Option Pricing in a Path Integral Framework

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Declaration

I, Gabriele Sorrentino, declare that the PhD thesis entitled ***Option Pricing in a Path Integral Framework*** is no more than 100,000 words in length including quotes and exclusive tables, figures, appendices, bibliography, references and footnotes. This thesis contains no material that has been submitted previously, in whole or part, for the award of any other academic degree or diploma. Except where otherwise indicated, this thesis is my own work.

Signature: Date:

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Abstract

This dissertation is an examination of methods for computing an option price using a path integral framework. The framework, developed by Chiarella, El-Hassan and Kucera, is based on the Black and Scholes paradigm. The path integral is backward recursive with the payoff known at expiry and has no closed form solution. Three specific financial derivatives are used in this dissertation, they are, European (call and put), American put and a down and out call (Barrier type) option.

The work in this dissertation examines three methods to approximate the option price. The first is a review of the spectral method offered by Chiarella et al. Their method involves the use of a Fourier-Hermite series expansion which represents the option value at each time step. The Hermite orthogonal polynomials and their associated properties are employed to create a set of recurrence relations so that a final option pricing polynomial is formed. A similar approach using normalised Hermite orthogonal polynomials is also presented. Similar methods and techniques are utilised to form a new set of recurrence relations. The accuracy obtained for both types of orthogonal polynomials are of the same magnitude.

In the other approaches, the path integral is transformed from an infinite interval integral to one of a finite interval with a bound on the resulting error. This is achieved by using the weight (in the form of a Gaussian) within the integrand of the path integral. Using an *a-priori* value, the tails of the Gaussian are eliminated to form the finite interval. Two numerical methods are used to approximate the option price namely, **mathematical interpolation** and various **quadrature (Newton-Cotes) rules**.

The interpolation approach takes a series of Hermite interpolation polynomials (of order 2) to represent the option price at each time step. Since there is no closed form solution of the path integral, converting the option price function to a series of polynomials allows an approximation of the option price to be found. By discretizing the underlying, a series of integrations are evaluated for each time step. Various discretization schemes are implemented including a fixed number of partitions (equally spaced over each time step), equally spaced partitions (over each time step) and an adaptive node distribution. In this final discretization scheme, the partitions are formed so that the errors caused by interpolation are controlled. The option price approximations are highly accurate with some discretization schemes working better than others.

The final approach takes the finite interval path integral and uses various quadrature (Newton-Cotes) rules. Endpoint, Midpoint, Trapezoidal and Simpson's rules are employed to approximate the option price. The underlying is discretized using a fixed number of partitions, equally spaced over all time steps for each of the rules implemented. The results obtained using the various rules are highly accurate for the European option and the down and out call option but require a large number of partitions to obtain the same accuracy as the other methods for the American put option.

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Chapter 1

Introduction

The pricing of derivative securities, such as options, has in the past three decades, encroached into the world of science. Many mathematicians, physicists and statisticians have contributed their methodologies and techniques to the world of finance. These methods, usually used in engineering and the physical sciences, have been aptly adapted to problems in the financial world.

The major issue confronting investors is security of their assets or financial position. A wheat grower may want to sell his/her crop in the future at a predetermined price and not wait until the crop is ready to sell (at a price below expectation). An investor would like to buy or sell shares in a company ABC for a predetermined price in the future.

Given these issues, pricing of derivative securities is not so simple. With different underlying assets to protect, the condition of financial markets changing from nation to nation, investor sentiments differing due to human feelings and other factors influencing security prices, mathematical modeling can be complex.

In creating a financial instrument involving the risk of an underlying asset, the following aspects must be taken into consideration when modeling;

i. An understanding of the underlying asset,

ii. The volatility associated with the underlying asset,

iii. Other conditions involved in the markets where the financial instrument and the underlying asset are traded. Examples of such conditions include trading periods, transaction costs and interest rates.

Later in this chapter, an investigation is made into various methods and techniques used to assist in the pricing of derivative securities. In an effort to combat the complexities of models designed, many authors place conditions and constraints such that solutions/approximations can be made.

The thesis will concentrate on the area of pricing using a path integral framework. The use of path integrals has been commonplace in science for many years since the creation of the path integral in Feynman (1942). Its application to finance, in particular the pricing of derivative securities, has been less common. The thesis will offer various alternative techniques to solve a particular path integral model. One of the major advantages of the methods presented is the high accuracy achieved, very efficiently and with relatively low computational effort.

The remainder of this introduction includes a section 1.1 of commonly used terms. Section 1.2 gives a brief summary of the basic concepts used in the pricing of options. An explanation of factors which affect Options and their pricing are also given. Section 1.2 also gives a thorough review of the literature for non path integral modeling of option pricing.

Section 1.3 reviews option price modeling with an emphasis on Path Integrals. It is hoped that the review in section 1.2 and 1.3 will allow the reader to appreciate the vastness of the topic at hand. We finally state the objectives and aims of this thesis in Section 1.4.

1.1 Common Terminology

The following section gives a very brief overview of the basic terms and concepts involved in option pricing. If further understanding of the basic areas of financial derivatives and the markets they trade in is required, then Hull (2006) and Wilmott (1999) are excellent resources. Most of the terms and concepts within this section are sourced from Atkinson (1989), Kreyszig (2006), Hull (2006) and Wilmott (1999).

The following is a list of commonly used terms within this thesis.

Commodities: Commodities are usually raw products such as precious metals, oil, food products etc.

Forward Contract: A forward contract is an agreement where one party promises to buy an asset from another party at some specified time in the future and at some specified price.

Futures Contract: A futures contract is similar to a forward contract with the only difference being that they are traded on an exchange and are marked to market.

Options: Gives one party the opportunity to buy or sell an asset from/to another party at a prearranged price.

Call Options: The holder has the right to buy an asset by a certain date for a certain pre-agreed price.

Put Options: The holder has the right to sell an asset by a certain date for a certain pre-agreed price.

European Options: Options that can only be exercised at the expiration date.

American Options: Options that can be exercised at any time up to the expiration date.

Barrier Options: Options of an exotic type, in which the payoff depends upon the reaching or crossing of a barrier (predetermined price) by the underlying. These options include call options and put options, and are similar to common options in many respects. Barrier options become active/inactive when the underlying crosses the barrier.

Underlying: The financial instrument on which the derivative value depends. The option payoff is defined as some function of the underlying asset at expiry.

Strike or Exercise Price: The amount at which the underlying can be bought (call) or sold (put).

Expiration or Expiry Date: Date on which the derivative can be exercised or date on which the option ceases to exist or give the holder any rights to act.

Intrinsic Value: The payoff that would be received if the underlying is at its current level when the derivative expires.

In the Money: An option with positive intrinsic value.

Out of the Money: An option with no intrinsic value, only time value.

At the Money: A call or put with a strike that is close to the current asset value.

Hedging: A strategy to Establish a guaranteed future price of a commodity.

Speculating: Investors wishing to take a position in the market. Either they are betting that the price will go up or they are betting that it will go down.

Arbitrage: Involves locking in a riskless profit by simultaneously entering into transactions in two or more markets.

Volatility: Is the term given to represent the standard deviation of the instantaneous return of the underlying.

Fourier Analysis and Series: Fourier Analysis concerns the study of periodic phenomena. Fourier Series is a series which represents complicated functions in terms of simple periodic functions.

Mathematical Interpolation: Mathematical interpolation is the selection of a function $p(x)$ from a given class of functions satisfying some smoothness conditions in such a way that the graph of $y = p(x)$ passes through a finite set of given data points.

Quadrature: The quadrature of a geometric figure is the determination of its area.

Gaussian (Distribution): Is another term used for the Normal Distribution.

1.2 Options and Option Pricing

To appreciate the content of the following thesis, an introduction to some of the basic concepts is worthwhile. The concepts covered in this section include aspects of option pricing and the mathematics presented throughout the thesis. To a mathematician some of the methods used in the thesis are quite novel. But to understand the problem at hand, an introduction to terms and concepts used in option pricing may be required.

The term Risk Management is sometimes used to describe the security of investments. As people insure their valuable possessions such as houses, cars and boats, investors need to secure their assets and/or financial position by using financial instruments such as options (contingent claims).

Within the financial world, there are various assets, and many variants that affect the value of an asset. Some examples of assets that can be secured and the factors that affect the value of them, include:

- Shares
- Commodities such as Wheat, Wool, Sheep, Electricity, etc
- Bonds
- Stock Exchange Indices
- Foreign Exchange
- Interest Rates
- Volatility.

Given the nature of assets and the variants, the pricing of financial instruments such as options is sometimes complex and time consuming. Adams, Booth, Bowie & Freeth (2003) states various factors that affect the pricing of options. The factors include:

- Exercise Price
- Underlying Asset Price
- Time to Expiry
- Volatility
- Interest Rates
- Incomes & Dividends.

Adams et al. (2003) briefly explains the meaning of each factor but also describes how each factor affects the value of the option (Put and Call). In later chapters, we explain and analyse the effects of these factors on the price of options.

Options are common financial instruments which allow one party to buy/sell assets from another party for a particular price. As described, many factors influence the value of the option. The remainder of this section will take a detailed look at the modeling of options as well as the techniques used to determine the value of an option.

Since the development of the pricing of derivative securities by Black & Scholes (1973) and Merton (1973), the literature has become vast. This area of finance has developed to the point where science has taken a grasp and influenced the creation of various models and the techniques to solve them. With the Black, Scholes and Merton developments of their formula to the development of models which incorporate Jump Diffusion parameters, science and especially mathematics, have been at the forefront of pricing financial instruments (options).

The literature provides a variety of techniques to solve various option prices. Some of the major methods used include (in no particular order):

1. Lattice Structures (Trees)

2. Monte Carlo Simulation

3. Quadrature

4. Solutions to partial differential equations (PDE's)

5. Martingales and other probabilistic methods.

With the development of the Black and Scholes partial differential equation (PDE) and the analytic solution (formula), the mathematical/scientific world became involved. The further development and extensions of the Black-Scholes PDE and the creation of other types of options (that is, exotic, barrier and path dependent options) has led to other mathematical methods for their modeling and analysis. Chapter 2 gives a detailed presentation of the Black and Scholes paradigm and the development of the PDE leading to the Black and Scholes formula.

Since Black & Scholes (1973) and Merton (1973), the literature for pricing derivative securities has flourished. The techniques and methodologies employed are numerous and varied. The most common techniques used include simulation, particularly Monte Carlo and discretization methods like binomial and trinomial trees, and finite differences. The varying techniques employed are dependent on the equations to be solved. The most common form of equations used are differential equations. However, in recent times, the use of path integrals has increased and various techniques to solve these integrals have been developed.

Other techniques are also employed due to the creation of other types of securities. These securities are sometimes complex compared to the original warrants described by Black, Scholes and Merton. However, some of these securities are based on the Black and Scholes paradigm. They are based on similar assumptions and conditions as described in Chapter 2.

This section will present the influential and relevant works in the option pricing world. Some of the methods and techniques developed over the years have shown the multitude of mathematical adaptations used to procure an option price. This part of the review shows the vastness of the modeling, the techniques and the advancement of option pricing.

The ground breaking and defining work by Black, Scholes and Merton, paved the way for many changes in the management and modeling of risk. Many subsequent authors have gone on to extend and modify the early work of Black, Scholes and Merton. Along with these new works, has been the creation of new financial instruments (and options) based on the models and theories of these authors.

Another influential paper is that of Cox, Ingersoll & Ross (1985) who present a theory of the term structure of interest rates. This paper is of great importance to the financial world, as it has led to other types of modeling in finance, not just those related to Black, Scholes and Merton's work. They explain the term structure of interest rate as a relationship among the yields on default-free securities, that differ only in their term to maturity. By offering a complete schedule of predicted interest rates across time, the term structure embodies the markets' anticipations of future events.

The authors present a description of the previous works on the term structure of interest rates. Cox, Ingersoll and Ross incorporate general equilibrium theory, in combination with the previous studies to develop their term structure of interest rates. It is worth mentioning the work of Maghsoodi (1996) who extends the Cox, Ingersoll and Ross model to incorporate time-varying parameters. The work by Cox, Ingersoll and Ross and related authors shows that not all risk management and financial instrument modeling revolves around early methods and techniques of Black, Scholes and Merton, and that there are other methods and techniques to investigate and that model financial risk.

The rest of this section will describe the modeling of other authors who have based their works mainly around that of Black and Scholes, and Merton. Most of the modeling is based on extensions and alternatives of their basic models. Other models are described which include exotic options and American options. Also reviewed are some models with solutions to financial instruments using numerical methods, especially for American options. In reviewing these extended and modified models, the various types of methods and techniques used are clear. The authors presented various differences to the earlier models. Popular methods included the relaxation of assumptions, the introduction of real market occurrences and various differing methods and techniques to solve the old models. The following paragraphs are grouped in such a way that these variations are made clear.

An appropriate extension/modification to the work of Black & Scholes (1973) was devised by Hyland, McKee & Waddell (1999) to incorporate time-dependent interest rates and volatility. The authors present some interest rate and volatility models to illustrate their work. These models are very general time-dependent equations and are not indicative of the typical interest rate and volatility structures.

Silverman (1999) and Garven (1986) present alternative methods to find a solution to the Black and Scholes PDE, namely

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0. \tag{1.1}$$

where V is the option price, t is time, σ is the volatility associated with the asset which has a value S and r is the interest rate.

Silverman's involves the use of Green's function and Garven's presentation is in view of the risk neutrality arguments presented by both Cox & Ross (1976) and Rubinstein (1976). It is clear that there are alternative methods to solve (1.1) other than the conversion to the heat transfer equation method used by Black and Scholes.

In the following paragraphs, a summary of various types of European option models will be made. These descriptions will show the types of modifications and extensions to option pricing models that have been performed over the years, with particular reference to the Black-Scholes equation. These models present changes to the Black and Scholes paradigm. Conditions are altered or dispensed with, with the objective of modeling options for real market scenarios. As stated previously, the advantage of the Black, Scholes and Merton model is that the option price is easily found. Even though these models are more realistic, they do require extensive computational effort. In some cases, exact solutions are difficult to find.

Jennergren & Naslund (1996) and Merton (1976) present an extended Black-Scholes model to incorporate a class of option with stochastic lives (options which may be canceled but the underlying stock retain their value). This is an appropriate modification, as options may cease to exist due to company mergers, bankruptcy, and employee resignations (for an employee class option) as examples. The introduction of arbitrage is a useful modification to the modeling of financial risk. Ilinski (1999) allows the possibility of virtual arbitrage in his modified Black-Scholes model.

However, by allowing arbitrage possibilities, one would have to be realistic and have to consider the cost(s) involved in obtaining such a riskless position. So, another popular method of extending the Black-Scholes equation (or any other financial instrument model) is the introduction of transaction costs or fees. There have been various modified models presented over the years to incorporate the effects of transaction costs. One of the first and most popular works in regards to transaction costs was that of Hodges & Neuberger (1989). Later, Davis, Panas & Zariphopoulou (1993) developed a model for European options with transaction costs, with Davis & Zariphopoulou (1995) presenting a similar model for American options. Whalley & Wilmott (1997) produced an efficient and simpler hedging strategy to be calculated. One of the main problems in analysing these types of models is, that they may be too complex and the question as to whether there is a method to find a simpler solution. Chao, Jing-Yang & Sheng-Hong (2007) use a Markov chain approximation to compute Barrier option prices with transaction costs. Given new methodologies and techniques, instead of finding a precise solution for a complex model, determining an imprecise result, together with an estimate of the imprecision, will allow these real world models to be applied in practice.

Another assumption that can be modified/manipulated is the structure of volatility. The original Black and Scholes model used a constant volatility for the stock price, which was used for the purpose of simplification. However, stock/asset volatilities have complex structures and it would be appropriate to represent these complex volatility structures (i.e. stochastic volatility) in the option pricing models. One of the most popular models developed was by Heston (1993). The Heston Model is used by many authors when comparing their own models and techniques involving stochastic volatility. Heston shows there is a bias between volatility and the spot asset price. Therefore, he incorporates this into his model. Finally, analytical forms are found for the various PDE's using characteristic functions which are easy to compute.

Some other models presented to incorporate varying volatility structures worth mentioning include Chalupa (1997), Sircar & Papanicolaou (1998), Kurpiel & Roncalli (1998) and Zuhlsdorff (2001). In recent times, Medvedev & Scaillet (2007) derive implied volatilities for options under a two-factor jump-diffusion stochastic volatility. Hilber, Matache & Schwab (2005) offer a unique approach to solving option prices under stochastic volatility. They offer an algorithm based on a sparse wavelet space discretization.

Given the extensive works by the previous authors in modifying or extending the work of Black and Scholes, and Merton, there have been presentations of other financial instrument models (and in particular, other option pricing models). One of these is the modeling of Exotic options. These options are non-standard options, and have been examined extensively. This thesis will examine exotic (American and Barrier types) along with the Vanilla (European type) options.

There have been numerous exotic option pricing models presented including that of Carr, Ellis & Gupta (1998) who develop static hedges for several exotic options using standard options. In this area, the work of Neuberger & Hodges (2000) in which they devise a model for exotic options in the spirit of the Merton (1973) approach for rational bounds on the pricing of exotic options is worthy of examination. Penaud, Wilmott & Ahn (1998) extend a Vanilla Passport option to add various exotic features to that option. The authors present seven different types of exotic passport options, using the same assumptions as used in deriving the Black-Scholes equation. Schoutens & Symens (2003) present a Monte Carlo simulation method to price exotic options with stochastic volatility.

An interesting exotic option pricing model is proposed by Geman (2001). The author develops a technique to find the price of a type of exotic option called an Asian option (there is also the development of a Barrier option). The technique offered involves the use of Laplace transforms and of a time-changed squared Bessel process. Geman presents some numerical work, comparing the author's results to an equivalent Monte Carlo simulation.

Finally, some of the more recent techniques used in approximating financial instrument pricing models is via the use of Martingale methods and game theory. Musiela & Rutkowski (1997) present numerous financial instrument models via the use of Martingale Methods. Prigent, Renault & Scaillet (2004) also address the problem of option pricing (with discrete re-balancing) using Martingale measures. Henderson (2005) presents some Martingale measures to incomplete stochastic volatility models. The use of Martingale methods and game theory reiterates that the modeling of financial instrument (option) pricing is open to various methods and techniques. Olsder (2000) develops a technique for the pricing of options using game theory. The author offers one model for a two player system, with the players being *nature* and the investor. The second model consists of three players, being *nature*, the investor and the bank (whose presence forces the introduction of transaction costs).

So far the review has presented models for corporate liabilities, European options and exotic options. One of the most common financial instruments (and option) is the American option. These options allow the owner to buy or sell the underlying asset at any time up to the maturity date. There has been a vast amount of literature in the mathematical modeling of American options, with the main issue concerning when to exercise an option. This problem is known as the early exercise option.

One of the first American option pricing models to be presented is that of Brennan & Schwartz (1977). Their work has also been extended and modified over the years. Another two relevant American option pricing models presented are by Geske & Johnson (1984) and Kim (1990). The following paragraphs will summarise their work.

Brennan & Schwartz (1977) confirms that the American put option obeys the Black-Scholes equation. The authors then describe and state a numerical method to solve the Black-Scholes equation for an American option. The solution for the American option differs greatly to the European option, as an American option can be exercised at any time up to the exercise date. Brennan and Schwartz apply their model against some historical data. They compare the put result against the equivalent Black-Scholes European put option. This comparison seems to be irrelevant, as a European option can only be exercised on the exercise date. Cox, Ross & Rubinstein (1979) offers a Binomial tree approach to various options, including an American put. They argue that their alternative approach to Brennan & Schwartz (1977) is simpler and in most cases computationally more efficient.

Geske & Johnson (1984) developed an analytical approximation for an American put option. They argue that numerical solutions are expensive, which may have been the case in the 70's and 80's. The analytical solution presented by Geske and Johnson is

$$P = Xw_2 - Sw_1 \tag{1.2}$$

where w_1 and w_2 may be represented as a Taylor series, X is the exercise price and S is the stock price.

In devising this solution, Geske and Johnson determine at each instant, dt, the put will be exercised if, the put has not already been exercised and the payoff from exercising the put equals or exceeds the value of the put if it is not exercised. The authors go on to present formula evaluations and applications, comparing their results to those of Parkinson (1977) and Cox & Rubinstein (1984). In comparing their results, the authors state that the option values yielded are within one penny of each other. They also note that the European value is close to the American value, where the American option would be more valuable when the early exercise option is taken. They also indicate that the analytical solution they offer is faster to compute by a factor of 10 compared to the standard numerical methods. At the time of the model presentation, the analytical approximation may have been faster. Analytical approximations has its advantages as prices can be evaluated precisely and can be used to compare against other methods and techniques. But with high-speed computers and efficient numerical methods, the argument of analytical approximations being faster to calculate is now out-dated, however analytic solutions do provide valuable insight.

Kim (1990) offers a differing analytical evaluation of an American put via the use of numerical methods. Kim questions the Geske & Johnson (1984) solution, as Kim states *it is yet to be shown that an analytical solution to an American put value can be obtained as the sum of an infinite series of functions.*

The integral equation presented in Kim (1990) cannot be solved explicitly, however, it can be solved numerically. In determining the optimal exercise boundary, $B(s)$, the computation of the American put value is achieved by straight forward numerical integrations. Some of the techniques offered in this thesis may be applied to the integral equation presented in Kim (1990).

There has also been modeling of American options using various other methods and techniques. Jaillet, Lamberton & Lapeyre (1990) verify the modeling of Brennan & Schwartz (1977) with the use of variational inequalities. El Karoui & Karatzas (1995) describe a model for an American put option using Martingale methods. Part of their work is an extension of Bensoussan (1984). As discussed previously in this review, Davis & Zariphopoulou (1995) present a model for American options with transaction fees. Coleman, Li & Verma (1999) offer a Newton method for American option pricing. Their work is based around improving the work of Brennan & Schwartz (1977). These models show that there are various mathematical methods and techniques that can be applied to the pricing of American options.

Other models and solutions using numerical methods worth noting are Siddiqi, Manchanda & Kočvara (2000), who define an application of an efficient algorithm for a numerical solution for American options. The solution, like that of many authors previously, is based on the Black-Scholes equation. Stamicar, Ševčovič & Chadam (1999) find a numerical approximation for an early exercise boundary for an American put option near expiry. Zhao, Davison & Corless (2007) design a compact finite difference method for pricing American options. The authors offer three types of finite difference methods and the results compare favourably to the existing Crank-Nicolson methods.

Sullivan (2000) uses Gaussian quadrature to evaluate the price of an American put option. Initially the author presents approximations for a European put option using a Binomial Tree, Trapezoidal, Simpson's and Gauss-Legendre methods, with the Simpson and Gauss-Legendre methods working quite well. The Gauss-Legendre quadrature is then applied to the American put option using Chebyshev approximations. Thorough analysis of convergence, accuracy and speed are presented and comparisons to analytical methods are made. Some of the quadrature described in Sullivan (2000) will be applied to a path integral representation of various types of options in the thesis (Chapter 5 and 6).

In describing these models in the last couple of paragraphs, it is clear that the modeling of American options is more complex than the modeling of European options since American options can be exercised at any time up to the expiry date. Calculating the early exercise boundary (the point at which the American option should be exercised) is just as important as the value of the option itself.

In collating this review of pricing of financial instruments like options, it is clear that financial instruments are becoming complex to model and to price. This review was presented to give an overview of the changing landscape of option pricing. An area that has not been presented thus far is the use of Path Integrals which is the main emphasis of the thesis.

Path Integrals have been used in the area of science for many years. In the world of option pricing it has only been in the last decade or so that the path integral has been used to model the price of an option. The following section will give a review of the literature presented so far. It is envisaged that the reader has some basic knowledge and understanding of path integrals.

1.3 Option Pricing and Path Integrals

The use of path integrals has developed into a viable option pricing model representation in the past decade or so. Since the creation of the Black-Scholes PDE and the various techniques to solve (1.1), authors have attempted to model vanilla and non vanilla options in alternative forms. Path integrals has been one of the alternative methods.

Path integrals have been used in various areas of science over the years, especially in quantum physics. One of the advantages of using path integrals is the variety of techniques used to solve them. From Monte Carlo simulation to various quadrature methods, the techniques have been developed and applied to finance.

