where $\{u\}$ and $\{u^j\}$ are the displacement vectors due to the real load $\{P\}$ and the virtual unit load $\{F^j\}$, respectively; $\{u^i\}$ and $\{u^{ij}\}$ contain the entries from $\{u\}$ and $\{u^j\}$ which are related to the ith element. The value Δu_j can be positive or negative, which implies that u_j may change in opposite directions. It is observed in many cases that when the element size is reduced, i.e. when material is removed from the element, the stiffness of the element reduces, which also reduces the stiffness of the structure. As a result, the displacement will increase in the absolute value.

It has been found in Chapter 3 (see equations (3.7) and (3.8)) that the change in the specified displacement u_j due to the removal of the whole element is equal to the element virtual energy, that is

$$\alpha_{ij} = \{u^{ij}\}^{\mathrm{T}}[K^i]\{u^i\} \qquad (i = 1, n) \tag{7.4}$$

and

$$u_j = \sum_{i=1}^{n} \alpha_{ij} \tag{7.5}$$

where α_{ij} is also referred to as the element contribution. With an assumption that the changes in the displacement vectors are small due to sizing an element, the change in the element virtual energy can be determined approximately as

$$\Delta\alpha_{ij} = \alpha_{ij}^{new} - \alpha_{ij} = \{u^{ij}\}^{\mathrm{T}}[K^i]^{new}\{u^i\} - \{u^{ij}\}^{\mathrm{T}}[K^i]\{u^i\} \quad (i = 1, n) \tag{7.6}$$

Comparing equations (7.3) and (7.6) we have

$$\Delta u_j = -\{u^{ij}\}^{\mathrm{T}}[\Delta K^i]\{u^i\} = -\Delta\alpha_{ij} \qquad (i = 1, n) \tag{7.7}$$

which states that, *in absolute values, the change in the specified displacement is equal to the change in the element virtual energy due to changing the element sizes.*

Equation (7.7) gives the estimated change in the specified displacement due to changing the element sizes. It is obvious that reducing the element whose $\Delta\alpha_{ij}$ is close to zero or $|\Delta\alpha_{ij}|$ is the lowest will result in the minimum change in the displacement. When all elements are allowed to be reduced by equal weight portions, reduction of the element with the lowest $|\Delta\alpha_{ij}|$ is always the best choice, because the weight of structure is reduced by the same amount while the least change in the specified displacement results. However, when elements are reduced by different weight portions, the efficiency of reduction of an element depends also on how much change in its own weight w_i. Comparing two elements whose reductions result in the same $|\Delta\alpha_{ij}|$, it is obvious that reduction of the element, which gives a larger reduction in weight, will result in a lighter structure with an equal response. This means that reduction of the element, which has lower value of the ratio $|\Delta\alpha_{ij}|/|\Delta w_i|$, is more efficient. Using exactly the same Lagrange multiplier approach, as given in Chapter 3, the sensitivity number for element reduction can be derived for the problem with a single and multiple constraints.

7.2.3 Sensitivity numbers for element reduction

7.2.3.1 Single displacement constraint

For the problem with the objective of minimizing the weight of a structure

$$W = \sum_{i=1}^{n} w_i \tag{7.8}$$

subject to a constraint imposed on a specified displacement given in the form

$$|u_j| - u_j^* \leq 0 \tag{7.9}$$

where n is number of elements, w_i is the weight of the ith element and u_j^* is the prescribed limit for the absolute value of the displacement u_j, the Lagrangian is defined as

$$L = W - \lambda(|u_j| - u_j^*) \tag{7.10}$$

where λ is a Lagrange multiplier.

Considering w_i (i = 1, n) as design variables, the optimality conditions for the above problem are

$$\frac{\partial L}{\partial w_i} = \frac{\partial W}{\partial w_i} - \lambda\left|\frac{\partial u_j}{\partial w_i}\right| = 1 - \lambda\left|\frac{\partial u_j}{\partial w_i}\right| = 0 \qquad (i = 1,\ n) \qquad (7.11)$$

which can be approximated by

$$1 - \lambda\left|\frac{\Delta u_j}{\Delta w_i}\right| = 0 \qquad (i = 1,\ n) \qquad (7.12)$$

Recalling equations (7.1) and (7.7), which give the change in the element weight and the change in the specified displacement due to the element reduction, the optimality conditions (7.12) become

$$\gamma_i = \frac{|\Delta\alpha_{ij}|}{|\Delta w_i|} = \frac{1}{\lambda} = constant \qquad (i = 1,\ n) \qquad (7.13)$$

Equation (7.13) represents an optimality criterion that, *at the optimum the absolute ratio of the change in the element virtual energy to the change in the element weight is equal for all elements*. This is a specialised interpretation of "the correct optimality criteria statement valid for a general class of optimization problems with diminishing return on investment" (Berke and Khot, 1988). When the design variables are continuous, various recurrence formulae have been suggested to search for the optimum where γ_i are equal for all elements. Thus, the weight of the structure is indirectly minimized. By means of selecting element sizes it is not always possible to reach such an uniform state. However, a more uniform values of γ_i in the resulting structure can be achieved by reducing sizes of the elements with the lowest γ_i. An iterative process is used to reduce the sizes of a small number of elements with the lowest γ_i whilst the specified response is slowly approaching the prescribed limit, and thus the minimum weight is reached at the same time. It is seen that the value γ_i is an essential measure of efficiency of the material.

Thus, *the sensitivity number for element reduction for a single displacement constraint* is defined as

$$\gamma_i = \gamma_{ij} = \frac{|\Delta\alpha_{ij}|}{|\Delta w_i|} = \frac{|\{u^{ij}\}^{\mathrm{T}}[\Delta K^i]\{u^i\}|}{|\Delta w_i|} \qquad (i = 1,\ n) \tag{7.14}$$

This sensitivity number can be easily calculated using the results available following the finite element analysis of the static problem (1) for both the real load $\{P\}$ and the virtual unit load $\{F^j\}$. It is not difficult to calculate the change in the element stiffness matrix and the change in the element weight since the new thickness or section is specified as the next lower thickness or next smaller section in the given set.

7.2.3.2 Multiple displacement constraints

With the objective of minimizing the weight of a structure subject to multiple displacement constraints given in the form

$$|u_j| - u_j^* \le 0 \qquad (j = 1,\ m) \tag{7.15}$$

where m is number of constraints and u_j^* is the given limit for $|u_j|$, the Lagrangian is

$$L = W - \sum_{j=1}^{m} \lambda_j (|u_j| - u_j^*) \tag{7.16}$$

where λ_j is the Lagrange multiplier for the jth constraint. The optimality conditions for the problem are

$$\frac{\partial L}{\partial w_i} = \frac{\partial W}{\partial w_i} - \sum_{j=1}^{m} \lambda_j \left|\frac{\partial u_j}{\partial w_i}\right| = 1 - \sum_{j=1}^{m} \lambda_j \left|\frac{\partial u_j}{\partial w_i}\right| = 0 \qquad (i = 1,\ n) \tag{7.17}$$

where $\lambda_j > 0$ for the active constraints $|u_j| - u_j^* = 0$, and $\lambda_j = 0$ for the passive constraints $|u_j| - u_j^* < 0$. The optimality conditions (7.17) can be approximated by

$$1-\sum_{j=1}^{m}\lambda_j\left|\frac{\Delta u_j}{\Delta w_i}\right|=0 \qquad (i=1,\ n) \tag{7.18}$$

Using equation (7.7), the optimality conditions (7.18) become

$$1-\sum_{j=1}^{m}\lambda_j\left|\frac{\Delta \alpha_{ij}}{\Delta w_i}\right|=0 \qquad (i=1,\ n) \tag{7.19}$$

or

$$\eta_i=\sum_{j=1}^{m}\lambda_j\left|\frac{\Delta \alpha_{ij}}{\Delta w_i}\right|=1 \qquad (i=1,\ n) \tag{7.20}$$

Equation (7.20) represents an optimality criterion for multiple constraints that, *at the optimum the weighted sum of the absolute ratio of the change in the element virtual energy to the change in the element weight is equal to unity for all elements, where the weighting parameters are the Lagrange multipliers.* Reducing the elements with the lowest values of η_i will result in more uniform distribution of η_i among elements, hence a more efficient design. The value η_i can be viewed as a measure of the efficiency of reduction of an element in the case of multiple constraints.

Therefore, *the sensitivity number for element reduction for multiple displacement constraints* is defined as

$$\eta_i=\sum_{j=1}^{m}\lambda_j\left|\frac{\Delta \alpha_{ij}}{\Delta w_i}\right| \qquad (i=1,\ n) \tag{7.21}$$

Using (7.14), the above equation can be rewritten as

$$\eta_i=\sum_{j=1}^{m}\lambda_j\gamma_{ij} \qquad (i=1,\ n) \tag{7.22}$$

which implies that *the sensitivity number of an element for multiple constraints is the weighted sum of its sensitivity numbers corresponding to each constraint, where the weighting parameters are the Lagrange multipliers.*

The same approach as given in Chapter 3 can be used to evaluate the Lagrange multipliers. The recurrence and the ratio formulae will have exactly the same forms as equations (3.28) and (3.29). Due to the different form of the optimality conditions given in (7.20), the linear equations will be slightly different and can be also derived from the optimality criterion (7.20) assuming that the constraints are active, i.e. conditions (7.15) become equality. By using the constraint limit scaling procedure where the scaling parameter for the displacement limits is determined as

$$\varphi = \max_{j=1,\,m} \varphi_j = \max_{j=1,\,m} \left(\frac{|u_j|}{u_j^*} \right) \tag{7.23}$$

the corresponding formulae for the Lagrange multipliers will have the following forms:

the recurrence formulae

$$\lambda_j^{new} = \lambda_j^{old} \left(\frac{|u_j|}{\varphi u_j^*} \right)^{\frac{1}{b}} \qquad (j = 1,\, m) \tag{7.24}$$

the ratio formulae

$$\lambda_j = \left(\frac{|u_j|}{\varphi u_j^*} \right)^{\frac{1}{b}} \qquad (j = 1,\, m) \tag{7.25}$$

and the linear equations

$$\sum_{p=1}^{m} \lambda_p \left(\sum_{i=1}^{n} \frac{|\Delta\alpha_{ip}||\alpha_{ij}|}{|\Delta w_i|} \right) = \varphi u_j^* \qquad (j = 1,\, m) \tag{7.26}$$

or

$$\sum_{p=1}^{m} \lambda_p \left(\sum_{i=1}^{n} \frac{|\Delta\alpha_{ip}||\alpha_{ij}|}{|\Delta w_i|} \right) = (1+r)u_j - r\varphi u_j^* \qquad (j = 1,\, m) \tag{7.27}$$

where b and r are step control parameters. Using these formulae, the Lagrange multipliers are calculated only for constraints in the active set. The use of linear equations (7.26) and (7.27) needs a procedure to revise the set of active constraints to obtain positive values for corresponding Lagrange multipliers.

7.2.4 Optimization procedure for weight minimization

Similar to the ESO method for topology problems, there can be two different optimization procedures depending on whether the design variable scaling or constraint limit scaling is used to keep constraints active. In discrete sizing problems, it is required to keep the design variable within the given sets of values. Therefore, the constraint limit scaling should be employed. The ESO procedure for sizing problems is exactly the same as for topology problems with only one difference that the step for maintaining non-singularity of the stiffness matrix is no longer needed for sizing problems.

Using the constraint limit scaling procedure, the evolutionary procedure for size selection to minimize the weight of structures subject to displacement constraints can be stated as follows:

Step 1: Model the structure by finite elements with maximum available sizes.

Step 2: Analyse the structure for the given loads and virtual unit loads.

Step 3: If any constraints is violated, go to step 7. Otherwise, go to step 4.

Step 4: Calculate the sensitivity numbers γ_{ij} for each element corresponding to each constraint. For multiple constraints, scale the constraint limits, select the active set of constraints, calculate the Lagrange multipliers for the active set and the weighted sensitivity numbers η_i for each element.

Step 5: If the sensitivity numbers are uniform, go to step 7. Otherwise, go to step 6.

Step 6: Select lower sizes for a number of elements which have the lowest sensitivity numbers. Go back to step 2.

Step 7: Stop. Scaling actual weights to get the objective weights and select the optimum solution among the available designs.

Initially, all elements are assigned their maximum available thicknesses or sections. If after analysis this initially chosen design violates the displacement constraints, then the optimization process is terminated because no further feasible design can be obtained by removing material from the structure. Sensitivity numbers are calculated assuming that the element size will change to the next lower value. During the optimization process elements whose thicknesses or sections have been reduced to their minimum thicknesses or sections cannot be further reduced. The optimum solution is the one which has the lowest objective weight among the available designs obtained during the optimization process.

In the proposed procedure, the number of elements subjected to size reduction at each iteration needs to be specified. This number can be prescribed by one of the following options:

(1) by an integer number;

(2) by the ratio of the number of elements subjected to size changing to the total number of elements, called the *element changing ratio* (ECR); or

(3) by the ratio of the weight (material) to be removed in each iteration to the total weight (material), called the *material removal ratio* (MRR).

The first and the second options are always equivalent because the total number of elements is unchanged. When the initial weight is considered as the total weight and equal weight portions can be removed from elements, the number of elements subjected to reduction is a constant. Only in this case, three options are equivalent.

In the third option, when the current weight is considered as the total weight and equal weight portions can be removed from elements, the number of elements subjected to sizing will decrease as the actual weight of structure reduces. In general, when the material removal ratio (MRR) is prescribed and elements are changing differently, the number of elements subjected to size reduction varies from iteration to iteration.

To get an accurate solution either the number of elements subjected to sizing, the value of ECR or MRR should be small enough to ensure a smooth change in the stiffness of the structure and consequently, a smooth change in displacements. It is observed that the level of change in displacements depends not only on the value of ECR which defined the number of elements subjected to size reduction, but also on how much the change in the size is or how large the step size is. For bar or plate elements, the reduction in cross-section areas or in thicknesses, respectively, is considered as a step size. For beam elements more than one step size may be needed. The change in the weight of elements is also an important factor, which reflects a change in any cross-sectional dimension of elements. So the decrement in the element weight should be considered as a common step size. As the step size increases, more material is removed from elements and the total change in displacements will be larger even if the same ECR is used. To reduce the change in the displacements, a smaller number of elements subjected to sizing (or smaller ECR) should be used. It is found that the material removal ratio (MRR) more closely relates to the level of change in displacements between two adjacent designs. Therefore, MRR is preferably used to control the change between two design steps. In the case where equal weight portions can be removed from elements, the number of elements subjected to reduction (or ECR) can be determined from the prescribed value of the MRR. It is noted that in the ESO method for topology and shape optimization, the element removal ratio (ERR) of 1% to 4% are used for continuum structures. When regular finite element meshes and an equal step size are used, the element removal ratio (ERR) and the material removal ratio (MRR) are equivalent. So these values are also adopted for MRR in sizing problems.

7.2.5 Computer implementation for weight minimization

A computer programs called ESOSZPLT is developed by the candidate for weight minimization of plate/shell structures and a similar program is developed for truss/frame structures. The program ESOSZPLT calculates the sensitivity numbers for all elements and reduces thicknesses of a specified number of elements. A batch file is set up to link each program to the finite element program STRAND6 and a loop is created to carry out the iterative process of optimization.

The given discrete thicknesses or sections for each element or each group of elements are prescribed by using different element properties. They are input either in ascending or descending order as the element property number increases. The number of properties for a plate element is equal to the number of given thicknesses. Similarly, the number of properties for a beam element is equal to the number of available sections. It should be noted that for beam elements any change in one or more cross-sectional dimensions will be taken into account by using different section properties. Each member and each group of members can have its own set of properties. Initially all elements are assigned the properties with maximum thicknesses or sections. After calculation of sensitivity numbers, elements with the lowest sensitivity numbers are reduced by changing their properties to the next properties with lower thicknesses or smaller sections. All properties of elements are put together ranging from their lower to upper property numbers. These lower and upper properties contain the minimum and maximum thicknesses. They are specified in data file. Elements which have reached their minimum thicknesses or sections cannot be further reduced. It is possible to keep a number of elements unchanged by specifying their properties as non-design properties

Input data for the optimization program ESOSZPLT include the number of real load cases, the number of constraints, the virtual load factor, the material removal ratio MRR or the element changing ratio ECR, the parameter indicating the order (ascending or descending) of thicknesses input in the plate property file, the parameter indicating the symmetry, the lower and upper property numbers with either maximum or minimum thicknesses, non-design properties, the type of formulae and step control parameter for the Lagrange multipliers, and the limits of displacements.

Output files of the program ESOSZPLT provide information including the actual weight, the values of constrained displacements, the maximum and minimum sensitivity numbers at each step. At the end of each iteration, the plate element connection file is copied and stored in a separate file for later recovery of the element thicknesses according to the element property numbers.

7.2.6 Examples of weight minimization

This section illustrates the capability of the proposed method for selecting element sizes from given sets of available discrete values to minimize the weight of structures subject to displacement constraints.

7.2.6.1 *Plate in bending*

Consider the problem of least weight design for a plate in bending with discrete thicknesses. A simply supported square plate with sides of 8 m is carrying at the centre a point load P = 100 kN normal to its plane as shown in Fig. 7.1. The Young's modulus E = 30 GPa and the Poisson's ratio ν = 0.2 are assumed. The plate is designed for different set of discrete thicknesses ranged from the minimum 0.2 m to the maximum 0.4 m. Because of the symmetry, only a quarter of the plate is modelled using 400 four-node quadrilateral plate elements. Initially, all elements are assigned the maximum thickness 0.4 m and the initial out-of-plane displacement at the centre is 1.78 mm. The limit of 2.1 mm and then 2.5 mm is imposed on the displacement at the centre.

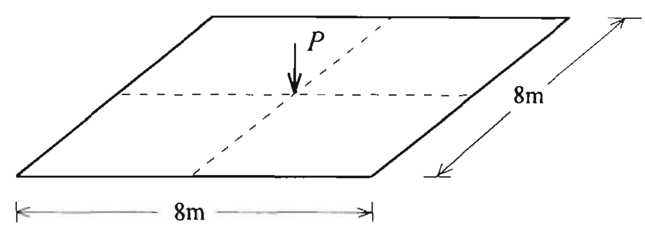


Fig. 7.1 Uniform thickness design for plate in bending.

Using MRR = 1% of the initial weight, solutions are obtained for the following cases:

a) *Design for 2 thicknesses*: step size in thickness Δt = 0.2 m, ECR = 2% (i.e. 8 elements subjected to thickness reduction in each iteration).

b) *Design for 3 thicknesses*: step size in thickness Δt = 0.1 m, ECR = 4% (i.e. 16 elements subjected to thickness reduction in each iteration).

c) *Design for 5 thicknesses*: step size in thickness Δt = 0.05 m, ECR = 8% (i.e. 32 elements subjected to thickness reduction at each iteration).

The optimization history for the limit of 2.5 mm for these cases are given in Fig. 7.2. It is seen in Fig. 7.2a that by using the same MRR (1% of the initial weight), the pattern of the change in the displacement in all cases are very similar. The displacement is almost unchanged in the first several iterations and then gradually increases in the following steps. This illustrates the efficiency of the proposed sensitivity numbers. Small differences are observed in the last few iterations.

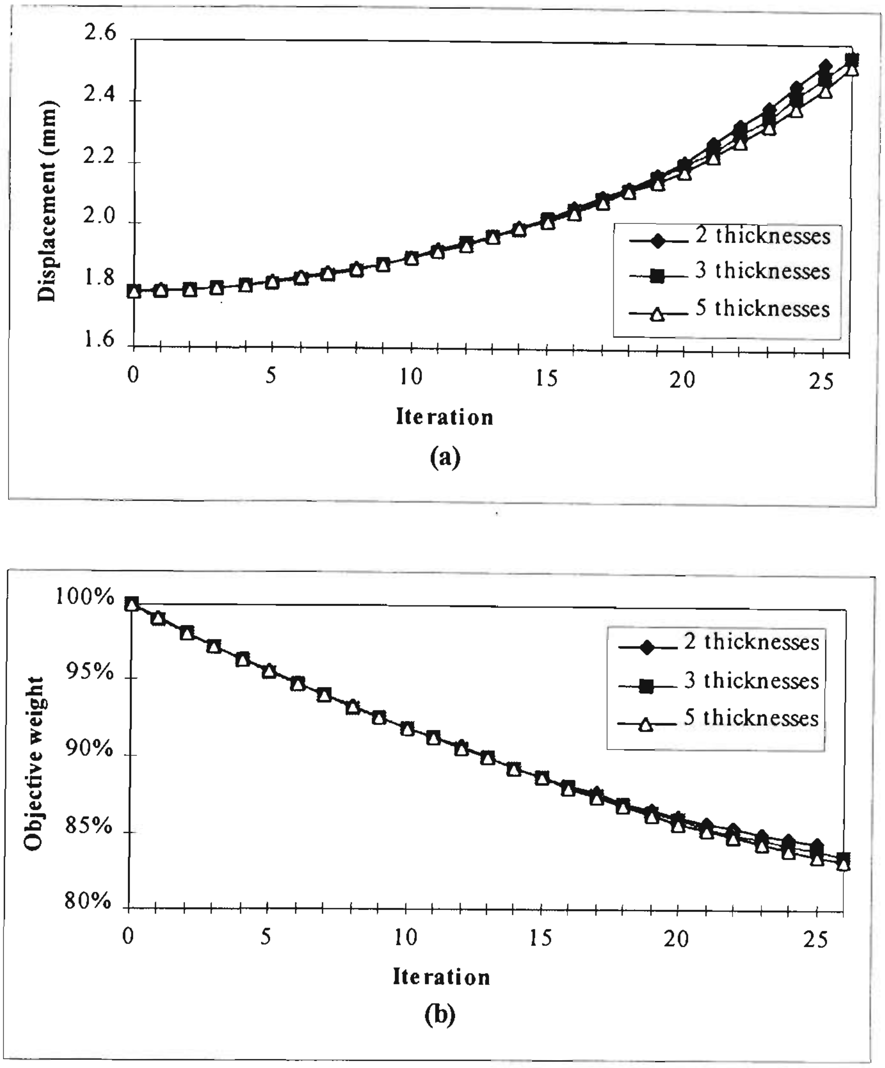


Fig. 7.2 Optimization history for plate in bending:

(a) displacement; (b) objective weight.

Table 7.1 Objective weight of optimal designs for plate in bending.

	Objective weight (with reference to the initial value)		
Limit (mm)	2 thicknesses	3 thicknesses	5 thicknesses
2.1	87.61%	87.53%	87.40%
2.5	84.42%	83.46%	83.18%

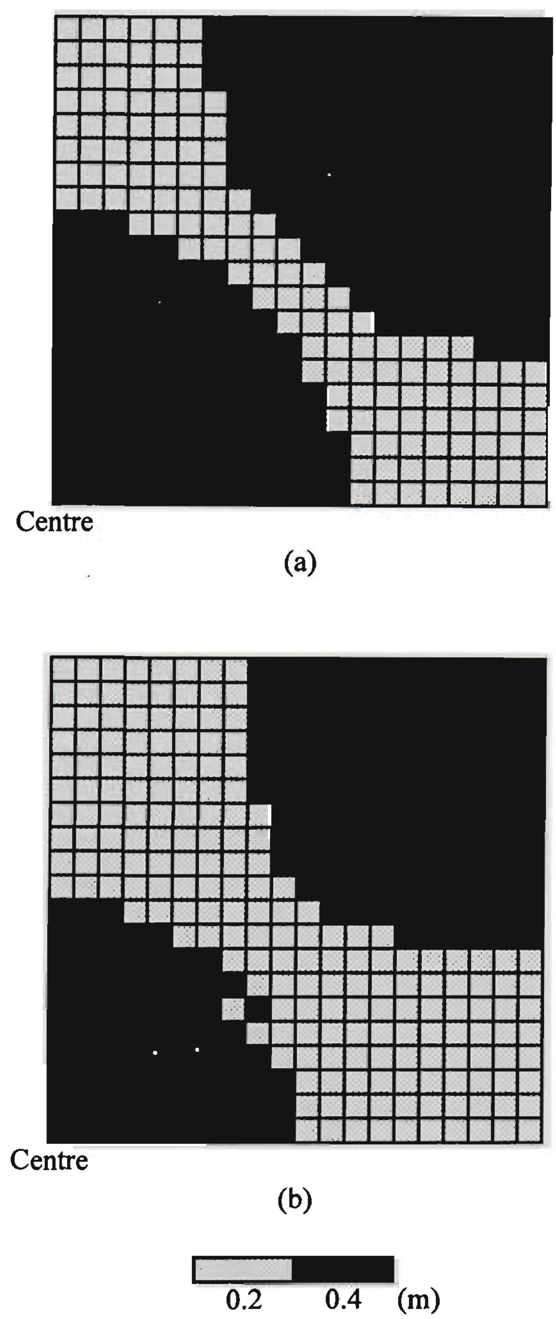


Fig. 7.3 Optimal designs for a quarter of the plate with 2 thicknesses and MRR = 1%: (a) limit of 2.1 mm; (b) limit of 2.5 mm.