The following review will present the use of path integrals to model and the techniques to evaluate option prices. One of the early uses of a path integral in derivative security pricing was from Makivic (1994). The author presents a Monte Carlo approach (using the Metropolis algorithm) to price a security.

Makivic also states that the main advantages of a path integral approach are:

(1) partial derivatives of the price with respect to all of the input parameters can be computed in a *single* simulation,

(2) results for *multiple sets* of parameters can be computed in a single simulation, and

(3) suitability for implementation on a parallel or distributed computing environment.

It must be said that his assertions are correct for a path integral approach using Monte Carlo simulation to evaluate the price. The best results show errors of order 10^{-4}.

Baaquie (1997) presents a path integral approach to option pricing with stochastic volatility. Baaquie generalises the results of Hull & White (1987) for cases when the stock price and volatility have non-zero correlation. Ingber (2000) also presents a path integral approach to options with stochastic volatilities. The author uses an Adaptive Simulated Annealing approach to determine the behaviour of diffusion. This behaviour is determined by daily Eurodollar future prices and implied volatilities. An algorithm called PATHINT is used to evaluate prices.

Linetsky (1998) offers a path integral approach to financial modeling and option pricing. The author states that "the path integral formalism constitutes a convenient and intuitive language for stochastic modeling in finance". Linetsky presents various path integrals, including a framework for the Black-Scholes paradigm path dependent options and multi-asset derivatives. The author finally develops evaluations for various options using analytical approximations and numerical methods (Monte Carlo simulation and/or discretization schemes).

Some authors have investigated the use of path integrals to model path dependent options. Matacz (2000) uses a partial averaging method to price path dependent options (Asian options and occupation time derivatives). The method of partial averaging reduces the dimension of the integral. The evaluation can be performed by Monte Carlo simulation methods. Baaquie, Corianò & Srikant (2003) also offer a path integral approach to solve for path dependent options. They build their model using the Black-Scholes paradigm and then extend it to create more complex securities such as exotic and path dependent options. Baaquie et al. (2003) evaluate the option prices by Monte-Carlo simulation. Bormetti, Montagna, Moreni & Nicrosini (2006) also present a path integral framework to evaluate (via Monte Carlo simulation) prices for various path dependent options.

An interesting application using a path integral approach is offered by Otto (1999). The author presents a model to price interest rate derivatives. Path integrals for the short term and bond option are developed. Otto suggests two techniques to solve these derivatives, they are a lattice method or the use of Monte Carlo simulation.

Bennati, Rosa-Clot & Taddei (1999) develop a path integral approach for various stochastic equations that best represent financial markets. The path integrals are designed to cater for one and multi dimensional cases. The authors then present some analytic results for various models such as Black-Scholes, Cox-Ingersoll-Ross and others. Rosa-Clot & Taddei (2002) offer numerical methods to price some of the derivative securities presented in Bennati et al. (1999). Rosa-Clot and Taddei use two methods to evaluate prices, Monte Carlo simulation and deterministic evaluations (quadrature rules). The deterministic evaluations has its advantages in low dimensional problems but in high dimensions the technique has issues with large matrix dimensions. Various options (European options , Zero-coupon bonds, Caplets, American options and Bermudan swaptions) are priced.

Some authors have investigated the use and evaluation of path integrals to price options using unique and less common techniques. Kleinert (2002) presents a Natural Martingale for non-Gaussian fluctuations of the underlying. Decamps, De Schepper & Goovaerts (2006) develop a path integral approach to asset-liability management. Chiarella, El-Hassan & Kucera (1999) present an evaluation of a European and American option in a path integral framework. The novel approach to the evaluation is the use of a Fourier-Hermite series. The technique takes into consideration the form of the integrand of the path integral (1.3),

$$f^{k-1}(\xi_{k-1}) = \frac{e^{-r\Delta t}}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-(\xi_k - \mu(\xi_{k-1}))^2} f^k(\sqrt{2\Delta t}\,\xi_k)\, d\xi_k. \tag{1.3}$$

The Gaussian in the integrand is in the form of the weight of a Hermite orthogonal polynomial. The price function, $f^k(\xi_k)$, is expanded into a Fourier-Hermite series. This series is continuous and is a differentiable representation of the underlying. Given the form of the Fourier-Hermite series, the Deltas are easily found as well as the option price.

In Chapter 2 we present the development of the path integral (1.3). Chapter 3, in this thesis, gives a thorough overview of the technique used to find the option price. In this overview of the technique, errors were found in the formulation and in the results presented. The path integral is formed using an application of Ito's Lemma. Chapter 4 offers a modification to the technique used to evaluate the option price. The alternative method uses normalised Hermite orthogonal polynomials. The use of the normalised polynomials has its advantages, especially when a large number of basis functions are used.

An extension of the previous approach is offered by Chiarella, El-Hassan & Kucera (2004) to incorporate the evaluation of point barrier option prices. The path integral is very similar with the only difference being the integral domain. The path integral (1.3) with a finite domain, namely,

$$f^{k-1}(\xi_{k-1}) = \frac{e^{-r\Delta t}}{\sqrt{\pi}} \int_{z_{k,l}}^{z_{k,u}} e^{-(\xi_k - \mu(\xi_{k-1}))^2} f^k(\sqrt{2\Delta t}\,\xi_k)\, d\xi_k, \tag{1.4}$$

where

$$z_{k,l} = \frac{\ln\left(b_{k,l}\right)}{\sigma_k\sqrt{2\Delta t_k}}, \; and \; z_{k,u} = \frac{\ln\left(b_{k,u}\right)}{\sigma_k\sqrt{2\Delta t_k}}, \tag{1.5}$$

for $k = K-1, \ldots, 1$ with $b_{k,l}$ and $b_{k,u}$ being the lower and upper barriers respectively, at time step k.

Chapters 5 and 6 offer alternative techniques to evaluate the same path integral framework (1.3) and (1.4). Prices are approximated for European, American and Barrier options. The techniques take into account the form of the integrand such that interpolation polynomials and various quadrature rules can be used. The techniques employed are highly accurate and very fast to compute.

Given the literature review presented in this thesis, it is clear that the methods and techniques used in evaluating the option price are vast. From the early days of Black, Scholes and Merton to the introduction of many scientific approaches, option pricing is a growing area in both finance and mathematics. Path integrals in finance is relatively new in comparison, with the last decade seeing an increase in activity. Path integrals have been used in areas such as quantum physics for many years since the initial work by Feynman (1942).

1.4 Thesis Objectives

The thesis is based around the path integral framework offered by Chiarella et al. (1999). In their method, the underlying is expanded into a Fourier-Hermite series. At each time step, the coefficients of the series are determined in a backward recursive manner, using recurrence relations. These relationships are formed utilising the orthogonal properties of Hermite orthogonal polynomials. In Chapter 3, an analysis of the method described by Chiarella et al. is presented. This will assist in understanding the remaining chapters and comparison of techniques used to solve the same problem.

The first approach is similar to that offered by Chiarella et al. The main difference being the use of normalised Hermite orthogonal polynomials. A set of recurrence relations are formed, as with the previous method. The benefits of using the normalised polynomials are the form of the recurrence relations as well as the speed to find accurate results (especially for the European option). Some relations have one less exponential term. Given this fact, the speed of computation should be improved for a large number of basis functions.

The next approach, using the same path integral framework, also converts the underlying price at each time step. The price is represented by a series of interpolation polynomials. In this method, integration is performed only once, at the beginning of the process. Using the result of the integration and the interpolation polynomial coefficients found, the option price is evaluated. This process is repeated at each time step. The method requires no transformations and is quite straight forward to implement. The path integral framework is converted from an infinite interval to a finite interval.

The major issues arising from this method include, the determination of the interval of integration and the node point allocation. The problem of the interval of integration is solved via the properties of the Gaussian in the integral. Node allocation or distribution will vary depending on the derivative security being priced. Similar to Chiarella et al., the resultant derivative security price is continuous and infinitely differentiable allowing for fast and accurate evaluation of the hedge ratios (if required). The major advantage of this method is the very high accuracy obtained and the easy adaptation for American and Barrier type options.

The final approach uses traditional quadrature rules such as the trapezoid and Simpson's rule. Using a similar set up to that of the previous technique, a quadrature scheme is formed to represent the derivative security price at each time step. The rules used show that accurate results can be found in relatively quick time. Issues as those that have arisen in the previous approach such as node allocation also exists in this approach. The quadrature rules can also be easily applied to American and Barrier type options.

The thesis is a numerical investigation of the path integral framework. The thesis will emphasise the performance and accuracy of each of the methods for the framework and particular parameters. Trade offs between accuracy and computational effort are addressed. The ease of implementation (in the case of the European options) allows an insight into the behaviour and performance of the method for the path integral framework and more complex options like, American put and down and out call options.

Chapter 2

The Black and Scholes Paradigm

This chapter shows the evolution of the Black & Scholes (1973) paradigm. It begins with the major assumptions in which a derivative security like an option is modeled and priced. We present the formulation of the Black and Scholes equation (a partial differential equation) using a replicating portfolio. In deriving the Black and Scholes equation, a formula is presented for both a European Call and a Put option. Finally, the development of the Chiarella et al. (1999) path integral is presented, which is constructed based on the Black and Scholes paradigm.

2.1 Introduction

Prior to presenting the path integral framework used in this thesis for option pricing, an understanding of the Black & Scholes (1973) paradigm is required. Since many option pricing models are based on this paradigm, the chapter will describe the fundamentals of the assumptions, equations and the derivation of the formula. We initially present the major assumptions on which a model using the paradigm must satisfy. There are many assumptions which exist and continue to be used since the creation of the Black and Scholes formula well over three decades ago.

Following the assumptions, we present a summary version of the creation of the Black and Scholes equation (a partial differential equation) using a *replicating portfolio*. The presented method is based on that in Wilmott (1999). The partial differential equation (pde), is derived using a portfolio containing a long position in the option and a short position in a quantity of the underlying. The portfolio is replicating because it changes continuously with respect to time and a change in value of the underlying. The pde is also derived using common financial principles of delta hedging and no arbitrage.

We finally present the formulation of the path integral framework based on Chiarella et al. (1999). This is the framework which is central to this thesis. The framework developed uses the assumptions and ideas described in this paradigm. The framework is built based on the technique of path integrals in statistical physics.

2.2 The Black-Scholes Assumptions

Understanding of the modeling of an option price based on the Black and Scholes paradigm, requires a list of assumptions and conditions to be followed. Since the creation of the Black and Scholes formula over three decades ago, these assumptions have extended to cater for the changing evolution of the financial world. Here are a list of the major assumptions in the Black and Scholes paradigm.

1. The underlying asset follows a log-normal random walk and the variance is known and constant.

2. The risk-free interest rate is a known function of time.

3. The underlying pays no dividends and is fungible.

4. Options can only be exercised at Maturity (Vanilla Options).

5. There are no transaction costs.

6. There are no arbitrage opportunities.

7. An investor can borrow any amount of money to purchase the security, at the short-term interest rate.

8. There is no credit risk.

There are other assumptions which can be included in the paradigm. Given these assumptions, a Black and Scholes model can be created.

2.3 Replicating Portfolio

We can develop the Black and Scholes equation (a partial differential equation) by creating a portfolio of one long position in the option and a short position in a quantity of the underlying. If we denote the option price as $V(S,t)$, the quantity Δ of the underlying S, then the value of the portfolio is given by

$$\Pi = V(S,t) - \Delta S, \tag{2.1}$$

where S is the value of the underlying and t is time. If we assume that the underlying follows a log-normal random walk

$$dS = \mu S dt + \sigma S dW, \tag{2.2}$$

where μ is the drift parameter, σ is the volatility associated with the underlying and W represents the Brownian motion. For a thorough investigation of Brownian motion, we refer the reader to Chapter 3.3 in Ross (2003). A portfolio value therefore changes with respect to time

$$d\Pi = dV - \Delta dS. \tag{2.3}$$

We now introduce Ito's Lemma, the reader is referred to Wilmott, Dewynne & Howison (2000) for a thorough investigation of the Lemma (Chapter 2.3).

$$dV = \frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial S} dS + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} dt \tag{2.4}$$

and we substitute (2.4) into (2.3) so that the portfolio changes by

$$d\Pi = \frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial S} dS + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} dt - \Delta dS. \tag{2.5}$$

In (2.5) there are two terms which involve risk. They are $\frac{\partial V}{\partial S} dS$ and ΔdS. To eliminate this risk we let

$$\Delta = \frac{\partial V}{\partial S}. \tag{2.6}$$

This elimination is commonly known as *Delta Hedging* giving from (2.5) and (2.6)

$$d\Pi = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt, \tag{2.7}$$

where as a consequence of assumption (2.6), the change in the portfolio is now riskless.

Since the change in the portfolio value is risk free, it must earn the risk free rate of interest otherwise riskless arbitrage opportunities would exists. Namely

$$d\Pi = r\Pi dt. \tag{2.8}$$

Therefore, substituting (2.8) into (2.7) and using (2.1) and (2.6) gives

$$\left(\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}\right)dt = r\Pi dt$$

$$= r\big(V - S\frac{\partial V}{\partial S}\big)dt \tag{2.9}$$

and with the rearrangement of (2.9) gives

$$\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}+rS\frac{\partial V}{\partial S}-rV=0. \tag{2.10}$$

The partial differential equation (2.10) is the Black and Scholes equation that is commonly referred to in the literature.

The pde (2.10) is of a parabolic form, which are usually called diffusion equations. These equations have been used to model many areas of science. The simplest form of the diffusion equation is

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}, \tag{2.11}$$

which is the heat equation for the temperature in a bar. In (2.11), u is the temperature, x is the spatial coordinate and t is time. This is achievable by changing variables in (2.10) to reduce to the form of (2.11).

2.4 The Black-Scholes Formula

In this thesis, the techniques used, are initially applied to the European call and put options and so, a presentation of how the Black and Scholes formula (for a European call and put option) is derived from (2.10) follows. The boundary conditions will determine the type of options to be considered. The derivation of the Black and Scholes formula will also assist in the understanding of the derivation of the path integral of Chiarella et al. (1999).

Equation (2.10) is a backward equation since we are valuing an option for an underlying with some future value at time T. Discounting for interest rate, r gives

$$V(S,t) = e^{-r(T-t)}U(S,t) \tag{2.12}$$

which upon substitution into (2.10) gives

$$\frac{\partial U}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 U}{\partial S^2} + rS\frac{\partial U}{\partial S} = 0. \tag{2.13}$$

With the substitutions,

$$\tau = T - t, \quad z = \ln(S), \quad \frac{\partial}{\partial S} = e^{-z}\frac{\partial}{\partial z} \quad and \quad \frac{\partial^2}{\partial S^2} = e^{-2z}\frac{\partial^2}{\partial z^2} - e^{-2z}\frac{\partial}{\partial z},$$

in (2.13) gives, after some algebra,

$$\frac{\partial U}{\partial \tau} = \frac{1}{2}\sigma^2\frac{\partial^2 U}{\partial z^2} + (r - \frac{1}{2}\sigma^2)\frac{\partial U}{\partial z}. \tag{2.14}$$

A final substitution

$$x = z + (r - \frac{1}{2}\sigma^2)\tau \quad and \quad U = W(x,t)$$

reduces the Black and Scholes equation (2.14) to,

$$\frac{\partial W}{\partial \tau} = \frac{1}{2}\sigma^2\frac{\partial^2 W}{\partial x^2}. \tag{2.15}$$

This simplified version of the Black and Scholes equation, is in a similar form to the diffusion (heat) equation (2.11).

At this point we direct the reader to Wilmott (1999) for a step by step solution to (2.15). The solution offered by Wilmott (1999) is,

$$V(S,t) = \frac{e^{-r(T-t)}}{\sigma\sqrt{2\pi(T-t)}}\int_0^\infty e^{-(\ln(\frac{S}{S^*}) + (r-\frac{1}{2}\sigma^2)(T-t))^2/2\sigma^2(T-t)}P(S^*)\frac{dS^*}{S^*} \tag{2.16}$$

where S^* is an arbitrary constant and $P(S^*)$ is the payoff function for various options which can be applied in a fairly straight forward fashion. The payoff function $P(S^*)$, is the boundary condition, which varies depending on the type of option being considered.

2.4.1 European Call and Put Option

For a European call option, $E_c(S,t)$, with a payoff function

$$P(S) = (S - E, 0)^+ = max(S - E, 0), \tag{2.17}$$

where E is the strike price. Equation (2.16) is re-written as

$$E_c(S,t) = \frac{e^{-r(T-t)}}{\sigma\sqrt{2\pi(T-t)}} \int_E^\infty e^{-(\ln(\frac{S}{S^*}) + (r - \frac{1}{2}\sigma^2)(T-t))^2/2\sigma^2(T-t)} (S^* - E)\frac{dS^*}{S^*}, \tag{2.18}$$

and using the change of variable $x^* = \ln(S^*)$, (2.18) becomes

$$E_c(S,t) = \frac{e^{-r(T-t)}}{\sigma\sqrt{2\pi(T-t)}} \int_{\ln(E)}^\infty e^{-(-x^* + \ln(S) + (r - \frac{1}{2}\sigma^2)(T-t))^2/2\sigma^2(T-t)} (e^{x^*} - E) dx^*,$$

$$= \frac{e^{-r\Delta t}}{\sigma\sqrt{2\pi\Delta t}} \int_{\ln(E)}^\infty e^{-(-x^* + \ln(S) + (r - \frac{1}{2}\sigma^2)\Delta t)^2/2\sigma^2\Delta t} e^{x^*} dx^*$$

$$- \frac{Ee^{-r\Delta t}}{\sigma\sqrt{2\pi\Delta t}} \int_{\ln(E)}^\infty e^{-(-x^* + \ln(S) + (r - \frac{1}{2}\sigma^2)\Delta t)^2/2\sigma^2\Delta t} dx^*, \tag{2.19}$$

where $\Delta t = T - t$. Since the two integrals in (2.19) are in the form

$$\frac{w}{\sqrt{2\pi}} \int_d^\infty e^{-\frac{1}{2}x^2} dx, \tag{2.20}$$

the European call option price can be written in terms of the cumulative distribution frequency of the Normal Distribution, $N(.)$. Namely,

$$E_c = SN(d_1) - Ee^{-r\Delta t}N(d_2), \tag{2.21}$$

where

$$d_1 = \frac{\ln\left(\frac{S}{E}\right) + (r + \frac{1}{2}\sigma^2)\Delta t}{\sigma\sqrt{\Delta t}}, \tag{2.22}$$

and

$$d_2 = \frac{\ln\left(\frac{S}{E}\right) + (r - \frac{1}{2}\sigma^2)\Delta t}{\sigma\sqrt{\Delta t}}$$

$$= d_1 - \sigma\sqrt{\Delta t}. \tag{2.23}$$

The European put option, $E_p(S,t)$, is similarly derived using the following payoff function,

$$P(S) = (E - S, 0)^+ = max(E - S, 0). \tag{2.24}$$

Since the payoff for a European put is $E - S$, the above steps are used to find,

$$E_p = -SN(-d_1) + Ee^{-r\Delta t}N(-d_2), \tag{2.25}$$

where d_1 and d_2 are as given in (2.22) and (2.23).
So, this is the derivation of the Black and Scholes formula using the various assumptions and a partial differential equation formed using a replicating portfolio. To finalise this chapter on the Black and Scholes paradigm, we will look at the pricing problem in a path integral framework.

2.5 Path Integral Framework

So far in this chapter we have given a presentation of the Black and Scholes paradigm and the development of their equation and formula. The investigation is a good stepping stone in understanding the motivation of the thesis. The proceeding chapters present some of the previous work and introduce new techniques to the pricing of options in a path integral framework. The path integral framework developed is based on the Black and Scholes paradigm and uses some of the ideas presented so far in this chapter.

The path integral used in this thesis was developed by Chiarella et al. (1999). The derivative security or option price $f(x,t)$ is given by a Feynman-Kac formula

$$f(x_0,t_0) = e^{-r(T-t_0)} E_{t_0}[g(x_T)] \tag{2.26}$$

where x is the log of the underlying, t is time, T is the expiry date, $g(.)$ is the payoff function and E_{t_0} is the expectation at t_0, generated by

$$dx = rdt + \sigma dW(t) \tag{2.27}$$

where $W(t)$ is standard Brownian motion.

Since E_{t_0} is the expectation with respect to the transition probability distribution function from (x_0,t_0) to (x,T), represented by $\pi(x,T|x_0,t_0)$, (2.26) becomes

$$f(x_0,t_0) = e^{-r(T-t_0)} \int g(x)\pi(x,T|x_0,t_0)dx. \tag{2.28}$$

The interval (t_0,T) can be subdivided into K intervals, $t_0,t_1,\ldots,t_{k-1},t_k,\ldots,t_K$, so that (2.28) can be related to the option price over the subinterval t_{k-1} to t_k, namely

$$f(x_{k-1},t_{k-1}) = e^{-r(t_k-t_{k-1})} \int f(x_k,t_k)\pi(x_k,t_k|x_{k-1},t_{k-1})dx_k. \tag{2.29}$$

At this point Chiarella et al. (1999) observe that time has been discretized but the price dependence is continuous and so (2.29) can be rewritten as

$$f^{k-1}(x_{k-1}) = e^{-r(t_k-t_{k-1})} \int f^k(x_k)\pi(x_k,t_k|x_{k-1},t_{k-1})dx_k, \tag{2.30}$$

with time dependence of f denoted by the superscript k. Since $\pi(x_k,t_k|x_{k-1},t_{k-1})$ satisfies the Chapman-Kolmogorov equation, a repeated Chapman-Kolmogorov equation is used in Chiarella et al. (1999) namely,

$$\pi(x_n,t_n|x_1,t_1) = \int \pi(x_n,t_n|x_2,t_2)\pi(x_2,t_2|x_1,t_1)dx_2, \tag{2.31}$$

to formulate the expectation as a path integral. Therefore, (2.28) is transformed to a path integral by multiple use of the Chapman-Kolmogorov equation. As stated earlier, the time interval t_0 to T is subdivided into K intervals of length $\Delta t = (T - t_0)/K$, with $t_k = t_0 + k\Delta t$ and the transition probabilities become

$$\begin{aligned} \pi(x_K, t_K | x_0, t_0) = & \int dx_{K-1} \int dx_{K-2} \ldots \int dx_{K-k} \ldots \int dx_2 \int dx_1 \\ & \times \pi(x_K, t_K | x_{K-1}, t_{K-1}) \times \pi(x_{K-1}, t_{K-1} | x_{K-2}, t_{K-2}) \ldots \\ & \times \pi(x_{K-k}, t_{K-k} | x_{K-(k+1)}, t_{K-(k+1)}) \ldots \\ & \times \pi(x_2, t_2 | x_1, t_1) \times \pi(x_1, t_1 | x_0, t_0). \end{aligned} \tag{2.32}$$

For small time intervals, Δt, the transition probability density for (2.27) is approximated by a normal density so that,

$$\pi(x, t + \Delta t | x^*, t) = \frac{1}{\sqrt{2\pi\sigma^2 \Delta t}} e^{-((x - x^*) - r\Delta t)^2 / 2\sigma^2 \Delta t}. \tag{2.33}$$

Substituting (2.33) into (2.32) gives

$$\begin{aligned} \pi(x_K, t_K | x_0, t_0) = & \frac{1}{\sqrt{2\pi\sigma^2 \Delta t}} \int_{K-1} \ldots \int e^{-((x_1 - x_0) - r\Delta t)^2 / 2\sigma^2 \Delta t} \\ & \times e^{-\lambda} \prod_{k=1}^{K-1} \frac{dx_k}{\sqrt{2\pi\sigma^2 \Delta t}} \end{aligned} \tag{2.34}$$

where,

$$\lambda = -\sum_{k=1}^{K-1} ((x_{k+1} - x_k) - r\Delta t)^2 / 2\sigma^2 \Delta t.$$

Equation (2.34) becomes the path integral expression for $\pi(x_K, t_K | x_0, t_0)$ as $n \to \infty$ in the limit of finite dimensional integrals, therefore, the option pricing formula becomes

$$\begin{aligned} f(x_0, t_0) &= e^{-r(t_K - t_0)} \int g(x_K)\pi(x_K, t_K|x_0, t_0)dx_K \\ &= e^{-r(t_K - t_0)} \int\int \ldots \int \pi(x_1, t_1|x_0, t_0) \times \pi(x_2, t_2|x_1, t_1)\ldots \\ &\times \pi(x_{K-1}, t_{K-1}|x_{K-2}, t_{K-2}) \times \pi(x_K, t_K|x_{K-1}, t_{K-1}) \\ &\times g(x_K)dx_K, dx_{K-1}\ldots dx_2 dx_1. \end{aligned} \tag{2.35}$$

Integrating successively with respect to each x_k, where $k = K, K-1, \ldots, 1$, equation (2.35), reduces to

$$f^{k-1}(x_{k-1}) = e^{-r\Delta t} \int_{-\infty}^{\infty} \pi(x_k, t_k|x_{k-1}, t_{k-1}) f^k(x_k) dx_k, \tag{2.36}$$

where $f^{k-1}(x_{k-1}) \equiv f(x_{k-1}, t_{k-1})$ and $f^K(x_K)$ denotes the payoff function $g(x_K)$.