The efficiency of the ESO designs can be evaluated by scaling the actual weight to get the objective weight for the prescribed limit. The same equation (5.1)-(5.3) as given in Chapter 5 (section 5.4.4) are used to calculate the objective weight at any step. It has been pointed out that the pattern of change in the objective weight with respect to its initial value is the same for any limit. With the same changes in the actual weight

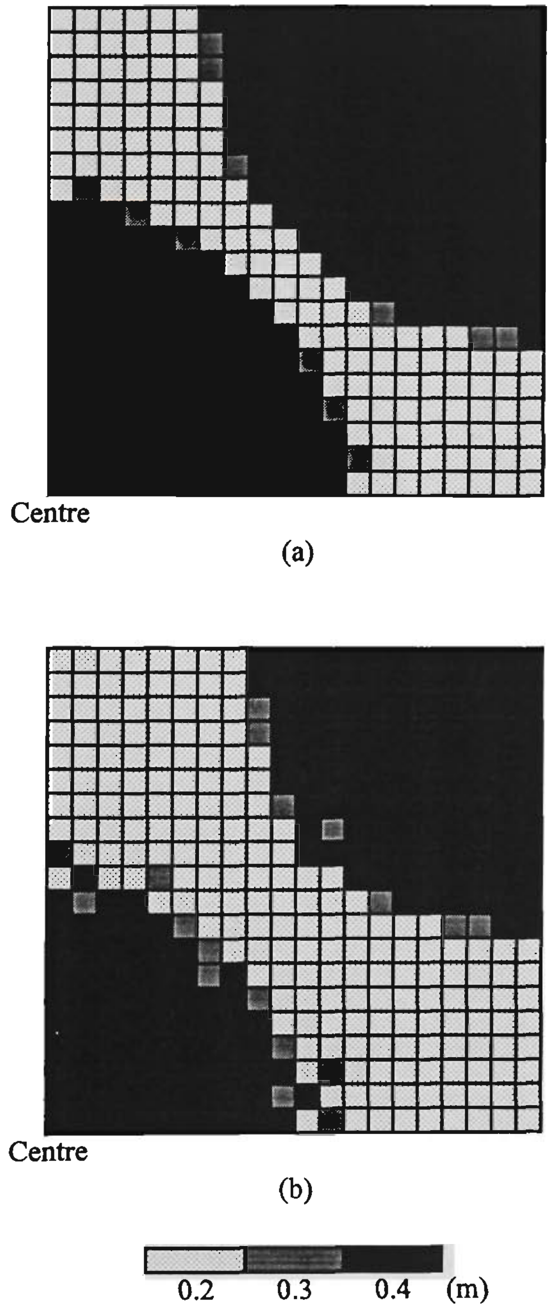


Fig. 7.4 Optimal designs for a quarter of the plate with 3 thicknesses and MRR = 1%: (a) limit of 2.1 mm; (b) limit of 2.5 mm.

and similar changes in the displacement, the changes in the objective weight in all cases are almost the same as shown in Fig. 7.2b. Only some differences occur in the last few iterations. In all cases the objective weight have been significantly reduced. Table 7.1 presents the objective weights for the limits of 2.1 mm and 2.5 mm. In this example, the smaller the step size is, the lower the minimum weight will be.

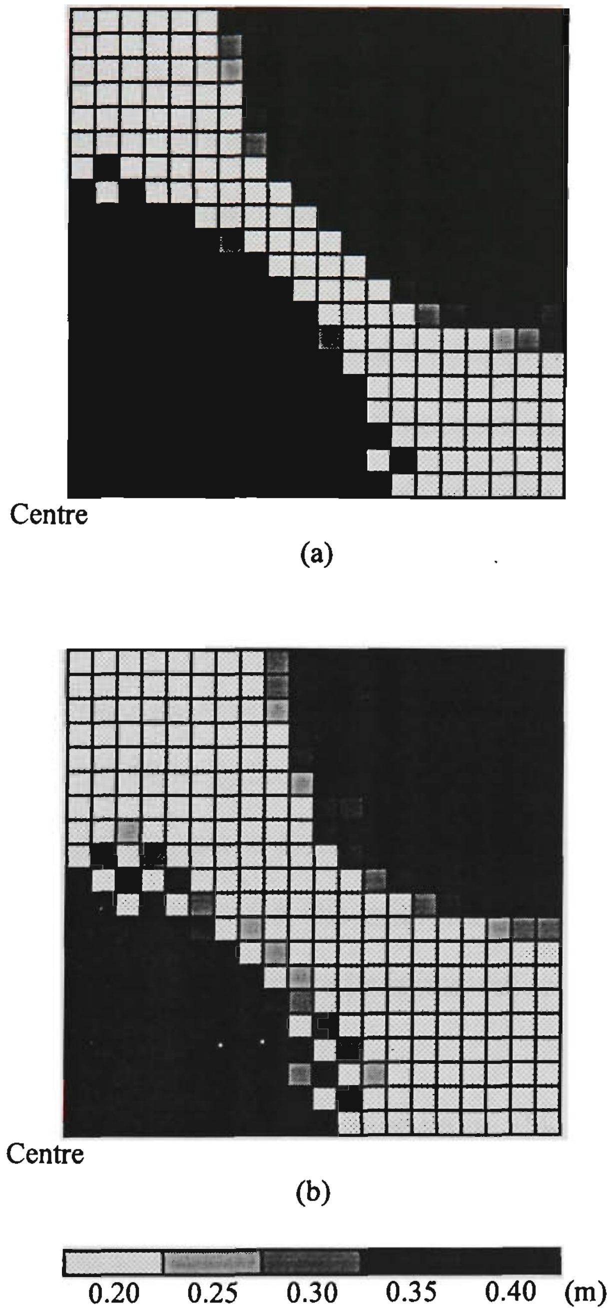


Fig. 7.5 Optimal designs for a quarter of the plate with 5 thicknesses and MRR = 1%: (a) limit of 2.1 mm; (b) limit of 2.5 mm.

The optimal designs for 2, 3 and 5 thicknesses are shown in Figs. 7.3, 7.4 and 7.5 for the limits of 2.1 and 2.5 mm. It is seen that the thickness distributions are similar for the same limit. The thicknesses of elements are mainly at the maximum and minimum values. The areas with minimum thickness in all cases are located along the elastic hinge lines, that have been reported in Tenek and Hagiwara (1994).

It is seen from the trend of the change in the objective weight (Fig. 7.2b), the optimal design for each case is the last design, where the displacement reaches the limit. It should be noted that these designs are optimum for the specified set of fixed thicknesses and the prescribed limit of the displacement.

In the case where the thicknesses are allowed to change proportionally, i.e. they can be scaled by the same factor, the optimum thickness distribution may be obtained beyond the specified limit. In this case, a larger limit for the displacement should be used to continue the optimization process until no further improvement can be achieved. A close look at these cases reveals that the objective weight will reach a minimum value at a certain step and then start rising. This is due to the fact that the process starts from the uniform maximum thickness design. As more material is removed later from the design, more elements are assigned the minimum thickness. Sooner or later a uniform design with the minimum thickness will be obtained by which the objective weight will get back to its initial value.

It is worth pointing out that the use of the same value 1% for MRR in all three cases gives very similar change in the displacement as seen in Fig. 7.2a despite substantial differences in Δt (0.20, 0.10 and 0.05m) and large differences in ECR (2, 4, 8%). This illustrates that the material removal ratio is much more important than the element changing ratio in the evolutionary method for sizing optimization.

7.2.6.2 *Plate under multiple displacement constraints*

This serves as an example of structures subject to multiple displacement constraints. A rectangular plate with dimensions L_x = 4 m and L_y = 2 m is clamped at one long edge. Three point loads P = 20 kN, normal to the plate, are acting at the free corners and in the middle of the free edge as shown in Fig. 7.6. The Young's modulus $E = 30$ GPa and the Poison's ratio $\nu = 0.2$ are assumed.

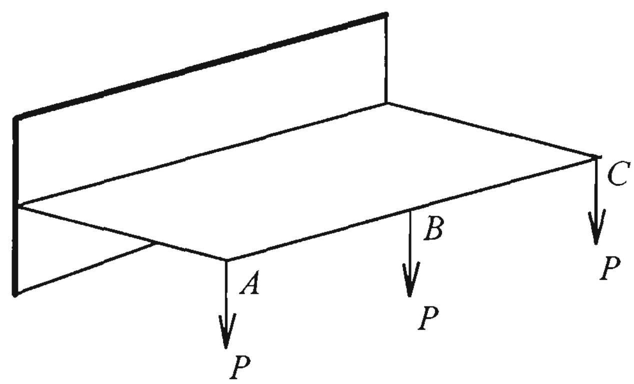


Fig. 7.6 Initial uniform design for a plate under multiple constraints.

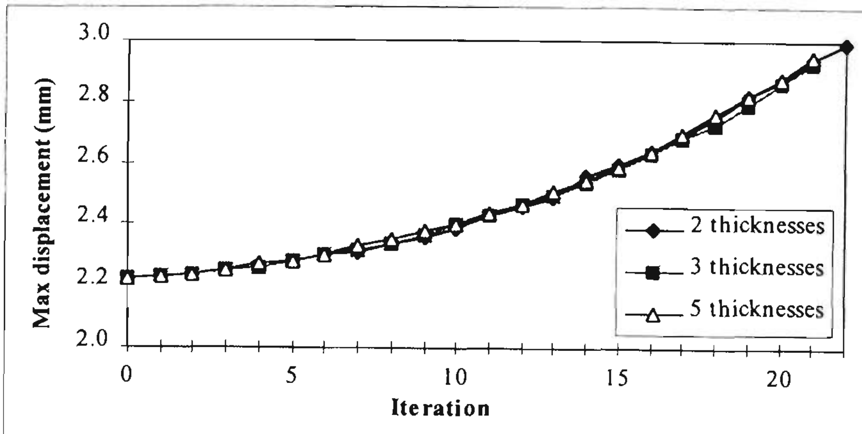
The plate is being designed for different sets of thicknesses with the maximum of 0.2 m and the minimum of 0.1 m. Due to symmetry, only a half of the plate is analysed using 400 four-node quadrilateral plate elements. Initially, all elements are assigned the maximum thickness $t_1 = 0.2$ m and the initial maximum out-of-plane displacement is 2.22 mm. A common limit of 2.5 mm and then of 3.0 mm is imposed on displacements at three loaded points.

All constraints are included into the active constraint set. The ratio formulae (3.38) are used with $b = 1$ for the Lagrange multipliers. Using MRR = 1% of the initial weight, solutions are obtained for the following cases:

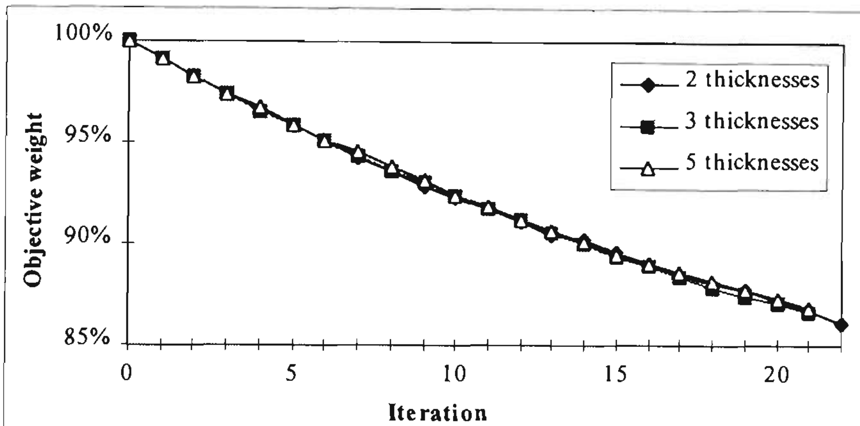
a) *Design for 2 thicknesses*: $\Delta t = 0.1$ m, ECR = 2% (for a half plate, 8 elements are subjected to thickness reduction at each iteration).

b) *Design for 3 thicknesses*: $\Delta t = 0.05$ m, ECR = 4% (for a half plate, 16 elements are subjected to thickness reduction at each iteration),

c) *Design for 5 thicknesses*: $\Delta t = 0.025$ m, ECR = 8% (for a half plate, 32 elements are subjected to thickness reduction at each iteration).



(a)



(b)

Fig. 7.7 Optimization history for the plate under multiple constraints:

(a) maximum displacements; (b) objective weights.

Table 7.2 Objective weight of optimal designs for plate under multiple constraints

	Objective weight (with reference to the initial value)		
Limit (mm)	2 thicknesses	3 thicknesses	5 thicknesses
2.5	90.39%	91.19%	91.19%
3.0	86.14%	86.66%	86.85%

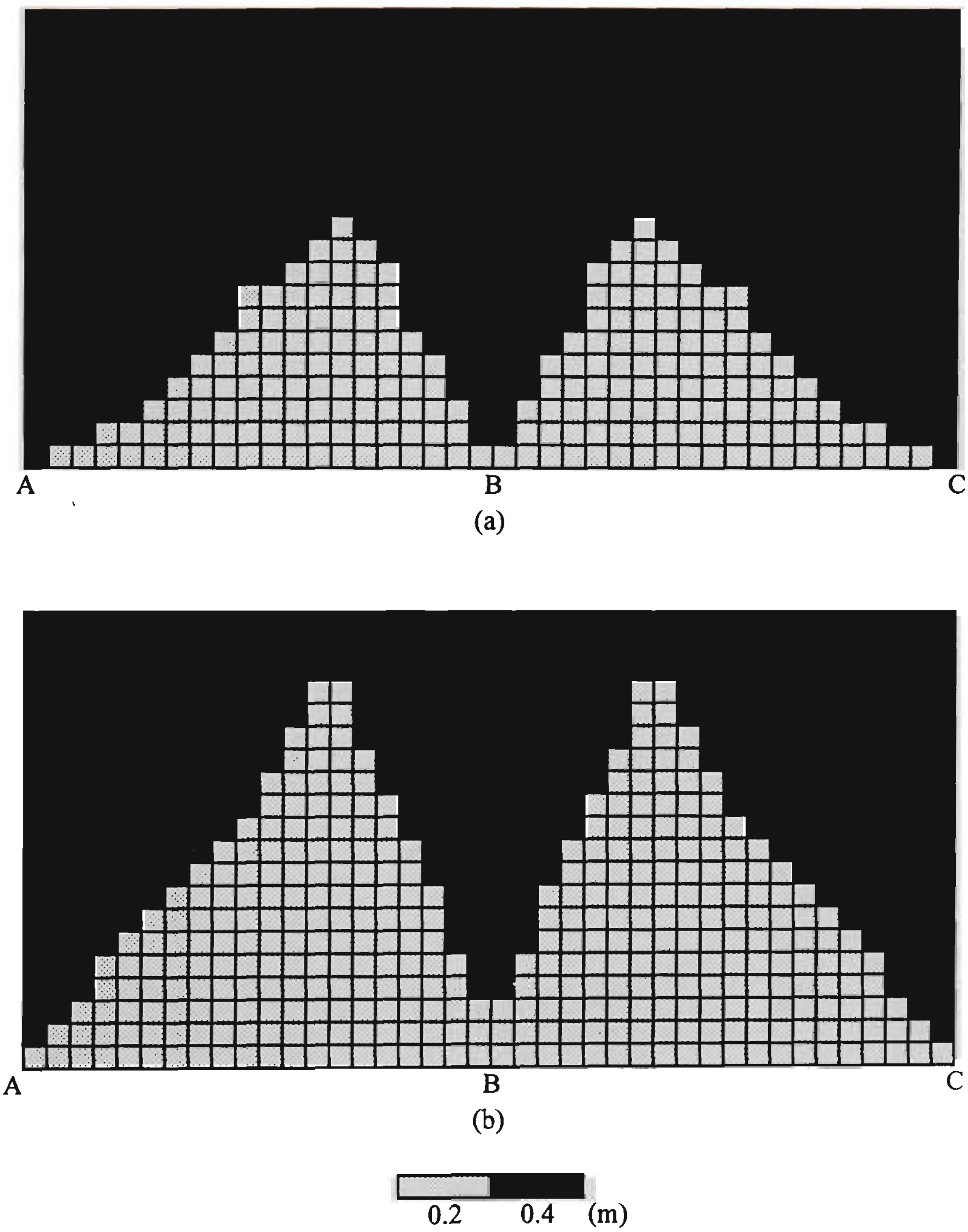


Fig. 7.8 Optimal designs for the plate under multiple constraints with 2 thicknesses and MRR = 1%: (a) for limit of 2.5 mm; (b) for limit of 3.0 mm.

The optimization history for all cases is given in Fig. 7.7. It is found in Fig 7.7a that the change in the maximum displacement in all cases are almost the same. These changes are very small in the first several iterations and then gradually increases in the following steps while an equal amount of material is removed in each step. This illustrates the effectiveness of the sensitivity numbers for multiple constraints.

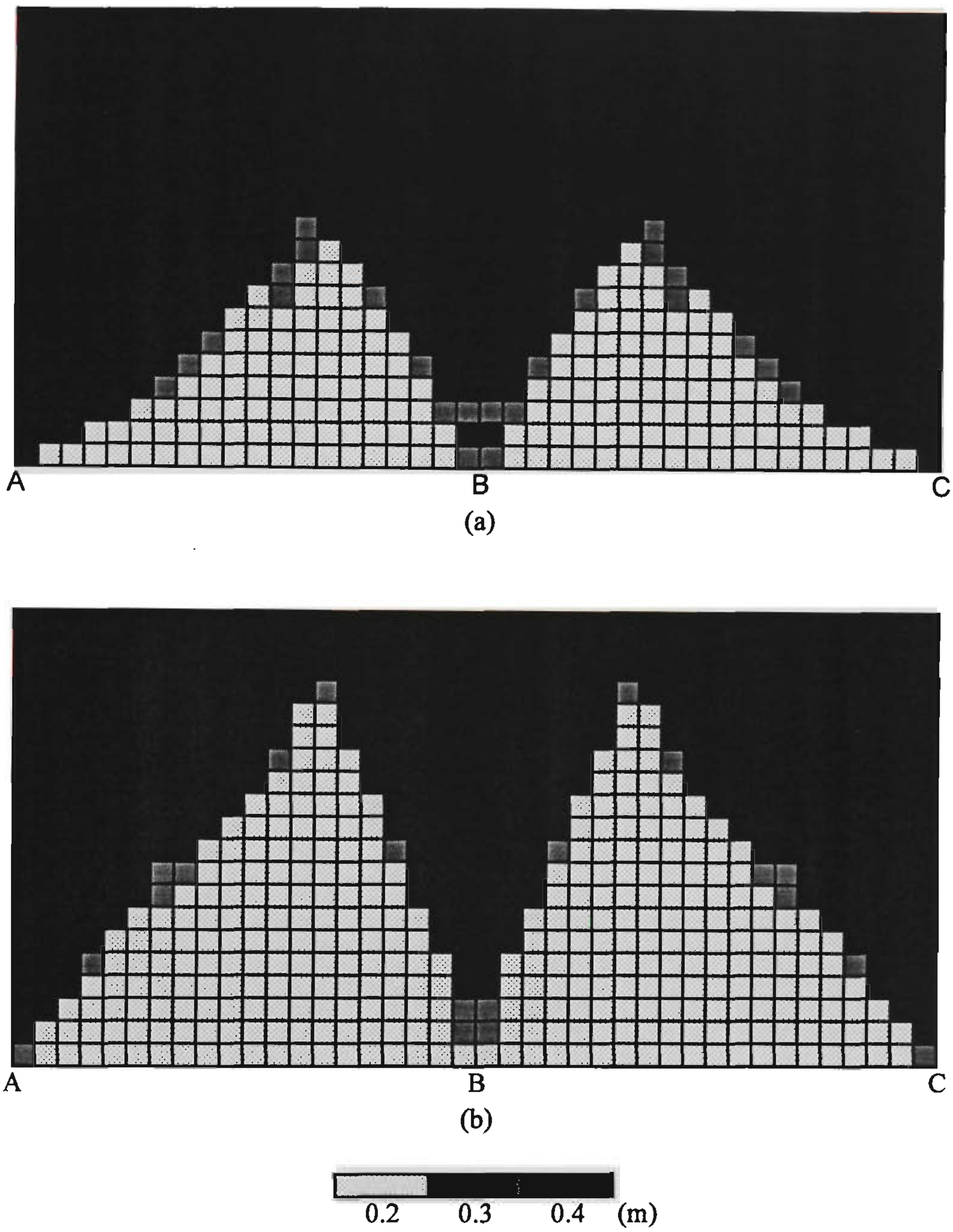
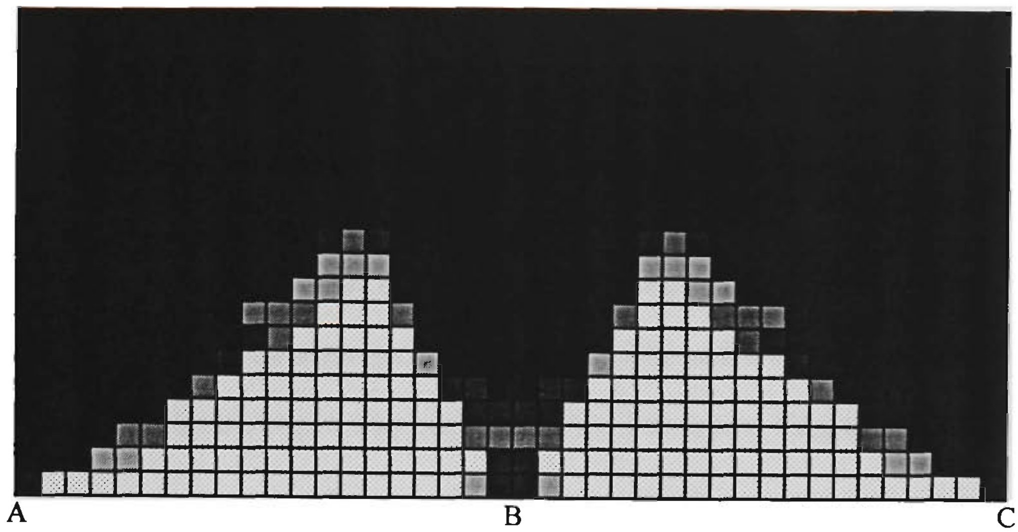
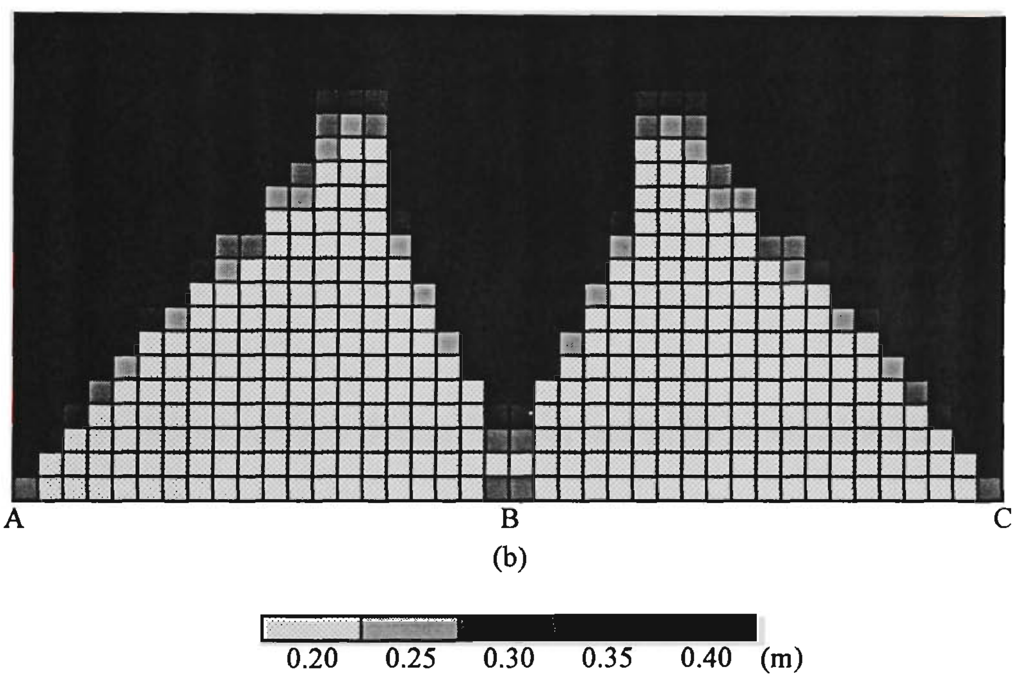


Fig. 7.9 Optimal designs for the plate under multiple constraints with 3 thicknesses and MRR = 1%: (a) for limit of 2.5 mm; (b) for limit of 3.0 mm.

The changes in the objective weight in all cases are almost the same. The objective weight continuously reduces until the displacement reaches the limit. The minimum weight for the specified limit is obtained in the last step where the displacement reaches the limit. The optimal designs are shown in Figs. 7.8, 7.9 and 7.10. It is observed that the shapes of the optimal designs in all cases are very similar. The



(a)



(b)

Fig. 7.10 Optimal designs for the plate under multiple constraints with 5 thicknesses and MRR = 1%: (a) for limit of 2.5 mm; (b) for limit of 3.0 mm.

objective weights of the optimal designs are given in Table 7.2. The designs for 2 thicknesses result in a slightly lighter objective weight. The designs for 3 and 5 thicknesses have the very similar objective weights. All optimal designs obtained by the proposed method are more efficient in comparison with the initial uniform design.

7.2.6.3 *Plate under torsional loading*

A plate of dimensions 0.20 m × 0.10 m is clamped along one short edge and is under a torsional loading, which is created by applying two point loads of the same magnitude P = 1 N but opposite direction at the corner nodes of the free edge as shown in Fig. 7.11. The Young's modulus E = 90 GPa and the Poison's ratio ν = 0.3 are assumed. A similar example was considered by Tenek and Hagiwara (1994).

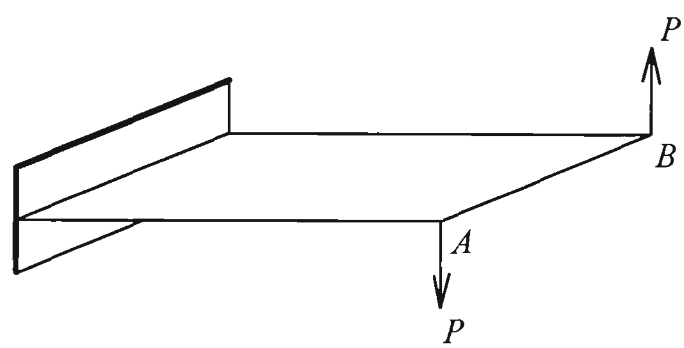


Fig. 7.11 Initial uniform design for the plate under torsional loading.

The plate is designed for the set of 10 thicknesses ranged from the minimum of 0.1 mm to the maximum of 1 mm with the step size Δt = 0.1 mm. The plate is modelled by 800 four-node quadrilateral plate elements. All elements are initially given the maximum thickness. The initial vertical displacements at the corners (points A & B) are 0.80 mm. The limit of 1.5 mm is imposed on the displacements at the corners.

Using MRR = 1% of the initial material, for the given step size the element changing ratio is ECR = 10% (i.e. 80 elements are subjected to thickness reduction at each iteration). The ratio formulae (3.38) are used for the Lagrange multipliers. The optimization history is given in Fig. 7.12 where the maximum displacement slowly

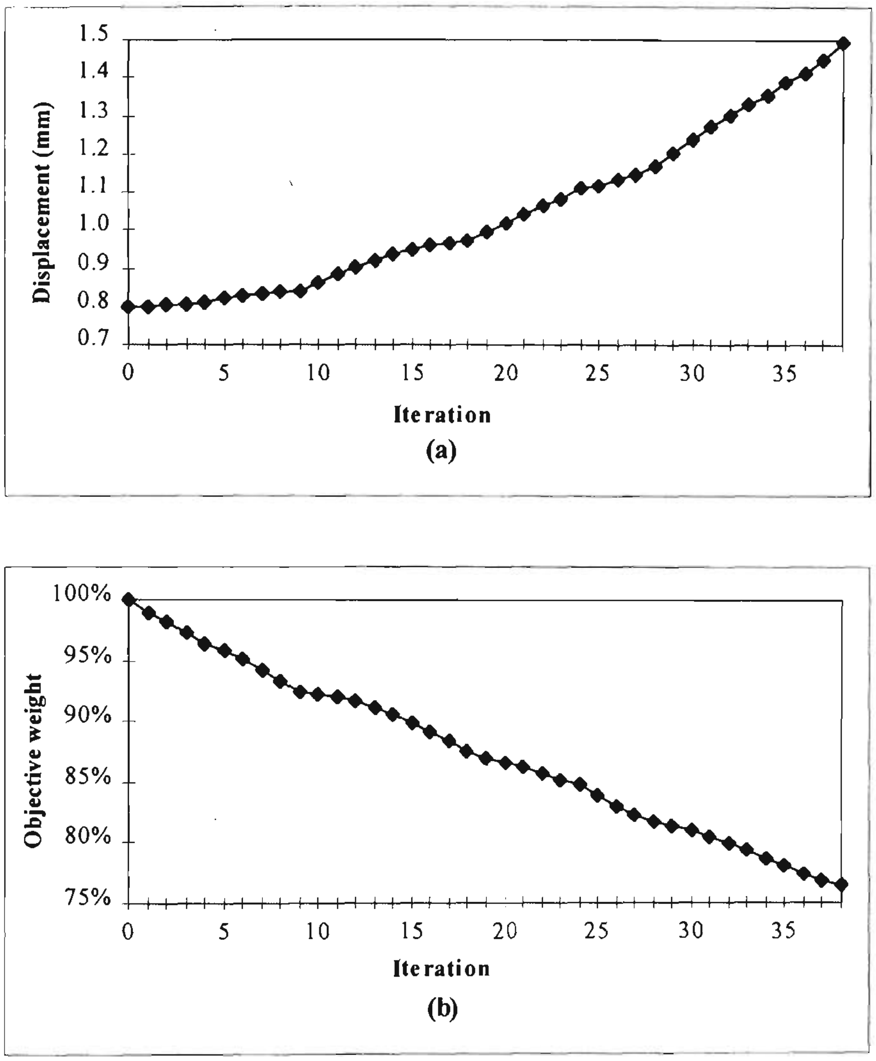


Fig. 7.12 Optimization history for the plate under torsional loading: (a) maximum displacement; (b) objective weight.

increases to the limit. At the same time, the objective weight steady reduces to the minimum value of 76.37% of the initial value when the displacement reaches the limit. The optimal design of the plate for the given set of thicknesses and limit is shown in Fig. 7.13, where it is seen that the elements with maximum and minimum thicknesses are dominant. Based on the trend of the change in the objective in the last few iterations, it is possible to obtain a lower minimum objective weight in the case where the thicknesses can be changed proportional to their values in the initially given set.

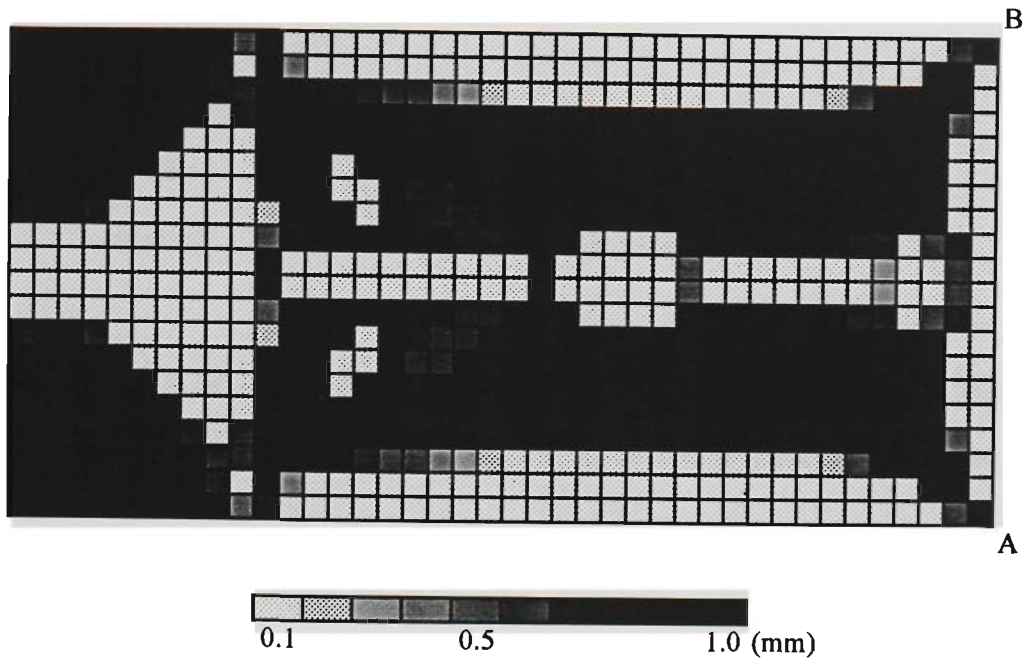


Fig. 7.13 Optimal design for the plate under torsional loading, MRR = 1% and limit of 1.5 mm.

7.3 Minimum displacement design subject to a weight constraint

7.3.1 Problem formulation

Consider the problem of minimizing a displacement component at a point of a structure for a given weight. It is realised that, in contrast to removal of material, when material is added to an element by increasing its thickness, the element becomes stiffer and consequently, it increases the overall stiffness of the structure. As a result, the displacement is generally reduced in the absolute value. So the displacement can be significantly reduced when the material is shifted from the locations where it makes a small effect on the increase in the displacement to the locations where it has a large effect on the reduction of the displacement.

The problem can be stated as follows:

By selecting the sizes for elements,

minimize *the specified displacement of a structure*

subject to *a constraint on the weight.*

Shifting material from one element to the other can be done by reducing the thickness of the first element and increasing the thickness of the second one by the same amount of material. In general, when elements vary differently, it may not possible to shift the same amount of material from one element to another. However, it is possible to shift the same amount material from a number of elements to other elements. In some cases, additional scaling design variables is needed to keep weight unchanged. At any step, every element can be either reduced or increased. Therefore, to solve this problem we need to evaluate the effects on the displacement by removing and adding material due to decrease and increase in the element thickness, respectively. Similar type of sensitivity numbers, as defined for the problem of minimum weight design, can be formulated for the above stated problem. For simplicity, the discussion is given for plate structures. However it is equally applicable to the frame or truss structures.

7.3.2 Sensitivity analysis

Assume that the structure to be optimized is modelled by elements with thicknesses chosen from the given set $t_i = \{t_1, t_2, \ldots, t_{s-1}, t_s, t_{s+1}, \ldots, t_r\}$. Each element or each group of elements can have it own set of thicknesses. For convenience, thicknesses in each set are put in ascending order. Suppose the current thickness of an element is t_s so that the next lower and higher thicknesses are t_{s-1} and t_{s+1}, respectively. Material can be removed from or added to an element by selecting the next lower or higher thickness in the given set. Similarly, the sensitivity numbers for material removal and addition, can be formulated using the Lagrange multipliers approach.

For the problem of minimizing the absolute value of the displacement $|u_j|$ subject to a weight constraint

$$W = \sum_{i=1}^{n} w_i = W^* \tag{7.28}$$

the Lagrangian is defined as

$$L = |u_j| - \lambda(\sum_{i=1}^{n} w_i - W^*) \tag{7.29}$$

where λ is a Lagrange multiplier and W^* is the prescribed weight for the structure. Taking w_i as the design variables, the optimality condition for the problem is

$$\frac{\partial L}{\partial w_i} = \left|\frac{\partial u_j}{\partial w_i}\right| - \lambda = 0 \qquad (i = 1,\ n) \tag{7.30}$$

which can be approximated by

$$\left|\frac{\Delta u_j}{\Delta w_i}\right| - \lambda = 0 \qquad (i = 1,\ n) \tag{7.31}$$

Recalling equation (7.7), which implies that the change in the specified displacement, in the absolute value, is equal to the change in the element virtual energy due to sizing the element, the above optimality condition becomes

$$\left|\frac{\Delta \alpha_{ij}}{\Delta w_i}\right| - \lambda = 0 \qquad (i = 1,\ n) \tag{7.32}$$

or

$$\gamma_i = \left|\frac{\Delta \alpha_{ij}}{\Delta w_i}\right| = \lambda = constant \qquad (i = 1,\ n) \tag{7.33}$$

The optimality criterion (7.33) states that *at the optimum the absolute ratio of the change in the element virtual energy and the change in the element weight is equal for all element.* This optimality criterion is exactly the same as the one obtained for the problem of weight minimization with a single displacement constraint, i.e. equation (7.13). The value γ_i in (7.33) can be viewed as a measure of effectiveness of the material within the portion to be removed from or added to an element. It is obvious that the most effective way of removing material is to remove the material from the element with the lowest value of γ_i because it will have the smallest effect on the displacement. In contrary, the most effective way of adding material is to add the material to the element with the highest value of γ_i because this amount of material will have a largest effect on the displacement. This sizing strategy can result in designs with more uniform values of γ_i and the specified displacement is significantly reduced. In some special cases, it possible to reach the uniform state when the optimality criterion (7.33) is satisfied for all element.

Thus, the sensitivity number for element sizing for the displacement minimization problem of is defined as

$$\gamma_i = \frac{|\Delta\alpha_{ij}|}{|\Delta w_i|} = \frac{|\{u^{ij}\}^{\mathrm{T}}[\Delta K^i]\{u^i\}|}{|\Delta w_i|} \qquad (i = 1,\ n) \qquad (7.34)$$

The sensitivity number in (7.34) is identical to the sensitivity number defined by (7.14) for the weight minimization problem subject to displacement constraints.

Based on the sensitivity numbers defined by (7.34), different optimization procedures can be employed to obtain the solution. One way is to use the material removal (or element reduction) technique, as given for weight minimization problem, to solve this problem. Using (7.34), the sensitivity numbers for element reduction are calculated for all elements. Material are removed from elements with the lowest sensitivity numbers by changing their thicknesses to the next lower values. As a result, the weight of the structure decreases. In this case scaling the design variables is needed to satisfy the weight constraint (7.28). The disadvantage of this procedure is that by using the design variable scaling, the thicknesses of elements will not have the exact values as given in the sets of discrete values. It is preferable to use a procedure which will keep the thicknesses within the given sets.

It is possible to keep the weight of the structure unchanged, without involvement of design variable scaling, is to shift material between elements, i.e. the same amount of material, which has been removed from some elements, is added to others. To do this, the best elements to remove material from and the best elements to add material to need to be identified. Because the increase and decrease in thicknesses can have different effects, two sensitivity numbers need to be calculated for each element using equation (7.34).

Due to reduction and increase in the ith element thickness, the changes in the element weight are as

$$(\Delta w_i)^- = w_i(t_{s-1}) - w_i(t_s) \qquad (i = 1,\ n) \tag{7.35}$$

and

$$(\Delta w_i)^+ = w_i(t_{s+1}) - w_i(t_s) \qquad (i = 1,\ n) \tag{7.36}$$

Correspondingly, the changes in stiffness matrix of the element are given as

$$[\Delta K^i]^- = [K^i(t_{s-1})] - [K^i(t_s)] \qquad (i = 1,\ n) \tag{7.37}$$

and

$$[\Delta K^i]^+ = [K^i(t_{s+1})] - [K^i(t_s)] \qquad (i = 1,\ n) \tag{7.38}$$

where the superscript "-" denotes thickness reduction and "+" denotes thickness increase. The change in the element virtual energy can be calculated using (7.7), which gives

$$(\Delta\alpha_{ij})^- = \{u^{ij}\}^T[\Delta K^i]^- \{u^i\} \qquad (i = 1,\ n) \tag{7.39}$$

$$(\Delta\alpha_{ij})^+ = \{u^{ij}\}^T[\Delta K^i]^+ \{u^i\} \qquad (i = 1,\ n) \tag{7.40}$$

Therefore, for each element two following sensitivity numbers are calculated

$$\gamma_i^- = \left|\frac{(\Delta\alpha_{ij})^-}{(\Delta w_i)^-}\right| = \frac{|\{u^{ij}\}^T[\Delta K^i]^- \{u^i\}|}{|(\Delta w_i)^-|} \qquad (i = 1,\ n) \tag{7.41}$$

$$\gamma_i^+ = \left|\frac{(\Delta\alpha_{ij})^+}{(\Delta w_i)^+}\right| = \frac{|\{u^{ij}\}^T[\Delta K^i]^+ \{u^i\}|}{|(\Delta w_i)^+|} \qquad (i = 1,\ n) \tag{7.42}$$

It is obvious that removing material from elements will generally increase the displacement and adding material will reduce the displacement in absolute values. Reducing the thickness of the element with the smallest γ_i^- will result in the minimum increase in the objective displacement. Conversely, increasing the thickness of the element with largest γ_i^+ will make the maximum decrease in the objective displacement. If the maximum value of γ_i^+ is much larger than the minimum value of γ_i^- the specified displacement will be significantly reduced when the material is shifted from the element with minimum γ_i^- to the element with maximum γ_i^+.

7.3.3 Optimization procedure for displacement minimization

To get accurate solution an iterative procedure, where only a small amount of material is shifted in each iteration, should be employed. The evolutionary optimization procedure for the displacement minimization problem is as follows:

Step 1: Model the structure by finite elements with intermediate thicknesses.

Step 2: Analyse the structure for the given real load and the virtual unit load corresponding to the objective displacement.

Step 3: If convergence in the objective displacement has been reached, go to step 7. Otherwise, go to step 4.

Step 4: Calculate the sensitivity numbers γ_i^- and γ_i^+ for each element.

Step 5: If the sensitivity numbers are uniform, go to step 7. Otherwise, go to step 6.

Step 6: Shift a specified amount of material in the structure by selecting the next lower thicknesses for a number of elements which have the lowest γ_i^- and the next larger thicknesses for other elements with highest γ_i^+. Go back go to step 2.

Step 7: Stop.

For the proposed evolutionary procedure two parameters, tolerance and the amount of material to be shifted at each iteration, need to be specified. Tolerance is used to check the convergence of the objective displacement. Convergence is reached when the relative change in the objective displacement between two successive iterations is less than the given tolerance δ, i.e.

$$\left|\frac{u_j^{\text{new}} - u_j^{\text{old}}}{u_j^{\text{new}}}\right| \leq \delta \tag{7.43}$$

The optimization process will also be terminated when the sensitivity numbers are uniform, by which the optimality conditions are satisfied.

The amount of material to be shifted can be prescribed by *the material shifting ratio* (MSR), which is defined as the ratio of the portion of the weight (material) to be shifted at each iteration to the total weight (material) of the structure. This parameter, similar to the material removal ratio (MRR), controls the change of the design in each step. Depending on the step size, it controls the number of elements subjected to thickness decrease and increase. In the most general case where elements can change differently, the number of elements subjected to thickness reduction and the number of elements subjected to thickness increase are different and vary from iteration to iteration. When all elements can change by equal weight portions, as in the case where all elements are identical and equal step sizes are used, the number of elements subjected to decrease equal to the number of elements subjected to increase and is constant. In this case, an equivalent parameter called *the element shifting ratio* (ESR), defined by the ratio of the number of elements to be reduced (or increased) at each iteration to the total number of elements, can be used. The values of 1% or 2% can be adopted for MSR although the influence of the MSR on the final design needs further investigation.

During the optimization process, elements having their thicknesses changed to the maximum or minimum values are not allowed to be further increased or decreased. It is noted that the thicknesses initially chosen for all elements must be other than maximum or minimum thicknesses, otherwise shifting the material within the structure cannot be carried out. The initial thicknesses for all elements are determined from the given weight for the structure.

7.3.4 Computer implementation for displacement minimization

A computer program ESODOPLT has been developed by the candidate to calculate the sensitivity numbers and to shift material from elements to elements. A batch file is set up to link this program to the finite element program STRAND6 and a loop is created to carry out the iterative process of optimization. The thicknesses t_s ($s = 1, r$) are input in either descending or ascending order as the property number increases.

Initially an intermediate thickness or sections are assigned to all the elements so that the volume is equal to the given value. Removing or adding material from or to an element is done by substituting the current material property by the preceding or following material property. During the optimization process, elements having their thicknesses changed to the maximum or minimum values are not allowed to be further increased or decreased.

Input data file for the optimization program include the value for tolerance, the material shifting ratio MSR (or element shifting ratio ERR), the parameter indicating the order of thicknesses (ascending or descending order) input in the member property file, the parameter for maintaining symmetry, lower and upper design property numbers (properties with the minimum and maximum thicknesses or sections). It also includes the maximum number of iterations that is allowed for a particular problem.

7.3.5 Example of minimum displacement design for a plate in bending

A plate with the same dimensions, material properties, loading and boundary condition, as given in section 7.2.6.1 (Fig. 7.1), is designed for minimum displacement at the centre, subject to a constant weight. A mesh of 400 four-node quadrilateral elements for a quarter of the plate is used. The minimum thickness is 0.2 m and the maximum thickness is 0.4 m. Initially, all element are assigned the intermediate thickness of 0.3 m which defines the weight of the plate. The initial displacement at the centre is 4.22 mm.

Using $MSR = 1\%$ the plate is designed for two cases:

1) *Three thicknesses*: $t = \{0.2\text{ m}, 0.3\text{ m}, 0.4\text{ m}\}$, the step size $\Delta t = 0.1$ m, ESR = 3% (i.e. for a quarter of the plate 12 elements having thickness reduced and 12 elements having thickness increased);

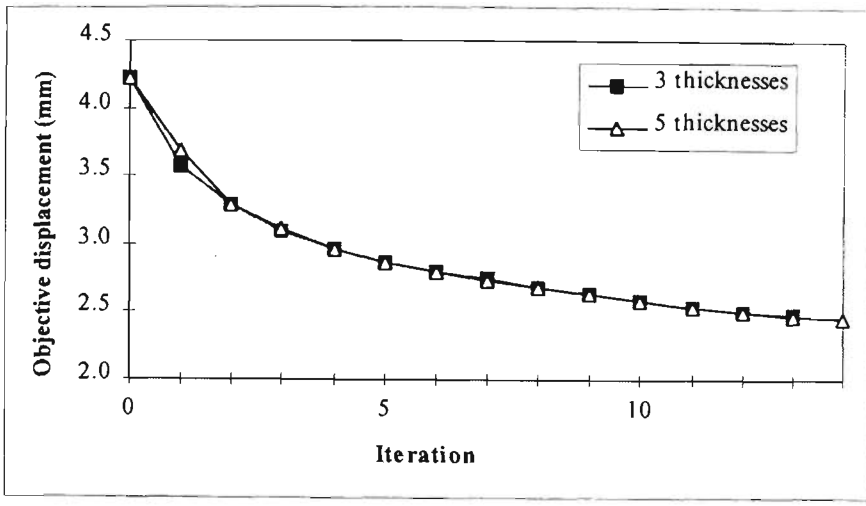


Fig. 7. 14 Optimization history for the plate in bending with a constant weight.

2) *Five thicknesses*: $t = \{0.2 \text{ m}, 0.25 \text{ m}, 0.3 \text{ m}, 0.35 \text{ m}, 0.4 \text{ m}\}$, the step size Δt = 0.05 m, ESR = 6% (i.e. for a quarter of the plate 24 elements having thickness reduced and 24 elements having thicknesses increased).

History of the optimization process for the tolerance $\delta = 0.01$ for two cases is given in Fig. 7.14. It is seen that the changes in the objective displacement (out-of-plane displacement at the centre) in two cases are almost the same. The displacement is sharply reduced in first several iterations. The rate of displacement reduction decreases in later iterations. This trend illustrates the effectiveness of the proposed sensitivity numbers and the efficiency of the proposed ESO method. The structural responses in both cases have been significantly improved compared to the initial uniform design while the weight remains the same. In the case of three thicknesses, after 13 iterations the objective displacement reduces to 2.49 mm, which is 59% of the initial value of 4.22 mm. For the case of five thicknesses, after 14 iterations the displacement reduces to 2.46 mm which is 58.3% of the initial value. The use of smaller step size in the case of five thicknesses provides a lower minimum displacement.

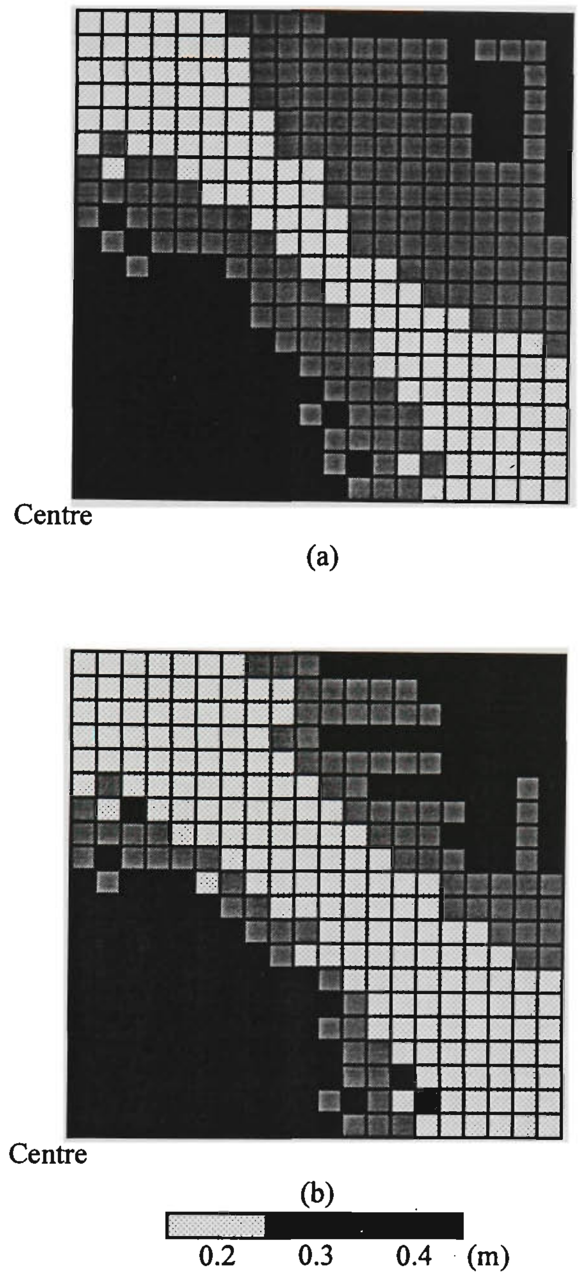


Fig. 7.15 Minimum displacement design for the plate with a constant weight, 3 thicknesses and MSR = 1%: (a) $\delta = 0.02$; (b) $\delta = 0.01$.