Given that the underlying S follows a geometric Brownian motion represented by

$$dS = rSdt + \sigma SdW \tag{2.37}$$

for $0 \leq t \leq T$ and σ is a constant volatility, Chiarella et al. (1999) transforms (2.37) to incorporate unit diffusion coefficient and an infinite interval.

Firstly, the underlying is normalised by the exercise price, namely $S \equiv S/X$ and with

$$\xi = \int \frac{1}{\sigma S} dS = \frac{1}{\sigma} \ln(S). \tag{2.38}$$

giving a representation of the underlying on an infinite interval. On applying Ito's Lemma,

$$d\xi = \frac{1}{\sigma}(r - \frac{1}{2}\sigma^2)dt + dW(t), \tag{2.39}$$

ξ can now be written as a time dependent variable, namely

$$\xi_t = \xi_0 + \frac{1}{\sigma}(r - \frac{1}{2}\sigma^2)t + W_t, \tag{2.40}$$

from which we conclude that

$$\xi_t \sim N(\xi_0 + \frac{1}{\sigma}(r - \frac{1}{2}\sigma^2)t, t). \tag{2.41}$$

Using the fact that ξ_t is normally distributed, the transition probability density function (2.34) becomes

$$\pi(\xi_T, T|\xi_t, t) = \frac{1}{\sqrt{2\pi(T-t)}} e^{-((\xi_T - \sqrt{2(T-t)}\mu(\xi_t, T-t))^2/2(T-t))}, \tag{2.42}$$

where

$$\mu(\xi_t, T-t) = \frac{1}{\sqrt{2(T-t)}}\big[\xi_t + \frac{1}{\sigma}(r - \frac{1}{2}\sigma^2)(T-t)\big].$$

So, substituting (2.42) into (2.36) gives (1.3) namely,

$$f^{k-1}(\xi_{k-1}) = \frac{e^{-r\Delta t}}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-(\xi_k - \mu(\xi_{k-1}, \Delta t))^2} f^k(\sqrt{2\Delta t}\xi_k) d\xi_k, \tag{2.43}$$

where $\Delta t = t_k - t_{k-1}$ and $\mu(\xi_{k-1}, \Delta t)$ is given by (2.42).

This is the path integral that is going to be used throughout the thesis. The exponential in (2.43) is in a form very similar to properties associated with *Hermite orthogonal polynomials*. The path integral (2.43), has only an analytical solution at t_K (the first time step), when the payoff function, $f^K(\xi_K)$ is known. All subsequent time steps need to be solved using non-analytical methods.

As described in the previous chapter, Chiarella et al. (1999) use a Fourier-Hermite series expansion to represent the option pricing function $f^k(\xi_k)$. Using the recursive nature of this method, the option price, $f^0(\xi_0)$ is found by solving for the coefficients of the Fourier-Hermite series. Chapter 3 gives a presentation of the method and chapter 4 details a normalised version of the same technique.

The subsequent chapters utilise the same path integral (2.43) using interpolation polynomials and various quadrature rules. These methods offer an alternative to the

Fourier-Hermite method. These new methods are applied to a European, American and Barrier option.

Chapter 3

Fourier-Hermite Series Evaluation

The approach used in this chapter closely follows and is a summary of the method presented in Chiarella et al. (1999) which is crucial to the understanding of further developments in the current work. This spectral method is based on the particular form of the integrand of the appropriate path integral of the problem at hand. The method links the function representing the underlying by using a Fourier-Hermite series expansion, with the coefficients of the series from one time step linked to the coefficients of the next time step. The process is repeated until the final time step, at which stage the final option price is evaluated using a pricing polynomial.

3.1 Introduction

A presentation and analysis of the method and techniques used by Chiarella et al. (1999) is made in this chapter. The method presented involves the use of Hermite orthogonal polynomials and Fourier series.

The first part of the integrand in the path integral (2.43) (a weight function in the form of a Gaussian) is in a form appropriate to be applied to Hermite orthogonal polynomials and the properties associated with these types of polynomials. The aim of this method is to represent the underlying in a Fourier-Hermite series. As with most Fourier series, the objective is to find the coefficients of the series. These coefficients are determined by using the orthogonal properties of the Hermite polynomials. A set of recurrence relations are formed which are in turn used to explicitly evaluate the coefficients of the Fourier-Hermite series. The recurrence relations are expressed so that the coefficients of the polynomial, at the final step, is used to evaluate the option price.

One of the advantages of the method employed is that the underlying is represented by a polynomial. This allows for a set of option prices to be found for a set of model parameters. Most methods determine only a single option price whereas this spectral method allows for multiple prices. Also, approximating the hedge ratios are easily determined by differentiation, given the pricing polynomial approximation is smooth.

Section 3.2 introduces the Fourier-Hermite series as a representation of the underlying. With the use of various Hermite polynomial and their mathematical properties, a link between the coefficients is formed from one time step to the next. The link, in the form of a recurrence relation, is used to find the elements of a 2 dimensional matrix. This upper triangular matrix is used to modify the coefficients from one time step to the next, until the final option price can be evaluated.

In sub-sections 3.2.1 and 3.2.2, the relationship built in the previous section is applied to evaluate a call and put option price respectively. The implementation of both types of option prices differ due to the payoff function used at the expiry date of the option. Since the path integral is backward recursive, the payoff function is used firstly to allocate the first set of coefficients in the form of a vector. Given the difference in payoff functions, the initial coefficients will differ but all subsequent steps remain the same in the process of evaluating the option price.

Sub-section 3.2.3 will present a thorough analysis of the method applied to European options. The analysis will be based on the comparison of the Fourier-Hermite series expansion method with the analytical solutions obtained by Black and Scholes formula.

Section 3.3 investigates the same method applied to an American put option. The major difference is the path integral set-up. Since an American option can be exercised at any time during the life of the option, the integral is split into two parts. The two parts represents whether or not the option is being exercised. So, one of the issues in the evaluation of the option price is *where do we exercise*? In the implementation, the exercise component is derived using the payoff for a put option and the non-exercise part is similar to the European put, with the interval of integration being the difference. Sub-section 3.3.1 will present an analysis of the method applied to an American put option. The analysis will compare the results obtained by this method to those evaluated by a Binomial tree method. Some further results will also be presented in chapters 5 and 6.

3.2 European Options

The first derivative security price to be evaluated is a European option. A European option allows the holder the right (but not the obligation) to exercise the option at the final expiry date. The European option price therefore is evaluated depending on the time to expiry, T, the volatility of the underlying, σ and the short-term interest rate, r. The volatility and interest rate are constant throughout the life of the option.

The path integral, as presented in Chapter 2 for $k = K, K-1, \ldots, 1$, is given by

$$f^{k-1}(\xi_{k-1}) = \frac{e^{-r\Delta t}}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-(\xi_k - \mu(\xi_{k-1}))^2} f^k(\sqrt{2\Delta t}\, \xi_k)\, d\xi_k, \tag{3.1}$$

where

$$\mu(\xi_{k-1}) = \frac{\xi_{k-1} + b}{\sqrt{2\Delta t}}, \tag{3.2}$$

and

$$b = \frac{1}{\sigma}(r - \frac{1}{2}\sigma^2)\Delta t. \tag{3.3}$$

The aim of this spectral approach is to represent $f^{k-1}(\xi_{k-1})$ in a form that allows its use in subsequent time steps. The appropriateness of this method is due to the form of the exponential (Gaussian) in the integrand. Given the Gaussian form is similar to the weighting function of Hermite polynomials, the properties associated with these types of polynomials can be utilised.

Namely, the underlying, $f^{k-1}(\xi_{k-1})$, may be represented by a Fourier series of Hermite polynomials, with α_q^{k-1} being the coefficient of the Hermite polynomial term $H_q(\xi_{k-1})$. The series is fixed to a finite number of basis functions N. This representation, with the use of further substitutions and the properties of the Hermite polynomial, will form a set of polynomial representations for the underlying. So, the form of $f^{k-1}(\xi_{k-1})$ can be expressed as,

$$f^{k-1}(\xi_{k-1}) \backsimeq \sum_{q=0}^{N} \alpha_q^{k-1} H_q(\xi_{k-1}). \tag{3.4}$$

With the Hermite polynomial properties,

$$H_0(t) = 1, \quad H_1(t) = 2t, \tag{3.5}$$

$$H_n(\upsilon t+b)=2(\upsilon t+b)H_{n-1}(\upsilon t+b)-2(n-1)H_{n-2}(\upsilon t+b) \quad for \quad n>1, \tag{3.6}$$

$$\frac{d}{dt}H_n(\upsilon t+b)=2\upsilon nH_{n-1}(\upsilon t+b), \tag{3.7}$$

$$\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}H_0(t)e^{-t^2}\,dt=1, \tag{3.8}$$

$$\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}H_1(t)e^{-t^2}\,dt=0, \tag{3.9}$$

and

$$erfc(x)=\frac{2}{\sqrt{\pi}}\int_{x}^{\infty}e^{-t^2}\,dt=1-erf(x). \tag{3.10}$$

The use of the Hermite polynomials and various properties associated with them, allow an expression (recurrence relation) to be formed. This will enable the coefficients of one time step to be expressed in terms of the coefficients of the previous time step. The process begins with the allocation of the coefficients at the first time step. At this initial time step, the coefficients for $f^{K-1}(\xi_{K-1})$ are evaluated, with $f^K(\xi_K)$ being represented by the payoff function. The payoff function is in such a form that there is an analytic solution to the path integral (3.1) at this time step.

In the proceeding steps, a Fourier-Hermite series expansion for $f^k(\xi_k)$ in (3.1) will also be introduced to complete the expression. The coefficients found for $f^{K-1}(\xi_{K-1})$ are used to find the coefficients of the subsequent time steps until the coefficients of $f^0(\xi_0)$ are evaluated. The polynomial formed for $f^0(\xi_0)$ is the representation of the required option price. This polynomial can then be used to find any option price or hedge ratio.

To begin the process of determining the coefficients, (3.4) is substituted into (3.1), to produce,

$$\sum_{q=0}^{N} \alpha_q^{k-1} H_q(\xi_{k-1}) = \frac{e^{-r\Delta t}}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-(\xi_k - \mu(\xi_{k-1}))^2} f^k(\sqrt{2\Delta t}\, \xi_k)\, d\xi_k. \tag{3.11}$$

The aim is to determine the coefficients α_q^{k-1} in (3.11). Therefore, (3.11) needs transformation taking into consideration the Hermite polynomial $H_q(\xi_{k-1})$ and the form of the integrand.

To this end, the following orthogonality property of Hermite polynomials

$$m!2^m\sqrt{\pi} \int_{-\infty}^{\infty} e^{-t^2} H_m(t) H_n(t)\, dt = \delta_{m,n} = \begin{cases} 1, & m = n \\ 0, & otherwise \end{cases}, \tag{3.12}$$

may be utilised to determine the coefficients α_q^{k-1}. Thus, from (3.11) we have

$$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\xi_{k-1}^2} H_m(\xi_{k-1}) \sum_{q=0}^{N} \alpha_q^{k-1} H_q(\xi_{k-1})\, d\xi_{k-1}$$

$$= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\xi_{k-1}^2} H_m(\xi_{k-1}) \frac{e^{-r\Delta t}}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-(\xi_k - \mu(\xi_{k-1}))^2} f^k(\sqrt{2\Delta t}\, \xi_k)\, d\xi_k\, d\xi_{k-1},$$

and so the left hand side is further simplified by using property (3.12) to give

$$\alpha_m^{k-1} = \frac{e^{-r\Delta t}}{2^m m! \sqrt{\pi}} \int_{-\infty}^{\infty}$$

$$\left[\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-[(\xi_k - \mu(\xi_{k-1}))^2 + \xi_{k-1}^2]} H_m(\xi_{k-1})\, d\xi_{k-1} \right] f^k(\sqrt{2\Delta t}\, \xi_k)\, d\xi_k,$$

$$= \frac{e^{-r\Delta t}}{2^m m! \sqrt{\pi}} \int_{-\infty}^{\infty} f^k(\sqrt{2\Delta t}\, \xi_k) I_m(\xi_k)\, d\xi_k, \tag{3.13}$$

where,

Proposition 3.2.1. *The integrand term,*

$$I_m(\xi_k) = \frac{\sqrt{2\Delta t}\, e^{-(\frac{\sqrt{2\Delta t}\, \xi_k - b}{\upsilon})^2} H_m(\frac{\sqrt{2\Delta t}\, \xi_k - b}{\upsilon})}{\upsilon^{m+1}} \tag{3.14}$$

Proof. The analytical form (3.14) can be utilised so that the coefficients for the $(k-1)^{th}$ time step can be determined. This can be achieved by modifying (3.13), using the analytical form (3.14) and introducing a Fourier-Hermite series for time step k. This will create a relationship between the coefficients from one time step to the next. Given,

$$I_m(\xi_k) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-[(\xi_k - \mu(\xi_{k-1}))^2 + \xi_{k-1}^2]} H_m(\xi_{k-1})\, d\xi_{k-1}. \tag{3.15}$$

Finding the analytical form of (3.15) can be assisted by firstly completing the square of the index of the exponential within the integrand in (3.14). The reason for completing the square is to maintain the exponential within the integrand in a form suitable for the use of Hermite polynomials and their properties. This is achieved using some simple algebra (see A.1.1 for a step by step evaluation).

$$(\xi_k - \mu(\xi_{k-1}))^2 + \xi_{k-1}^2 = \left[\frac{\upsilon \xi_{k-1}}{\sqrt{2\Delta t}} - \frac{\sqrt{2\Delta t}\,\xi_k - b}{\upsilon\sqrt{2\Delta t}}\right]^2 + \left[\frac{\sqrt{2\Delta t}\,\xi_k - b}{\upsilon}\right]^2, \tag{3.16}$$

where $\mu(\xi_{k-1})$ and b are as defined by (3.2) and (3.3) respectively and

$$\upsilon = \sqrt{1 + 2\Delta t}. \tag{3.17}$$

Therefore, substituting (3.16) into (3.14) and rearranging gives

$$I_m(\xi_k) = \frac{e^{-(\frac{\sqrt{2\Delta t}\,\xi_k - b}{\upsilon})^2}}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-(\frac{\upsilon \xi_{k-1}}{\sqrt{2\Delta t}} - \frac{\sqrt{2\Delta t}\,\xi_k - b}{\upsilon\sqrt{2\Delta t}})^2} H_m(\xi_{k-1})\, d\xi_{k-1}, \tag{3.18}$$

where b is given by (3.3) and υ by (3.17).

To assist in obtaining an analytical form for I_m, the following substitution is required,

$$y = \frac{\upsilon\, \xi_{k-1}}{\sqrt{2\Delta t}},$$

and (3.18) is evaluated analytically as follows,

$$I_m(\xi_k) = \frac{e^{-(\frac{\sqrt{2\Delta t}\,\xi_k - b}{\upsilon})^2}}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-(y - \frac{\sqrt{2\Delta t}\,\xi_k - b}{\upsilon\sqrt{2\Delta t}})^2} H_m(\frac{y\sqrt{2\Delta t}}{\upsilon}) \, \frac{\sqrt{2\Delta t}}{\upsilon} \, dy,$$

$$= \frac{\sqrt{2\Delta t}\, e^{-(\frac{\sqrt{2\Delta t}\,\xi_k - b}{\upsilon})^2}}{\upsilon\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-(y - \frac{\sqrt{2\Delta t}\,\xi_k - b}{\upsilon\sqrt{2\Delta t}})^2} H_m(\frac{y\sqrt{2\Delta t}}{\upsilon}) \, dy,$$

$$= \frac{\sqrt{2\Delta t}\, e^{-(\frac{\sqrt{2\Delta t}\,\xi_k - b}{\upsilon})^2} H_m(\frac{\sqrt{2\Delta t}\,\xi_k - b}{\upsilon})}{\upsilon^{m+1}}, \tag{3.19}$$

$\square$

Since (3.14) is in an analytical form, (3.13) can be modified by substituting (3.14) and replacing $f^k(\xi_k)$ with a Fourier-Hermite series. This will transform (3.13) so that property (3.12) is applied, which allows in turn a recurrence relation to be obtained with $\boldsymbol{\alpha}^{k-1}$ expressed in terms of $\boldsymbol{\alpha}^k$. Given,

$$\alpha_m^{k-1} = \frac{e^{-r\Delta t}}{2^m m!\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{\sqrt{2\Delta t}\, e^{-(\frac{\sqrt{2\Delta t}\,\xi_k - b}{\upsilon})^2} H_m(\frac{\sqrt{2\Delta t}\,\xi_k - b}{\upsilon})}{\upsilon^{m+1}} f^k(\sqrt{2\Delta t}\,\xi_k) \, d\xi_k, \tag{3.20}$$

and by performing a further substitution,

$$z = \frac{\sqrt{2\Delta t}\,\xi_k - b}{\upsilon},$$

(3.20) is simplified to,

$$\alpha_m^{k-1} = \frac{e^{-r\Delta t}}{2^m m! \upsilon^m \sqrt{\pi}} \int_{-\infty}^{\infty} e^{-z^2} H_m(z) f^k(\upsilon z + b) \, dz. \tag{3.21}$$

The final step required to obtain a recurrence relation between $\boldsymbol{\alpha}^{k-1}$ and $\boldsymbol{\alpha}^k$, is to introduce a Fourier-Hermite series for $f^k(\upsilon z + b)$. This Fourier-Hermite series is similar to (3.4), the major difference being the coefficients are for time step k. Therefore,

$$f^k(\xi_k) \backsimeq \sum_{n=0}^{N} \alpha_n^k H_n(\xi_k), \tag{3.22}$$

and the Fourier-Hermite series (3.22) is substituted into (3.21). This will express $\boldsymbol{\alpha}^{k-1}$ in terms of $\boldsymbol{\alpha}^k$,

$$\alpha_m^{k-1} = \frac{e^{-r\Delta t}}{2^m m! \upsilon^m \sqrt{\pi}} \int_{-\infty}^{\infty} e^{-z^2} H_m(z) \sum_{n=0}^{N} \alpha_n^k H_n(\upsilon z + b)\, dz,$$

$$= \frac{e^{-r\Delta t}}{2^m m! \upsilon^m \sqrt{\pi}} \sum_{n=0}^{N} \alpha_n^k \int_{-\infty}^{\infty} e^{-z^2} H_m(z) H_n(\upsilon z + b)\, dz. \tag{3.23}$$

Proposition 3.2.2. *The coefficients α_m^{k-1} can be evaluated by the recurrence relation*

$$\alpha_m^{k-1} = e^{-r\Delta t} \sum_{n=0}^{N} \alpha_n^k A_{m,n}, \tag{3.24}$$

where,

$$A_{m,n} = \frac{1}{2^m m! \upsilon^m \sqrt{\pi}} \int_{-\infty}^{\infty} e^{-z^2} H_m(z) H_n(\upsilon z + b)\, dz, \tag{3.25}$$

with the following relations define the elements $A_{m,n}$,

$$\begin{aligned}
&A_{0,0} = 1 \qquad A_{0,1} = 2b, \\
&A_{0,n} = 2bA_{0,n-1} + 2(n-1)(\upsilon^2 - 1)A_{0,n-2}, \quad \textit{for } n = 2, 3, \ldots, N \\
&A_{m,n} = \frac{n}{m} A_{m-1,n-1}, \quad \textit{for} \quad m = 1, 2, \ldots, N \quad \textit{and} \quad n = 2, 3, \ldots, N, \\
&A_{m,n} = 0 \quad \textit{for } m > n.
\end{aligned} \tag{3.26}$$

Proof. Since (3.24) is an expression that links the $\boldsymbol{\alpha}$'s from time step k to $k-1$, a recurrence relation is built. This relationship is created by finding the elements of the 2 dimensional matrix $\mathbf{A}$ from (3.25). The elements, $A_{m,n}$ are in a similar form to (3.12).

To find the elements of matrix $\mathbf{A}$, the initial elements are required. The matrix $\mathbf{A}$ is used to modify the coefficients from one time step to the next. Therefore, the coefficients $\boldsymbol{\alpha}^{K-1}$ are multiplied by $\mathbf{A}$ to give, $\boldsymbol{\alpha}^{K-2}$. This process is repeated for the proceeding time steps until $\boldsymbol{\alpha}^0$ is found. So, prior to evaluating any coefficients, the elements of matrix $\mathbf{A}$ require determination.

To start with, the element $A_{0,0}$ is given by,

$$\begin{aligned} A_{0,0} &= \frac{1}{2^0 0! \upsilon^0 \sqrt{\pi}} \int_{-\infty}^{\infty} e^{-z^2} H_0(z) H_0(\upsilon z + b)\, dz, \\ &= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-z^2}\, dz, \end{aligned}$$

and therefore using (3.8), $A_{0,0} = 1$ as given in (3.26).

Element $A_{0,1}$ is given by,

$$\begin{aligned} A_{0,1} &= \frac{1}{2^0 0! \upsilon^0 \sqrt{\pi}} \int_{-\infty}^{\infty} e^{-z^2} H_0(z) H_1(\upsilon z + b)\, dz, \\ &= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-z^2} H_1(\upsilon z + b)\, dz, \\ &= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} 2ze^{-z^2}\, dz + \frac{2b}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-z^2}\, dz, \end{aligned} \tag{3.27}$$

with the first integral in (3.27) being in the form of (3.9) and the second integral in (3.27) is in the form of (3.8). So, $A_{0,1} = 2b$ as given in (3.26).