The optimal designs obtained for two cases for the tolerance $\delta = 0.02$ and $\delta = 0.01$ are given in Figs. 7.15 and 7.16. It is seen that the thickness distribution for the same tolerance are similar for two cases. The areas with minimum thickness in all cases locate along the hinge lines, which have been reported in Tenek and Hagiwara (1994).

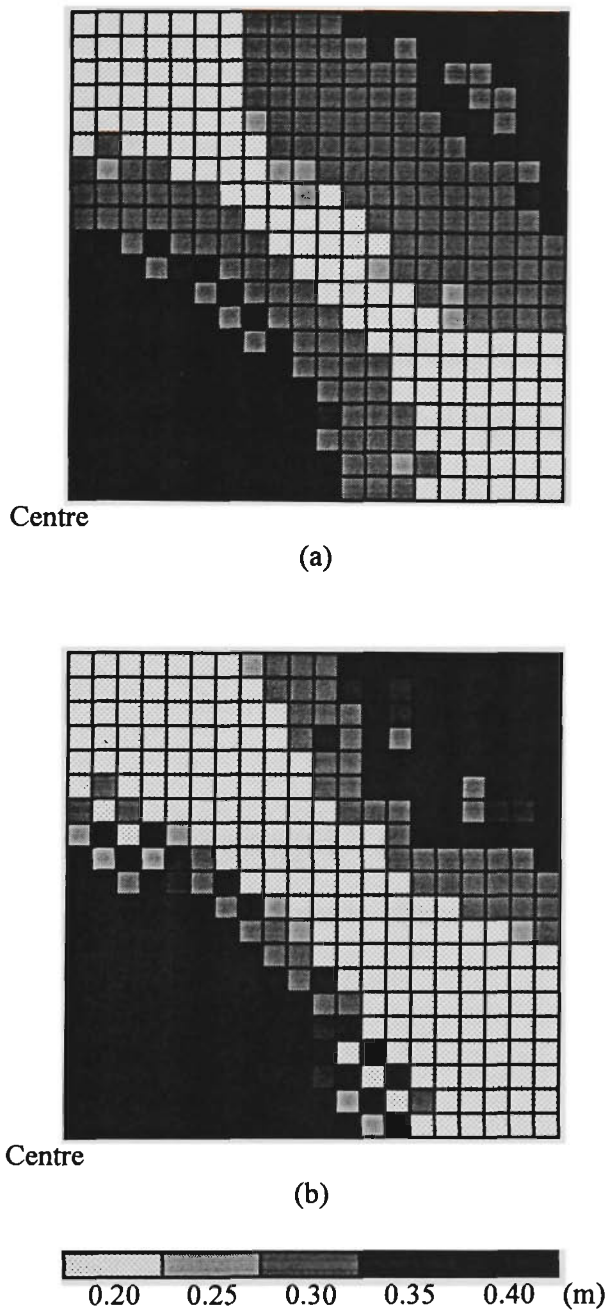


Fig. 7.16 Minimum displacement design for the plate with a constant weight, 5 thicknesses and MSR = 1%: (a) $\delta = 0.02$; (b) $\delta = 0.01$.

It is observed from this example (Fig. 7.14) that the value MSR = 1% gives similar change in the objective displacement despite the large differences in the step size (0.1 m and 0.05 m) and in ESR (3% and 6%). This illustrates that MSR plays a more important role in determining the change in the objective displacement in the proposed ESO method for sizing problems with constant weight than ESR.

7.4 Maximum stiffness design subject to a weight constraint

7.4.1 Problem formulation

The method for minimizing displacement of a structure given in the previous section can be easily applied to the problem of minimizing the strain energy of a structure, which is equivalent to maximizing the stiffness of the structure. The problem can be stated as follows:

By selecting the sizes for elements

minimize	*the strain energy of a structure*
subject to	*a constraint on the weight.*

Similarly, to keep the weight unchanged, i.e. to satisfy the weight constraint, the material is gradually shifted from elements to elements.

7.4.2 Sensitivity analysis

The effect of sizing elements on the strain energy and the sensitivity number for sizing elements are derived similarly as for the problem of minimizing the specified displacement, for which the strain energy S_k defined by (3.52) should be used instead of the displacement u_j and all the element virtual energies α_{ij} should be replaced by the element strain energies s_{ik} defined by (3.54). The similarity is in the sense that the virtual unit load introduced for determining the objective displacement is replaced by the real load to calculate the strain energy.

The strain energy of a structure is defined as

$$S = \tfrac{1}{2}\{P\}^{\mathrm{T}}\{u\} \tag{7.44}$$

where $\{u\}$ is the displacement due to the real load $\{P\}$. The objective here is to minimize the strain energy under only one loading condition, so the index k is not needed. The element strain energy is

$$s_i = \Delta S = \tfrac{1}{2}\{u^i\}^{\mathrm{T}}[K^i]\{u^i\} \qquad (i = 1,\ n) \tag{7.45}$$

where $\{u^i\}$ is the displacement of the ith element, which is obtained from $\{u\}$. Similar to the change in the displacement given by (7.7), the change in the strain energy of the structure is equal, with an opposite sign, to the change in the element strain energy due to the change in the element size, i.e.

$$\Delta S = -\tfrac{1}{2}\{u^i\}^{\mathrm{T}}[\Delta K^i]\{u^i\} = -\Delta s_i \qquad (i = 1,\ n) \tag{7.46}$$

For a weight constraint given by (7.28), the Lagrangian for the problem is defined as

$$L = S + \lambda(\sum_{i=1}^{n} w_i - W^*) \tag{7.47}$$

where λ is a Lagrange multiplier. The optimality conditions are

$$\frac{\partial L}{\partial w_i} = \frac{\partial S}{\partial w_i} + \lambda = 0 \qquad (i = 1,\ n) \tag{7.48}$$

which can be approximated by

$$\frac{\Delta S}{\Delta w_i} + \lambda = 0 \qquad (i = 1,\ n) \tag{7.49}$$

By using equation (7.46), the optimality conditions becomes

$$-\frac{\Delta s_i}{\Delta w_i} + \lambda = 0 \qquad (i = 1,\ n) \tag{7.50}$$

or

$$\gamma_i = \frac{\Delta s_i}{\Delta w_i} = \lambda = constant \qquad (i = 1,\ n) \tag{7.51}$$

From optimality conditions (7.51), the sensitivity number for sizing elements is

$$\gamma_i = \frac{\Delta s_i}{\Delta w_i} = \frac{\frac{1}{2}\{u^i\}^{\mathrm{T}}[\Delta K^i]\{u^i\}}{\Delta w_i} \qquad (i = 1,\ n) \qquad (7.52)$$

For shifting material within the structure to minimize the strain energy, the following two sensitivity numbers for each elements are calculated

$$\gamma_i^- = \frac{(\Delta s_i)^-}{(\Delta w_i)^-} = \frac{\frac{1}{2}\{u^i\}^{\mathrm{T}}[\Delta K^i]^-\{u^i\}}{(\Delta w_i)^-} \qquad (i = 1,\ n) \qquad (7.53)$$

$$\gamma_i^+ = \frac{(\Delta s_i)^+}{(\Delta w_i)^+} = \frac{\frac{1}{2}\{u^i\}^{\mathrm{T}}[\Delta K^i]^+\{u^i\}}{(\Delta w_i)^+} \qquad (i = 1,\ n) \qquad (7.54)$$

where the superscript "-" denotes thickness reduction and "+" denotes thickness increase. The changes in the element weight and the element stiffness matrix are calculated by equations (7.35)-(7.38).

It is obvious that removing material from elements will increase the strain energy and adding material will reduce the strain energy. It is the best to remove material from the elements with lowest γ_i^- because it results in the minimum increase in the strain energy. Conversely, it is the most effective to add material to the elements with the largest γ_i^+ because it makes the maximum reduction in the strain energy. Thus, to minimize the strain energy, the material should be shifted from the elements with lowest γ_i^- to the elements with largest γ_i^+.

7.4.3 Optimization procedure for stiffness maximization

The evolutionary optimization procedure for stiffness maximization is exactly the same as given for the problem of displacement minimization in section 7.3.3, except that here the structure is analysed for the real load only. The amount of material to be shifted is prescribed by the material shifting ratio MSR. Convergence is said to have reached when the relative change in the objective between two successive iterations is less than the given tolerance δ, i.e.

$$\left|\frac{S^{new} - S^{old}}{S^{new}}\right| \leq \delta \tag{7.55}$$

The computer program ESODOPLT is extended to the problem of minimizing the strain energy.

7.4.4 Examples of stiffness maximization

7.4.4.1 *Maximum stiffness design for a cantilever plate*

A cantilever plate with the same dimensions, material properties, loading and boundary conditions given in section 7.2.6.2, is considered. The plate is to be designed for minimum strain energy using different sets of thicknesses with the minimum of 0.1 m and the maximum of 0.2 m.

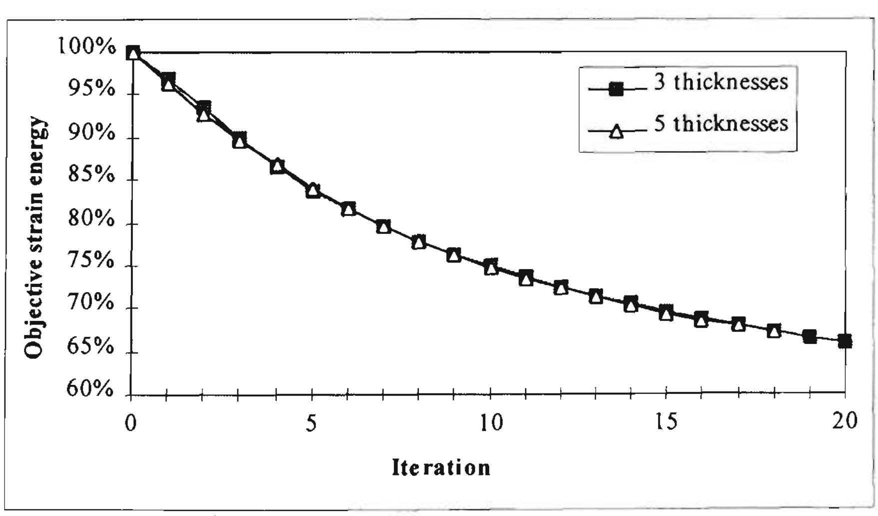


Fig. 7.17 Optimization history of the strain energy for the cantilever plate, MSR = 0.5% and $\delta = 0.01$.

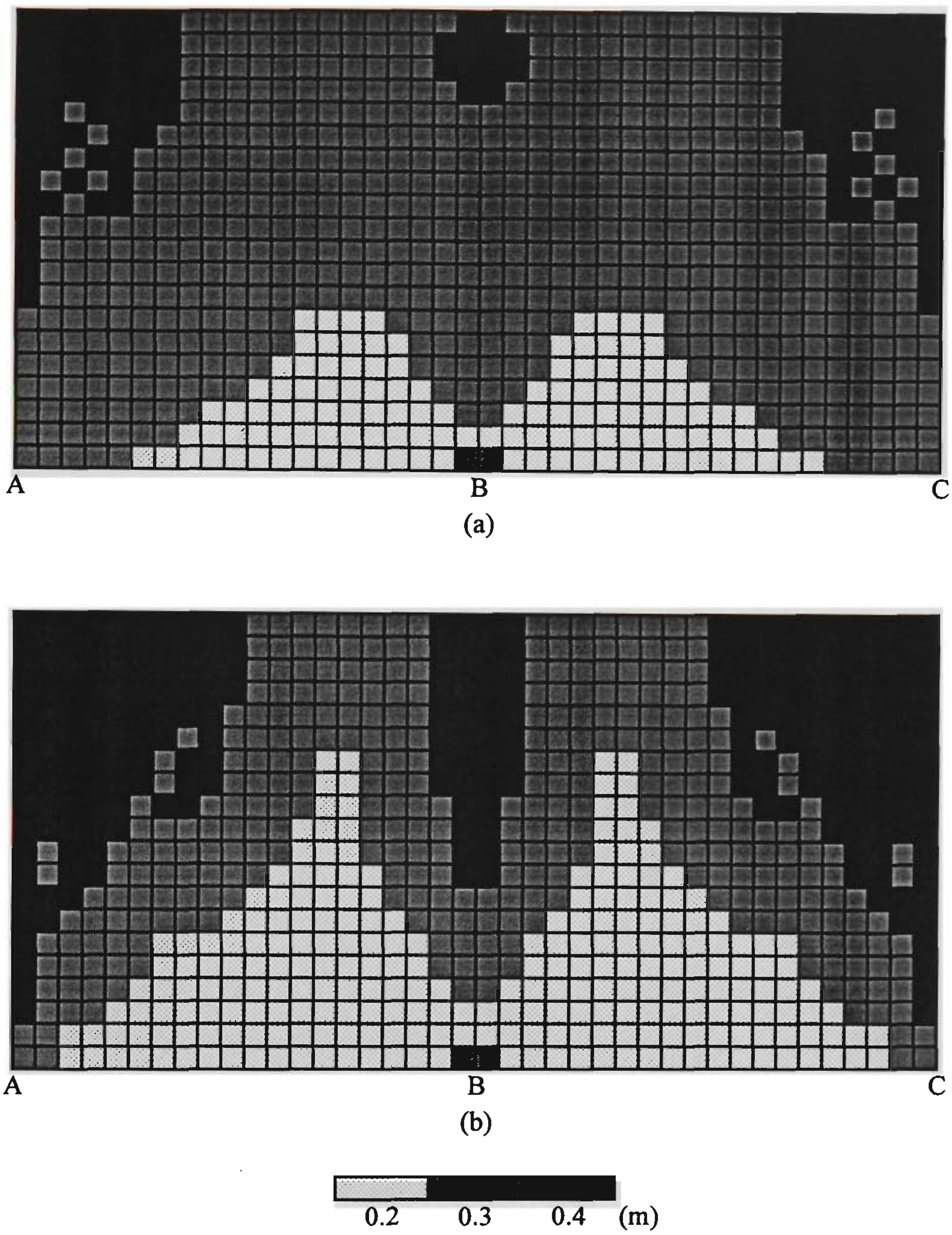


Fig. 7.18 Minimum strain energy design for the cantilever plate, 3 thicknesses, MSR = 0.5%, (a): $\delta = 0.02$, (b): $\delta = 0.01$.

Because of the symmetry, only a half of the plate is analysed using 400 four-node quadrilateral plate elements. Initially, all elements are assigned a thickness of 0.15 m, which defines the weight of the plate. The initial strain energy of the plate for the initially chosen thickness is 74.0 Nm.

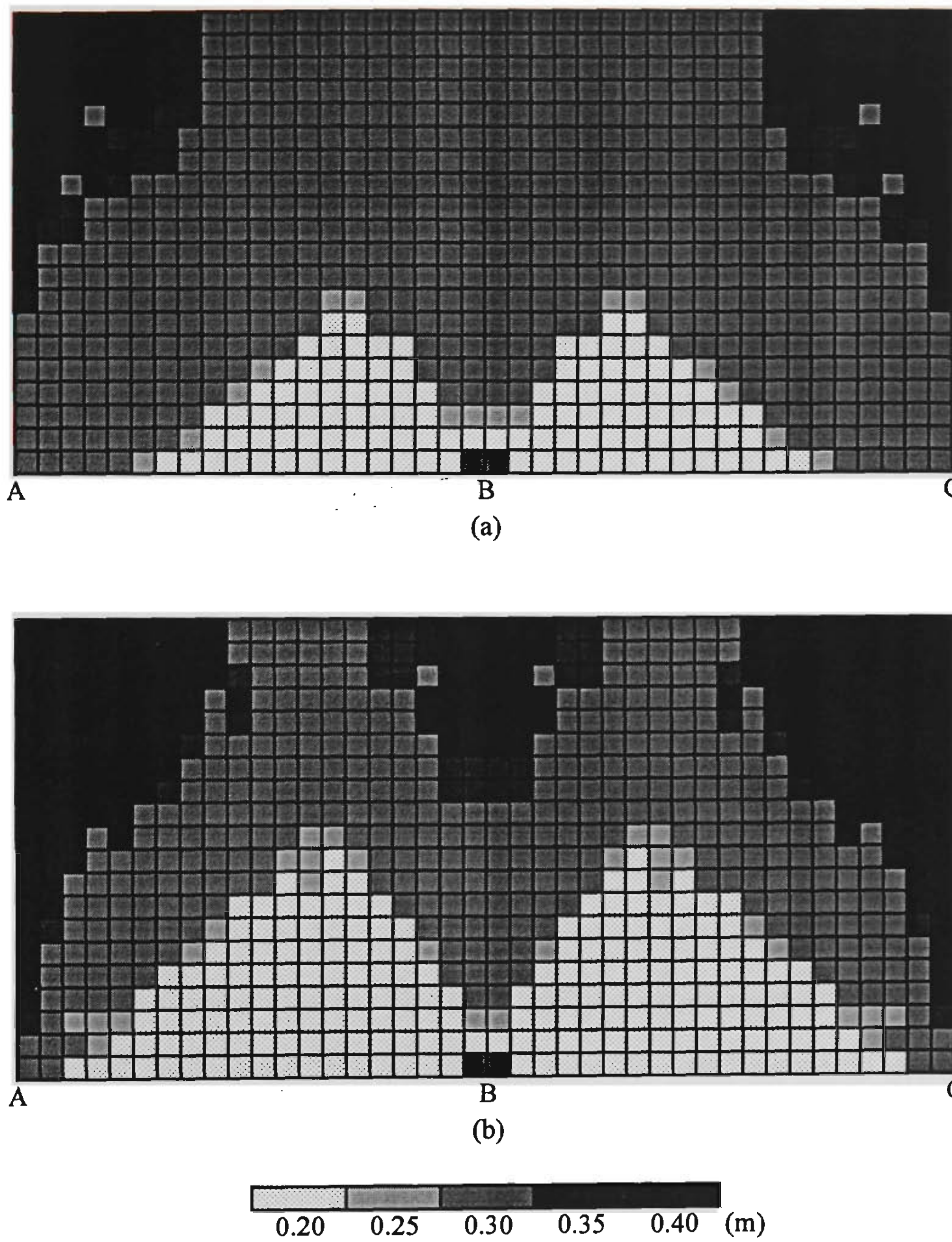


Fig. 7.19 Minimum strain energy design for the cantilever plate, 5 thicknesses and MSR = 0.5%: (a) $\delta = 0.02$, (b) $\delta = 0.01$.

Using MSR = 0.5%, the problem is considered in two cases:

1) *Design for 3 thicknesses*: $\Delta t = 0.05$ m, ESR = 1.5% (i.e. for a half of the plate 6 elements having thicknesses reduced and other 6 elements having thicknesses increased);

2) *Design for 5 thicknesses*: Δt = 0.025m, ESR = 3% (i.e. for a half of the plate 12 elements having thicknesses reduced and other 12 elements having thicknesses increased).

The optimization history for δ = 0.01 for both cases is given in Fig. 7.17. It is seen that the changes in the strain energy in both cases are almost the same. The strain energies in both cases are substantially reduced. The same trend of reduction in the objective, as shown in the previous example, is also observed in this example. Using the same value MSR = 0.5% gives almost the same change in the strain energy in both cases. For three thicknesses, the strain energy is reduced from 74.04 Nm to 55.48 Nm (for δ = 0.02) and then to 48.83 Nm (for δ = 0.01), which are 74.93% and 65.95% of the initial value. For five thicknesses, the strain energy is reduced to 55.38 Nm (for δ = 0.02) and to 49.77 Nm (for δ = 0.01), which are about 74.80% and 67.22% of the initial value, respectively. The optimal designs are shown in Fig. 7.18 and Fig. 7.19. The optimal designs for both cases are similar. For the same tolerance, the thickness distribution for two cases are very similar. The structural response of the optimal designs are much more improved in the sense that the strain energy is significantly reduced while the weight remains unchanged.

7.4.4.2 *Maximum stiffness design for a plate under torsional loading*

A plate under torsional loading given in section 7.2.6.3. is considered again for minimum strain energy design. The same dimensions, material properties, loading and boundary conditions are assumed. The plate is designed for a set of 10 thicknesses ranged from the minimum of 0.1 mm to the maximum of 1 mm with the step size Δt = 0.1 mm. The whole plate is modelled by 800 four-node quadrilateral plate elements. All elements are initially assigned given the thickness of 0.7 mm and the initial strain energy is equal to 2.22×10^{-3} Nm.

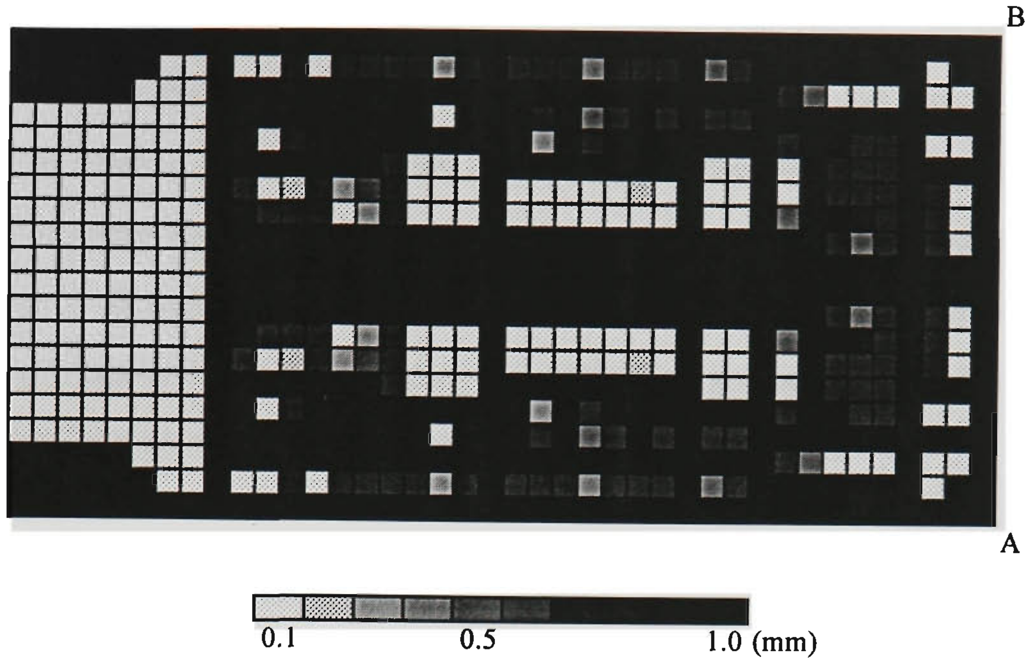


Fig. 7.20 Minimum strain energy design for the plate under torsional loading.

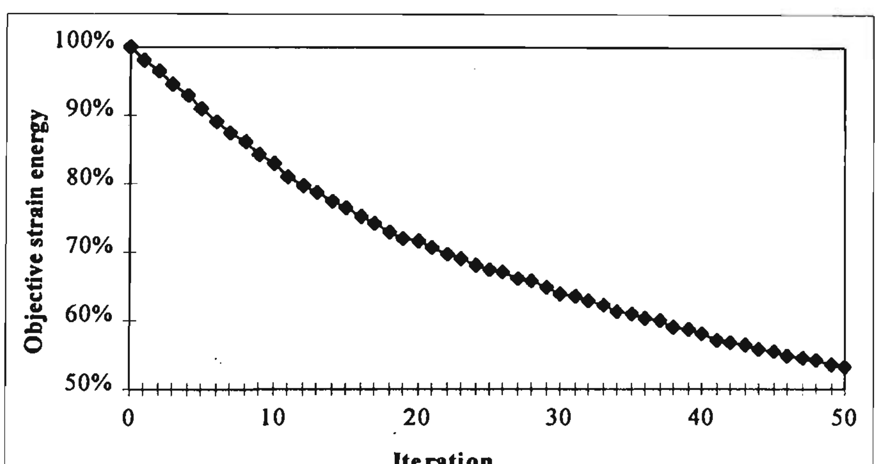


Fig. 7.21 Optimization history for the plate under a torsional loading.

Using MSR = 0.5% (i. e. ESR = 3.5%, 28 elements having thicknesses reduced and 28 elements having thicknesses increased at each iteration), the optimal design shown in Fig. 7.20 is obtained at iteration 50. Fig. 7.21 shows the change in the strain energy during the optimization process. The strain energy continuously reduces to the value of 1.18×10^{-3} Nm, which is only 53.15% of the initial value. The structure becomes stiffer in comparison to the initial uniform design for the same weight. Based on the trend of the change of the strain energy in the last few iterations, it is possible to obtain a lower value for the strain energy by continuing the process. It is seen in Fig. 7.20 that the thicknesses are mainly at the minimum and maximum values. There is a tendency for thickening along the edges and across the plate, which suggests that for such structures edge beams and stiffeners are useful.

7.5 Concluding remarks

It has been shown that the proposed ESO method can easily and directly deal with discrete design variables in sizing problems. The optimal solution is obtained by simply repeating the cycle of finite element analysis, calculation of sensitivity numbers and selecting sizes for elements. The thicknesses of elements are gradually changed to neighbouring values from the given set of discrete values according to their sensitivity numbers. Compared to other existing solution methods, where repeated analyses are needed in order to find a continuous solution and the problem is re-solved to get a discrete solution, the proposed evolutionary optimization method is very simple and efficient. Only one finite element analysis is required for each design step. The weight or structural response (displacement or strain energy) of the optimal designs obtained by the proposed ESO method is significantly improved in comparison to the initial design. Examples given in this chapter, where the number of design variables is in the range from 400 to 800, illustrate the capacity of the proposed method for discrete sizing optimization problems. Although the examples provided in this chapter are only two-dimensional continuum structures, the method can also be applied to the skeletal structures (trusses, frames). For beam elements, one can use a range of sections that are commercially available from manufacturers as a set of discrete sections for sizing.

It is found that in the proposed method the material removal ratio (MRR) and the material shifting ratio (MSR) are more important than the element changing ratio (ECR) and the element shifting ratio (ESR), respectively. They are more closely related to the change between any two adjacent designs. It is obvious that the smaller the value for the material removal ratio or for the material shifting ratio is, the more accurate the solution will be, but at higher computational costs. Although the values of 1% for MRR and 0.5% or 1% for MSR have been used in this chapter, which give desirable results, the influences of these parameters on the optimal designs need further investigation.

Chapter 8

CONCLUSIONS AND RECOMMENDATIONS FOR FURTHER INVESTIGATION

8.1 Conclusions

This study has proposed an ESO method for topology, shape and sizing optimization of structural systems with displacement constraints. With the proposed ESO method, structural optimization can be easily achieved by simply running standard finite element analysis followed by element modification repeatedly. The sensitivity numbers for element modification (removal or sizing) are formulated from the optimality criteria obtained by using the Lagrange multiplier approach for the corresponding optimization problems. The proposed evolutionary procedures can drive the solution, step by step, to the optimum where the optimality criterion is satisfied for all remaining elements and the objective is indirectly minimized. The method can be applied to other constraints on generalized displacement, stiffness, stress and frequency. Having based on a more rigorous mathematical basis, the method given in this thesis can be referred to as *an optimality criteria based approach to the ESO method.* This gives improved solutions than other formerly proposed evolutionary procedures where element responses (element stress, element virtual energy or element strain energy) are used as the sensitivity numbers for removing or sizing elements.

The proposed ESO method is very simple and easy to be implemented into any finite element analysis package with an additional subroutine for calculating sensitivity numbers and element modification. Only one finite element analysis is required for each iteration. The computational cost for calculating the sensitivity numbers is nominal compared to the cost of finite element analysis. Sensitivity numbers have been formulated for various types of optimization problems, including topology, shape and sizing problems with single displacement constraint, multiple displacement constraints and multiple load cases. Problems of minimizing a specified response (displacement, strain energy) of structures subject to weight constraint are also considered. The examples of structures, whose models range from hundreds to several thousand finite elements, prove the capability and efficiency of the proposed method.

For topology and shape optimization, the proposed method is capable of reproducing many structural optimization solutions obtained by other mathematically more complex methods. Some original optimum solutions have been obtained by the proposed ESO method. The performance of the method for topology optimization has been investigated in more detail. It has been shown that the element removal ratio (ERR) is an important parameter in the proposed method. It controls the change between design steps, playing a very similar role as the move limit in mathematical programming methods or step control parameter in traditional optimality criteria methods. It is found that the element removal ratio has little effect on the minimum objective weight, but has a considerable influence on the topology of optimal designs. Appropriate values for the element removal ratio are suggested for each type of structures to obtain desirable results in terms of weight and topology while keeping solution time to acceptable levels. As any other method using the ground structure approach, the solutions by the proposed method are also affected by the initial chosen ground structure. It has been shown that the ground structure (initial chosen grid or element connection) has significant influence on the optimal topology for truss structures. Recommendations for choosing better ground structures for truss topology optimization problems are provided. It is observed that the ground structures (mesh size, element type) has little effect on the minimum weight and the outer shape of the optimal solution for two-dimensional continuum structures, but has substantial effect on the details of topology in the inner part of the solution.

For sizing problems, the proposed ESO methods can directly deal with discrete design variables. In minimum weight design problem for structures subject to displacement constraints, it is found that the material removal ratio (MRR) is an important parameter and values of 1% or 2% are recommended. The element changing ratio should only be used when all elements can change by equal weight portions. In this case it is determined from the material removal ratio and the step size. In the problem of minimizing a specified displacement or strain energy of structure subject to constant weight, the material shifting ratio of 1% or less is recommended.

It should be noted that the sensitivity numbers are formulated for general static problems and the proposed method can be applied to all types of structures. Conclusions on the performance of the proposed method drawn from considering examples of topology and sizing optimization of two-dimensional continuum structures and skeletal structures are valid only for these types of structures. For other types of structures such as three-dimensional bodies, further investigation is required.

8.2 Recommendations for further investigation

1. From the formulation of the sensitivity numbers and the evolutionary optimization procedure for topology and shape problems, the change of the constrained displacement of the structure should be small enough to obtain an accurate

solution. It has been shown that the element removal ratio, in general, closely relates to the level of change in the displacements. However, it is recognised from the optimization history provided for some examples that there still exist some large changes in the displacements which always relate to large changes in topology of the structure at some iterations. These result in increases in the objective weight at some steps. Therefore, an additional investigation on features to facilitate a smooth transition between any two consecutive designs of the structure should be carried out.

2. It has been seen that for problem with multiple constraints, the use of different formulae for the Lagrange multipliers may result in different topological solutions. The ratio formulae have been mostly used. More investigation should be made to compare performances of different formulae for the Lagrange multipliers.
3. Study of the performance of the method for topology and shape optimization of three-dimensional continuum structures should be carried out.
4. Application of the proposed method to topology and sizing optimization of frame structures should be considered.
5. The performance of the method for optimization of structures subject to combinations of different constraints on displacement, stiffness, stress, frequency and stability needs to be investigated.
6. Further investigation on the suggested procedure for removing checker-board patterns should be continued.
7. Study of the performance of the proposed ESO method using higher-order elements.

The above-recommended topics for further investigation are expected to enhance and broaden the scope of the method given in this thesis.

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Appendix I

Appendix I

Analytical solution for the problem of the optimal configuration of two bar truss

The problem of the optimal configuration of two bar truss can be stated as: for the given vertical load P and the distance d from the load to the vertical line of support, find the configuration of the two bars that minimize the weight of the structure subject to a limit U^* on the vertical displacement of the loaded point.

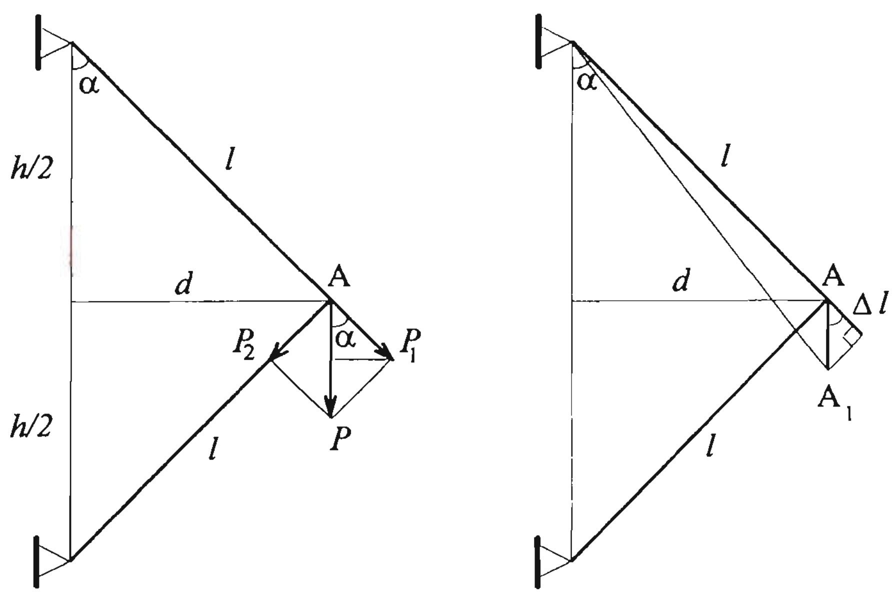

Figure 1 Figure 2

The problem can be solved by the analytical method. The weight of the structure is

$$W = 2\,\rho l A \tag{1}$$

where ρ is the mass density, l is the length of the bars and A is the cross-sectional area of bars. From Figure 1 we have the following relations:

$$l = \frac{d}{\sin\alpha} \quad (2)$$

and

$$P_1 = P_2 = \frac{P}{2\cos\alpha} \quad (3)$$

where α is the angle between the bar and the vertical line of support. The elongation of the upper bar can be calculated as

$$\Delta l = \frac{P_1 l}{EA} = \frac{Pd}{EA\sin 2\alpha} \quad (4)$$

where E is the Young's modulus.

Suppose that under the given load P the vertical displacement of the loaded point A is u, where $u = \overline{AA_1}$. From Figure 2 the relation between the bar elongation and the vertical displacement can be written as

$$u = \frac{\Delta l}{\cos\alpha} = \frac{Pd}{EA\sin 2\alpha\cos\alpha} \quad (5)$$

Therefore, the constraint on the displacement has the form

$$u = \frac{Pd}{EA\sin 2\alpha\cos\alpha} = U^* \quad (6)$$

or

$$A = \frac{Pd}{EU^*\sin 2\alpha\cos\alpha} \quad (7)$$

By substituting (2) and (7) into (1), the weight becomes a function of only one variable α as

$$W = 2\rho \times \frac{d}{\sin\alpha} \times \frac{Pd}{EU^*\sin 2\alpha\cos\alpha} = \frac{4\rho P d^2}{EU^*}\frac{1}{\sin^2 2\alpha} \quad (8)$$

The optimality condition for the weight is

$$\frac{\partial W}{\partial \alpha} = -\frac{16\rho P d^2}{EU^*}\frac{\cos 2\alpha}{\sin^3 2\alpha} = 0 \tag{9}$$

which gives

$$\cos 2\alpha = 0 \tag{10}$$

Therefore, the optimal configuration of two bar truss is achieved when

$$\alpha = 45^\circ \tag{11}$$

from which we can have

$$h = 2d \tag{12}$$

For the specified limit U^*, using (8), the weight of the structure at the optimum is found as

$$W'^{opt} = \frac{4\rho P d^2}{EU^*} \tag{13}$$

Using equations (1), (2), (3) and (5), one can easily solve other optimization problems such as minimizing the displacement or bar stress for a prescribed weight W_0. The same solution $\alpha = 45^\circ$ is also obtained for these problem. In fact, the constraint on the weight will have the form

$$W_0 = 2\rho l A \tag{14}$$

or

$$A = \frac{W_0}{2\rho l} = \frac{W_0 \sin\alpha}{2\rho d} \tag{15}$$

By substituting (15) to (5), the displacement will be

$$u = \frac{4P\rho d^2}{EW_0}\frac{1}{\sin^2 2\alpha} \tag{16}$$

The derivative of the displacement with respect to α is

$$\frac{\partial u}{\partial \alpha} = -\frac{16P\rho d^2}{EW_0}\frac{\cos 2\alpha}{\sin^3 2\alpha} = 0 \tag{17}$$

which gives the same equation as (10).

By using equations (3) and (15), the bar stress is determined as

$$\sigma = \frac{P_1}{A} = \frac{2\rho dP}{W_0}\frac{1}{\sin 2\alpha} \tag{18}$$

Taking the derivative of the stress, we will have

$$\frac{\partial \sigma}{\partial \alpha} = -\frac{4P\rho d}{W_0}\frac{\cos 2\alpha}{\sin^2 2\alpha} = 0 \tag{19}$$

which also leads to equation (10).

Appendix II

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AN EVOLUTIONARY PROCEDURE FOR STRUCTURAL OPTIMIZATION WITH DISPLACEMENT CONSTRAINTS

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ABSTRACT: This paper presents a simple evolutionary procedure to minimize the weight of a structure while satisfying displacement constraints. A sensitivity number is calculated for each element after the finite element analysis. Elements with the lowest sensitivity numbers are eliminated from the structure so that the weight of the structure is reduced while the displacement increments at specified locations are kept minimum. This process of finite element analysis and element elimination is repeated until the displacement constraints reach their given limits. It is shown that many existing solutions in structural optimization can be reproduced by the proposed simple procedure.

1. INTRODUCTION

Shape and layout optimization can greatly improve the performance of the structures. One of important developments in this area is the homogenization method proposed by Bendsøe and Kikuchi [1], in which the structure is modelled by material with microvoids and the optimization objective is to seek the optimal porosity of such a porous medium using optimality criteria. Many interesting results of the homogenization method have been reported in [1-3]. However, the homogenization method is mathematically complicated.

Atrek [4] reported a program called SHAPE for shape optimization of continuum structures by means of element removals, for which a linear maximization sub-problem for Lagrange multipliers needs to be solved.

Recently, Xie and Steven [5,6] proposed an evolutionary procedure for structural shape and layout optimization which was also based on element removals. For static problems, lowly stressed elements are removed from the structure after each finite element analysis. For dynamic problems, at the end of each eigenvalue analysis, a sensitivity number indicating the change of the frequency due to removal of the element is calculated and a number of elements with specific sensitivity numbers are removed so that the frequencies of the resulting structure will be shifted towards a desired direction [7]. A wide range of examples has been presented in [5-7] showing the capacity of such an evolutionary procedure for structural shape and layout optimization.

The purpose of this paper is to extend the evolutionary procedure to structural shape and layout optimization problems with displacement constraints.

2. THE SENSITIVITY NUMBER FOR ELEMENT REMOVAL

The static behaviour of a structure is represented by

$$[K]\{u\} = \{P\} \tag{1}$$

where $[K]$, $\{u\}$ and $\{P\}$ are stiffness matrix, displacement vector and load vector, respectively.

Suppose that the i^{th} element is removed from the structure. The change of the stiffness matrix $[K]$ due to such element removal is $[\Delta K]$ which is equal to $-[K^i]$, where $[K^i]$ is the i^{th} element stiffness matrix. We assume that the removal of the element has no effect on the load vector $\{P\}$. From equation (1) we can find the change in the displacement vector as

$$\{\Delta u\} = -[K]^{-1}[\Delta K]\{u\} \tag{2}$$

In order to extract the change of the j^{th} displacement component denoted by u_j, we introduce a unit load vector $\{F^j\}$, in which only j^{th} component equals unity and all the others equal zero. Multiplying equation (2) by the unit load vector $\{F^j\}$ gives the change in the displacement component u_j as

$$\Delta u_j = \{F^j\}^T\{\Delta u\} = -\{F^j\}^T[K]^{-1}[\Delta K]\{u\} = -\{u^j\}^T[\Delta K]\{u\} = \{u^{ij}\}^T[K^i]\{u^i\} = \alpha_{ij} \tag{3}$$

where

$$\alpha_{ij} = \{u^{ij}\}^T[K^i]\{u^i\} \qquad (i = 1, n) \tag{4}$$

in which $\{u^j\}$ is the solution of equation (1) for the unit load vector$\{F^j\}$; $\{u^i\}$ and $\{u^{ij}\}$ are the element displacement vectors containing the entries of $\{u\}$ and $\{u^j\}$, respectively, which are related to the i^{th} element. The value of α_{ij} can be positive or negative, which implies that u_j may change in two opposite directions.

Our aim is to find the lightest structure while satisfying the displacement constraints, typically given in the form of $|u_j| \leq u_j^*$, where u_j^* is the given limit for $|u_j|$. To achieve this goal through element removal, it is obviously most effective to remove the element whose α_{ij} is close to zero or $|\alpha_{ij}|$ is the lowest. We thus define

$$\alpha_i = |\alpha_{ij}| \tag{5}$$

as the *sensitivity number for element removal*, which indicates the change of the specified displacement due to the removal of an element. It should be noted that the sensitivity number can be easily calculated using the results available from the finite element analyses of the static problem (1) for both the actual load $\{P\}$ and the unit load $\{F^j\}$.

In the case of multiple displacement constraints we calculate the sensitivity numbers $|\alpha_{ij}|$ $(j = 1,m)$ for each constraint, where m is the number of constraints. Ideally, the element whose $|\alpha_{ij}|$ for all the constraints are the lowest should be removed. However such an element generally does not exist.

There is a need to treat each of the $|\alpha_{ij}|$ differently depending on whether the corresponding displacement is close to its limit or not. In order to take into account the relative importance of each constraint, we introduce weighting parameters λ_j and for each element we calculate the following new sensitivity number

$$\alpha_i = \sum_j \lambda_j |\alpha_{ij}| \tag{6}$$

where

$$\lambda_j = |u_j| / u_j^* \tag{7}$$

By using the above weighting parameters λ_j those constraints become less important if the corresponding displacement are far less than the limits.

3. EVOLUTIONARY OPTIMIZATION PROCEDURE

It is obvious that the optimum solution cannot be achieved in one step. An evolutionary procedure has to be adopted, ie. only a small number of elements should be removed from the structure. In this paper the rejection ratio (the ratio of the number of elements being removed at each step to the initial number of elements) of 1% or 2% is used. The evolutionary procedure for optimization with displacement constraints is as follows:

Step 1: Discretize the structure using a fine mesh of finite elements;
Step 2: Do static analysis for both the actual load case and the unit load cases corresponding constrained displacements;
Step 3: Calculate the sensitivity number for each element;
Step 4: Remove a number of elements (1% or 2%) which have the lowest sensitivity numbers;
Step 5: Repeat Step 2 to Step 4 until one of the constrained displacement reaches its limit.

The evolutionary procedure can also be terminated when a prescribed percentage of volume has been eliminated from the structure.

4. EXAMPLES

We shall illustrate the capability of the proposed evolutionary procedure for solution of structural shape and layout optimization with displacement constraints. Material can be removed from inside a structure or only from the boundaries. It is also possible to freeze some parts of the structure as the non-design domains.

4.1. Short cantilever

The cantilever beam shown in Fig. 1 is under plane stress conditions. The left side of the beam is fixed and a vertical load of 3kN is applied at the middle of the free end. The dimensions of the beam are L_x=0.16m, L_y=0.10m and the thickness t = 0.001m. The Young's modulus E = 2.07 x 10^{11} N/m^2 and the Poisson's ratio ν = 0.3 are assumed.

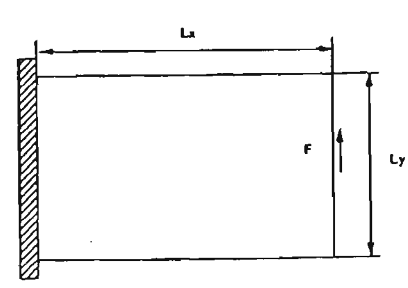


Fig. 1 Design domain for a cantilever

It is required that the vertical displacement at the middle of the free end be less than 0.00075m. The design domain is divided into 32 x 20 quadrilateral elements. Using rejection ratio of 2%, we obtain the optimum design with 40% of the initial volume as shown in Fig. 2(a). This result is obviously very similar to what Suzuki and Kikuchi [3] have obtained using the homogenization method, which is given in Fig. 2(b).

(a) (b)

Fig. 2 Optimal designs for the short cantilever. (a): by evolutionary procedure; (b): by homogenization method [3]

4.2. Two-dimensional structure supporting three loads

A structure is to be designed to support three loads, each at 10kN, under the given boundary conditions shown in Fig. 3. The dimensions for the design domain are $L_x = 0.20$m, $L_y = 0.10$m and thickness $t = 0.001$m. The Young's modulus $E = 2.07 \times 10^{11}$N/m^2, Poisson's ratio $\nu = 0.3$ are assumed.

Using the evolutionary procedure with rejection ratio of 1% and displacement limit of 0.00045m on the three loaded points in the vertical direction, we obtain the optimum design shown in Fig. 4(a), with the weight being reduced by 70%. In a similar example Yang and Chuang [8] used an empirical relationship between the Young's modulus and the density with the objective of minimizing the compliance of the structure. Their solution is shown in Fig. 4(b), which bears similarity to what we have obtained.

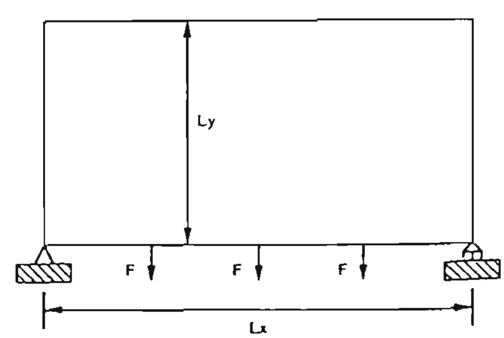
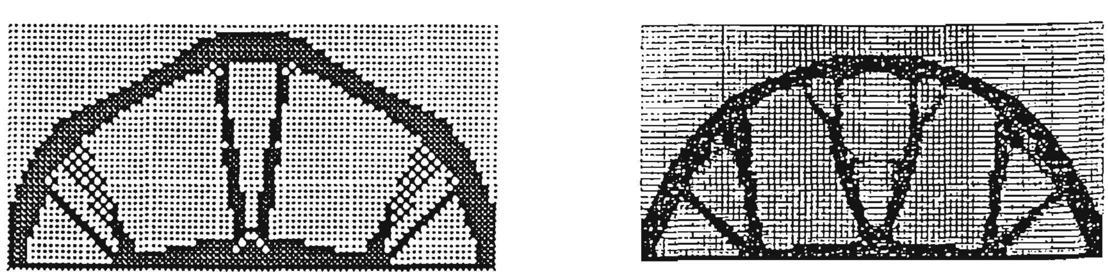


Fig. 3 Design domain for Example 4.2



(a) (b)

Fig. 4 Optimal designs for the two dimensional structure supporting three loads. (a): by evolutionary procedure; (b): by Yang and Chuang [8]

4.3. The Michell truss

In this example we test whether our simple procedure can reproduce the classical Michell truss [9]. A optimum structure is to be designed to transfer a vertical force P to the circular fixed support. The same design domain as used in [3] is adopted in this study, as shown in Fig. 5. The whole design domain is divided into 110 x 80 quadrilateral elements. Due to symmetry, only a half-model is analyzed. The constraint on the vertical displacement of the loaded point is imposed. The rejection ratio of 1% is used. The solution by the evolutionary optimization is given in Fig. 6(a), which is similar to result by Suzuki and Kikuchi [3], as shown in Fig. 6(b).

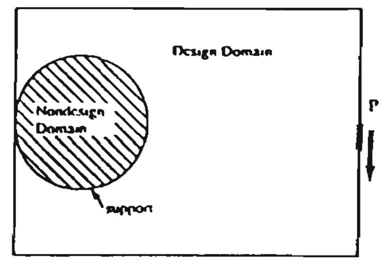
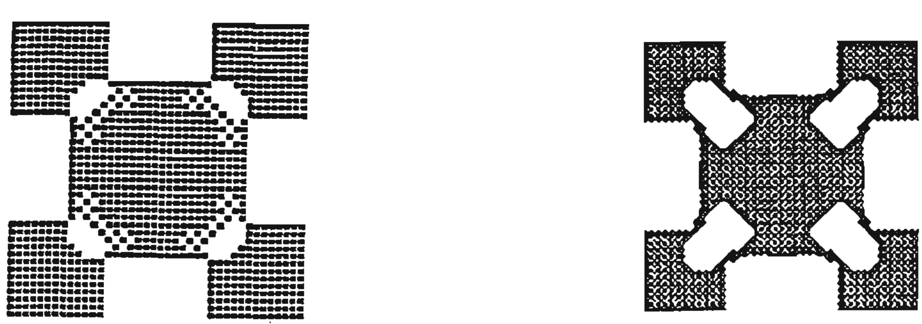


Fig. 5 Design domain for the Michell truss

(a) (b)

Fig. 6 Optimal designs for the Michell truss. (a): by evolutionary procedure; (b): by Suzuki and Kikuchi [3]



(a) (b)

Fig. 7 Optimal designs for the plate in bending. (a): by evolutionary procedure; (b): by Atrek [4]

4.4. Plate in bending

The simply supported square plate (0.20m x 0.20m x 0.0001m) is loaded at the centre by a point load P = 0.04N normal to its plane. The Young's modulus E = 1.7472 x 10^{11} N/m^2 and Poisson's ratio ν = 0.3 are assumed. The limit for the out-of-plane displacement at the centre is set at 0.0015m. One quarter of the plate is modelled with 20 x 20 quadrilateral plate elements. In order to keep symmetry, an even number of elements should be removed at each step. The result using rejection ratio of 2% is given in Fig. 7(a), which is very close to the solution obtained by Atrek [4], as shown in Fig. 7(b).