Given the elements $A_{0,0}$ and $A_{0,1}$, the subsequent elements $A_{0,n}$ are evaluated by,

$$\begin{aligned} A_{0,n} &= \frac{1}{2^0 0! \upsilon^0 \sqrt{\pi}} \int_{-\infty}^{\infty} e^{-z^2} H_0(z) H_n(\upsilon z + b)\, dz, \\ &= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-z^2} \Big[2(\upsilon z + b) H_{n-1}(\upsilon z + b) - 2(n-1) H_{n-2}(\upsilon z + b) \Big] dz, \\ &= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} 2\upsilon z e^{-z^2} H_{n-1}(\upsilon z + b)\, dz + 2bA_{0,n-1} - 2(n-1)A_{0,n-2}, \end{aligned} \tag{3.28}$$

where we have used (3.6) to transform $A_{0,n}$. The integral in (3.28) is evaluated using the property (3.7) and integration by parts, to give,

$$A_{0,n} = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \upsilon e^{-z^2} 2\upsilon(n-1) H_{n-2}(\upsilon z + b)\, dz + 2bA_{0,n-1} - 2(n-1)A_{0,n-2},$$

$$= 2\upsilon^2(n-1)A_{0,n-2} + 2bA_{0,n-1} - 2(n-1)A_{0,n-2},$$

and so

$$A_{0,n} = 2bA_{0,n-1} + 2(n-1)(\upsilon^2 - 1)A_{0,n-2} \quad \textit{for } n = 2, 3, \ldots, N. \tag{3.29}$$

The solution to the elements $A_{m,n}$ are derived using the Hermite polynomial properties (3.6) and (3.7). Also, to assist in the evaluation of elements $A_{m,n}$, the $H_m(\upsilon z+b)$ term in the integrand of (3.25) is replaced using (3.7). The reason for this replacement is to complement the method (integration by parts) of evaluation of $A_{m,n}$. A proof for elements $A_{m,n}$ can be found in appendix A.1.2. So,

$$A_{m,n} = \frac{1}{2^m m! \upsilon^m \sqrt{\pi}} \int_{-\infty}^{\infty} e^{-z^2} H_m(z) \left[\frac{d}{dz} \frac{1}{\upsilon} \frac{1}{2(n+1)} H_{n+1}(\upsilon z + b) \right] dz,$$

and using integration by parts, $A_{m,n}$ is transformed to,

$$A_{m,n} = \frac{1}{2^m m! \upsilon^m} \left[-\frac{1}{\upsilon} \frac{1}{2(n+1)} \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} H_{n+1}(\upsilon z + b) \left(\frac{d}{dz} e^{-z^2} H_m(z) \right) dz \right]. \tag{3.30}$$

The derivative in (3.30) can be solved using property (3.6) and the product rule. Therefore,

$$\left(\frac{d}{dz} e^{-z^2} H_m(z) \right) = 2m e^{-z^2} H_{m-1}(z) - 2z e^{-z^2} H_m(z),$$

$$= e^{-z^2} \left[2m H_{m-1}(z) - 2z H_m(z) \right],$$

$$= e^{-z^2} \left[-H_{m+1}(z) \right]. \tag{3.31}$$

So, to evaluate the element $A_{m,n}$, (3.31) is substituted into (3.30). Since (3.30) is expressed in a forward manner, rearrangement is required so that $A_{m,n}$ is expressed in terms of $A_{m-1,n-1}$. Therefore,

$$
\begin{aligned}
A_{m,n} &= \frac{1}{2^m m! \upsilon^m}\left[-\frac{1}{\upsilon}\frac{1}{2(n+1)}\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} H_{n+1}(\upsilon z+b)e^{-z^2}\Big(-H_{m+1}(z)\Big)\,dz\right],\\
&= \frac{1}{2^m m! \upsilon^m}\left[\frac{1}{\upsilon}\frac{1}{2(n+1)}\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-z^2}H_{m+1}(z)H_{n+1}(\upsilon z+b)\,dz\right],\\
&= \frac{1}{2^{m+1}(m+1)!\upsilon^{m+1}}\left[\frac{m+1}{n+1}\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-z^2}H_{m+1}(z)H_{n+1}(\upsilon z+b)\,dz\right],\\
&= \frac{m+1}{n+1}A_{m+1,n+1},
\end{aligned}
$$

and so

$$
A_{m+1,n+1} = \frac{n+1}{m+1}A_{m,n},
$$

giving

$$
A_{m,n} = \frac{n}{m}A_{m-1,n-1}. \tag{3.32}
$$

It must be noted that when $m > n$ elements $A_{m,n} = 0$.

□

The expression for $A_{0,n}$ and $A_{m,n}$ in (3.26) are different to those presented in Chiarella et al. (1999). Since there are no proofs in Chiarella et al. (1999) for the elements of matrix $\mathbf{A}$, it is difficult to ascertain where the differences have occurred.

The relation (3.25) gives the elements of an upper triangular matrix with leading diagonal elements being one. For particular model attributes, the elements of the Matrix $\mathbf{A}$ are evaluated by the relationships expressed. The next part of the process is to firstly find the values of $\boldsymbol{\alpha}^{K-1}$, since it is the first step in the backward recursive path integral framework. Using the values of the Matrix $\mathbf{A}$ and $\boldsymbol{\alpha}^{K-1}$, a recurrence relation can be found and as such, the value of a European call and put option price can be evaluated.

For each particular type of European option (call or put), the values of $\boldsymbol{\alpha}$ differ because of the payoff function. Therefore, the $\boldsymbol{\alpha}_m^{K-1}$ values will require separate evaluations. The next two sections will present the $\boldsymbol{\alpha}_m^{K-1}$ values for the call and put option respectively. These option pricing solutions are obtained by the following expression, which is derived from (3.24), and depending on the number of time steps K,

$$\boldsymbol{\alpha}^0 = e^{-r(K-1)\Delta t}\mathbf{A}^{K-1}\boldsymbol{\alpha}^{K-1}. \tag{3.33}$$

In implementing this method and taking into consideration the expression (3.33), the two major issues to ponder are the values of the matrix $\mathbf{A}$ and $\boldsymbol{\alpha}^{K-1}$. The coefficients of the option price polynomial ($\boldsymbol{\alpha}^0$) are determined by this matrix multiplication.

Since the elements of matrix $\mathbf{A}$ have been found in this section, the next two sections will describe the relationships to find the coefficients for the first time step $K-1$. These coefficients are determined using the payoff functions for a call or put option. Since the payoffs differ for each type of option, the coefficients are evaluated using different f^K and intervals of integration.

3.2.1 European Call Options

The next step in determining an option price using the spectral method, is to calculate the coefficients of the final time step ($\boldsymbol{\alpha}^0$). The values in $\boldsymbol{\alpha}^0$ are the coefficients of the option price polynomial (a Fourier-Hermite series). These coefficients are evaluated by (3.33). Since the elements of matrix $\mathbf{A}$ can be found using the relationships (3.26) determined from the previous section, the final requirement is to find the vector $\boldsymbol{\alpha}^{K-1}$.

To calculate the values of $\boldsymbol{\alpha}^{K-1}$, consideration must be given to the payoff function for the type of option being modeled. Therefore, for the first time step and recalling (3.13), namely

$$\alpha_m^{K-1} = \frac{e^{-r\Delta t}}{2^m m!\sqrt{\pi}} \int_{-\infty}^{\infty} f^K(\sqrt{2\Delta t}\,\xi_K) I_m(\xi_K)\, d\xi_K,$$

where the payoff function for a European Call option is given by

$$f^K(\xi_K) = (e^{\sigma\xi_K} - 1)^+, \tag{3.34}$$

then substituting the payoff function (3.34) into (3.13) gives

$$\alpha_m^{K-1} = \frac{e^{-r\Delta t}}{2^m m!\sqrt{\pi}} \int_{0}^{\infty} (e^{\sqrt{2\Delta t}\,\sigma\xi_K} - 1) I_m(\xi_K)\, d\xi_K, \tag{3.35}$$

where $I_m(\xi_K)$ is as given in (3.18) and explicitly in (3.19).

The interval of integration in (3.35) is now over $[0, +\infty)$ since the payoff only occurs for positive ξ_K. Given the form of (3.35), a recurrence relation will be created to link the coefficients of the first time step. The integral in (3.35) using (3.19) may be simplified to obtain an analytical form for the α_m^{K-1} for $m = 0$ and 1. All other subsequent values are derived by a recurrence relation linking α_m^{K-1} to α_{m-1}^{K-1}. So,

$$\begin{aligned}
\alpha_m^{K-1} =& \frac{e^{-r\Delta t}}{2^m m!\sqrt{\pi}} \left[\int_0^{\infty} e^{\sqrt{2\Delta t}\,\sigma\xi_K} I_m(\xi_K) d\xi_K - \int_0^{\infty} I_m(\xi_K)\, d\xi_K \right], \\
=& \frac{e^{-r\Delta t}}{2^m m!\sqrt{\pi}} \frac{\sqrt{2\Delta t}}{\upsilon^{m+1}} \left[\int_0^{\infty} e^{\sqrt{2\Delta t}\,\sigma\xi_K} e^{-(\frac{\sqrt{2\Delta t}\,\xi_K - b}{\upsilon})^2} H_m(\frac{\sqrt{2\Delta t}\,\xi_K - b}{\upsilon})\, d\xi_K \right. \\
& \left. - \int_0^{\infty} e^{-(\frac{\sqrt{2\Delta t}\,\xi_K - b}{\upsilon})^2} H_m(\frac{\sqrt{2\Delta t}\,\xi_K - b}{\upsilon})\, d\xi_K \right].
\end{aligned}$$

To simplify the above, the following substitution is required,

$$z = \frac{\sqrt{2\Delta t}\,\xi_K - b}{\upsilon},$$

which gives,

$$\alpha_m^{K-1} = \frac{e^{-r\Delta t}}{2^m m!\sqrt{\pi}} \frac{\sqrt{2\Delta t}}{\upsilon^{m+1}} \left[\int_{-\frac{b}{\upsilon}}^{\infty} e^{\sigma(\upsilon z+b)} e^{-z^2} H_m(z) \frac{\upsilon}{\sqrt{2\Delta t}}\, dz \right.$$
$$\left. - \int_{-\frac{b}{\upsilon}}^{\infty} e^{-z^2} H_m(z) \frac{\upsilon}{\sqrt{2\Delta t}}\, dz \right],$$

$$= \frac{e^{-r\Delta t}}{2^m m! \upsilon^m \sqrt{\pi}} \left[\int_{-\frac{b}{\upsilon}}^{\infty} e^{\sigma(\upsilon z+b)} e^{-z^2} H_m(z)\, dz - \int_{-\frac{b}{\upsilon}}^{\infty} e^{-z^2} H_m(z)\, dz \right],$$

$$= \frac{e^{-r\Delta t}}{2^m m! \upsilon^m \sqrt{\pi}} \left[e^{\sigma b} \int_{-\frac{b}{\upsilon}}^{\infty} e^{\sigma \upsilon z} e^{-z^2} H_m(z)\, dz - \int_{-\frac{b}{\upsilon}}^{\infty} e^{-z^2} H_m(z)\, dz \right]. \tag{3.36}$$

To find the values of α_m^{K-1}, redefining (3.36) to a neater form, with the following expressions will assist in the process of finding these values,

$$\Psi_m^c(-\frac{b}{\upsilon}) = \frac{1}{\sqrt{\pi}} \int_{-\frac{b}{\upsilon}}^{\infty} e^{\sigma \upsilon z} e^{-z^2} H_m(z)\, dz,$$

$$= \frac{e^{\frac{1}{4}\sigma^2 \upsilon^2}}{\sqrt{\pi}} \int_{-\frac{b}{\upsilon}}^{\infty} e^{-(z-\frac{\sigma \upsilon}{2})^2} H_m(z)\, dz, \tag{3.37}$$

and

$$\Omega_m^c(-\frac{b}{\upsilon}) = \frac{1}{\sqrt{\pi}} \int_{-\frac{b}{\upsilon}}^{\infty} e^{-z^2} H_m(z)\, dz. \tag{3.38}$$

The derivation of $\Psi_m^c(-\frac{b}{\upsilon})$ can be found in A.1.3. Therefore, (3.36) is redefined by,

$$\alpha_m^{K-1} = \frac{e^{-r\Delta t}}{2^m m! \upsilon^m} \left[e^{\sigma b} \Psi_m^c(-\frac{b}{\upsilon}) - \Omega_m^c(-\frac{b}{\upsilon}) \right]. \tag{3.39}$$

To determine all the values of α_m^{K-1}, a recurrence relationship for Ψ^c and Ω^c is built. In finding the relationships in Ψ^c and Ω^c, the various Hermite polynomial properties (3.5) - (3.10) described in the previous section are used. The relationship requires that the initial values are found first. It is clear that the analytical forms found for these initial terms are going to require attention when implemented.

Since the complementary error function,

$$erfc(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2}\, dt \tag{3.40}$$

is to be used, a suitable and accurate math library is needed. Computer algebra packages like ***Maple*** and ***Matlab*** have an extensive math library including the complementary error function. These packages also allows for very high accuracy. At this point it must be stated that the use of these computer packages are sometimes not the most efficient and quickest options in the implementation of this method. These packages are excellent to use for testing and for the accuracy of math library functions. However, due to the overheads associated with GUI and operating system constraints, speed of processing is decreased.

An alternative to using a computer algebra package, is to implement the method using a computer programming language like FORTRAN. This programming language, like others, only allow for double precision (16 digit accuracy) for the complementary error function (as well as various other math functions). Which means that accuracy is forsaken but speed of processing is increased markedly, since GUI is not as sophisticated.

Proposition 3.2.3. *So, to find the coefficients, $\boldsymbol{\alpha}^{K-1}$, the analytical form of Ψ^c and Ω^c are required. The proofs for Ψ^c and Ω^c can be found in Appendix A.1.3 and A.1.4.*

Since the Hermite polynomials are formed using a two term recurrence relation, Ψ_0^c and Ω_0^c require evaluation. Namely, they are given by (A.9) *and* (A.15) *as*

$$\Psi_0^c(-\frac{b}{\upsilon}) = \frac{e^{\frac{1}{4}\sigma^2\upsilon^2}}{\sqrt{\pi}} \int_{-\frac{b}{\upsilon}}^\infty e^{-(z-\frac{\sigma\upsilon}{2})^2} H_0(z)\, dz = \frac{e^{\frac{1}{4}\sigma^2\upsilon^2}}{2} erfc\big(-\frac{b}{\upsilon} - \frac{\sigma\upsilon}{2}\big), \tag{3.41}$$

and

$$\Omega_0^c(-\frac{b}{\upsilon}) = \frac{1}{\sqrt{\pi}} \int_{-\frac{b}{\upsilon}}^\infty e^{-z^2} H_0(z)\, dz = \frac{1}{2} erfc\big(-\frac{b}{\upsilon}\big). \tag{3.42}$$

The analytical form of Ψ_1^c and Ω_1^c are,

$$\Psi_1^c(-\frac{b}{\upsilon}) = e^{\frac{1}{4}\sigma^2\upsilon^2}\frac{\sigma\upsilon}{2}erfc\big(-\frac{b}{\upsilon} - \frac{\sigma\upsilon}{2}\big) + \frac{1}{\sqrt{\pi}}e^{-((\frac{b}{\upsilon})^2+\sigma b)}, \tag{3.43}$$

and

$$\Omega_1^c(-\frac{b}{\upsilon}) = \frac{1}{\sqrt{\pi}}\int_{-\frac{b}{\upsilon}}^{\infty} 2ze^{-z^2}\,dz = \frac{1}{\sqrt{\pi}}e^{-(\frac{b}{\upsilon})^2}. \tag{3.44}$$

The proofs for $\Psi_1^c(-\frac{b}{\upsilon})$ and $\Omega_1^c(-\frac{b}{\upsilon})$ are in Appendix A.1.3 and A.1.4 and are given by (A.13) *and* (A.17).

Finally, the general values $\Psi_m^c(-\frac{b}{\upsilon})$ and $\Omega_m^c(-\frac{b}{\upsilon})$ are evaluated, with proofs in Appendix A.1.3 and A.1.4 and given by (A.14) *and* (A.18) *namely,*

$$\begin{aligned}\Psi_m^c(-\frac{b}{\upsilon}) =& \frac{e^{\frac{1}{4}\sigma^2\upsilon^2}}{\sqrt{\pi}}\int_{-\infty}^{-\frac{b}{\upsilon}} e^{-(z-\frac{\sigma\upsilon}{2})^2}\Big[2zH_{m-1}(z) - 2(m-1)H_{m-2}(z)\Big]\,dz,\\ =&\Big[\sigma\upsilon\Psi_{m-1}^c(-\frac{b}{\upsilon}) - \frac{1}{\sqrt{\pi}}e^{-((\frac{b}{\upsilon})^2+\sigma b)}H_{m-1}(-\frac{b}{\upsilon})\Big],\end{aligned} \tag{3.45}$$

and

$$\begin{aligned}\Omega_m^c(-\frac{b}{\upsilon}) =& \frac{1}{\sqrt{\pi}}\int_{-\frac{b}{\upsilon}}^{\infty} e^{-z^2}\Big[2zH_{m-1}(z) - 2(m-1)H_{m-2}(z)\Big]\,dz,\\ =&\frac{1}{\sqrt{\pi}}e^{-(\frac{b}{\upsilon})^2}H_{m-1}(-\frac{b}{\upsilon}).\end{aligned} \tag{3.46}$$

Proposition 3.2.4. *Since we have solved the initial and general cases for Ψ^c and Ω^c, a recurrence relation for α_m^{K-1} can be formed, namely*

$$\alpha_m^{K-1} = \frac{\sigma}{2m}\Big[\frac{e^{-r\Delta t}}{2^{m-1}(m-1)!\upsilon^{m-1}\sqrt{\pi}}e^{-(\frac{b}{\upsilon})^2}H_{m-2}(-\frac{b}{\upsilon}) + \alpha_{m-1}^{K-1}\Big], \tag{3.47}$$

with

$$\alpha_0^{K-1} = \frac{e^{-r\Delta t}}{2}\Big[e^{\sigma b+\frac{1}{4}\sigma^2\upsilon^2}erfc(-\frac{b}{\upsilon} - \frac{\sigma\upsilon}{2}) - erfc(-\frac{b}{\upsilon})\Big], \tag{3.48}$$

and

$$\alpha_1^{K-1} = \frac{\sigma}{4} e^{-r\Delta t+\sigma b+\frac{1}{4}\sigma^2\upsilon^2} erfc(-\frac{b}{\upsilon} - \frac{\sigma\upsilon}{2}). \tag{3.49}$$

Proof. The elements α_m^{K-1} with $m = 1, 2, \ldots, N$ can be formed from (3.39) and using (3.45) and (3.46), viz

$$\begin{aligned}\alpha_m^{K-1} =& \frac{e^{-r\Delta t}}{2^m m! \upsilon^m}\Bigg[e^{\sigma b}\Bigg(\sigma\upsilon\Psi_{m-1}^c(-\frac{b}{\upsilon}) + \frac{1}{\sqrt{\pi}} e^{-((\frac{b}{\upsilon})^2+\sigma b)} H_{m-1}(-\frac{b}{\upsilon})\Bigg) \\ &- \frac{1}{\sqrt{\pi}} e^{-(\frac{b}{\upsilon})^2} H_{m-1}(-\frac{b}{\upsilon})\Bigg],\end{aligned} \tag{3.50}$$

and so

$$\alpha_m^{K-1} = \frac{e^{-r\Delta t}}{2^m m! \upsilon^m}\Bigg[\sigma\upsilon e^{\sigma b}\Psi_{m-1}^c(-\frac{b}{\upsilon})\Bigg]. \tag{3.51}$$

The next step is to find a relationship between α_m^{K-1} and α_{m-1}^{K-1}. Given (3.39) for coefficient $m-1$,

$$\alpha_{m-1}^{K-1} = \frac{e^{-r\Delta t}}{2^{m-1}(m-1)!\upsilon^{m-1}}\Bigg[e^{\sigma b}\Psi_{m-1}^c(-\frac{b}{\upsilon}) - \Omega_{m-1}^c(-\frac{b}{\upsilon})\Bigg],$$

therefore, rearranging α_{m-1}^{K-1} for,

$$e^{\sigma b}\Psi_{m-1}^c(-\frac{b}{\upsilon}) = \Bigg[\Omega_{m-1}^c(-\frac{b}{\upsilon}) + e^{r\Delta t}2^{m-1}(m-1)!\upsilon^{m-1}\alpha_{m-1}^{K-1}\Bigg], \tag{3.52}$$

and substituting (3.52) into (3.50) gives

$$
\begin{aligned}
\alpha_m^{K-1} =& \frac{e^{-r\Delta t}}{2^m m! \upsilon^m}\left[\sigma\upsilon\left(\Omega_{m-1}^c(-\frac{b}{\upsilon}) + e^{r\Delta t}\upsilon^{m-1}2^{m-1}(m-1)!\alpha_{m-1}^{K-1}\right)\right],\\
=& \frac{e^{-r\Delta t}}{2^m m! \upsilon^m}\left[\sigma\upsilon\Omega_{m-1}^c(-\frac{b}{\upsilon}) + \sigma\upsilon e^{r\Delta t}2^{m-1}(m-1)!\upsilon^{m-1}\alpha_{m-1}^{K-1}\right],\\
=& \sigma\left[\frac{e^{-r\Delta t}}{2^m m! \upsilon^{m-1}}\Omega_{m-1}^c(-\frac{b}{\upsilon}) + \frac{\alpha_{m-1}^{K-1}}{2m}\right],\\
=& \frac{\sigma}{2m}\left[\frac{e^{-r\Delta t}}{2^{m-1}(m-1)!\upsilon^{m-1}\sqrt{\pi}}e^{-(\frac{b}{\upsilon})^2}H_{m-2}(-\frac{b}{\upsilon}) + \alpha_{m-1}^{K-1}\right],
\end{aligned} \tag{3.53}
$$

where (3.46) has been used in the final step. So, (3.53) achieves a relationship between coefficient m and $m-1$. The expression (3.53) are the $\boldsymbol{\alpha}^{K-1}$ values for $m \geq 2$, with the following initial conditions,

$$
\begin{aligned}
\alpha_0^{K-1} =& e^{-r\Delta t}\left[e^{\sigma b}\Psi_0^c(-\frac{b}{\upsilon}) - \Omega_0^c(-\frac{b}{\upsilon})\right]\\
=& \frac{e^{-r\Delta t}}{2}\left[e^{\sigma b+\frac{1}{4}\sigma^2\upsilon^2}erfc(-\frac{b}{\upsilon}-\frac{\sigma\upsilon}{2}) - erfc(-\frac{b}{\upsilon})\right]
\end{aligned} \tag{3.54}
$$

and

$$
\begin{aligned}
\alpha_1^{K-1} =& \frac{e^{-r\Delta t}}{2\upsilon}\left[e^{\sigma b}\Psi_1^c(-\frac{b}{\upsilon}) - \Omega_1^c(-\frac{b}{\upsilon})\right],\\
=& \frac{e^{-r\Delta t}}{2\upsilon}\left[e^{\sigma b}\left(e^{\frac{1}{4}\sigma^2\upsilon^2}\frac{\sigma\upsilon}{2}\,erfc\big(-\frac{b}{\upsilon}-\frac{\sigma\upsilon}{2}\big) + \frac{1}{\sqrt{\pi}}e^{-\frac{1}{2}((\frac{b}{\upsilon})^2+2\sigma b)}\right),\right.\\
&\left. - \frac{1}{\sqrt{\pi}}e^{-(\frac{b}{\upsilon})^2}\right],\\
=& \frac{\sigma}{4}e^{-r\Delta t+\sigma b+\frac{1}{4}\sigma^2\upsilon^2}erfc(-\frac{b}{\upsilon}-\frac{\sigma\upsilon}{2}).
\end{aligned} \tag{3.55}
$$

□

So, with all values for $\boldsymbol{\alpha}^{K-1}$ determined, the European call option price can be found. The price is evaluated by solving (3.33). Subsection 3.2.3 will analyse the call option prices taking into consideration the number of basis functions and time steps used, along with the model parameters.

The next step is to evaluate the coefficients for a European put option.

3.2.2 European Put Options

To determine the recurrence relation for $\boldsymbol{\alpha}^{K-1}$ for a European put option, the mathematics and process involved are similar to that of the European call. The major difference is the payoff function for the put option namely,

$$f^K(\xi_K) = (1 - e^{\sigma\xi_K})^-. \tag{3.56}$$

The expression (3.13) differs due to the different payoff function for a put option for which the interval of integration is $(-\infty, 0]$. Therefore,

$$\alpha_m^{K-1} = \frac{e^{-r\Delta t}}{2^m m!\sqrt{\pi}} \int_{-\infty}^{0} (1 - e^{\sqrt{2\Delta t}\,\sigma\xi_K}) I_m(\xi_K)\, d\xi_K. \tag{3.57}$$

As with the European call option, substituting (3.19) into (3.57) gives

$$\alpha_m^{K-1} = \frac{e^{-r\Delta t}}{2^m m! \upsilon^m \sqrt{\pi}} \left[\int_{-\frac{b}{\upsilon}}^{-\infty} e^{-z^2} H_m(z)\, dz - e^{\sigma b} \int_{-\infty}^{-\frac{b}{\upsilon}} e^{\sigma\upsilon z} e^{-z^2} H_m(z)\, dz \right], \tag{3.58}$$

and redefining (3.58) with

$$\Psi_m^p(-\frac{b}{\upsilon}) = \frac{e^{\frac{1}{4}\sigma^2\upsilon^2}}{\sqrt{\pi}} \int_{-\infty}^{-\frac{b}{\upsilon}} e^{-(z-\frac{\sigma\upsilon}{2})^2} H_m(z)\, dz, \tag{3.59}$$

and

$$\Omega_m^p(-\frac{b}{\upsilon}) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{-\frac{b}{\upsilon}} e^{-z^2} H_m(z)\, dz. \tag{3.60}$$

where $\Psi_m^p(-\frac{b}{\upsilon})$ is in a similar form to $\Psi_m^c(-\frac{b}{\upsilon})$ in (3.37) and $\Omega_m^p(-\frac{b}{\upsilon})$ is similar to $\Omega_m^c(-\frac{b}{\upsilon})$ in (3.38). Therefore, (3.58) is defined by,

$$\alpha_m^{K-1} = \frac{e^{-r\Delta t}}{2^m m! \upsilon^m}\Big[\Omega_m^p(-\frac{b}{\upsilon}) - e^{\sigma b}\Psi_m^p(-\frac{b}{\upsilon})\Big]. \tag{3.61}$$

Similarly to the European call option, a recurrence relation is required. The only differences being the form of (3.61) and the interval of integration. The mathematics applied is similar, with use of the same properties and techniques. An alternative definition for the complementary function is also used in the evaluation of European put options. Namely

$$erfc(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-t^2}\, dt \tag{3.62}$$

Proposition 3.2.5. *The values of Ψ^p are,*

$$\Psi_0^p(-\frac{b}{\upsilon}) = \frac{e^{\frac{1}{4}\sigma^2\upsilon^2}}{2} erfc(\frac{b}{\upsilon} + \frac{\sigma\upsilon}{2}), \tag{3.63}$$

$$\Psi_1^p(-\frac{b}{\upsilon}) = \frac{\sigma\upsilon e^{\frac{1}{4}\sigma^2\upsilon^2}}{2} erfc(\frac{b}{\upsilon} + \frac{\sigma\upsilon}{2}) - \frac{1}{\sqrt{\pi}}e^{-((\frac{b}{\upsilon})^2+\sigma b)}, \tag{3.64}$$

$$\Psi_m^p(-\frac{b}{\upsilon}) = \sigma\upsilon\Psi_{m-1}^p(-\frac{b}{\upsilon}) - \frac{1}{\sqrt{\pi}}e^{-((\frac{b}{\upsilon})^2+\sigma b)}H_{m-1}(-\frac{b}{\upsilon}), \tag{3.65}$$

where the proofs for Ψ^p are found in A.1.5.

Proposition 3.2.6. *The values of Ω^p,*

$$\Omega_0^p(-\frac{b}{\upsilon}) = \frac{1}{2} erfc(\frac{b}{\upsilon}), \tag{3.66}$$

$$\Omega_1^p(-\frac{b}{\upsilon}) = -\frac{1}{\sqrt{\pi}}e^{-(\frac{b}{\upsilon})^2}, \tag{3.67}$$

$$\Omega_m^p(-\frac{b}{\upsilon}) = -\frac{1}{\sqrt{\pi}}e^{-(\frac{b}{\upsilon})^2}H_{m-1}(-\frac{b}{\upsilon}), \tag{3.68}$$

where the proofs for Ω^p are found in A.1.6.

Since the initial and general cases for Ψ^p and Ω^p have been solved, the following recurrence for α_m^{K-1} holds

$$\alpha_m^{K-1} = \frac{\sigma}{2m}\left[\frac{e^{-r\Delta t}}{2^{m-1}(m-1)!\upsilon^{m-1}\sqrt{\pi}}e^{-(\frac{b}{\upsilon})^2}H_{m-2}(-\frac{b}{\upsilon}) + \alpha_{m-1}^{K-1}\right] \tag{3.69}$$

for $m = 2, 3, \ldots N$, with initial conditions,

$$\alpha_0^{K-1} = \frac{e^{-r\Delta t}}{2}\left[erfc(\frac{b}{\upsilon}) - e^{\sigma b + \frac{1}{4}\sigma^2\upsilon^2}erfc(\frac{b}{\upsilon} + \frac{\sigma\upsilon}{2})\right], \tag{3.70}$$

$$\alpha_1^{K-1} = -\frac{\sigma}{4}e^{-r\Delta t + \sigma b + \frac{1}{4}\sigma^2\tau^2}erfc(\frac{b}{\upsilon} + \frac{\sigma\upsilon}{2}). \tag{3.71}$$

Given the values for $\boldsymbol{\alpha}^{K-1}$ and matrix $\mathbf{A}$, the option price polynomial is formed for both the put and call options. The following section will analyse the method. Consideration will be given to the issues which affect the accuracy and speed of evaluation of the option prices.

3.2.3 Results and Analysis

One of the advantages of the Fourier-Hermite expansion method is the explicit representation of the underlying. Since the path integral is based on the Black-Scholes paradigm, comparisons are easily evaluated. These comparisons can be made numerically and graphically.

Due to the underlying being represented by a polynomial (Fourier-Hermite series), the errors associated with this method will vary, due to the oscillatory nature of the Fourier-Hermite series, for different asset values. Figure 3.1 represents the Black-Scholes formula as a curve and the corresponding Fourier-Hermite expansion for a set of model parameters.

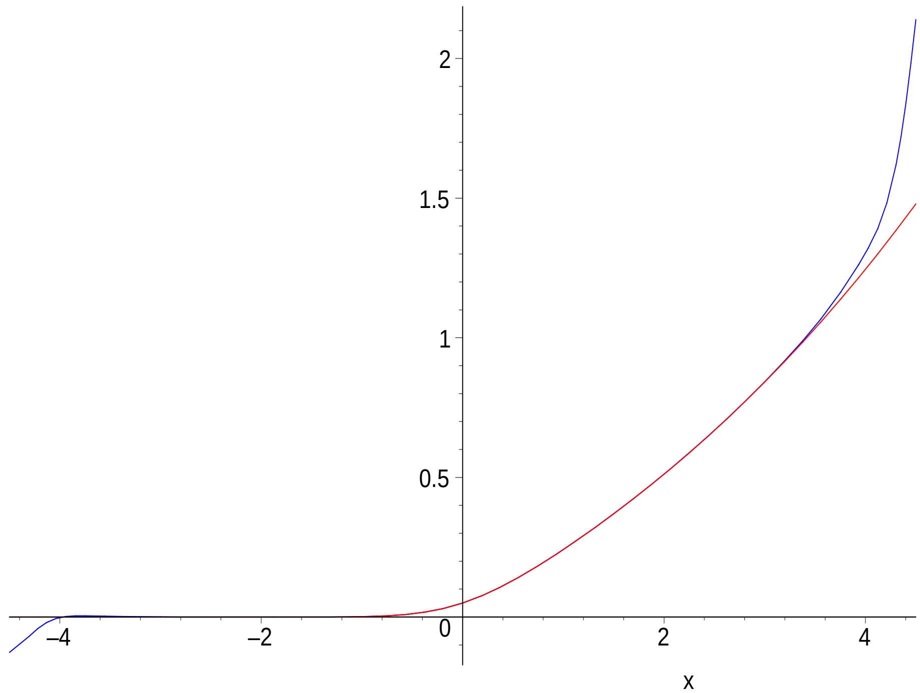

Figure 3.1: A Fourier-Hermite expansion (blue) and Black-Scholes formula (red) for a European call with $\sigma = 0.20$, $r = 0.08$, $T = 0.25$ and strike price, $X = \$100$. The Fourier-Hermite expansion was derived for 4 time steps and 32 basis functions.

The vertical axis is the option price and the horizontal axis represents the transformed variable ξ. Recalling that,

$$\xi = \frac{1}{\sigma} \ln (S),$$

where S is the normalised asset price (i.e. $S = \dfrac{AssetValue}{StrikePrice}$).

Figure 3.2 are the absolute errors when comparing the Fourier-Hermite expansion result to the Black-Scholes formula for a European call option.

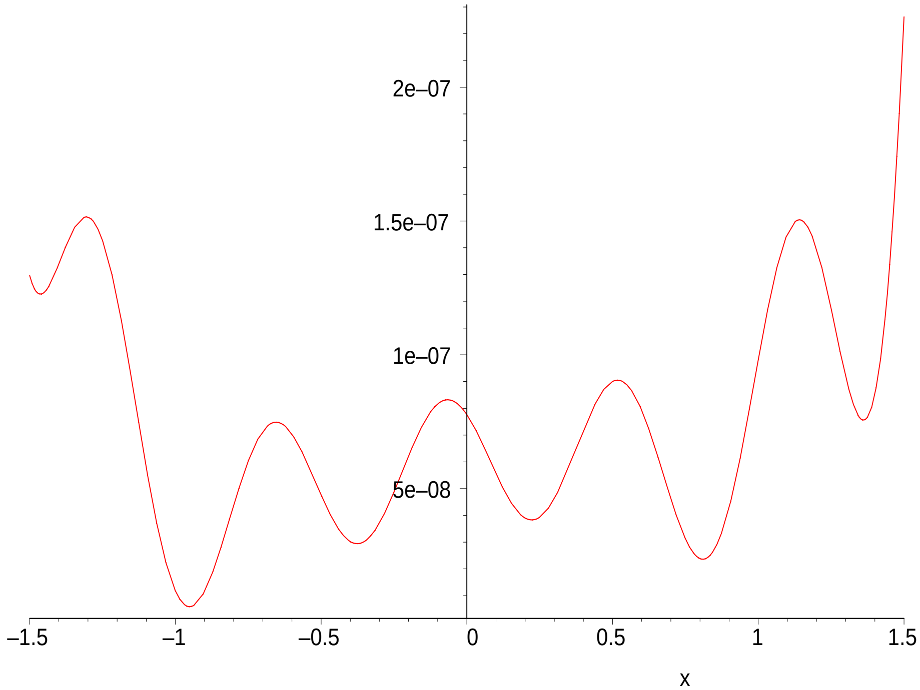

Figure 3.2: The absolute error of a Fourier-Hermite expansion vs Black-Scholes for a European call $K = 4$, $N = 32$, $\sigma = 0.20$, $r = 0.08$, $T = 0.25$ and strike price of \$100.

It is clear from figures 3.1 and 3.2 that as the asset price moves further away from the strike price, the Fourier-Hermite expansion method deteriorates, as expected. It must be also stated that the deterioration occurs for options that are rarely written. Figure 3.2 also shows the oscillating nature of the Fourier-Hermite series. As can be seen in figure 3.2, some asset values will give better approximations than others (refer to $-1.0 \leq \xi \leq -0.9$, compare to $\xi = 0$).

Table 3.1 shows a numerical representation of the Fourier-Hermite expansion method for a set of model parameters and 4 time steps. In the various tables presented in this thesis, the absolute error is used to measure the accuracy of the methods. The absolute error is calculated by evaluating the difference between the approximate value compared to the so called exact value.

Basis Functions	Asset Price ($)	Black-Scholes	Fourier-Hermite	Absolute Error
8	80	0.069017900	0.10042273	$3.14E{-2}$
16			0.068409100	$6.09E{-4}$
32			0.069023763	$5.86E{-6}$
64			0.068723077	$2.95E{-4}$
128			1.1449292	$1.08E0$
	90	1.0254530	1.0088962	$1.66E{-2}$
			1.0248461	$6.07E{-4}$
			1.0254590	$6.00E{-6}$
			1.0254570	$4.00E{-6}$
			1.0243229	$1.13E{-3}$
	100	5.0169820	5.0308067	$1.38E{-2}$
			5.0175159	$5.34E{-4}$
			5.0169880	$6.00E{-6}$
			5.0169860	$4.00E{-6}$
			5.0169575	$2.45E{-5}$
	110	12.620446	12.602872	$1.76E{-2}$
			12.620203	$2.43E{-4}$
			12.620456	$1.00E{-5}$
			12.620452	$6.00E{-6}$
			12.620058	$3.88E{-4}$
	120	22.066563	22.092199	$2.56E{-2}$
			22.066622	$5.90E{-5}$
			22.066564	$1.00E{-6}$
			22.066533	$3.00E{-5}$
			21.817455	$2.49E{-1}$

Table 3.1: Fourier-Hermite - European call option for 4 time steps, $\sigma = 0.20$, $r = 0.08$, $T = 0.25$, strike price of $100 and for various basis functions. Single precision was used to calculate the values.

Given the model parameters ($\sigma = 0.20$, $r = 0.08$, $T = 0.25$, strike of $100) and the number of time steps equal to 4, table 3.1 shows that approximately 32 to 64 basis functions gives the best results. Figure 3.3 shows the errors for three different expansions (varying basis functions).

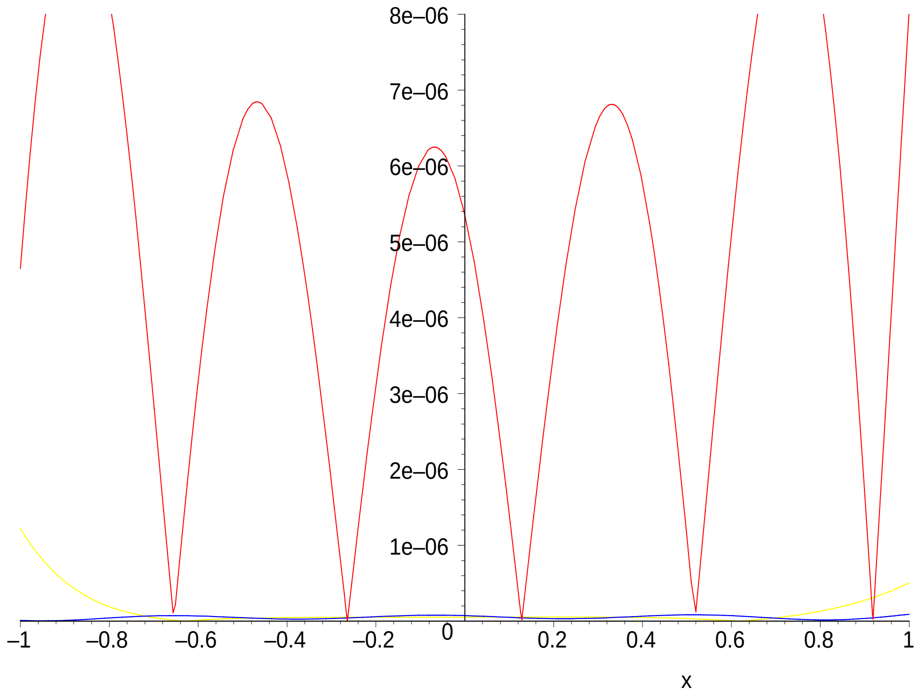

Figure 3.3: The absolute error of a Fourier-Hermite expansion vs Black-Scholes for a European call with $K = 4$, $N = 16$ (red), $N = 32$ (blue), $N = 64$ (yellow), $\sigma = 0.20$, $r = 0.08$, $T = 0.25$ and strike of \$100.

Figure 3.3 reinforces the earlier statement that as the asset price moves away from the strike price, the errors associated with Fourier-Hermite expansion method generally increase. Figures 3.2 and 3.3 also demonstrates the oscillatory nature of the series solution.

So far the analysis has looked at approximations using single precision (8 digits). By increasing the precision to double precision (16 digits) and given the form of the recurrence relations, a marked improvement is expected. Figure 3.4 shows the absolute errors for a particular expansion compared to the Black-Scholes formula.

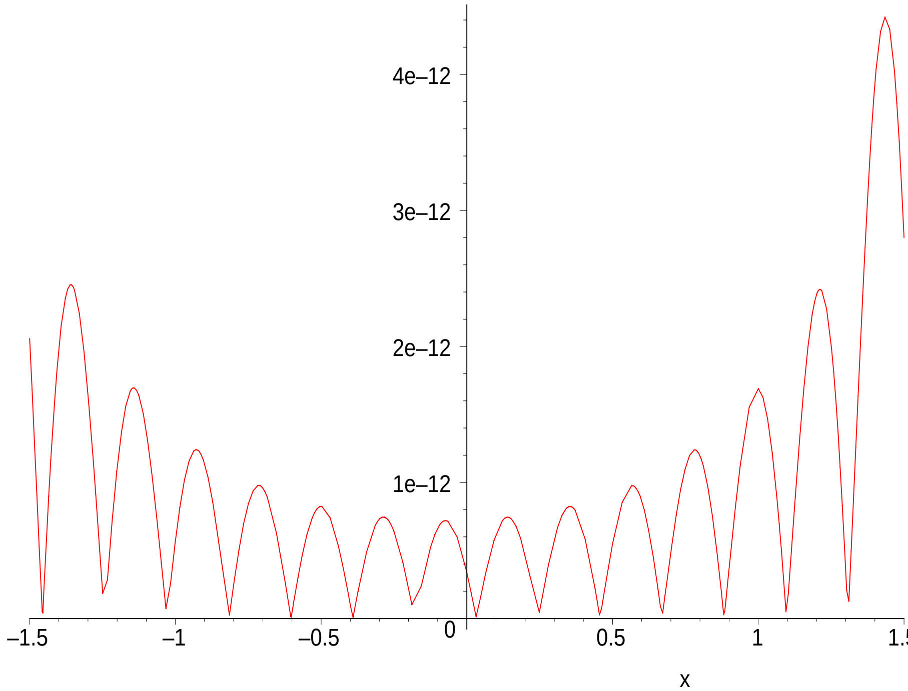

Figure 3.4: The absolute error of a Fourier-Hermite expansion vs Black-Scholes for a European call $K = 4$, $N = 64$, $\sigma = 0.20$, $r = 0.08$, $T = 0.25$ and strike of \$100. The comparison is made with double precision accuracy.

The errors pictured in figure 3.4 have the expansion within 10^{-12} of the actual (Black-Scholes) price. If we compare this to the data in table 3.1, there is an improvement of the order of one million. This improvement is achieved by just increasing the precision of the implementation. Figure 3.4 also shows the trend, as the asset price drifts away from the strike price, the approximation deteriorates. The magnitude of deterioration is relatively the same when comparing figures 3.2 and 3.4.

Table 3.2 presents some numerical results using the same model parameters as table 3.1, but is performed with double precision.

Basis Functions	**Asset Price ($)**	**Black-Scholes**	**Fourier-Hermite**	**Absolute Error**
8	80	0.06901773330119400	0.10041898548	$3.14E{-2}$
16			0.06840482653	$6.13E{-4}$
32			0.06901884376	$1.11E{-6}$
64			0.06901773314	$1.56E{-10}$
128			0.06901772269	$1.06E{-8}$
	90	1.025453734133940	1.008891303993	$1.66E{-2}$
			1.024841496754	$6.12E{-4}$
			1.025454269903	$5.36E{-7}$
			1.025453734209	$7.60E{-11}$
			1.025453734141	$8.02E{-12}$
	100	5.016980606262390	5.030800866969739	$1.38E{-2}$
			5.017510042390387	$5.29E{-4}$
			5.016982239341966	$1.63E{-6}$
			5.016980606296171	$3.38E{-11}$
			5.016980606262366	$2.40E{-14}$
	110	12.62044850198304	12.602865397914	$1.76E{-2}$
			12.620197548190	$2.51E{-4}$
			12.620451132365	$2.63E{-6}$
			12.620448501957	$2.58E{-11}$
			12.620448501979	$3.26E{-12}$
	120	22.06656020160710	22.09219252269	$2.56E{-2}$
			22.06661688376	$5.67E{-5}$
			22.06655832124	$1.88E{-6}$
			22.06656020154	$6.13E{-11}$
			22.06656020231	$7.32E{-10}$

Table 3.2: Fourier-Hermite - European call option for $K = 4$, $\sigma = 0.20$, $r = 0.08$, $T = 0.25$, strike of $100 and for various basis functions. Double precision was used to calculate the values.

Table 3.2 shows that the best results occur when 64 to 128 basis functions are used. Since the precision of the implementation has increased, the time taken to evaluate these prices increase. Also, the best prices in double precision seem to occur with more basis functions, this means that further computation is required.

In table 3.1 the best results occurred when 32 or 64 basis functions were used. If we compare the results for 8, 16 and 32 basis functions in tables 3.1 and 3.2, the errors obtained are slightly better in double precision. So, the question remains whether a small improvement in the evaluation of the option price is worth the extra computational time caused by an increase in the precision of implementation. It is clear though, with large number of basis function, that in double precision, prices are evaluated much more precisely.

Figure 3.5 graphs the absolute errors for various time step expansions, with the number of basis functions N fixed at 32.

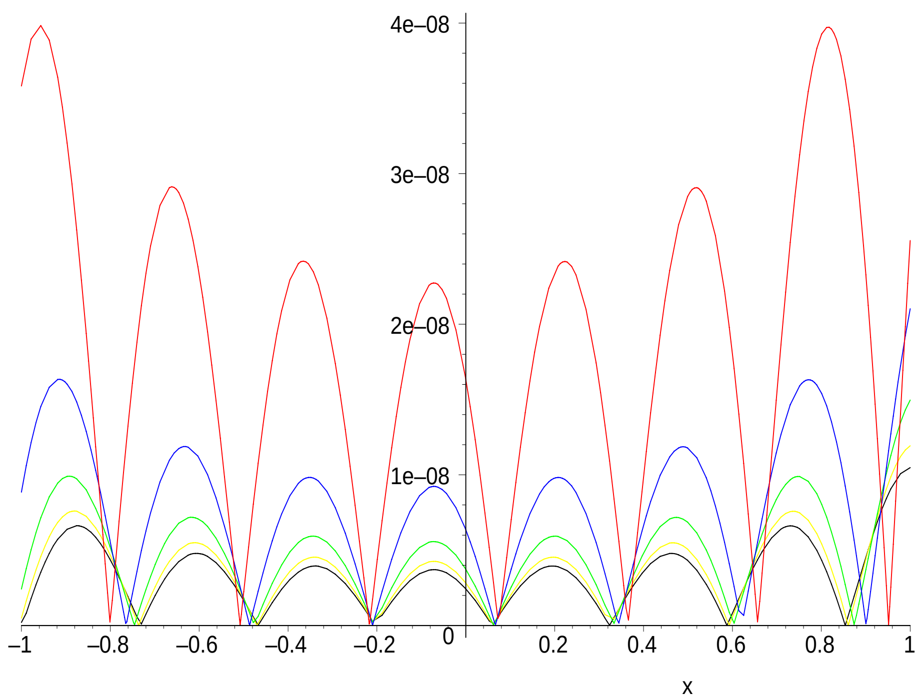

Figure 3.5: The absolute error of a Fourier-Hermite expansion vs Black-Scholes for a European call with $K = 4$ (red), $K = 8$ (blue), $K = 16$ (green), $K = 32$ (yellow), $K = 64$ (black), $N = 32$, $\sigma = 0.20$, $r = 0.08$, $T = 0.25$ and strike of \$100. The comparison is made with double precision accuracy.

Figure 3.5 shows that as the number of time steps increase, the errors tend to improve. However, it must be stated that some of the improvements are minor. It is also clear that as the number of time steps increase, so does the time taken to evaluate the option price. Therefore, one needs to determine whether the time required to obtain certain accuracy is beneficial.

Table 3.3 shows the prices for a European call with the strike price set to the asset value of \$100. The prices are for a varying number of time steps and 32 basis functions were used.

Time Steps	**Black-Scholes**	**Fourier-Hermite**	**Absolute Error**
4	5.016980606262390	5.016982239341966	1.63$E-6$
8		5.016981241524321	6.35$E-7$
16		5.016980979787151	3.74$E-7$
32		5.016980887803363	2.82$E-7$
64		5.016980849554562	2.43$E-7$
128		5.016980832150103	2.26$E-7$
256		5.016980823852529	2.18$E-7$

Table 3.3: Fourier-Hermite - European call option for 32 basis functions, $\sigma = 0.20$, $r = 0.08$, $T = 0.25$, asset price of \$100, strike of \$100 and for various time steps K. Double precision was used to calculate the values.

The data in table 3.3 reiterates the point that after 32 time steps, the improvement is marginal. Figure 3.6 also shows the absolute errors for various time step expansions, with the number of basis functions fixed at 64.

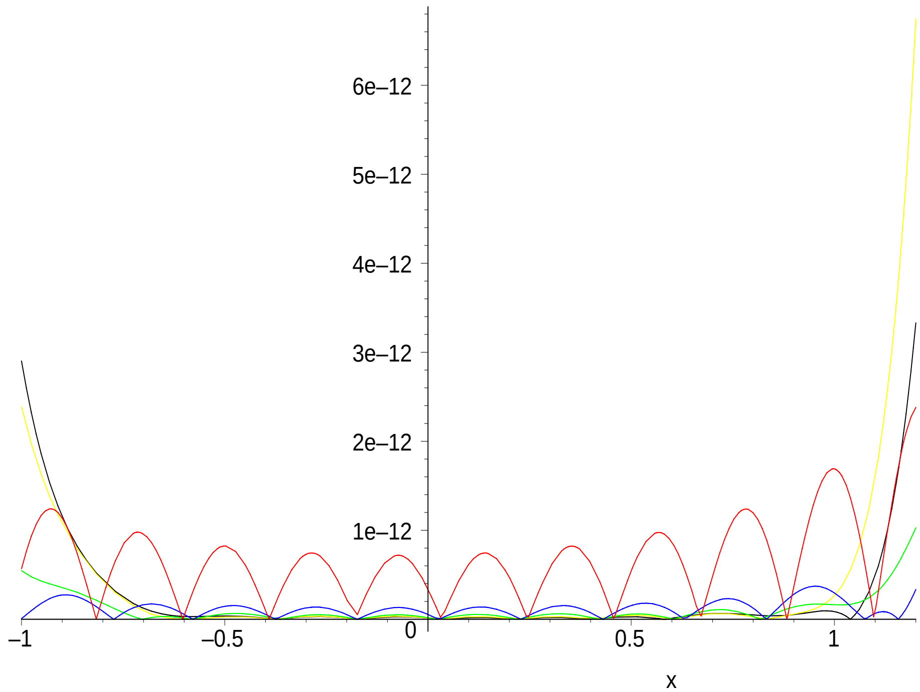

Figure 3.6: The absolute error of a Fourier-Hermite expansion vs Black-Scholes for a European call with $K = 4$ (red), $K = 8$ (blue), $K = 16$ (green), $K = 32$ (yellow), $K = 64$ (black), $N = 64$, $\sigma = 0.20$, $r = 0.08$, $T = 0.25$ and strike of \$100. The comparison is made with double precision accuracy.