5. CONCLUDING REMARKS

It has been shown that the proposed evolutionary structural optimization for displacement constraints is capable of reproducing many structural optimization solutions by other mathematically much more complicated methods. With the proposed procedure, structural optimization can be achieved by simply running standard finite element analysis repeatedly, with the additional calculation of the sensitivity numbers. The computational cost for these sensitivity numbers is nominal.

Although the rejection ratio of 1% or 2% gives good results, it is necessary to further investigate its effect on the final design. The effect of the size of the finite elements on the solution also needs further examination.

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Appendix III

ELSEVIER

Finite Elements in Analysis and Design 21 (1996) 239-251

FINITE ELEMENTS IN ANALYSIS AND DESIGN

Evolutionary structural optimization for problems with stiffness constraints

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Abstract

This paper presents a simple evolutionary procedure based on finite element analysis to minimize the weight of structures while satisfying stiffness requirements. At the end of each finite element analysis, a sensitivity number, indicating the change in the stiffness due to removal of each element, is calculated and elements which make the least change in the stiffness of a structure are subsequently removed from the structure. The final design of a structure may have its weight significantly reduced while the displacements at prescribed locations are kept within the given limits. The proposed method is capable of performing simultaneous shape and topology optimization. A wide range of problems including those with multiple displacement constraints, multiple load cases and moving loads are considered. It is shown that existing solutions of structural optimization with stiffness constraints can easily be reproduced by this proposed simple method. In addition some original shape and layout optimization results are presented.

1. Introduction

The stiffness of a structure is one of the major requirements a designer has to take into consideration to design structures such as buildings and bridges. It is often required that a structure be stiff enough so that the maximum deflection in the structure is within a prescribed limit.

Despite the significant effort directed towards structural optimization over the past three decades, most techniques developed so far are restricted to sizing optimization or shape optimization with fixed topology. The search for a general method capable of performing simultaneous shape and topology optimization has been a great challenge. An important recent development in this area was made by Bendsøe and Kikuchi [1] who proposed the homogenization method, where

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the structure was represented by a model with microvoids and the objective was to seek the optimal porosity of the porous medium using an optimality criterion. Some of the results of the homogenization method can be found in [1–4].

Recently, a simple method for shape and layout optimization, called Evolutionary Structural Optimization (ESO), has been proposed by Xie and Steven [5–7], which is based on the concept of gradually removing redundant elements to achieve an optimal design. Although the idea of element removal has been tried by other researchers including Maier [8], Rodriguez-Velazquez and Seireg [9] and Atrek [10], these studies have not resulted in a generalized method. The original idea of ESO is that the optimal shape and layout of a structure can be obtained by systematically removing lowly stressed elements from the structure. Some examples of the ESO method for problems with stress consideration can be found in [5, 6]. This idea has been further extended by Xie and Steven to frequency optimization problems [7], where a sensitivity number for element removal has been introduced and calculated for each element based on information available from the solution of the eigenproblem. By removing elements with special values of sensitivity numbers, the specified frequency of a structure can be shifted toward a desired value. It is seen that the ESO method may be applied to a whole range of structural optimization problems. Compared with other existing methods, the ESO method is much more straightforward. In fact it can be easily implemented into any general purpose finite element analysis (FEA) program. In contrast to most other methods, the ESO involves no mathematical programming techniques in the optimization process. A systematic investigation is under way on the full potential of such a simple but attractive idea of removing unwanted material. So far the results are very promising.

This paper presents a new development of ESO for shape and layout optimization problems with stiffness constraints. A new type of sensitivity number, which indicates the change in the overall stiffness or a specified displacement due to removal of an element, is formulated using results from a finite element analysis. Then a number of elements with the lowest sensitivity numbers will be eliminated from the structure. The optimal design of the structure will be obtained by repeating the cycle of finite element analysis, calculation of sensitivity numbers and element elimination until the overall stiffness or specified displacements reach their given limits. The proposed ESO method for stiffness constraints proves to be simple and capable of performing simultaneous shape and topology optimization. It is shown that many structural optimization solutions obtained by other mathematically much more complicated methods, can be reproduced by the proposed simple method.

2. The sensitivity number for element removal

2.1. The sensitivity number for problems with overall stiffness constraints

In the finite element method, the static behaviour of a structure is represented by

$$[K]\{u\} = \{P\}, \tag{1}$$

where $[K]$ is the global stiffness matrix, $\{u\}$ is the global nodal displacement vector and $\{P\}$ is the nodal load vector.

The strain energy of the structure, which is defined as

$$C = (1/2)\,\{P\}^{\mathrm{T}}\{u\} \tag{2}$$

is commonly used as the inverse measure of the overall stiffness of the structure. It is obvious that maximizing the overall stiffness is equivalent to minimizing the strain energy.

Consider the removal of the ith element from a structure comprising n finite elements. The stiffness matrix will change by $[\Delta K] = [K^*] - [K] = -[K^i]$, where $[K^*]$ is the stiffness matrix of the resulting structure after removal of the ith element and $[K^i]$ is the stiffness matrix of the ith element. It is assumed that the removal of the element has no effect on the load vector $\{P\}$. By ignoring a higher order term, we can find the change of the displacement vector from Eq. (1) as

$$\{\Delta u\} = -[K]^{-1}[\Delta K]\{u\}. \tag{3}$$

From Eqs. (2) and (3) we get

$$\Delta C = \tfrac{1}{2}\{P\}^{\mathrm{T}}\{\Delta u\} = -(\tfrac{1}{2})\{P\}^{\mathrm{T}}[K]^{-1}[\Delta K]\{u\} = \tfrac{1}{2}\{u^i\}^{\mathrm{T}}[K^i]\{u^i\}, \tag{4}$$

where $\{u^i\}$ is the displacement vector of the ith element. We thus define

$$\alpha_i = (\tfrac{1}{2})\{u^i\}^{\mathrm{T}}[K^i]\{u^i\} \quad (i = 1, n) \tag{5}$$

as *the sensitivity number for problems with an overall stiffness constraint*, which indicates the change in the strain energy due to the removal of the ith element. It should be noted that α_i is the element strain energy. Both C and α_i are always positive values.

The objective is to find the lightest structure while satisfying the stiffness constraint, typically given in the form $C \leqslant C^*$, where C^* is the prescribed limit for C. In general, when an element is removed, the stiffness of a structure reduces and correspondingly the strain energy increases. To achieve this objective through element removal, it is obviously most effective to remove the element which has the lowest value of α_i so that the increase in C is minimum.

When there are more than one stiffness constraint, $C_k \leqslant C_k^*$ $(k = 1, l)$, as in the case of multiple loading conditions, the sensitivity number of the ith element corresponding to the kth load case, is determined by

$$\alpha_{ik} = \tfrac{1}{2}\,\{u^{ik}\}^{\mathrm{T}}[K^i]\{u^{ik}\}, \tag{6}$$

where $\{u^{ik}\}$ is the element displacement vector for the kth load case. Ideally, the element whose all α_{ik} are lowest should be removed. However, this particular element generally does not exist. To overcome this difficulty, the simplest way is to evaluate the element by the sum of its values $\alpha_{ik}(k = 1, l)$ or alternatively by the average value of α_{ik}. Furthermore, there is a need of to treat each α_{ik} differently depending on whether the corresponding strain energy is close to its limit or not. In order to take this into account we introduce weighting parameters λ_k and for each element we calculate the following new sensitivity number

$$\alpha_i = \sum_k \lambda_k \alpha_{ik}, \tag{7}$$

where

$$\lambda_k = C_k / C_k^*. \tag{8}$$

By using these weighting parameters, those constraints become less important when the corresponding strain energies are far less than the given limits.

2.2. The sensitivity number for problems with displacement constraints

Assume that a constraint is imposed on the jth displacement component denoted by u_j, given in the form $|u_j| \leqslant u_j^*$, where u_j^* is the prescribed limit for $|u_j|$. In order to find the change of u_j due to an element removal, we introduce a unit load vector $\{F^j\}$, in which only the corresponding jth component is equal to unity and all the others are equal to zero. Multiplying Eq. (3) by $\{F^j\}^{\mathrm{T}}$ we obtain

$$\Delta u_j = \{F^j\}^{\mathrm{T}}\{\Delta u\} = -\{F^j\}^{\mathrm{T}}[K]^{-1}[\Delta K]\{u\} = -\{u^j\}^{\mathrm{T}}[\Delta K]\{u\} = \{u^{ij}\}^{\mathrm{T}}[K^i]\{u^i\}, \tag{9}$$

where $\{u^j\}$ is the solution of equation (1) for the unit load vector $\{F^j\}$, $\{u^i\}$ and $\{u^{ij}\}$ are the element displacement vectors containing the entries of $\{u\}$ and $\{u^j\}$, respectively, which are related to the ith element. The value

$$\alpha_{ij} = \{u^{ij}\}^{\mathrm{T}}[K^i]\{u^i\} \quad (i = 1, n) \tag{10}$$

indicates the change of the specified displacement component u_j due to the removal of the ith element. It should be noted that, unlike α_i in Eq. (5), α_{ij} can be positive or negative, which implies that u_j may change in opposite directions. In this case, it is best to remove the element whose α_{ij} is close to zero or $|\alpha_{ij}|$ is the lowest. We thus define

$$\alpha_i = |\alpha_{ij}| \tag{11}$$

as *the sensitivity number for problems with a displacement constraint*. This sensitivity number can be easily calculated using the results available from the finite element analyses of the static problem (1) for both the actual load $\{P\}$ and the unit load $\{F^j\}$.

The simplest case is when there is only one point load Q acting on a structure and a constraint is imposed on the displacement at the same location and in the direction of the load. For this special case $\{u^j\} = (1/Q)\{u\}$, Eq. (1) needs not to be solved for the unit load. The sensitivity number for this problem is almost the same as for the problems with stiffness constraint as discussed in the previous section.

In the case of multiple displacement constraints, we calculate the values of α_{ij} using Eq. (10) for all displacement constraints, assuming m constraints. Similar to the problem with multiple stiffness constraints, the new sensitivity number is defined as

$$\alpha_j = \sum_j \lambda_j |\alpha_{ij}| \quad (i = 1, n; j = 1, m), \tag{12}$$

where

$$\lambda_j = |u_j| / u_j^*. \tag{13}$$

A more general case is when a structure is designed for multiple load cases $\{P^k\}$ and is subject to multiple displacement constraints $|u_j^k| \leqslant u_j^{k*}$ $(j = 1, m; k = 1, l)$. For each element with respect to the jth displacement constraint and the kth load case we calculate values

$$\alpha_{ijk} = \{u^{ij}\}^{\mathrm{T}}[K^i]\{u^{ik}\} \quad (i = 1, n; j = 1, m; k = 1, l), \tag{14}$$

where $\{u^{ik}\}$ is the ith element displacement vector due to the load case $\{P^k\}$. The new sensitivity number is determined as

$$\alpha_i = \sum_k \sum_j \lambda_{jk} |\alpha_{ijk}|, \tag{15}$$

where

$$\lambda_{jk} = |u_j^k| / u_j^{k*}. \tag{16}$$

3. Evolutionary optimization procedure

It is obvious that the optimum solution cannot be achieved in one step. An evolutionary procedure has to be adopted, i.e. only a small number of elements should be removed from the structure at each iteration. The number of elements to be removed at each iteration can be prescribed by its ratio to the total number of elements of the initial or the current FEA model. This ratio is called *the removal ratio*. For the purposes of this paper a removal ratio of 1 or 2% has been adopted. The influence of the removal ratio on the final solution will be discussed in detail in a separate paper. However, it is obvious that the accuracy in the solution will improve with decreasing removal ratios, at the expense of higher computational costs.

The evolutionary procedure for optimization with overall stiffness or displacement constraints is as follows:

Step 1: Discretize the structure using a fine mesh of finite elements;
Step 2: Analyse the structure for the given loads;
Step 3: Calculate the sensitivity number for each element;
Step 4: Remove elements which have the lowest sensitivity numbers;
Step 5: Repeat Step 2 to Step 4 until one of the constraints reaches its limit.

It should be noted that for the problems with displacement constraints, additional solutions for unit loads corresponding to the constrained displacements need to be included in Step 2. In Step 3 only one of the formulae from Eqs. (5), (7), (11), (12) and (15) is used depending on the type and the number of constraints involved. The number of elements to be removed in Step 4 is determined by the removal ratio times the number of elements in the initial or the current FEA model, rounded off to the nearest integer. In the case where the symmetry of a structure needs to be maintained, an even number of elements should be eliminated. In Step 5 the evolutionary procedure can also be terminated when a prescribed percentage of volume has been eliminated from the structure.

The proposed procedure can be easily implemented into any general purpose FEA program. A sub-program has been written for calculating sensitivity numbers and removing elements. A batch file is set up to create a loop for the iterative process of optimization. Elements may be removed from any part of the structure or only from the boundaries. It is also possible to freeze some parts of the structure as non-design domains.

4. Examples

We shall illustrate the capability of the proposed evolutionary procedure for solving structural shape and layout optimization problems with displacement constraints. These examples are analysed using a 486DX2/66MHz personal computer. Typically, one example takes a few hours.

4.1. Short cantilever

The cantilever beam shown in Fig. 1 is under plane stress conditions. The left-hand side of the beam is fixed and a vertical load of 3 kN is applied at the middle of the free end. The dimensions of the beam are $L_x = 0.16$ m, $L_y = 0.10$ m and the thickness $t = 0.001$ m. The Young's modulus $E = 207$ GPa and the Poisson's ratio $\nu = 0.3$ are assumed. The initial vertical displacement at the middle of the free end is 0.33 mm.

The design domain is divided into 32×20 quadrilateral elements. Using the removal ratio of 2% of initial elements, the optimal designs with volumes of 56.87, 39.37 and 30.00% of the initial volume are obtained for the limits of 0.50, 0.75 and 1.0 mm, respectively. It can be seen that the shape and topology of these results, shown in Fig. 2, are very similar to solutions obtained by Suzuki and Kikuchi [3] using the homogenization method.

One way of looking at the efficiency of the optimal design is to compare its volume with the volume of the optimized initial shape, which is obtained by simply reducing the thickness of the initial design until the maximum displacement reaches the same limit. The volume of the optimized initial rectangular shape are 66, 44 and 33% of the initial volume for the same given limits of 0.50, 0.75 and 1.0 mm, respectively. So the volume reductions of the optimal designs are 13.8, 10.5 and 9.0% with respect to the corresponding optimized initial rectangular shapes.

4.2. The Michell truss

In this example we test whether our simple procedure can reproduce the Michell truss [11]. An optimum structure is to be designed to transfer a vertical force P to the circular fixed support.

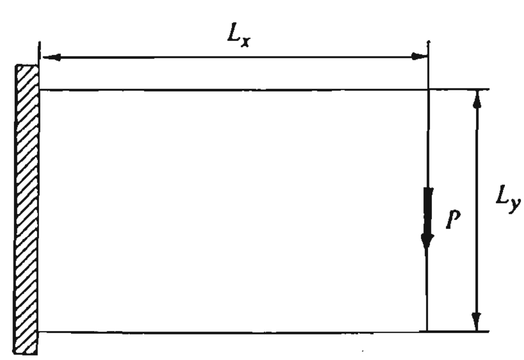

Fig. 1. Design domain for a short cantilever.

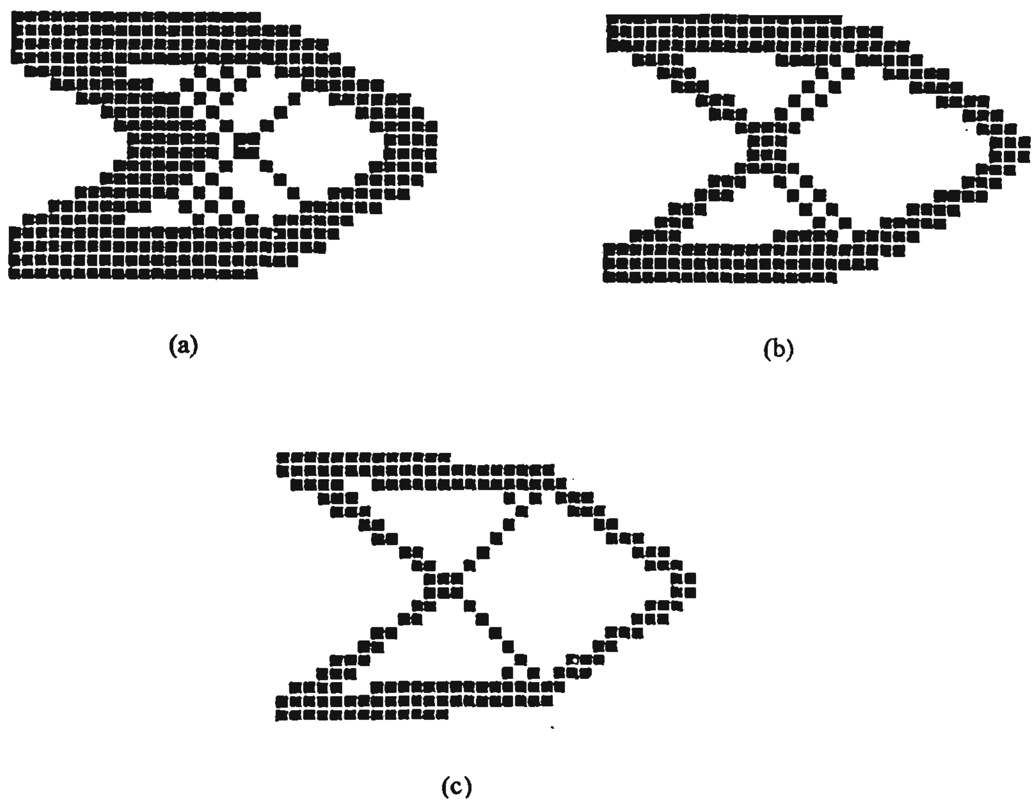

Fig. 2. Optimal designs for the short cantilever for limits: (a) 0.50 mm; (b) 0.75 mm; (c) 1.00 mm.

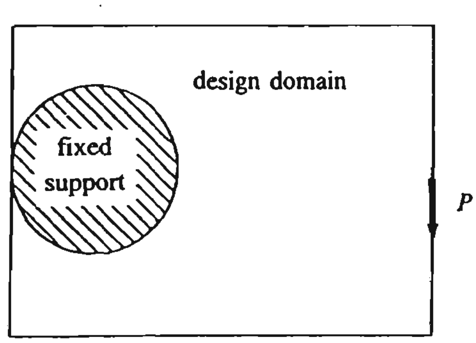

Fig. 3. Design domain for the Michell truss.

The rectangular design domain (0.55 m × 0.4 m) similar to what is used in [3] is adopted in this study as shown in Fig. 3. The radius of circular fixed support is 0.1 m. The Young's modulus $E = 205$ GPa and the Poisson's ratio $v = 0.3$ are used. For the thickness $t = 0.001$ m and the force $P = 50$ kN, the initial displacement of the loaded point in the vertical direction is 2.87 mm.

The whole design domain is divided into 110 × 80 quadrilateral elements. Approximation is used to represent the circular fixed support. Due to symmetry, only a half-model is analysed. The constraint on the vertical displacement of the loaded point is imposed. A removal ratio of 1% of the initial elements is used. The solutions, using the evolutionary optimization method for different

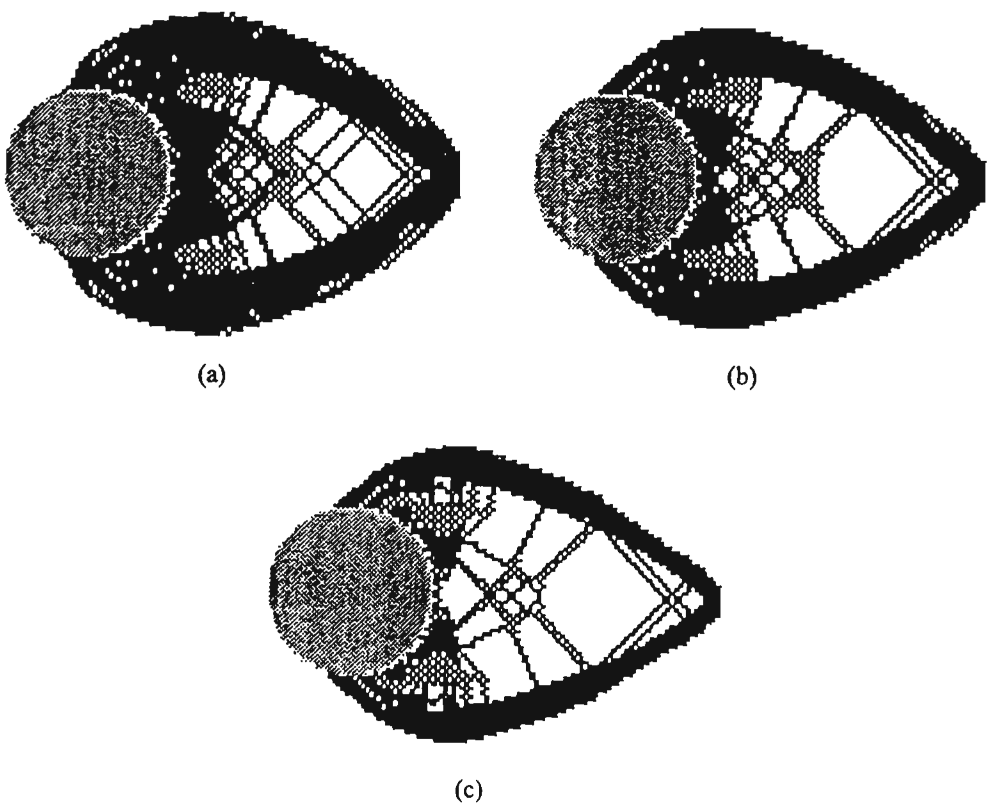

Fig. 4. Optimal designs for the Michell truss for limits: (a) 5 mm; (b) 7 mm; (c) 9 mm.

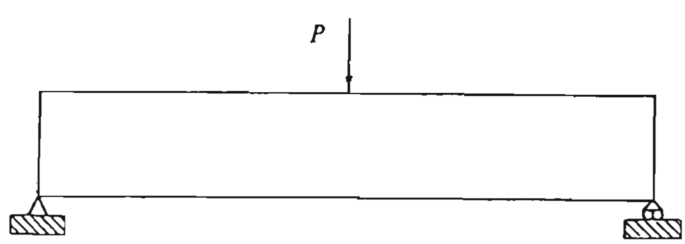

Fig. 5. Design domain for the MBB beam.

displacement limits of 5, 7 and 9 mm, are given in Fig. 4. which have the volumes of 47.45, 36.26 and 29.26% of the initial volume. These optimal shapes are similar to the results from the homogenization method [3]. Compared to the corresponding optimized initial shapes, whose volumes are 57.5, 41 and 31.9% of the initial volume, the optimal designs have the volume reductions of 17.5, 11.4 and 8.3%, respectively.

4.3. The MBB beam

The initial geometry with loads and boundary condition of the MBB beam is shown in Fig. 5. This problem has previously been solved by several researchers, e.g. Olhoff et al: [12], Rozvany and

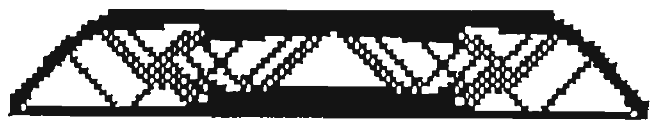

Fig. 6. Optimal design for the MBB beam.

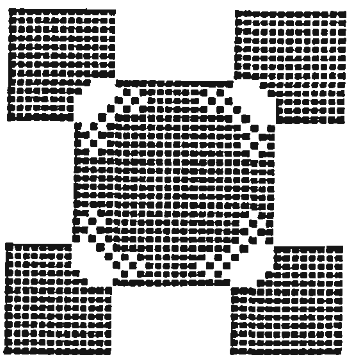

Fig. 7. Optimal design for a plate in bending.