Figure 3.6 shows similar patterns as those described previously. They include the peaks and troughs in the errors and the deterioration of the approximations as the asset prices move away from the strike price.

Table 3.4 shows the prices for a European call with the strike price set to the asset value of \$100. The prices are for varying number of time steps and 64 basis functions were used.

Time Steps	**Black-Scholes**	**Fourier-Hermite**	**Absolute Error**
4	5.016980606262390	5.016980606296171	$3.38E{-}11$
8		5.016980606267894	$5.50E{-}12$
16		5.016980606264234	$1.84E{-}12$
32		5.016980606263993	$1.60E{-}12$
64		5.016980606264063	$1.67E{-}12$
128		5.016980606262296	$9.40E{-}14$
256		5.016980606260944	$1.45E{-}12$

Table 3.4: Fourier-Hermite - European call option for 64 basis functions, $\sigma = 0.20$, $r = 0.08$, $T = 0.25$, asset price of \$100, strike of \$100 and for various time steps K. Double precision was used to calculate the values.

Figure 3.6 and table 3.4 also show that errors can improve with an increase in the number of time steps. However, in the case of $N = 64$ when ξ is less than -0.7, the absolute error for $K = 16$ is better than for $K = 32$ and $K = 64$. Table 3.4 shows small improvements as the number of time steps increase. The issue again arises whether the time taken to evaluate the price for the minimal improvement is worthwhile.

We now investigate the Fourier-Hermite expansion method for the evaluation of European put option prices. Figure 3.7 shows the errors for an expansion with 4 time steps and 64 basis functions.

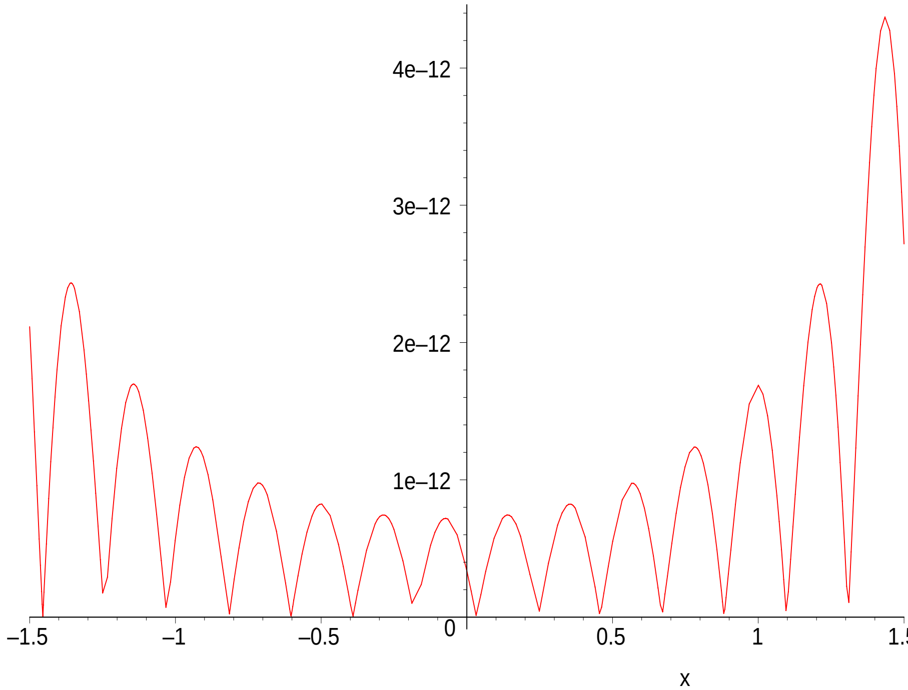

Figure 3.7: The absolute error of a Fourier-Hermite expansion vs Black-Scholes for a European put $K = 4$, $N = 64$, $\sigma = 0.20$, $r = 0.08$, $T = 0.25$ and strike of \$100. The comparison is made with double precision accuracy.

Table 3.5 shows the prices and errors for expansions of 4 time steps and varying number of basis functions.

Basis Functions	Asset Price ($)	Black-Scholes	Fourier-Hermite	Absolute Error
8	80	18.08888506397669	18.12028631642989	$3.14E{-2}$
16			18.08827215721178	$6.13E{-4}$
32			18.08888617444142	$1.11E{-6}$
64			18.08888506382045	$1.56E{-10}$
128			18.08888505394765	$1.00E{-8}$
	90	9.045321064809460	9.028758634532931	$1.66E{-2}$
			9.044708827429788	$6.12E{-4}$
			9.045321600578635	$5.36E{-7}$
			9.045321064885542	$7.61E{-11}$
			9.045321064820661	$1.12E{-11}$
	100	3.036847936937940	3.050668197707927	$1.38E{-2}$
			3.037377373065952	$5.29E{-4}$
			3.036849570017532	$1.63E{-6}$
			3.036847936971743	$3.38E{-11}$
			3.036847936937936	$4.00E{-15}$
	110	0.6403158326585500	0.6227327286612299	$1.76E{-2}$
			0.6400648788657842	$2.51E{-4}$
			0.6403184630408953	$2.63E{-6}$
			0.6403158326328689	$2.57E{-11}$
			0.6403158326560198	$2.53E{-12}$
	120	0.08642753228261400	0.1120598529337736	$2.56E{-2}$
			0.08648421444485786	$5.67E{-5}$
			0.08642565191656440	$1.88E{-6}$
			0.08642753222133237	$6.13E{-11}$
			0.08642753317810308	$8.95E{-10}$

Table 3.5: Fourier-Hermite - European put option for 4 time steps, $\sigma = 0.20$, $r = 0.08$, $T = 0.25$, strike of \$100 and for various basis functions. Double precision was used to calculate the values.

The absolute errors for the European put options in table 3.5 are very similar to the corresponding call option. There are some minor differences for prices evaluated using 128 basis functions. The similarity can be justified by analysing the payoff functions (3.34) and (3.56). Since the payoffs are similar in form, the only two differences in the evaluation in α_0^{K-1} and α_1^{K-1}. The recurrence relations (3.53) and (3.69) to find the other $\boldsymbol{\alpha}^{K-1}$ values are the same for a call and put option. The matrix $\mathbf{A}$ is the same for a call and put and the coefficients for each option, $\boldsymbol{\alpha}^0$ are evaluated using (3.33).

3.3 American Put Options

The American put option differs greatly to the European options presented so far. An American option allows the holder to exercise his/her right *at any time* during the life of the option. The major issue with an American option is, when is the best time to exercise?

In the path integral framework (3.1) presented previously, the interval of integration was over an infinite domain. For the American put, (3.1) will need to be partitioned to take into account the point (barrier) where the option is to be exercised. Therefore, the path integral will be split into two, with the first integral representing the payoff or early exercise area and the second being the non-exercise value of the American put option.

To help to distinguish the difference between the American put and the European option, (3.1) becomes

$$V^{k-1}(\xi_{k-1}) = \frac{e^{-r\Delta t}}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-(\xi_k - \mu(\xi_{k-1}))^2} F^k(\sqrt{2\Delta t}\,\xi_k)\,d\xi_k, \tag{3.72}$$

where V is the value of the American put option unexercised. If we denote F^{k-1} as the value of the option at time t_{k-1} and since an American put option can be exercised at any time, ξ^*_{k-1} is introduced to denote the optimal exercise point, then

$$F^{k-1}(\xi_{k-1}) = \begin{cases} V^{k-1}(\xi_{k-1}), & \xi^*_{k-1} < \xi_{k-1} < \infty \\ 1 - e^{\sigma \xi_{k-1}}, & -\infty < \xi_{k-1} < \xi^*_{k-1} \end{cases}, \tag{3.73}$$

Since the American put option can be exercised at any time, the path integral (3.72) is split into two parts,

$$V^{k-1}(\xi_{k-1}) = h^{k-1}(\xi_{k-1}) + \frac{e^{-r\Delta t}}{\sqrt{\pi}} \int_{\frac{\xi_k^*}{\sqrt{2\Delta t}}}^{\infty} e^{-(\xi_k - \mu(\xi_{k-1}))^2} V^k(\sqrt{2\Delta t}\,\xi_k)\, d\xi_k, \tag{3.74}$$

where

$$h^{k-1}(\xi_{k-1}) = \frac{e^{-r\Delta t}}{\sqrt{\pi}} \int_{-\infty}^{\frac{\xi_k^*}{\sqrt{2\Delta t}}} e^{-(\xi_k - \mu(\xi_{k-1}))^2} (1 - e^{\sigma\sqrt{2\Delta t}\,\xi_k})\, d\xi_k. \tag{3.75}$$

The integral in (3.75) is the payoff or early exercise component of the path integral (3.74). The early exercise point ξ_{k-1}^* is the value of ξ which solves

$$V^{k-1}(\xi) = 1 - e^{\sigma\xi}. \tag{3.76}$$

Equation (3.76) is an important part of evaluating the American put option price. At each time step the value of ξ is determined such that (3.76) is satisfied.

As with the path integral for European options, the following Fourier-Hermite series expansions are introduced,

$$V^{k-1}(\xi_{k-1}) \backsimeq \sum_{q=0}^{N} \alpha_q^{k-1} H_q(\xi_{k-1}), \tag{3.77}$$

and

$$h^{k-1}(\xi_{k-1}) \backsimeq \sum_{q=0}^{N} \gamma_q^{k-1} H_q(\xi_{k-1}). \tag{3.78}$$

With the American put option, the most appropriate manner in evaluating V^0 is to treat the two integrals separately. Once recurrence relations are determined for the coefficients of the Fourier-Hermite series, the two parts are joined for final evaluation of the American put option price. Therefore, substituting (3.77) into (3.74) gives

$$\sum_{q=0}^{N} \alpha_q^{k-1} H_q(\xi_{k-1}) = h^{k-1}(\xi_{k-1}) + \frac{e^{-r\Delta t}}{\sqrt{\pi}} \int_{\frac{\xi_k^*}{\sqrt{2\Delta t}}}^{\infty} e^{-(x_k - \mu(x_{k-1}))^2} V^k(\sqrt{2\Delta t}\xi_k)\, dx_k, \tag{3.79}$$

and substituting (3.78) into (3.75) gives

$$\sum_{q=0}^{N} \gamma_q^{k-1} H_q(\xi_{k-1}) = \frac{e^{-r\Delta t}}{\sqrt{\pi}} \int_{-\infty}^{\frac{\xi_k^*}{\sqrt{2\Delta t}}} e^{-(x_k-\mu(x_{k-1}))^2}(1 - e^{\sigma\sqrt{2\Delta t}\,\xi_k})\, dx_k. \tag{3.80}$$

So, (3.79) becomes

$$\begin{aligned} &\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\xi_{k-1}^2} H_m(\xi_{k-1}) \sum_{q=0}^{N} \alpha_q^{k-1} H_q(\xi_{k-1})\, d\xi_{k-1} \\ &= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\frac{\xi_k^*}{\sqrt{2\Delta t}}} e^{-\xi_{k-1}^2} H_m(\xi_{k-1}) \frac{e^{-r\Delta t}}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-(\xi_k-\mu(\xi_{k-1}))^2}(1 - e^{\sigma\sqrt{2\Delta t}\,\xi_k})\, d\xi_k\, d\xi_{k-1} \\ &+ \frac{1}{\sqrt{\pi}} \int_{\frac{\xi_k^*}{\sqrt{2\Delta t}}}^{\infty} e^{-\xi_{k-1}^2} H_m(\xi_{k-1}) \frac{e^{-r\Delta t}}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-(\xi_k-\mu(\xi_{k-1}))^2} V^k(\sqrt{2\Delta t}\,\xi_k)\, d\xi_k\, d\xi_{k-1}, \end{aligned} \tag{3.81}$$

Prior to forming the evaluation of $\boldsymbol{\alpha}^{k-1}$, the coefficients of $\boldsymbol{\gamma}^k$ require generation. The values of $\boldsymbol{\gamma}^k$ are found recursively, with $\boldsymbol{\gamma}^{K-1} = 0$ since the early exercise boundary is at 0 at the first time step. To assist in the evaluation of the elements in $\boldsymbol{\gamma}^k$, the following Hermite polynomial and their mathematical properties are used,

$$H_0(t) = 1, \quad H_1(t) = 2t, \tag{3.82}$$

$$H_n(vt+b) = 2(vt+b)H_{n-1}(vt+b) - 2(n-1)H_{n-2}(vt+b), \quad \text{for } n > 1, \tag{3.83}$$

$$\frac{d}{dt} H_n(vt+b) = 2vnH_{n-1}(vt+b), \tag{3.84}$$

$$\frac{1}{\sqrt{\pi}} \int_{x}^{\infty} H_1(t) e^{-t^2}\, dt = \frac{e^{-x^2}}{\sqrt{\pi}}, \tag{3.85}$$

$$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{x} H_1(t) e^{-t^2}\, dt = -\frac{e^{-x^2}}{\sqrt{\pi}}, \tag{3.86}$$

$$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} H_0(t) e^{-t^2}\, dt = 1, \tag{3.87}$$

and

$$erfc(x) = \frac{2}{\sqrt{\pi}} \int_{-\infty}^{-x} e^{-t^2}\, dt \tag{3.88}$$

So,

$$\begin{aligned}
&\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\xi_{k-1}^2} H_m(\xi_{k-1}) \sum_{q=0}^{N} \gamma_q^{k-1} H_q(\xi_{k-1})\, d\xi_{k-1} \\
&= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\frac{\xi_k^*}{\sqrt{2\Delta t}}} e^{-\xi_{k-1}^2} H_m(\xi_{k-1}) \frac{e^{-r\Delta t}}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-(\xi_k - \mu(\xi_{k-1}))^2} (1 - e^{\sigma\sqrt{2\Delta t}\,\xi_k})\, d\xi_k\, d\xi_{k-1},
\end{aligned}$$

which simplifies to

$$\begin{aligned}
\gamma_m^{k-1} =& \frac{e^{-r\Delta t}}{2^m m! \sqrt{\pi}} \int_{-\infty}^{\frac{\xi_k^*}{\sqrt{2\Delta t}}} \\
& \left[\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-[(\xi_k - \mu(\xi_{k-1}))^2 + \xi_{k-1}^2]} H_m(\xi_{k-1})\, d\xi_{k-1} \right] (1 - e^{\sigma\sqrt{2\Delta t}\,\xi_k})\, d\xi_k, \\
=& \frac{e^{-r\Delta t}}{2^m m! \sqrt{\pi}} \int_{-\infty}^{\frac{\xi_k^*}{\sqrt{2\Delta t}}} I_m(\xi_k)(1 - e^{\sigma\sqrt{2\Delta t}\,\xi_k})\, d\xi_k,
\end{aligned} \tag{3.89}$$

where,

$$I_m(\xi_k) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-[(\xi_k - \mu(\xi_{k-1}))^2 + \xi_{k-1}^2]} H_m(\xi_{k-1})\, d\xi_{k-1}. \tag{3.90}$$

Using the analytical solution to $I_m(\xi_k)$ as presented in subsection 3.2 gives,

$$\gamma_m^{k-1} = \frac{e^{-r\Delta t}}{2^m m! \sqrt{\pi}} \int_{-\infty}^{\frac{\xi_k^*}{\sqrt{2\Delta t}}} \frac{\sqrt{2\Delta t}\, e^{-(\frac{\sqrt{2\Delta t}\,\xi_k - b}{\upsilon})^2} H_m(\frac{\sqrt{2\Delta t}\,\xi_k - b}{\upsilon})}{\upsilon^{m+1}} (1 - e^{\sigma\sqrt{2\Delta t}\,\xi_k})\, d\xi_k, \tag{3.91}$$

and by performing a further substitution,

$$z = \frac{\sqrt{2\Delta t}\,\xi_k - b}{\upsilon},$$

(3.91) is simplified to,

$$\gamma_m^{k-1} = \frac{e^{-r\Delta t}}{2^m m! \upsilon^m \sqrt{\pi}} \int_{-\infty}^{\frac{\xi_k^*-b}{\upsilon}} e^{-z^2} H_m(z)(1 - e^{\sigma\upsilon z+\sigma b})\, dz$$

$$= \frac{e^{-r\Delta t}}{2^m m! \upsilon^m \sqrt{\pi}} \left[\int_{-\infty}^{z_k} e^{-z^2} H_m(z)\, dz - e^{\sigma b} \int_{-\infty}^{z_k} e^{-z^2+\sigma\upsilon z} H_m(z)\, dz \right], \qquad (3.92)$$

and

$$z_k = \frac{\xi_k^* - b}{\upsilon}. \qquad (3.93)$$

Given (3.92), the values of vector $\boldsymbol{\gamma}^{k-1}$ can be evaluated. Beginning with $m = 0$,

$$\gamma_0^{k-1} = \frac{e^{-r\Delta t}}{2^0 0! \upsilon^0 \sqrt{\pi}} \left[\int_{-\infty}^{z_k} e^{-z^2} H_0(z)\, dz - e^{\sigma b} \int_{-\infty}^{z_k} e^{-z^2+\sigma\upsilon z} H_0(z)\, dz \right],$$

$$= \frac{e^{-r\Delta t}}{\sqrt{\pi}} \left[\int_{-\infty}^{z_k} e^{-z^2}\, dz - e^{\sigma b} \int_{-\infty}^{z_k} e^{-z^2+\sigma\upsilon z}\, dz \right]. \qquad (3.94)$$

Using property (3.88) and completing the square of the Gaussian in the right integral, (3.94) becomes

$$\gamma_0^{k-1} = e^{-r\Delta t} \left[\frac{1}{2} erfc(-z_k) - \frac{e^{\sigma b + \frac{\sigma^2\upsilon^2}{4}}}{\sqrt{\pi}} \int_{-\infty}^{z_k} e^{-(z-\frac{\sigma\upsilon}{2})^2}\, dz \right],$$

$$= \frac{e^{-r\Delta t}}{2} \left[erfc(-z_k) - e^{\sigma b + \frac{\sigma^2\upsilon^2}{4}} erfc\left(\frac{\sigma\upsilon}{2} - z_k \right) \right]. \qquad (3.95)$$

For $m = 1$,

$$\gamma_1^{k-1} = \frac{e^{-r\Delta t}}{2^1 1! \upsilon^1 \sqrt{\pi}} \left[\int_{-\infty}^{z_k} e^{-z^2} H_1(z)\, dz - e^{\sigma b} \int_{-\infty}^{z_k} e^{-z^2+\sigma\upsilon z} H_1(z)\, dz \right],$$

$$= \frac{e^{-r\Delta t}}{2\upsilon\sqrt{\pi}} \left[\int_{-\infty}^{z_k} 2z e^{-z^2}\, dz - e^{\sigma b + \frac{\sigma^2\upsilon^2}{4}} \int_{-\infty}^{z_k} 2z e^{-(z-\frac{\sigma\upsilon}{2})^2}\, dz \right]. \qquad (3.96)$$

Using properties (3.86) and (3.88), (3.96) becomes

$$\gamma_1^{k-1} = \frac{e^{-r\Delta t}}{2\upsilon}\left[-\frac{1}{\sqrt{\pi}}e^{-z_k^2} + \frac{e^{\sigma b - z_k^2 + \sigma\upsilon z_k}}{\sqrt{\pi}} - \frac{\sigma\upsilon e^{\sigma b + \frac{\sigma^2\upsilon^2}{4}}}{2}\, erfc\left(\frac{\sigma\upsilon}{2} - z_k\right)\right]. \tag{3.97}$$

A proof of (3.97), can be found in appendix A.2.1.

For $m = 2, 3, \ldots, N$,

$$\gamma_m^{k-1} = \frac{e^{-r\Delta t}}{2^m m! \upsilon^m}\left[\Theta_m^{k-1} - \Phi_m^{k-1}\right], \tag{3.98}$$

where

$$\Theta_m^{k-1} = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{z_k} e^{-z^2} H_m(z)\, dz, \tag{3.99}$$

and

$$\Phi_m^{k-1} = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{z_k} e^{-z^2+\sigma\upsilon z} H_m(z)\, dz. \tag{3.100}$$

and using,

$$H_n(\upsilon t + b) = 2(\upsilon t + b)H_{n-1}(\upsilon t + b) - 2(n-1)H_{n-2}(\upsilon t + b), \quad \mathit{for}\ n > 1,$$

becomes

$$\Theta_m^{k-1} = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{z_k} e^{-z^2}(2zH_{m-1}(z) - 2(m-1)H_{m-2}(z))\, dz, \tag{3.101}$$

and

$$\Phi_m^{k-1} = \frac{e^{\sigma b + \frac{\sigma^2\upsilon^2}{4}}}{\sqrt{\pi}}\int_{-\infty}^{z_k} e^{-(z-\frac{\sigma\upsilon}{2})^2}(2zH_{m-1}(z) - 2(m-1)H_{m-2}(z))\, dz. \tag{3.102}$$

An analytical form for (3.101) is determined using properties, (3.84) and (3.85), and along with integration by parts gives

$$\Theta_m^{k-1} = -\frac{1}{\sqrt{\pi}}e^{-z_k^2}H_{m-1}(z_k), \tag{3.103}$$

where z_k is given by (3.93). The proof for (3.103) can be found in Appendix A.2.2 leading to the evaluation (A.26). For Φ, a recurrence relation is built using properties, (3.84) and (3.85), and along with integration by parts gives

$$\Phi_m^{k-1} = -\frac{e^{\sigma b+\frac{\sigma^2\upsilon^2}{4}}}{\sqrt{\pi}} e^{-(z_k-\frac{\sigma\upsilon}{2})^2} H_{m-1}(z_k) + \sigma\upsilon e^{\sigma b+\frac{\sigma^2\upsilon^2}{4}} \Phi_{m-1}^{k-1}. \tag{3.104}$$

The proof for (3.104) can also be found in Appendix A.2.3 leading to evaluation (A.27). Therefore,

$$\begin{aligned}\gamma_m^{k-1} =& \frac{e^{-r\Delta t}}{2^m m!\upsilon^m}\Bigg[-\frac{1}{\sqrt{\pi}} e^{-z_k^2} H_{m-1}(z_k) + \frac{e^{\sigma b - z_k^2 + \sigma\upsilon z_k}}{\sqrt{\pi}} H_{m-1}(z_k) \\ & - \sigma\upsilon e^{\sigma b+\frac{\sigma^2\upsilon^2}{4}} \Phi_{m-1}^{k-1}\Bigg].\end{aligned} \tag{3.105}$$

To obtain a recurrence relation for γ_m^{k-1}, Φ_{m-1}^{k-1} is replaced with γ_{m-1}^{k-1} by rearranging

$$\gamma_{m-1}^{k-1} = \frac{e^{-r\Delta t}}{2^{m-1}(m-1)!\upsilon^{m-1}}\Bigg[\Theta_{m-1}^{k-1} - \Phi_{m-1}^{k-1}\Bigg], \tag{3.106}$$

for Φ_{m-1}^{k-1} and substituting into (3.105). Therefore, (3.105) becomes

$$\gamma_m^{k-1} = \frac{\sigma}{2m}\gamma_{m-1}^{k-1} + \frac{e^{-r\Delta t - z_k^2}}{2^m m!\upsilon^m\sqrt{\pi}}\Bigg[H_{m-1}(z_k)(e^{\sigma b+\sigma\upsilon z_k} - 1) + \sigma\upsilon H_{m-2}(z_k)\Bigg]. \tag{3.107}$$

The proof for (3.107) can be found in Appendix A.2.4 leading to the evaluation (A.30).

Therefore, given (3.107), (3.81) becomes

$$\begin{aligned}&\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-\xi_{k-1}^2} H_m(\xi_{k-1}) \sum_{q=0}^{N} \alpha_q^{k-1} H_q(\xi_{k-1})\, d\xi_{k-1} = \gamma_m^{k-1} \\ &+ \frac{1}{\sqrt{\pi}}\int_{\frac{\xi_k^*}{\sqrt{2\Delta t}}}^{\infty} e^{-\xi_{k-1}^2} H_m(\xi_{k-1}) \frac{e^{-r\Delta t}}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-(\xi_k - \mu(\xi_{k-1}))^2} V^k(\sqrt{2\Delta t}\,\xi_k)\, d\xi_k\, d\xi_{k-1},\end{aligned}$$

which simplifies to,

$$
\begin{aligned}
\alpha_m^{k-1} =& \gamma_m^{k-1} + \frac{e^{-r\Delta t}}{2^m m! \sqrt{\pi}} \int_{\frac{\xi_k^*}{\sqrt{2\Delta t}}}^{\infty} \\
&\left[\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-[(\xi_k - \mu(\xi_{k-1}))^2 + \xi_{k-1}^2]} H_m(\xi_{k-1}) d\xi_{k-1} \right] V^k(\sqrt{2\Delta t}\, \xi_k)\, d\xi_k, \\
=& \gamma_m^{k-1} + \frac{e^{-r\Delta t}}{2^m m! \sqrt{\pi}} \int_{\frac{\xi_k^*}{\sqrt{2\Delta t}}}^{\infty} V^k(\sqrt{2\Delta t}\, \xi_k) I_m(\xi_k)\, d\xi_k, \qquad (3.108)
\end{aligned}
$$

where,

$$
I_m(\xi_k) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-[(\xi_k - \mu(\xi_{k-1}))^2 + \xi_{k-1}^2]} H_m(\xi_{k-1})\, d\xi_{k-1}. \qquad (3.109)
$$

Using the analytical solution to $I_m(\xi_k)$ as presented in sub-section 3.2 gives

$$
\alpha_m^{k-1} = \gamma_m^{k-1} + \frac{e^{-r\Delta t}}{2^m m! \sqrt{\pi}} \int_{\frac{\xi_k^*}{\sqrt{2\Delta t}}}^{\infty} \frac{\sqrt{2\Delta t}\, e^{-(\frac{\sqrt{2\Delta t}\,\xi_k - b}{\upsilon})^2} H_m(\frac{\sqrt{2\Delta t}\,\xi_k - b}{\upsilon})}{\upsilon^{m+1}} V^k(\sqrt{2\Delta t}\, \xi_k)\, d\xi_k \qquad (3.110)
$$

and by performing a further substitution,

$$
z = \frac{\sqrt{2\Delta t}\, \xi_k - b}{\upsilon},
$$

(3.110) is simplified to,

$$
\alpha_m^{k-1} = \gamma_m^{k-1} + \frac{e^{-r\Delta t}}{2^m m! \upsilon^m \sqrt{\pi}} \int_{\frac{\xi_k^* - b}{\upsilon}}^{\infty} e^{-z^2} H_m(z) V^k(\upsilon z + b)\, dz. \qquad (3.111)
$$

Finally, a Fourier-Hermite series is introduced for V^k to complete the relationship between $\boldsymbol{\alpha}^{k-1}$ and $\boldsymbol{\alpha}^k$,

$$
V^k(\xi_k) \backsimeq \sum_{n=0}^{N} \alpha_n^k H_n(\xi_k), \qquad (3.112)
$$

and substituting (3.112) into (3.110) gives

$$\alpha_m^{k-1} = \gamma_m^{k-1} + \frac{e^{-r\Delta t}}{2^m m! \upsilon^m \sqrt{\pi}} \int_{\frac{\xi_k^* - b}{\upsilon}}^{\infty} e^{-z^2} H_m(z) \sum_{n=0}^{N} \alpha_n^k H_n(\upsilon z + b)\, dz,$$

$$= \gamma_m^{k-1} + \frac{e^{-r\Delta t}}{2^m m! \upsilon^m \sqrt{\pi}} \sum_{n=0}^{N} \alpha_n^k \int_{\frac{\xi_k^* - b}{\upsilon}}^{\infty} e^{-z^2} H_m(z) H_n(\upsilon z + b)\, dz. \tag{3.113}$$

Proposition 3.3.1. *The expression* (3.113) *can be rewritten into*

$$\alpha_m^{k-1} = \gamma_m^{k-1} + \sum_{n=0}^{N} \alpha_n^k A_{m,n}^k, \tag{3.114}$$

where

$$A_{m,n}^k = \frac{e^{-r\Delta t}}{2^m m! \upsilon^m \sqrt{\pi}} \int_{z_k}^{\infty} e^{-z^2} H_m(z) H_n(\upsilon z + b)\, dz, \tag{3.115}$$

and

$$z_k = \frac{\xi_k^* - b}{\upsilon}. \tag{3.116}$$

With the elements of $\mathbf{A}^k$ *being*

$$A_{0,0}^k = \frac{e^{-r\Delta t}}{2} erfc(z_k), \tag{3.117}$$

$$A_{0,1}^k = e^{-r\Delta t}\left[b\, erfc(z_k) + \frac{\upsilon}{\sqrt{\pi}} e^{-z_k^2}\right], \tag{3.118}$$

$$A_{1,0}^k = \frac{e^{-r\Delta t}}{2\upsilon\sqrt{\pi}} e^{-z_k^2}, \tag{3.119}$$

for $m = 0$ *and* $n = 2, 3, \ldots, N$*,*

$$A_{0,n}^k = \frac{\upsilon e^{-r\Delta t}}{\sqrt{\pi}} e^{-z_k^2} H_{n-1}(\upsilon z_k + b) + 2bA_{0,n-1}^k + 2(\upsilon^2 - 1)(n-1)A_{0,n-2}^k, \tag{3.120}$$

and for $m > 1$ *and* $n = 0$*,*

$$A_{m,0}^k = \frac{e^{-r\Delta t}}{2^m m! \upsilon^m \sqrt{\pi}} e^{-z_k^2} H_{m-1}(z_k). \tag{3.121}$$

For general m and n,

$$A^k_{m,n} = \frac{n}{m} A^k_{m-1,n-1} + \frac{e^{-r\Delta t}}{2^m m! \upsilon^m \sqrt{\pi}} e^{-z_k^2} H_{m-1}(z_k) H_n(\upsilon z_k + b). \tag{3.122}$$

Proof. As with the European options, the elements of matrix $\mathbf{A}^k$ require evaluation. The major difference with the matrix $\mathbf{A}^k$ to the European option matrix is that for each time step, the elements will change because the early exercise point, ξ^*_k, will differ. Therefore, for each time step, the coefficients, $\boldsymbol{\alpha}^{k-1}$ are found. In the European option, the coefficients of the option price series, $\boldsymbol{\alpha}^0$, are evaluated by using matrix $\mathbf{A}$ and $\boldsymbol{\alpha}^{K-1}$ only.

So element $A^k_{0,0}$ is given by

$$\begin{aligned} A^k_{0,0} &= \frac{e^{-r\Delta t}}{2^0 0! \upsilon^0 \sqrt{\pi}} \int_{z_k}^{\infty} e^{-z^2} H_0(z) H_0(\upsilon z + b)\, dz \\ &= \frac{e^{-r\Delta t}}{\sqrt{\pi}} \int_{z_k}^{\infty} e^{-z^2}\, dz, \end{aligned} \tag{3.123}$$

therefore using (3.88),

$$A^k_{0,0} = \frac{e^{-r\Delta t}}{2} erfc(z_k). \tag{3.124}$$

The next element $A^k_{0,1}$ is given by

$$\begin{aligned} A^k_{0,1} &= \frac{e^{-r\Delta t}}{2^0 0! \upsilon^0 \sqrt{\pi}} \int_{z_k}^{\infty} e^{-z^2} H_0(z) H_1(\upsilon z + b)\, dz \\ &= \frac{e^{-r\Delta t}}{\sqrt{\pi}} \left[\upsilon \int_{z_k}^{\infty} 2z e^{-z^2} dz + 2b \int_{z_k}^{\infty} e^{-z^2}\, dz \right]. \end{aligned} \tag{3.125}$$

Using properties (3.85) and (3.88), (3.125) becomes,

$$A^k_{0,1} = e^{-r\Delta t} \left[b\, erfc(z_k) + \frac{\upsilon}{\sqrt{\pi}} e^{-z_k^2} \right]. \tag{3.126}$$

For $m = 0$ and $n = 2, 3, \ldots, N$,

$$
\begin{aligned}
A_{0,n}^k &= \frac{e^{-r\Delta t}}{2^0 0! \upsilon^0 \sqrt{\pi}} \int_{z_k}^{\infty} e^{-z^2} H_0(z) H_n(\upsilon z + b)\, dz \\
&= \frac{e^{-r\Delta t}}{\sqrt{\pi}} \int_{z_k}^{\infty} e^{-z^2} H_n(\upsilon z + b)\, dz, \qquad (3.127)
\end{aligned}
$$

and using property (3.83), (3.127) can be expressed as,

$$
\begin{aligned}
A_{0,n}^k &= \frac{e^{-r\Delta t}}{\sqrt{\pi}} \int_{z_k}^{\infty} e^{-z^2} \Big[2(\upsilon z + b) H_{n-1}(\upsilon z + b) - 2(n-1) H_{n-2}(\upsilon z + b) \Big]\, dz, \\
&= \frac{e^{-r\Delta t}}{\sqrt{\pi}} \int_{z_k}^{\infty} 2\upsilon z e^{-z^2} H_{n-1}(\upsilon z_k + b)\, dz + 2b A_{0,n-1}^k - 2(n-1) A_{0,n-2}^k, \\
&= \frac{\upsilon e^{-r\Delta t}}{\sqrt{\pi}} e^{-z_k^2} H_{n-1}(\upsilon z_k + b) + 2b A_{0,n-1}^k + 2(\upsilon^2 - 1)(n-1) A_{0,n-2}^k. \qquad (3.128)
\end{aligned}
$$

The proof to (3.120) can be found in Appendix A.2.5 leading to evaluation (A.33).

For the value of $m = 1$ and $n = 0$,

$$
\begin{aligned}
A_{1,0}^k &= \frac{e^{-r\Delta t}}{2^1 1! \upsilon^1 \sqrt{\pi}} \int_{z_k}^{\infty} e^{-z^2} H_1(z) H_0(\upsilon z + b)\, dz, \\
&= \frac{e^{-r\Delta t}}{2\upsilon\sqrt{\pi}} \int_{z_k}^{\infty} 2z e^{-z^2}\, dz, \qquad (3.129)
\end{aligned}
$$

and using property (3.85), (3.129) becomes

$$
A_{1,0}^k = \frac{e^{-r\Delta t}}{2\upsilon\sqrt{\pi}} e^{-z_k^2}. \qquad (3.130)
$$

For $m > 1$ and $n = 0$ and using property (3.83) gives,

$$
\begin{aligned}
A_{m,0}^{k} &= \frac{e^{-r\Delta t}}{2^m m! \upsilon^m \sqrt{\pi}} \int_{z_k}^{\infty} e^{-z^2} H_m(z) H_0(\upsilon z + b)\, dz, \\
&= \frac{e^{-r\Delta t}}{2^m m! \upsilon^m \sqrt{\pi}} \int_{z_k}^{\infty} e^{-z^2} \Big[2z H_{m-1}(z) - 2(m-1) H_{m-2}(z) \Big]\, dz, \\
&= \frac{e^{-r\Delta t}}{2^m m! \upsilon^m \sqrt{\pi}} \int_{z_k}^{\infty} 2z e^{-z^2} H_{m-1}(z)\, dz - 2(m-1) A_{m-2,0}^{k}. \qquad (3.131)
\end{aligned}
$$

Using integration by parts, (3.131) reduces to,

$$
A_{m,0}^{k} = \frac{e^{-r\Delta t}}{2^m m! \upsilon^m \sqrt{\pi}} e^{-z_k^2} H_{m-1}(z_k). \qquad (3.132)
$$

For general m and n and using property (3.83) for $H_m(z)$ gives,

$$
\begin{aligned}
A_{m,n}^{k} &= \frac{e^{-r\Delta t}}{2^m m! \upsilon^m \sqrt{\pi}} \int_{z_k}^{\infty} e^{-z^2} H_m(z) H_n(\upsilon z + b)\, dz, \\
&= \frac{e^{-r\Delta t}}{2^m m! \upsilon^m \sqrt{\pi}} \int_{z_k}^{\infty} e^{-z^2} H_n(\upsilon z + b) \Big[2z H_{m-1}(z) - 2(m-1) H_{m-2}(z) \Big]\, dz, \\
&= \frac{e^{-r\Delta t}}{2^m m! \upsilon^m \sqrt{\pi}} \int_{z_k}^{\infty} 2z e^{-z^2} H_{m-1}(z) H_n(\upsilon z + b)\, dz - 2(m-1) A_{m-2,n}^{k}. \qquad (3.133)
\end{aligned}
$$

Using integration by parts, (3.133) reduces to,

$$
A_{m,n}^{k} = \frac{n}{m} A_{m-1,n-1}^{k} + \frac{e^{-r\Delta t}}{2^m m! \upsilon^m \sqrt{\pi}} e^{-z_k^2} H_{m-1}(z_k) H_n(\upsilon z_k + b). \qquad (3.134)
$$

$\square$

Prior to finding all coefficients, as with the European options, the coefficients for the first time step $\boldsymbol{\alpha}^{K-1}$ are evaluated. Since the American put and European put at the first time step are equivalent, the expression (3.69) and (3.70) are used. Namely,

$$
\alpha_m^{K-1} = \frac{\sigma}{2m} \left[\frac{e^{-r\Delta t}}{2^{m-1}(m-1)! \upsilon^{m-1} \sqrt{\pi}} e^{-(\frac{b}{\upsilon})^2} H_{m-2}(-\frac{b}{\upsilon}) + \alpha_{m-1}^{K-1} \right] \qquad (3.135)
$$

for $m = 2, 3, \ldots N$, with initial conditions,

$$\alpha_0^{K-1} = \frac{e^{-r\Delta t}}{2}\left[erfc(\frac{b}{\upsilon}) - e^{\sigma b + \frac{1}{4}\sigma^2\upsilon^2} erfc(\frac{b}{\upsilon} + \frac{\sigma\upsilon}{2})\right], \tag{3.136}$$

$$\alpha_1^{K-1} = -\frac{\sigma}{4} e^{-r\Delta t + \sigma b + \frac{1}{4}\sigma^2\tau^2} erfc(\frac{b}{\upsilon} + \frac{\sigma\upsilon}{2}). \tag{3.137}$$

Therefore, the coefficients $\boldsymbol{\alpha}^{k-1}$ are evaluated by,

$$\alpha_m^{k-1} = \gamma_m^k + \sum_{n=0}^{N} \alpha_n^k A_{m,n}^k \qquad (k = K-1, K-2, \ldots, 1), \tag{3.138}$$

The recurrence relation (3.138) is evaluated similar to (3.33) in the European options section. The major difference being that for the American put, the coefficients $\boldsymbol{\alpha}^k$ are evaluated for each time step since the early exercise point varies from one time step to the next. So, (3.138) is used at each time step until $k = 1$.

3.3.1 Results and Analysis

In Chiarella et al. (1999), the results presented used a high number of time steps for both the European and American options. As was presented in Section 3.2.3, the number of time steps required to achieve an accurate result was not as large as envisaged. However, initial investigation of the American put option showed that large time steps were required to achieve some accurate results.

Initial investigations also show that the oscillating approximations shown in figures 3.2, 3.3 and 3.4 for the European options are typical and also apply for the American put case. Therefore, a parameter set (K,N) may be an optimal approximation for one particular underlying asset value but may not give the same accuracy for another underlying value. Table 3.6 shows some results for American puts when the number of basis functions used is 40.

Asset Price ($)	**Binomial Method**	**F-H 60 steps**	**F-H 80 Steps**	**F-H 100 Steps**
80	20.000000	20.000000	20.000000	20.000000
90	10.037663	10.051996	10.062199	10.068956
100	3.224898	3.176885	3.194857	3.205842
110	0.665410	0.627337	0.637849	0.644346
120	0.088795	0.085086	0.088416	0.090615

Table 3.6: Fourier-Hermite - American put option for various time steps and 40 basis functions with $\sigma = 0.20$, $r = 0.08$, $T = 0.25$, strike of \$100. Double precision was used to calculate the values. The Binomial Method prices are optimal approximations (using various N) and were calculated based on Cox et al. (1979).

Table 3.6 shows that the optimal approximation will vary when the number of basis functions is fixed. For instance, when the asset value is \$90, the number of time steps required to find the best approximation is less than 60. For an asset value of \$110, the number of times steps is greater than 100. Therefore the computational effort required to find the optimal option price is greater for \$110 than \$90. In a computer algebra package like **Maple**, this can be quite time consuming, even with a search algorithm like a bi-section. An improvement could be made with an efficient search algorithm. A bi-section was chosen because of the ease of implementation. Table 3.7 shows the best number of time steps required for various underlying asset values to give optimal approximations.

Asset Price ($)	**Time Steps**	**Binomial Method**	**F-H 40 Basis Functions**
90	43	10.037663	10.037439
100	172	3.224898	3.224875
110	363	0.665410	0.665417
120	83	0.088795	0.088800

Table 3.7: Fourier-Hermite - American put option for 40 basis functions and the best number of time steps with $\sigma = 0.20$, $r = 0.08$, $T = 0.25$, strike of \$100. Double precision was used to calculate the values. The Binomial Method prices are optimal approximations (using various N) and were calculated based on Cox et al. (1979).

Clearly there is a great discrepancy in results presented in Table 3.7. The oscillating nature of the Fourier-Hermite expansion/series explains the differences between the number of time steps to find the optimal approximation. The issue arising from the results present in this table is the time required to find the optimal result. As we can see when the asset value is \$110, the number of time steps required is 363. As the number of time steps increases, so does the time and computational effort required. Table 3.8 presents results for the number of basis functions required to find the best approximation when the number of time steps are fixed to 100.

Asset Price (\$)	**Basis Functions**	**Binomial Method**	**F-H 40 steps**
90	35	10.037663	10.037604
100	12	3.224898	3.254261
110	20	0.665410	0.631602
120	13	0.088795	0.089520

Table 3.8: Fourier-Hermite - American put option for 40 time steps and the best basis functions with $\sigma = 0.20$, $r = 0.08$, $T = 0.25$, strike of \$100. Double precision was used to calculate the values. The Binomial Method prices are optimal approximations (using various N) and were calculated based on Cox et al. (1979).

Again the same issues arise as with table 3.7, as the number of basis functions (N) or the number of time steps (K) increase, so does the computational effort required. In the case of N increasing, the calculations of exponentials and factorials are an issue. Luckily, technology allows as to evaluate these functions much faster today. Further analysis of results will also be presented in Chapters 4, 5 and 6.

3.4 Conclusion

Chiarella et al. (1999) offer a unique approach to evaluate the price of an option in a path integral framework. The use of a Fourier-Hermite series to represent the underlying allows the final option price polynomial to be formed by using recurrence relations. These relations allow the coefficients of the Fourier series (the price polynomial) to be evaluated. The Fourier-Hermite series is used due to the form of the Gaussian within the integrand of the path integral (3.1). The recurrence relations are formed using the orthogonality properties of the Hermite polynomials, analytical integration methods and some algebra.

The results using this method are quite good, especially for the European options. For the American put, precise results can be obtained. However the computational effort to evaluate a good approximation can be long especially when using a computer algebra package like ***Maple***. Given the oscillatory nature of the Fourier-Hermite series, the parameter set, K (the number of time steps) and N (the number of basis functions), may be precise for certain asset values but not necessarily for others. Therefore, to obtain accurate results for a particular asset value, a search such as bi-section, may be required to find the best parameter set.

In investigating this method, it was clear that some of the equations/relations and results were not accurately stated. The results obtained are quite different to those presented in Chiarella et al. (1999). It must be said however that accurate results are possible but require some computational effort. One of the main advantages of this method is the fact that more than one option price may be calculated at any given time. This was very advantageous for the European option. Due to the oscillatory nature of the Fourier-Hermite series, some option prices were more accurate than others. In the case of the European option where errors were in a trough, the errors were as low as 10^{-14}. Even approximate prices where the errors peak, using the same Fourier-Hermite series had errors in the order of 10^{-11}, which is still very accurate. For the American put option, the results were not as accurate and therefore, as shown in section 3.3.1, one parameter set may give an accurate result for a certain asset price but was not so accurate for another.

One of the issues with this method is the orthogonality property (3.12) of the Hermite polynomial. This property contains an exponential and factorial with respect to N. So, even with sophisticated computing, the recurrence relations will require time to compute for large values of N. To combat this problem, chapter 4 will present a modified version of the Fourier-Hermite method. The modified method uses *normalised Hermite polynomials* in a Fourier series expansion. The method is very similar to that presented in this Chapter, with the creation of recurrence relations to find the coefficients of the Fourier series. The main advantage of the modification is in that the orthogonality property for the normalised Hermite does not have an exponential or factorial involved.

Chapter 4

Normalised Fourier-Hermite Series Evaluation

This chapter offers an alternative Fourier series method to the one presented in Chapter 3. The alternative uses a *normalised* Fourier-Hermite series to represent the underlying. The major difference in this method is the form of the Hermite orthogonal polynomial and their mathematical properties. The method is similar to that presented in the previous Chapter 3 with some of the recurrence relations formed being identical.

4.1 Introduction

In this chapter we offer a similar approach to the previous chapter with the major difference being the use of *normalised Hermite polynomials*. The approach is similar, with the properties associated to the normalised polynomials being different to those presented in Chapter 3.

The use of these normalised Hermite polynomials will offer an alternative to those presented in Chapter 3. One of the advantages envisaged by using these normalised polynomials is that the recurrence relations involving the coefficients $\boldsymbol{\alpha}^{k-1}$ may eliminate the use of the exponential (2^m term). This is achieved because of the form of the orthogonality property for the normalised Hermite polynomial, involving the Kronecker δ function

$$\sqrt{2\pi}n! \int_{-\infty}^{\infty} e^{-\frac{t^2}{2}} H_m^*(t) H_n^*(t)\, dt = \delta_{m,n} = \begin{cases} 1, & m = n \\ 0, & otherwise \end{cases}. \tag{4.1}$$

This δ function, (4.1), does not have an exponential term in its coefficient. It is hoped that this will improve the efficiency and speed of evaluation of the options being priced, especially for large N (the number of basis function). As described in the previous chapter, an improvement to the use of Hermite polynomials was required since as the number of basis functions increases, the evaluation of option prices became inefficient. It turns out that using normalised Hermite polynomials ameliorates this issue.