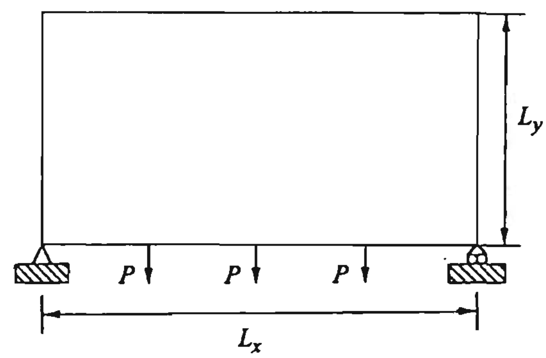

Fig. 8. Design domain for a structure under multiple displacement constraints.

Zhou [13] and Zhou and Rozvany [14]. We shall see that a similar result for this beam can be obtained using the evolutionary procedure.

The beam is 2400 mm long and 400 mm deep with a point load of 20 kN acting at the middle. The Young's modulus $E = 200$ GPa and the Poisson's ratio $\nu = 0.3$ are assumed. The initial volume is 1.07×10^6 mm^3 and the initial displacement at the loaded point is 6.3 mm.

Due to symmetry, only half of the structure is modelled with 60×20 quadrilateral elements. Fig. 6 shows our result of the MBB beam for a displacement limit of 9.4 mm. In this example a removal ratio of 1% of the current number of elements is used. The volume of the optimal design is 50.33% of the initial volume. The volume of the optimized initial shape is 67% of the initial volume. So the optimal design has a volume reduction of 24.88% with respect to the optimized initial shape.

4.4. Plate in bending

The simply supported square plate (0.20 m $\times$ 0.20 m $\times$ 0.0001 m) is loaded at the centre by a point load $P = 0.04$ N normal to its plane. The Young's modulus $E = 174.7$ GPa and Poisson's ratio $\nu = 0.3$ are assumed. The initial out-of-plane displacement at the centre is 1.16 mm. The limit for the out-of-plane displacement at the centre is set at 1.6 mm. One quarter of the plate is modelled with 20×20 quadrilateral plate elements. In order to keep symmetry, an even number of elements should be removed at each step. The result using a removal ratio of 2% of the initial elements is given in Fig. 7, which is close to the solution obtained by Atrek [10]. Similar to the results reported in [15, 16] hinge lines are formed between the central part and the four corners.

4.5. Structure under multiple displacement constraints

A structure is to be designed to support three point loads, each at 10 kN, under the given boundary conditions shown in Fig. 8. The dimensions for the design domain are $L_x = 0.20$ m, $L_y = 0.10$ m and thickness $t = 0.005$ m. The Young's modulus $E = 207$ GPa, Poisson's ratio $\nu = 0.3$ are assumed. The maximum initial vertical displacement is 0.22 mm.

Due to symmetry only half of the structure is analysed using a mesh of 40×40 quadrilateral elements. Using the evolutionary procedure with a removal ratio of 1% of the initial elements, we obtain optimal designs shown in Fig. 9 for the different limits of 0.30, 0.35 and 0.45 mm on all loaded points in the vertical direction. The black areas represent the remaining elements. The light grey areas represent the nodes of the initial FEA model. The optimal designs have volumes of

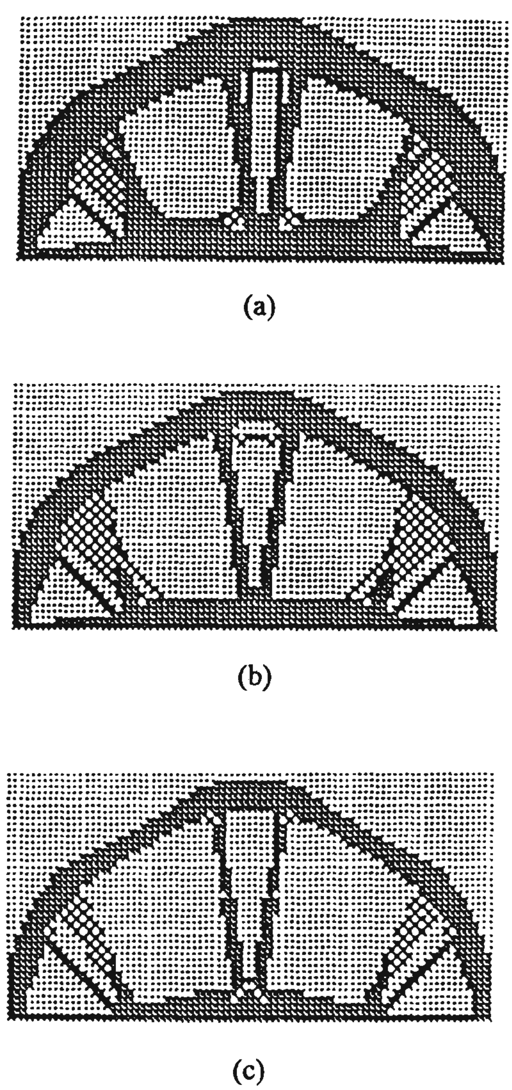

Fig. 9. Optimal designs for the structure under multiple displacement constraints for limits: (a) 0.30 mm; (b) 0.35 mm; (c) 0.45 mm.

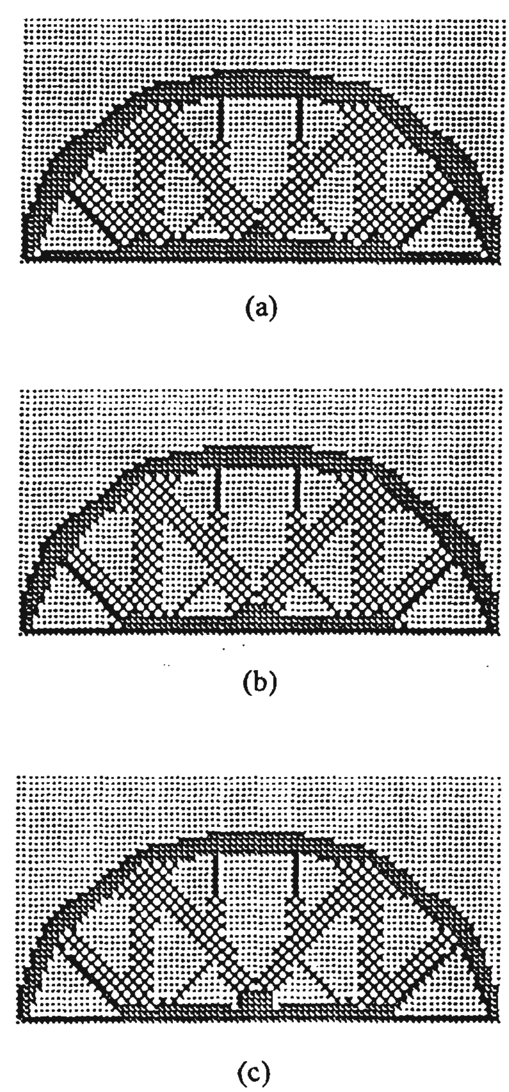

Fig. 10. Optimal designs for the structure under multiple load cases for limits: (a) 1.1 mm; (b) 1.3 mm; (c) 1.5 mm.

50, 39 and 30% of the initial volume. With respect to the volumes of the corresponding optimized initial rectangular shapes, which have the volumes of 73, 63 and 50% of the initial volume, the optimal designs have volume reductions of 32, 38 and 40% respectively. It is worth pointing out that by using the proposed weighting parameters the displacements in each optimum design at the three loaded points are all very close to the limit despite the fact that they are considerably different in the initial design. This explains why the volume reduction is much higher than that for previous examples.

Similar solutions to this example have been obtained by Díaz and Bendsøe [4] using the homogenization method and by Yang and Chuang [17] using a method based on an empirical relationship between the Young's modulus and the density.

4.6. Structure subjected to multiple load cases

Consider the same structure as shown in Fig. 8, but here each of the three given loads acts at different time, i.e. the structure is subjected to three load cases, each load case consists of a single point load. For the thickness $t = 0.001$ m, the maximum initial vertical displacement for the three load cases is 0.49 mm.

Due to asymmetry of the loads the whole structure has to be analysed using a mesh of 80×40 quadrilateral elements. The optimal designs by evolutionary procedure with the removal ratio of 1% of the initial elements for different displacement limits of 1.1, 1.3 and 1.5 mm are given in Fig. 10. An even number of elements have been removed. The meaning of black and light grey areas is the same as given in the example of Section 4.5. These results are similar to the solutions obtained by Díaz and Bendsøe [4] using the homogenization method. The volumes of our optimal designs are 34, 30, and 27% of the initial volume. The corresponding optimized initial rectangular shapes have volumes of 44.5, 37.7 and 32.66% of the initial volume. The optimal designs have the volume reductions of 25.4, 19.9 and 19.8% with respect to the volumes of corresponding optimized initial rectangular shapes.

4.7. A bridge with a moving load

The initial design for the bridge is shown in Fig. 11. The body of the bridge, with dimensions of $16\ \text{m} \times 5\ \text{m} \times 0.1$ m, is supported by four solid piers underneath. A point load $P = 1000$ kN, travelling from the left to the right of the bridge on the top surface, is approximated by nine load cases with an equal distance of 1.75 m between each other. The Young's modulus $E = 30$ GPa and the Poisson's ratio $\nu = 0.2$ are assumed. The maximum initial vertical displacement for all load cases is 1.39 mm.

The whole structure is modelled by a mesh of 64×40 quadrilateral elements and each solid pier is approximately represented by four fixed nodes. A minimum thickness of 0.5 m is required at the top of the bridge, which is specified as a non-design domain. Using a removal ratio of 1% of the initial number of elements, we obtain the optimal designs for different limits of 1.5, 1.7 and 2.0 mm on vertical displacements, as shown in Fig. 12, when an even number of elements have been removed only from bottom boundary. These optimal designs have the volumes of 77.66, 66.48 and 60.39% of the initial volume. The volumes of the corresponding optimized initial shapes are of 92.66, 81.76 and 69.5%. Thus, the optimal designs have the volume reductions of 16.19, 18.69 and 13.11%.

A similar example was considered by Xie and Steven in [6] and a slightly different optimal design was obtained by removing the lowly stressed elements from the structure.

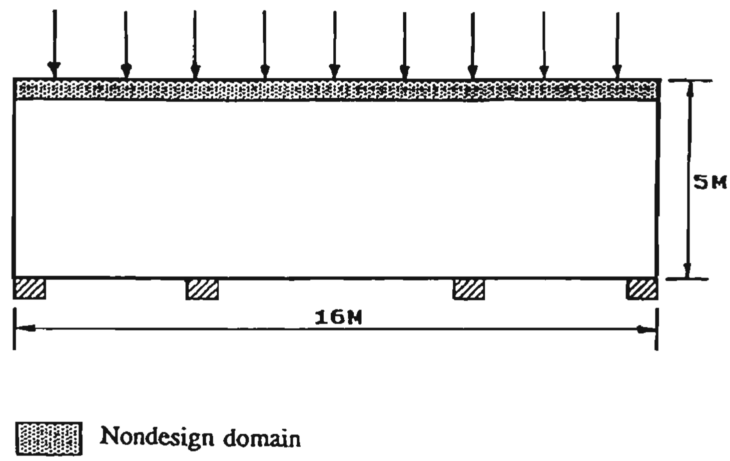

Fig. 11. Design domain for a bridge with a moving load.

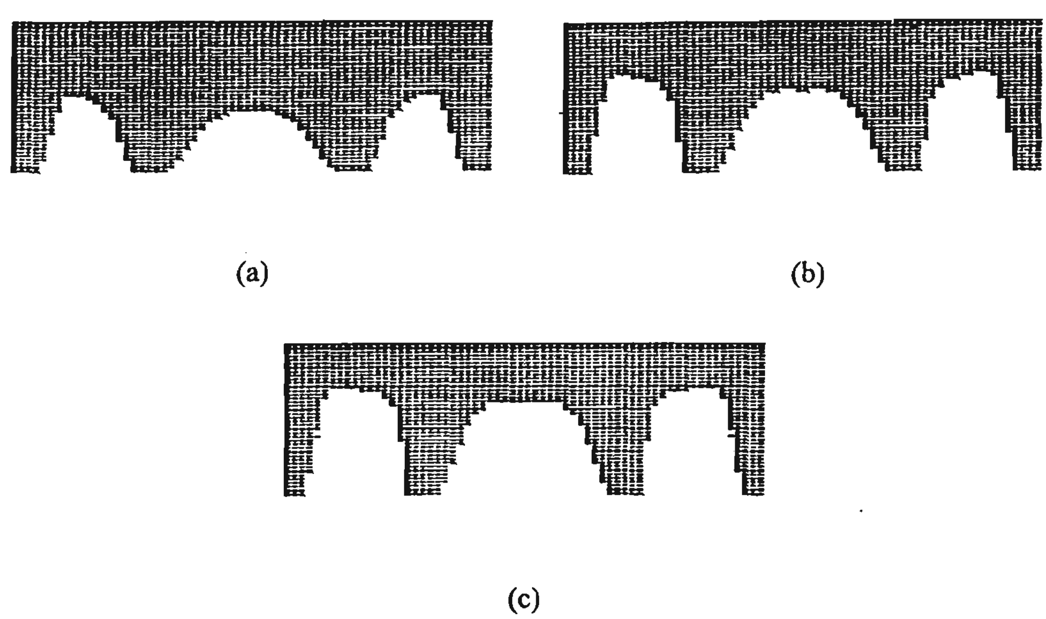

Fig. 12. Optimal design for the bridge with a moving load for limits: (a) 1.5 mm; (b) 1.7 mm; (c) 2.0 mm.

5. Concluding remarks

In this paper, it has been shown that the proposed ESO method for stiffness constraints is capable of reproducing many structural shape and layout optimization solutions previously obtained by other mathematically much more complicated methods. These optimal shapes result in significant volume reductions. Structural optimization using ESO method, can be achieved by simply running standard finite element analysis repeatedly, with additional calculation of the sensitivity numbers. We have presented the sensitivity numbers for various types of optimization problems, including those with multiple load cases and multiple displacement constraints. The computational costs for calculating these sensitivity numbers are nominal. Although the examples presented in this paper are two-dimensional structures, the proposed method can be applied to the shape and topology optimization problems of three-dimensional bodies using brick elements or to the topology optimization problems of frames using beam elements.

The removal ratio is an important parameter, which plays a similar role as the move limit or step size in mathematical programming and optimality criteria methods. The influences of the removal ratio, the mesh size and the type of elements on the final designs are now under investigation and will be reported in the near future.

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Appendix IV

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Finite Elements in Analysis and Design 24 (1997) 197–212

FINITE ELEMENTS IN ANALYSIS AND DESIGN

On various aspects of evolutionary structural optimization for problems with stiffness constraints

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Abstract

An evolutionary structural optimization (ESO) method for problems with stiffness constraints which is capable of performing simultaneous shape and topology optimization has been recently presented. This paper discusses various aspects of this method such as influences of the element removal ratio, the mesh size and the element type on optimal designs.

1. Introduction

During shape optimization the geometry of a structure is continuously changing. This often requires that a finite element model used to represent a structure be changed in the design process in order to maintain the accuracy of analysis. For this purpose an automatic remeshing needs to be implemented into shape optimization programs. Most structural optimization methods are restricted to problems with a fixed topology, i.e. the topology of the final design is equal to the topology of the initial design domain. They are incapable of producing internal holes without prior knowledge of their existence.

It is realized that topology optimization can greatly improve the performance of a structure. A generalized method capable of performing simultaneous shape and topology optimization, is a subject to be investigated. An important development in this area is the homogenization method proposed by Bendsøe and Kikuchi [1], where a structure is represented by a model with microvoids and the objective is to seek the optimal porosity of the porous medium using an optimality criterion. The use of a fixed finite element model for the design domain avoids the necessity of remeshing. Although the homogenization method is mathematically more complex, it offers a tool for simultaneous shape and topology optimization [1–4].

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Recently, a simple method for shape and layout optimization, called Evolutionary Structural Optimization (ESO), has been proposed by Xie and Steven [5–7] which is based on the concept of gradually removing redundant material to achieve an optimal design. Although the idea of material removal has been tried by other researchers including Maier [8], Rodriguez-Velazquez and Seireg [9] and Atrek [10], these studies have not resulted in a generalized method. The ESO method was developed by Xie and Steven first for problems with stress consideration [5, 6] and then it was extended by the same authors to frequency optimization [7]. An ESO method for problems with stiffness constraints has been presented recently by Chu et al. [11, 12]. The ESO method is simple and straightforward. It uses a fixed model with standard finite elements to represent the initial design domain. The optimum design of a structure is found as a subset of the initial set of finite elements. A design cycle consists of analysis, calculation of sensitivity numbers and element removal. The ESO method can be easily implemented into any general purpose finite element analysis (FEA) program. Like the homogenization method, the use of a fixed FEA model for the design domain by the ESO method results in non-smooth boundaries, but it avoids the necessity of remeshing and allows to predict the optimal topology of the structure. Many structural optimization solutions obtained by other mathematically more complex methods, have been reproduced by the ESO methods [5–7, 11, 12].

This paper presents a discussion on various aspects of the ESO method with stiffness constraints, such as influences of the element removal ratio, mesh size and element type on optimal designs.

2. Evolutionary structural optimization method with stiffness constraints

The main idea of ESO is to obtain an optimal shape and topology of a structure by gradual removal of unnecessary elements from the structure. The key point in this method is to work out an appropriate criterion which allows to assess the contribution of each element to the specified behaviour (response) of the structure and elements with the least contribution are subsequently removed. The optimization problem with stiffness constraints can be stated as follows:
By means of element removal,

find	the lightest design of the structure
subject to	constraints on overall stiffness or specified displacements.

Obviously, the elements whose removals make the least changes in the stiffness (the overall stiffness or specified displacement) of the structure, should be removed. The study on the effect of element removal on the stiffness is referred to as the sensitivity analysis and this has been given in detail in Chu et al. [11, 12]. *Sensitivity numbers*, which indicate the change in the overall stiffness (strain energy) or in a specified displacement due to removal of an element, have been formulated using information available from a static finite element analysis. The sensitivity number of the ith element is found to be

$$\alpha_i = (1/2)\{u^i\}^T[K^i]\{u^i\} \quad (i = 1, n) \tag{1}$$

for an overall stiffness constraint and

$$\alpha_i = |\alpha_{ij}| = |\{u^{ij}\}^T[K^i]\{u^i\}| \quad (i = 1, n) \tag{2}$$

for a displacement constraint, where $[K^i]$ is the ith element stiffness matrix, $\{u^i\}$ is the displacement vector of the ith element due to the given real load, $\{u^{ij}\}$ is the displacement vector of the ith element due to the virtual unit load corresponding to the specified jth displacement component, and n is the number elements of FEA model for the structure. In the case of multiple constraints new sensitivity numbers are determined by a weighted sum of the sensitivity numbers defined by Eq. (1) or Eq. (2) corresponding to all constraints [11, 12].

The evolutionary procedure for optimization with overall stiffness or displacement constraints is as follows:

Step 1: Discretize the structure using a fine mesh of finite elements.
Step 2: Analyse the structure for the given loads (and virtual unit loads if needed).
Step 3: Calculate the sensitivity number for each element.
Step 4: Remove a number of elements which have the lowest sensitivity numbers.
Step 5: Repeat Step 2–4 until one of the constraints reaches its given limit.

In the proposed procedure, the relative order of the sensitivity number of each element is more important than its absolute value. The number of elements to be removed at each iteration is prescribed by its ratio to the total number of elements of the initial or the current FEA model. This ratio is called as the *Element Removal Ratio* (*ERR*). Thus, the number of elements to be removed at each iteration is equal to *ERR* times the number of elements of the initial or the current FEA model, rounded off to the nearest integer (or to the nearest even integer in case where the symmetry of the structure needs to be maintained). The values of 1% and 2% for *ERR* have been used by Chu et al. [11, 12], giving results compared to existing solutions obtained by other methods.

It is worth noting that removal of elements may result in the case where one or more remaining elements do not have sufficient connectivities with other elements. For instance, triangular or quadrilateral elements may have connection to other elements by only one of its nodes. These elements which obviously do not contribute to the structure may result in singularity of the stiffness matrix, so they should be removed. An additional feature is included at the end of Step 4, for checking the connectivities of each element and removing any element which has insufficient connectivities. Therefore, the number of removed elements at some iterations may be greater than that determined by the specified value of the element removal ratio.

The element removal ratio is an important parameter in the proposed method and its influence on solutions is a subject to be discussed in detail. Furthermore, because finite element models are used to represent the structure, the influences of mesh size and element type on solutions are also subjects to be examined. We will show how the weight, shape and topology of the final designs will change by varying one of these factors while keeping the others unchanged. The computation time, which greatly depends on these factors, is also compared.

3. Influence of the element removal ratio (ERR)

It is expected that, the smaller the value of the element removal ratio used, the more accurate is the final design, at the expense of larger computation time. The use of larger element removal ratio will reduce the number of elements of the resulting design more rapidly, so the time for each subsequent iteration will sharply decrease. The element removal ratio in the proposed method plays the similar role as the move limit in mathematical programming and the step size in optimality criteria methods. In this study the values of 1%, 2% and 4% for the element removal ratio are used. For each value of the element removal ratio, two *Removal Strategies* (*RS*) can be used:

(1) Removal strategy A ($RS = A$): The element removal ratio is considered with respect to the current number of elements (the number of elements to be removed decreases).

(2) Removal strategy B ($RS = B$): The element removal ratio is considered with respect to the initial number of elements (the number of elements to be removed is constant).

Therefore, the influence of the elements removal ratio is examined considering the following cases:

Case 1A: $ERR = 1\%,\ RS = A$;
Case 1B: $ERR = 1\%,\ RS = B$;
Case 2A: $ERR = 2\%,\ RS = A$;
Case 2B: $ERR = 2\%,\ RS = B$;
Case 3A: $ERR = 4\%,\ RS = A$;
Case 3B: $ERR = 4\%,\ RS = B$.

The study is carried out using several examples including structures subject to multiple displacement constraints under multiple loading conditions. All examples given in this paper are solved using a 486DX2/66 MHz personal computer.

3.1. The MBB beam

This problem is an example of a structure subject to a single displacement constraint to be used for examining the influence of the element removal ratio. The structure is designed to carry the floor in the fuselage of an Airbus passenger carrier. This support beam is produced by MBB in Germany. The problem has previously been solved by several researchers, e.g. Olhoff et al. [13], Rozvany and Zhou [14] and Zhou and Rozvany [15], using the homogenization method and the traditional boundary variation method. The simply supported beam has a span of 2400 mm and is 400 mm deep, with a point load $P = 20$ kN acting at the midspan as shown in the Fig. 1. The Young's modulus $E = 200$ GPa and the Poisson's ratio $\nu = 0.3$ are assumed. The initial volume is 1.07×10^6 mm^3 and the initial displacement at the loaded point is 6.3 mm. A single displacement constraint is imposed at the loaded point. Due to symmetry, only half of the structure is modelled using 60×20 quadrilateral elements.

The weights of the final designs for different displacement limits using different element removal ratios and removal strategies (Cases 1A, 1B, 2A, 2B, 3A and 3B) are given in Table 1. The results clearly show that for all cases the weights of optimal designs are significantly reduced. For all given limits, $ERR = 4\%$ results in heavier solutions. For the displacement limit of 8.5 mm and above, the smaller element removal ratio gives the lighter designs if the same removal strategy is used. Using the same element removal ratio, the removal strategy A gives lighter weight than the removal strategy B. The weight differences among the optimal designs for each particular limit vary from 4.54% to 7.17%.

The number of iterations and the solution time required for the displacement limit of 9.4 mm are given in Table 2. It is observed that by using larger element removal ratios the number of iterations

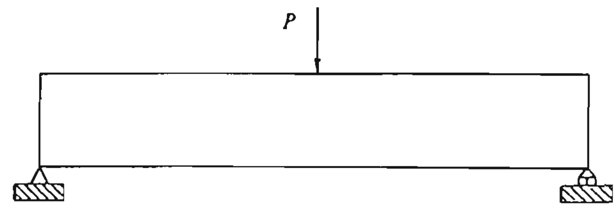


Fig. 1. Design domain for the MBB beam.

Table 1
Weight of the optimal designs for the MBB beam

	Weight of optimal designs (in percentages of the initial weight)					
Limit (mm)	Case 1A	Case 1B	Case 2A	Case 2B	Case 3A	Case 3B
7.0	75.42	76.00	75.33	78.00	78.33	80.00
7.5	66.83	67.00	68.08	68.00	72.17	72.00
8.0	61.00	65.00	61.58	64.00	66.50	68.00
8.5	57.50	59.75	58.00	60.00	61.25	64.00
9.0	52.83	53.75	54.58	56.00	56.50	60.00
9.4	50.33	51.75	52.42	54.00	54.25	56.00

Table 2
Number of iterations and time for the MBB beam with the limit of 9.4 mm

	Case 1A	Case 1B	Case 2A	Case 2B	Case 3A	Case 3B
Number of iterations	69	49	33	24	16	12
Time (hs)	4.3	3.1	1.5	1.1	0.9	0.6

and the computation time are dramatically reduced. For the same removal ratio, strategy *B* requires less time than strategy *A*.

The shape of the optimal designs for the limit of 9.4 mm obtained for all cases are shown in Fig. 2. It shows that the outer shapes of optimal designs in all cases are very similar. The element removal ratio has a greater influence on the details of the inner parts, however the main pattern and the orientation of these details are similar.

3.2. Structure under multiple displacement constraints

Consider a two-dimensional structure supporting three point loads, each at 10 kN, under the given boundary conditions shown in Fig. 3. The dimensions for the design domain are $L_x = 0.20$ m, $L_y = 0.10$ m and thickness $t = 0.005$ m. The Young's modulus $E = 207$ GPa and Poisson's ratio $\nu = 0.3$ are assumed. The maximum initial vertical displacement is 0.22 mm. Constraints are

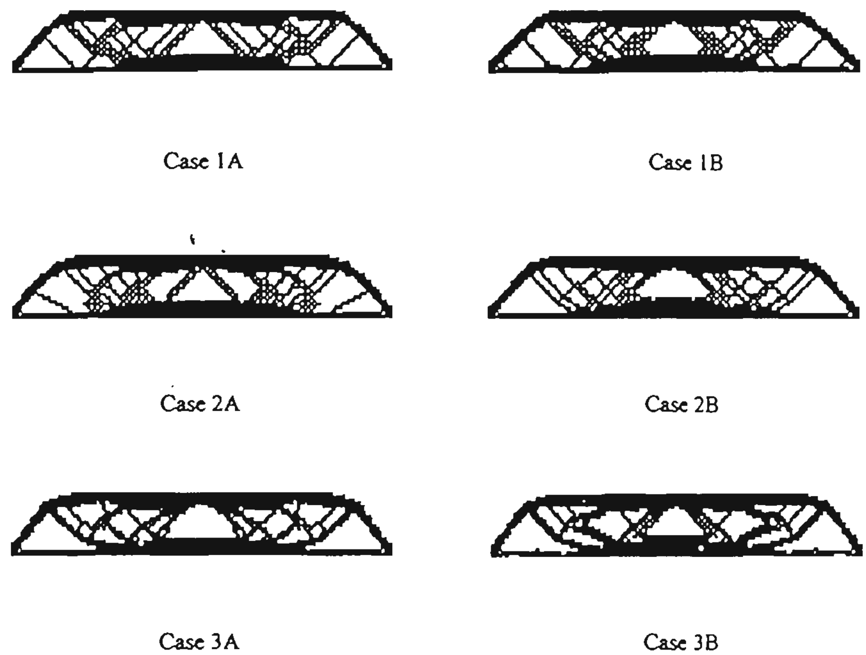


Fig. 2. Optimal designs for the MBB beam for the limit of 9.4 mm.

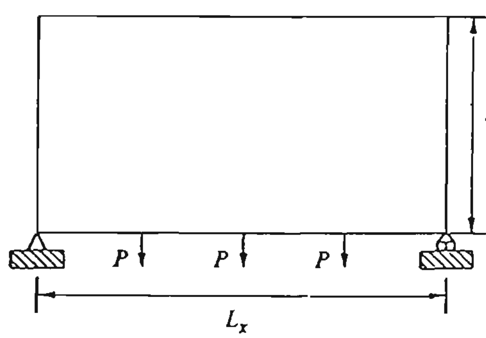


Fig. 3. Design domain for a structure under multiple displacement constraints.

imposed on the vertical displacements at the three loaded points. Due to symmetry only half of the structure is analysed using a mesh of 40×40 quadrilateral elements.

Weights of the optimal designs for limits of 0.30, 0.35 and 0.45 mm are given in Table 3. The weight differences among the optimal designs for each particular limit vary from 4.00% to 9.69%. It is seen that for the case of multiple displacement constraints, there is no clear relationship between the value of ERR and the resulting weight reductions. This is due to the fact that all the constrained displacements of the optimal designs are not always close to their limits (see Table 4). It can be seen from Tables 3 and 4 that, for the limit of 0.45 mm the optimal design in Case 2A, where all constrained displacements are closer to the limit, has lighter weight.

The number of iterations and the solution time required for the displacement limit of 0.45 mm are given in Table 5. Similar trend as in the previous example is seen in this example.

Table 3
Weight of optimal designs for the structure under multiple constraints

	Weight of optimal designs (in percentages of the initial weight)					
Limit (mm)	Case 1A	Case 1B	Case 2A	Case 2B	Case 3A	Case 3B
0.30	49.19	48.87	49.94	49.62	53.75	53.37
0.35	39.50	40.81	40.00	41.56	43.50	43.37
0.45	30.50	36.81	28.75	31.56	38.44	35.12

Table 4
Displacements of optimal designs for the structure under multiple constraints

	Displacements at three loaded points in optimal designs (mm)					
Limit (mm)	Case 1A	Case 1B	Case 2A	Case 2B	Case 3A	Case 3B
	0.27	0.27	0.26	0.27	0.26	0.27
0.30	0.30	0.30	0.30	0.30	0.29	0.29
	0.27	0.27	0.26	0.27	0.26	0.27
	0.32	0.31	0.32	0.31	0.30	0.32
0.35	0.35	0.34	0.34	0.34	0.34	0.35
	0.32	0.31	0.32	0.31	0.30	0.32
	0.41	0.36	0.44	0.42	0.33	0.40
0.45	0.43	0.36	0.45	0.40	0.37	0.42
	0.41	0.36	0.44	0.42	0.33	0.40

Table 5
Number of iterations and time for the structure under multiple constraints

	Case 1A	Case 1B	Case 2A	Case 2B	Case 3A	Case 3B
Number of iterations	119	64	61	35	24	17
Time (hs)	14.7	9.6	5.7	4.8	3.5	1.3

Optimal designs for the limit of 0.45 mm using different element removal ratios and removal strategies are shown in Fig. 4. It is seen that the outer shapes of these optimal designs are removal. There are some differences between the inner parts. The optimal designs for the Case 1B, 2A, 2B and 3B have very similar topologies.

3.3. Bridge with a moving load

This serves as an example of a structure subject to multiple constraints under multiple loading conditions. The initial design for the bridge is shown in Fig. 5. The body of the bridge, with

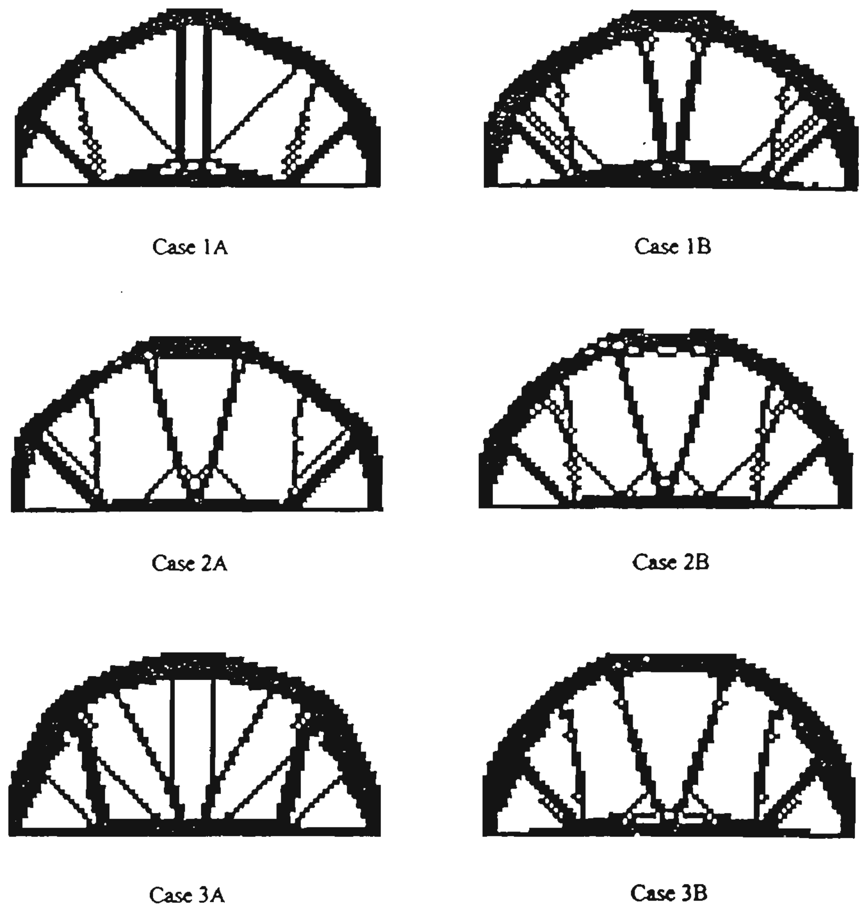


Fig. 4. Optimal designs for the structure under multiple displacement constraints for limits 0.45 mm.

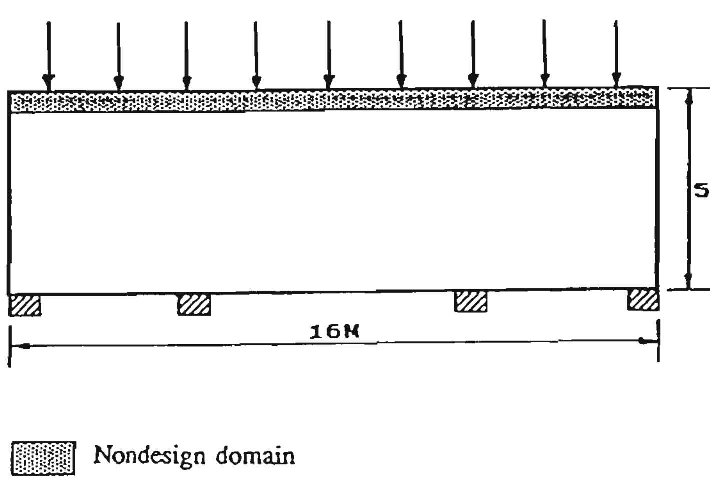


Fig. 5. Design domain for a bridge with a moving load.

Table 6
Weight of optimal designs for the bridge with a moving load

Limit (mm)	Weight (in percentages of the initial weight)		
	Case 1B	Case 2B	Case 3B
1.5	76.64	77.66	80.08
1.7	67.50	67.50	68.12
2.0	58.36	59.37	60.16

Table 7
Number of iterations and time for the bridge with limits of 2.0 mm

	Case 1B	Case 2B	Case 3B
Number of iterations	42	21	11
Time (hs)	124	60	27.2

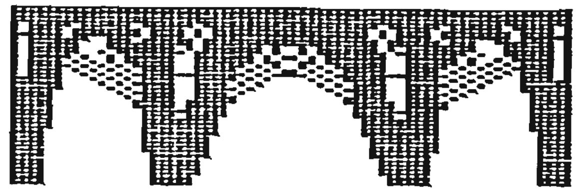
dimensions of $16\,\mathrm{m} \times 5\,\mathrm{m} \times 0.1\,\mathrm{m}$, is supported by four solid piers underneath. A point load $P = 1000\,\mathrm{kN}$, travelling from the left to the right of the bridge on the top surface, is approximated by 9 load cases with an equal distance of 1.75 m between each other. The Young's modulus $E = 30\,\mathrm{GPa}$ and the Poisson's ratio $\nu = 0.2$ are assumed. The whole structure is modelled by a mesh of 64×40 quadrilateral elements and each solid pier is approximately represented by four fixed nodes. A minimum thickness of 0.5 m is required at the top of the bridge, which is specified as a non-design domain. The left and right sides are kept unchaged so the elements on these two sides are also specified as non-design domain. The maximum initial vertical displacement for all load cases is 1.39 mm. Nine constraints are imposed on the vertical displacements at the loaded points.

The weights of the optimal designs for the limit of 1.5, 1.7 and 2.0 mm are given in Table 6 only for Cases 1B, 2B and 3B. It is seen that the smaller the element removal ratio, the lighter the weight obtained. For a larger limit, the weights of the optimal designs are closer. The weight differences among the optimal designs for a particular limit vary from 0.62% to 3.44%. For this example the element removal ratio has little effect on the weight.

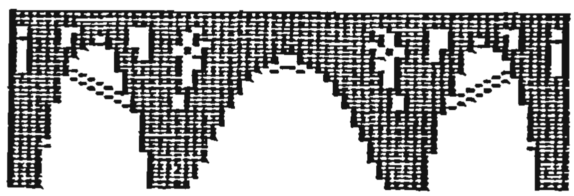
The number of iterations and the solution time for the limit of 2.0 mm are given in Table 7. It is observed that the time for solution is dramatically reduced when larger element removal ratios are used. Optimal designs for the limit 2.0 mm are obtained for only Cases 1B, 2B and 3B as shown in Fig. 6. It can be seen that the resulting shapes and topologies are similar.

4. Influence of mesh size

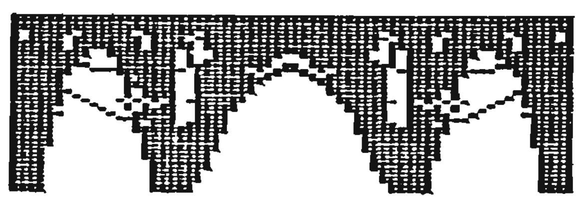
In this section the influence of mesh size on the final design is examined. Obviously more computation time is required when more elements are used in the initial FEA model. Solution for a short cantilever using different meshes is considered for this investigation. This example



Case 1B



Case 2B



Case 3B

Fig. 6. Optimal designs for the bridge with limits of 2.0 mm.

was used by Suzuki and Kikuchi [3] to show the convergence property of the homogenization method.

The cantilever beam shown in Fig. 7 is under plane stress conditions. The left-hand side of the beam is fixed and a vertical load of 3 kN is applied at the middle of the free end. The dimensions of the beam are $L_x = 0.16$ m, $L_y = 0.10$ m and the thickness $t = 0.001$ m. The Young's modulus $E = 207$ GPA and the Poisson's ratio $v = 0.3$ are assumed. The initial vertical displacement at the middle of the free end is 0.33 mm.

The designed domain is modelled by 32×20, 48×30 and 64×40 quadrilateral elements, and the element removal ratio of 2% of the initial number of elements is used. The weights of the optimal designs for these meshes are given in Table 8. It is seen that, for the same limit, finer mesh gives

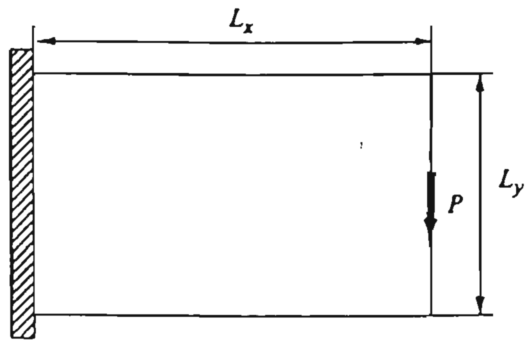


Fig. 7. Design domain for a short cantilever.

Table 8
Weight of optimal designs for the short cantilever

	Weight (in percentages of the initial weight)		
Limit (mm)	Mesh 32 × 20	Mesh 48 × 30	Mesh 64 × 40
0.50	56.87	59.17	59.37
0.75	39.37	41.67	41.09
1.00	30.00	30.00	30.94

slightly heavier solutions. However, for the larger limit, the weights of the final designs are almost the same. The weight difference among the optimal designs for each particular limit varies from 0.94% to 2.50%. So in this example the influence of the mesh size on the weight of final designs is very small.

The optimal designs for the three mesh sizes and limits of 0.50, 0.75 and 1.0 mm, are shown in Fig. 8. It is seen that the size of elements has little effect on the outer shape but considerably affects the inner part of the optimal designs. An interesting observation in the results obtained is that, truss-like frame structures are formed when displacement limit is considerable bigger than the initial value of the specified displacement of the design domain. For the limit closer to the initial displacement, curved frames are generated, and more continuum-like shapes are created.

5. Influence of element type

Finally, the influence of element type on final designs is studied using an example of a plate in bending. A simply supported square plate (0.20 m × 0.20 m × 0.0001 m) is loaded at the center by a point load $P = 0.04$ N normal to its plane. The Young's modulus $E = 174.7$ GPa and Poisson's ratio $v = 0.3$ are assumed. The initial out-of-plate displacement at the center is 1.16 mm.

Three models, representing a quarter of the plate, are considered using 400 linear quadrilateral plate elements, 800 one-way and 1600 two-way constant stress triangular plate elements which

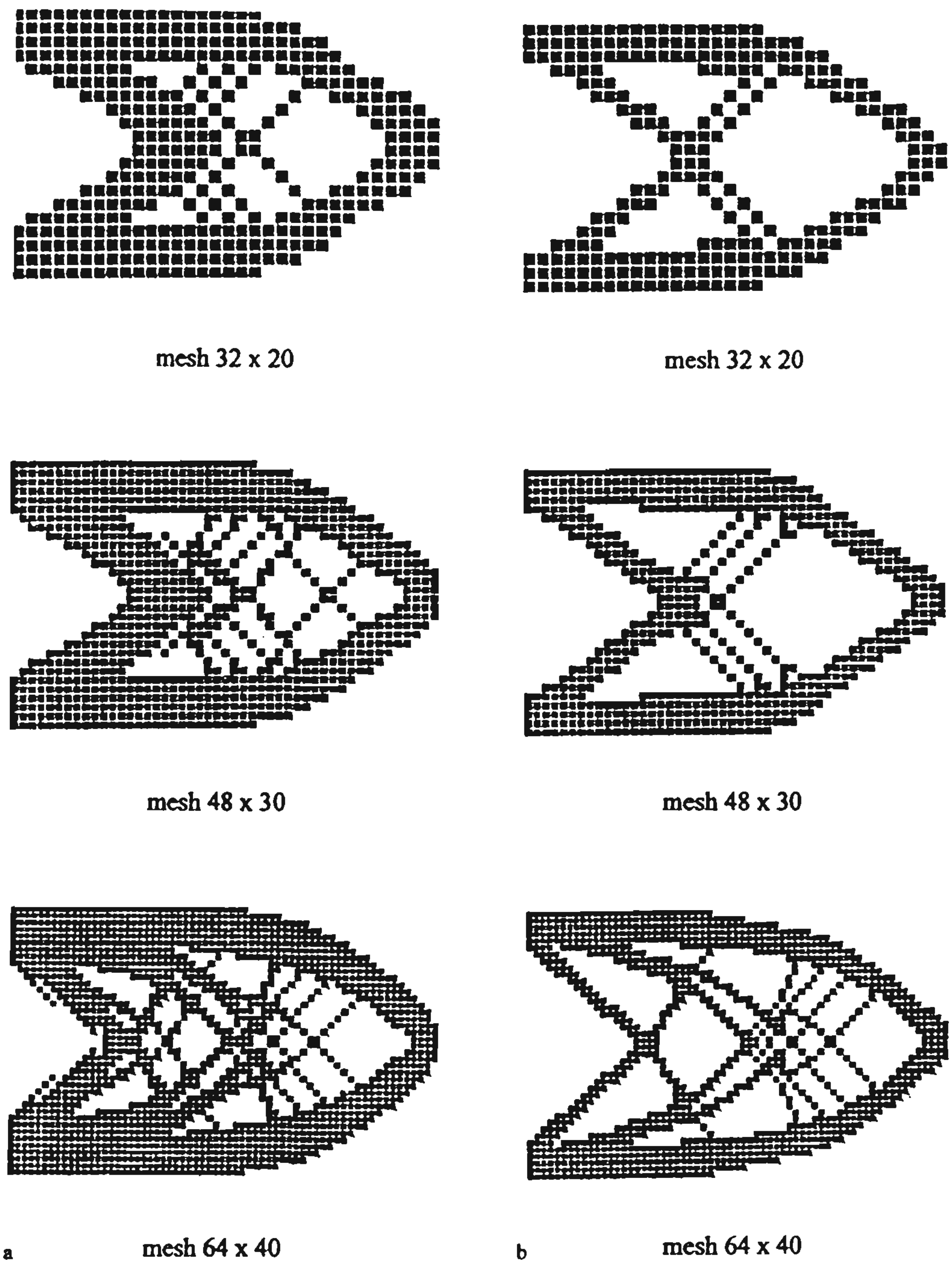
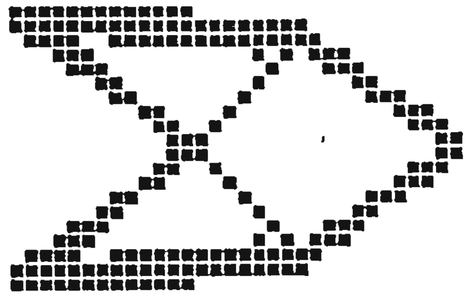


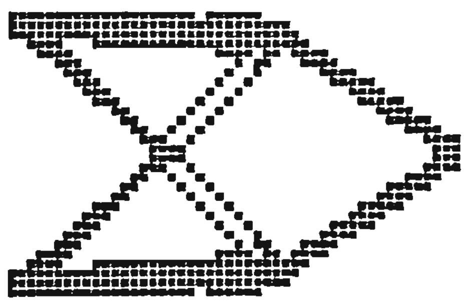
Fig. 8a. Optimal designs for the short cantilever for limits of 0.5 mm.

Fig. 8b. Optimal designs for the short cantilever for limits of 0.75 mm.

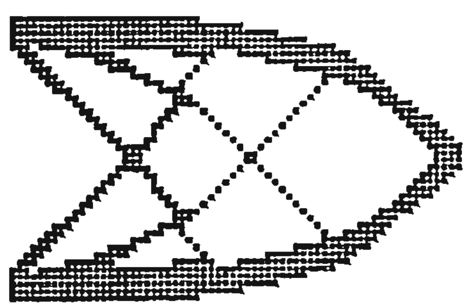
have similar size. For the limit of 1.6 mm on the displacement at the center, three corresponding optimal designs using an element removal ratio of 2% of the initial elements are obtained as shown in Fig. 9. The weight of the optimal designs are 64%, 64% and 63.87% of the initial weight, which are almost the same. The use of quadrilateral and two-way triangular elements gives almost



mesh 32 x 20



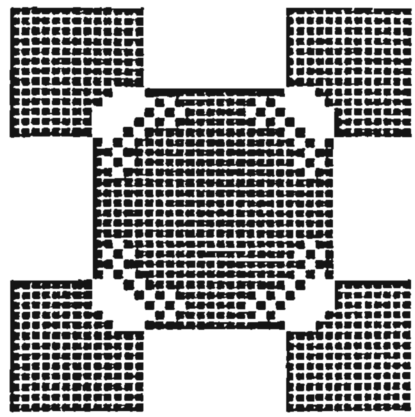
mesh 48 x 30



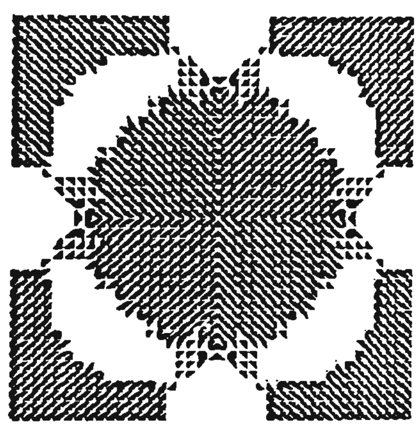
c mesh 64 x 40

Fig. 8c. Optimal designs for the short cantilever for limits of 1.0 mm.

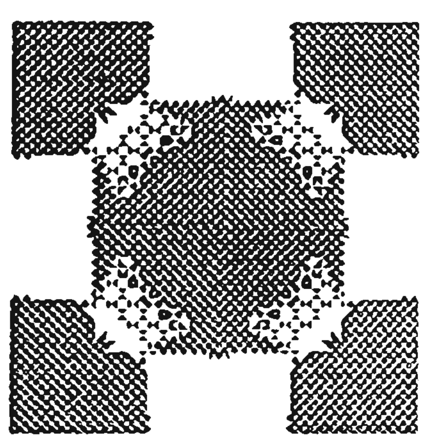
identical shapes. It is observed in all cases that hinges are formed along the hinge lines reported in [16, 17]. The solutions in Fig. 9 are similar to each other although the hinges appear at different locations along the hinge lines. These three optimal solutions have the same weight and response to the given load.



a) quadrilateral elements



b) one-way triangular elements



c) two-way triangular elements

Fig. 9. Optimal designs for a plate in bending.

6. Conclusion

The following conclusions can be drawn from the above discussions:
(1) When the element removal ratio varies from 1% to 4%, it has little effect on the weight and the outer shape of the optimal design. The element removal ratio does affect the details of the inner parts, however the main pattern and orientation of these inner parts are similar. It is suggested that one could use an element removal ratio as high as 4% to obtain optimal shape and topology with sufficient accuracy and significant time saving.
(2) The mesh size has little effect on the weight, even though it affects the details of the final design. However, even coarse mesh can provide a rough idea of the shape and topology of the optimal design.
(3) The type of elements with similar sizes has almost no effect on the weight however it has minor effects on the shape and topology of the optimal design.

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