4.2 European Options

As with the Fourier-Hermite method, we firstly transform the path integral (3.1), so that a recurrence relation can be built to link the coefficients of the normalised Fourier-Hermite polynomials from one time step to the next. Recalling the path integral,

$$f^{k-1}(\xi_{k-1}) = \frac{e^{-r\Delta t}}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-(\xi_k - \mu(\xi_{k-1}))^2} f^k(\sqrt{2\Delta t}\ \xi_k)\ d\xi_k, \tag{4.2}$$

where

$$\mu(\xi_{k-1}) = \frac{\xi_{k-1} + b}{\sqrt{2\Delta t}}, \tag{4.3}$$

and

$$b = \frac{1}{\sigma}(r - \frac{1}{2}\sigma^2)\Delta t. \tag{4.4}$$

Given the normalised Fourier-Hermite expansion,

$$f^{k-1}(\xi_{k-1}) \backsimeq \sum_{q=0}^{N} \alpha_q^{k-1} H_q^*(\xi_{k-1}), \tag{4.5}$$

where $H_q^*(\xi_{k-1})$ is a normalised Hermite polynomial then, substituting (4.5) into (4.2), the path integral is transformed to,

$$\sum_{q=0}^{N} \alpha_q^{k-1} H_q^*(\xi_{k-1}) = \frac{e^{-r\Delta t}}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-(\xi_k - \mu(\xi_{k-1}))^2} f^k(\sqrt{2\Delta t}\ \xi_k)\ d\xi_k. \tag{4.6}$$

Utilising the orthogonality property of normalised Hermite polynomials given by (4.1) and following the method used for the non-normalised Fourier-Hermite of section 3.2 produced upon simplification

$$\alpha_m^{k-1} = \frac{e^{-r\Delta t}}{m!\sqrt{\pi}} \int_{-\infty}^{\infty} f^k(\sqrt{2\Delta t}\ \xi_k) I_m^*(\xi_k) d\xi_k. \tag{4.7}$$

where,

$$I_m^*(\xi_k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}[2(\xi_k - \mu(\xi_{k-1}))^2 + \xi_{k-1}^2]} H_m^*(\xi_{k-1})\ d\xi_{k-1}. \tag{4.8}$$

We note that the exponential in the integrand of (4.8) has been modified to accommodate the use of normalised Hermite polynomials. Completing the square as with the Fourier-Hermite method (see Appendix B.1.1 for a step by step evaluation) gives

$$2(\xi_k - \mu(\xi_{k-1}))^2 + \xi_{k-1}^2 = \big[(\frac{\xi_{k-1}\tau}{\sqrt{\Delta t}} - \frac{\sqrt{2\Delta t}\ \xi_k - b}{\tau\sqrt{\Delta t}})^2 + (\frac{\sqrt{2\Delta t}\ \xi_k - b}{\tau})^2\big], \tag{4.9}$$

where $\mu(\xi_{k-1})$ and b are as defined in (4.3) and (4.4) respectively and

$$\tau = \sqrt{1+\Delta t}. \tag{4.10}$$

Therefore, substituting (4.9) into (4.8) and rearranging to give

$$I_m^*(\xi_k) = \frac{e^{-\frac{1}{2}(\frac{\sqrt{2\Delta t}\,\xi_k - b}{\tau})^2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left[(\frac{\tau\xi_{k-1}}{\sqrt{\Delta t}} - \frac{\sqrt{2\Delta t}\,\xi_k - b}{\tau\sqrt{\Delta t}})^2\right]} H_m^*(\xi_{k-1})\, d\xi_{k-1}, \tag{4.11}$$

where b is given by (4.4) and τ by (4.10).

(4.11) is evaluated analytically,

$$I_m^*(\xi_k) = \frac{\sqrt{\Delta t}\, e^{-\frac{1}{2}(\frac{\sqrt{2\Delta t}\,\xi_k - b}{\tau})^2} H_m^*(\frac{\sqrt{2\Delta t}\,\xi_k - b}{\tau})}{\tau^{m+1}}. \tag{4.12}$$

At this point we need to transform (4.7), so that a Fourier series for time step k can be introduced. Therefore, substituting (4.12) into (4.7) produces

$$\alpha_m^{k-1} = \frac{e^{-r\Delta t}}{m!\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{\sqrt{\Delta t}\, e^{-\frac{1}{2}(\frac{\sqrt{2\Delta t}\,\xi_k - b}{\tau})^2} H_m^*(\frac{\sqrt{2\Delta t}\,\xi_k - b}{\tau})}{\tau^{m+1}} f^k(\sqrt{2\Delta t}\,\xi_k)\, d\xi_k, \tag{4.13}$$

and on making the substitution,

$$z = \frac{\sqrt{2\Delta t}\,\xi_k - b}{\tau},$$

simplifies (4.13) to,

$$\alpha_m^{k-1} = \frac{e^{-r\Delta t}}{m!\tau^m\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{z^2}{2}} H_m^*(z) f^k(\tau z + b)\, dz. \tag{4.14}$$

The following normalised Fourier-Hermite series is introduced for $f^k(\tau z + b)$ to complete the recurrence relationship between $\boldsymbol{\alpha}^{k-1}$ and $\boldsymbol{\alpha}^k$, therefore

$$f^k(\xi_k) \backsimeq \sum_{n=0}^{N} \alpha_n^k H_n^*(\xi_k) \tag{4.15}$$

and the series (4.15) is substituted into (4.14) so that,

$$\alpha_m^{k-1} = \frac{e^{-r\Delta t}}{m!\tau^m\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{z^2}{2}} H_m^*(z) \sum_{n=0}^{N} \alpha_n^k H_n^*(\tau z + b)\, dz$$

$$= \frac{e^{-r\Delta t}}{m!\tau^m\sqrt{2\pi}} \sum_{n=0}^{N} \alpha_n^k \int_{-\infty}^{\infty} e^{-\frac{z^2}{2}} H_m^*(z) H_n^*(\tau z + b)\, dz. \tag{4.16}$$

The expression (4.16) can be rewritten into

$$\alpha_m^{k-1} = e^{-r\Delta t} \sum_{n=0}^{N} \alpha_n^k A_{m,n}^*, \tag{4.17}$$

where,

$$A_{m,n}^* = \frac{1}{m!\tau^m\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{z^2}{2}} H_m^*(z) H_n^*(\tau z + b)\, dz. \tag{4.18}$$

Since (4.17) is an expression that links the $\boldsymbol{\alpha}'s$ from time step k to $k-1$, a recurrence relation is built. This relationship is created by finding the elements of the 2 dimensional matrix $\mathbf{A}^*$ from (4.18). These elements, $A_{m,n}^*$ are found using the same methods and similar properties to those in the Fourier-Hermite section 3.2.

To find the elements of matrix $\mathbf{A}^*$, the initial elements are required. The following normalised Hermite polynomial and mathematical properties are used to assist in the evaluation of these elements.

$$H_0^*(x) = 1, \quad H_1^*(x) = x, \tag{4.19}$$

$$H_n^*(ax+b) = (ax+b)H_{n-1}^*(ax+b) - (n-1)\; H_{n-2}^*(ax+b), \tag{4.20}$$

$$\frac{d}{dt} H_n^*(ax+b) = a\, n\, H_{n-1}^*(ax+b), \tag{4.21}$$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} 2te^{-\frac{t^2}{2}}\, dt = 0, \tag{4.22}$$

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{t^2}{2}}\,dt = 1, \tag{4.23}$$

and

$$erfc(x) = \frac{2}{\sqrt{\pi}}\int_{x}^{\infty} e^{-t^2}\,dt = 1 - erf(x). \tag{4.24}$$

The first element $A^*_{0,0}$ is given by,

$$A^*_{0,0} = \frac{1}{0!\tau^0\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{z^2}{2}} H^*_0(z) H^*_0(\tau z + b) dz,$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{z^2}{2}} dz, \tag{4.25}$$

and so from (4.23) $A^*_{0,0} = 1$.

Element $A^*_{0,1}$ is given by,

$$A^*_{0,1} = \frac{1}{0!\tau^0\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{z^2}{2}} H^*_0(z) H^*_1(\tau z + b) dz,$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{z^2}{2}} H^*_1(\tau z + b) dz,$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} 2ze^{-\frac{z^2}{2}} dz + \frac{b}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{z^2}{2}} dz, \tag{4.26}$$

with the first integral in (4.26) being in the form of (4.22) and the second integral in the form of (4.23) and $A^*_{0,1} = b$.

Given the elements $A^*_{0,0}$ and $A^*_{0,1}$, the subsequent elements $A^*_{0,n}$ are evaluated by,

$$A^*_{0,n} = \frac{1}{0!\tau^0\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{z^2}{2}} H^*_0(z) H^*_n(\tau z + b) dz,$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \tau z e^{-\frac{z^2}{2}} H^*_{n-1}(\tau z + b) dz + b\, A^*_{0,n-1} - (n-1)\, A^*_{0,n-2}, \tag{4.27}$$

where we have used (4.20) to transform $A^*_{0,n}$. The integral in (4.27) is evaluated using property (4.21) and integration by parts to give,

$$A^*_{0,n} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tau^2 (n-1) e^{-\frac{z^2}{2}} H^*_{n-2}(\tau z + b) dz + b\, A^*_{0,n-1} - (n-1)\, A^*_{0,n-2},$$

$$= \tau^2 (n-1)\, A^*_{0,n-2} + b\, A^*_{0,n-1} - (n-1)\, A^*_{0,n-2},$$

and so

$$A^*_{0,n} = b\, A^*_{0,n-1} - (n-1)\,(\tau^2 - 1) A^*_{0,n-2} \quad \textit{for } n = 2, 3, \ldots, N. \tag{4.28}$$

As with the derivation for Hermite polynomials, the normalised properties (4.20) and (4.21) are used to evaluate elements $A^*_{m,n}$. Namely,

$$A^*_{m,n} = \frac{1}{m! \tau^m \sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{-z^2}{2}} H^*_m(z) \left[\frac{d}{dz} \frac{1}{\tau} \frac{1}{n+1} H^*_{n+1}(\tau z + b) \right] dz,$$

and using integration by parts, $A^*_{m,n}$ is transformed to,

$$A^*_{m,n} = \frac{1}{m! \tau^m} \left[-\frac{1}{\tau} \frac{1}{n+1} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} H^*_{n+1}(\tau z + b) \left(\frac{d}{dz} e^{-\frac{z^2}{2}} H^*_m(z) \right) dz \right]. \tag{4.29}$$

The derivative in (4.29) can be solved using property (4.20) and the product rule, to produce

$$\left(\frac{d}{dz} e^{-\frac{z^2}{2}} H^*_m(z) \right) = m e^{-\frac{z^2}{2}} H^*_{m-1}(z) - z e^{-\frac{z^2}{2}} H^*_m(z),$$

$$= e^{-\frac{z^2}{2}} \left[m\, H^*_{m-1}(z) - z H^*_m(z) \right],$$

$$= e^{-\frac{z^2}{2}} \left[-H^*_{m+1}(z) \right]. \tag{4.30}$$

Substitution of (4.30) in (4.29) produces

$$A^*_{m,n} = \frac{1}{m!\tau^m}\left[\frac{1}{\tau}.\frac{1}{n+1}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{z^2}{2}}[-H^*_{m+1}(z)]H^*_{n+1}(\tau z+b)\,dz\right],$$

$$= \frac{1}{(m+1)!\tau^{m+1}}\left[\frac{m+1}{n+1}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{z^2}{2}}H^*_{m+1}(z)H^*_{n+1}(\tau z+b)\,dz\right], \tag{4.31}$$

and so from (4.18)

$$A^*_{m,n} = \frac{m+1}{n+1}\,A^*_{m+1,n+1},$$

and so rearrangement produces

$$A^*_{m,n} = \frac{n}{m}\,A^*_{m-1,n-1}. \tag{4.32}$$

It must be noted that when $m > n$ element $\mathbf{A}^*_{m,n} = 0$.

Therefore in summary

$$\begin{aligned}
&A^*_{0,0} = 1,\\
&A^*_{0,1} = b,\\
&A^*_{0,n} = b\,A^*_{0,n-1} + (n-1)(\tau^2-1)\,A^*_{0,n-2}, \quad n = 2,3,\ldots,N,\\
&A^*_{m,n} = \frac{n}{m}\,A^*_{m-1,n-1}, \quad m = 1,2,\ldots,N; \quad n = 1,2,\ldots,N,\\
&A^*_{m,n} = 0 \quad for \quad m > n.
\end{aligned} \tag{4.33}$$

We note that the term $A^*_{0,1}$ and recurrence relation $A*_{0,n}$ differ by a multiple of 2 to those presented in the non-normalised method. The $A^*_{0,0}$ term and $A^*_{m,n}$ recurrence relation being identical to the other method. It must said that the technique to find the elements of $\mathbf{A}^*$ were very similar to find as those in the non-normalised method. The only difference being the mathematical identities and properties used for this Gaussian.

Given the elements of $\mathbf{A}^*$, the next step is to evaluate α_m^{K-1} values for the call and put option. Once the α_m^{K-1} are found, as with the non-normalised method, the following expression is used

$$\boldsymbol{\alpha}^0 = e^{-r(K-1)\Delta t}\mathbf{A}^{*K-1}\boldsymbol{\alpha}^{K-1}. \tag{4.34}$$

The $\boldsymbol{\alpha}^0$ values are the coefficients of the option price polynomial. We derive the values of $\boldsymbol{\alpha}^0$ for the non-normalised Hermites next, using the techniques shown in Chapter 3.

4.2.1 European Call Option Pricing

With a recurrence relations built to determine the elements of the Matrix $\mathbf{A}^*$ from (4.17), the values of $\boldsymbol{\alpha}^0$ can now be determined for a European call option price. Given the expression (4.34), a recurrence relationship is required to determine the values of $\boldsymbol{\alpha}^{K-1}$ such that the values of $\boldsymbol{\alpha}^0$ are found and in doing so, evaluating the European call option price.

Substituting the payoff function (3.34) into (4.7) gives

$$\alpha_m^{K-1} = \frac{e^{-r\Delta t}}{m!\tau^m\sqrt{2\pi}}\left[e^{\sigma b}\int_{-\frac{b}{\tau}}^{\infty} e^{\sigma\tau z}e^{-\frac{z^2}{2}}H_m^*(z)\,dz - \int_{-\frac{b}{\tau}}^{\infty} e^{-\frac{z^2}{2}}H_m^*(z)\,dz\right]. \tag{4.35}$$

Defining

$$\Psi_m^*(-\frac{b}{\tau}) = \frac{e^{\frac{1}{2}\sigma^2\tau^2}}{\sqrt{2\pi}} \int_{-\frac{b}{\tau}}^{\infty} e^{-\frac{1}{2}(z-\sigma\tau)^2} H_m^*(z)dz,$$

and

$$\Omega_m^*(-\frac{b}{\tau}) = \frac{1}{\sqrt{2\pi}} \int_{-\frac{b}{\tau}}^{\infty} e^{-\frac{1}{2}z^2} H_m^*(z)dz,$$

gives (4.35) as

$$\alpha_m^{K-1} = \frac{e^{-r\Delta t}}{\tau^m}\Big[e^{\sigma b}\Psi_m^*(-\frac{b}{\tau}) - \Omega_m^*(-\frac{b}{\tau})\Big].$$

The important values of Ψ^* and Ω^* are evaluated in Appendices B.1.2 and B.1.3 and are provided in terms of the well known $erfc(.)$ function by the following relationships,

$$\Psi_0^*(-\frac{b}{\tau}) = \frac{e^{\frac{1}{2}\sigma^2\tau^2}}{2} erfc\big(-\frac{b}{\sqrt{2}\tau} - \frac{\sigma\tau}{\sqrt{2}}\big),$$

$$\Omega_0^*(-\frac{b}{\tau}) = \frac{1}{2} erfc\big(-\frac{b}{\sqrt{2}\tau}\big),$$

$$\Psi_1^*(-\frac{b}{\tau}) = e^{\frac{1}{2}\sigma^2\tau^2}\frac{\sigma\tau}{2} erfc\big(-\frac{b}{\sqrt{2}\tau} - \frac{\sigma\tau}{\sqrt{2}}\big) + \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}((\frac{b}{\tau})^2+2\sigma b)},$$

$$\Omega_1^*(-\frac{b}{\tau}) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(\frac{b}{\tau})^2},$$

$$\Psi_m^*(-\frac{b}{\tau}) = \sigma\tau\Psi_{m-1}^*(-\frac{b}{\tau}) + \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}((\frac{b}{\tau})^2+2\sigma b)} H_{m-1}^*(-\frac{b}{\tau}),$$

and

$$\Omega_m^*(-\frac{b}{\tau}) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(\frac{b}{\tau})^2} H_{m-1}^*(-\frac{b}{\tau}). \tag{4.36}$$

Since we know the initial and general cases for Ψ^* and Ω^*, a recurrence relation for α_m^{K-1} for $m = 1, 2, \ldots, N$ can be formed (and is evaluated in Appendix B.1.4), namely

$$\alpha_m^{K-1} = \frac{\sigma}{m}\left[\frac{e^{-r\Delta t}}{(m-1)!\tau^{m-1}\sqrt{2\pi}} e^{-\frac{1}{2}(\frac{b}{\tau})^2} H_{m-2}^*(-\frac{b}{\tau}) + \alpha_{m-1}^{K-1}\right]. \tag{4.37}$$

The equation (4.37) are the $\boldsymbol{\alpha}^{K-1}$ values for $m = 2, 3, \ldots N$, with the following initial conditions,

$$\alpha_0^{K-1} = \frac{e^{-r\Delta t}}{2}\left[e^{\sigma b+\frac{1}{2}\sigma^2\tau^2} erfc(-\frac{b}{\sqrt{2}\tau} - \frac{\sigma\tau}{\sqrt{2}}) - erfc(-\frac{b}{\sqrt{2}\tau})\right], \tag{4.38}$$

and

$$\alpha_1^{k-1} = \frac{\sigma}{2} e^{-r\Delta t+\sigma b+\frac{1}{2}\sigma^2\tau^2} erfc(-\frac{b}{\sqrt{2}\tau} - \frac{\sigma\tau}{\sqrt{2}}). \tag{4.39}$$

With recurrence relations built for $\mathbf{A}$ and $\boldsymbol{\alpha}^{k-1}$, European call options can be evaluated.

4.2.2 European Put Option Pricing

With a recurrence relation built to determine the elements of the Matrix $\mathbf{A}^*$ from (4.17), the values of $\boldsymbol{\alpha}^0$ can now be determined for a European Put option price. Given the expression (4.34), a recurrence relationship is required to determine the values of $\boldsymbol{\alpha}^{K-1}$ such that the values of $\boldsymbol{\alpha}^0$ are found and in doing so, evaluating the European put option price.

Substituting the payoff function (3.56) into (4.7) gives

$$\alpha_m^{K-1} = \frac{e^{-r\Delta t}}{m!\tau^m\sqrt{2\pi}}\left[\int_{-\infty}^{-\frac{b}{\tau}} e^{-\frac{1}{2}z^2} H_m^*(z).dz - e^{\sigma b}\int_{\infty}^{-\frac{b}{\tau}} e^{\sigma\tau z} e^{-\frac{1}{2}z^2} H_m^*(z)dz\right]. \tag{4.40}$$

Defining anew for the put option

$$\hat{\Psi}^*_m(-\frac{b}{\tau}) = \frac{e^{\frac{1}{2}\sigma^2\tau^2}}{\sqrt{2\pi}} \int_{-\infty}^{-\frac{b}{\tau}} e^{-\frac{1}{2}(z-\sigma\tau)^2} H^*_m(z)dz.$$

and

$$\hat{\Omega}^*_m(-\frac{b}{\tau}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-\frac{b}{\tau}} e^{-\frac{1}{2}z^2} H^*_m(z)dz,$$

produces the expression from (4.40)

$$\alpha_m^{K-1} = \frac{e^{-r\Delta t}}{m!\tau^m}\Big[\hat{\Omega}^*_m(-\frac{b}{\tau}) - e^{\sigma b}\hat{\Psi}^*_m(-\frac{b}{\tau})\Big].$$

The important values of $\hat{\Psi}^*$ and $\hat{\Omega}^*$ are evaluated in Appendices B.1.5 and B.1.6 and are also provided in terms of the well known $erfc(.)$ function by the following relationships,

$$\hat{\Psi}^*_0(-\frac{b}{\tau}) = \frac{e^{\frac{1}{2}\sigma^2\tau^2}}{2} erfc(\frac{b}{\sqrt{2}\tau} + \frac{\sigma\tau}{\sqrt{2}}),$$

$$\hat{\Omega}^*_0(-\frac{b}{\tau}) = \frac{1}{2} erfc(\frac{b}{\sqrt{2}\tau}),$$

$$\hat{\Psi}^*_1(-\frac{b}{\tau}) = e^{\frac{1}{2}\sigma^2\tau^2}\frac{\sigma\tau}{2} erfc(\frac{b}{\sqrt{2}\tau} + \frac{\sigma\tau}{\sqrt{2}}) - \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}((\frac{b}{\tau})^2+2\sigma b)},$$

$$\hat{\Omega}^*_1(-\frac{b}{\tau}) = -\frac{1}{\sqrt{2\pi}} e^{(-\frac{b}{\tau})^2},$$

$$\hat{\Psi}^*_m(-\frac{b}{\tau}) = \sigma\tau\hat{\Psi}^*_{m-1}(-\frac{b}{\tau}) - \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}((\frac{b}{\tau})^2+2\sigma b)} H^*_{m-1}(z)$$

and

$$\hat{\Omega}^*_m(-\frac{b}{\tau}) = -\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(\frac{b}{\tau})^2} H^*_{m-1}(-\frac{b}{\tau}). \qquad (4.41)$$

A recurrence relation for α_m^{K-1} for $m = 1, 2, \ldots, N$ can be formed, namely

$$\alpha_m^{K-1} = \frac{\sigma}{m}\left[\alpha_{m-1}^{K-1} + \frac{e^{-r\Delta t}}{(m-1)!\tau^{m-1}\sqrt{2\pi}} e^{-\frac{1}{2}(\frac{b}{\tau})^2} H_{m-2}^{*}(-\frac{b}{\tau})\right]. \tag{4.42}$$

The equation (4.42) are the $\boldsymbol{\alpha}^{K-1}$ values for $m = 2, 3, \ldots, N$, with the following initial conditions,

$$\alpha_0^{K-1} = \frac{e^{-r\Delta t}}{2}\left[erfc(\frac{b}{\sqrt{2}\tau}) + e^{\sigma b + \frac{1}{2}\sigma^2\tau^2} erfc(-\frac{b}{\sqrt{2}\tau} - \frac{\sigma\tau}{\sqrt{2}})\right], \tag{4.43}$$

and

$$\alpha_1^{K-1} = -\frac{\sigma}{2} e^{-r\Delta t + \sigma b + \frac{1}{2}\sigma^2\tau^2} erfc(\frac{b}{\sqrt{2}\tau} + \frac{\sigma\tau}{\sqrt{2}}). \tag{4.44}$$

Now European put option prices can be evaluated.

4.2.3 Results and Analysis

The normalised Fourier-Hermite method has the same representation as the Fourier-Hermite method. One of the differences in obtaining the polynomial form are the form are the recurrence relations. The Delta function used in the normalised Fourier-Hermite approach does not include the exponential (2^m) function. Therefore, the issue of large **N** (the number of basis functions) that arise in the previous method, may be alleviated, to a certain extent, in the normalised approach.

As with the previous method, we can compare the normalised method against the Black Scholes formula.

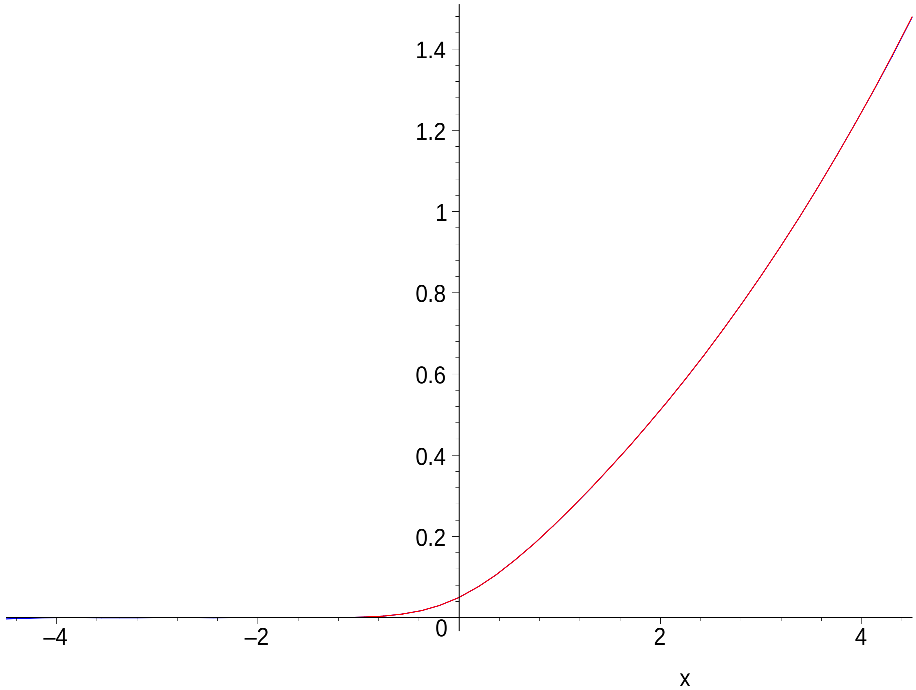

Figure 4.1: A normalised Fourier-Hermite expansion (blue curve) and Black Scholes formula (red curve) for a European call with $\sigma = 0.20$, $r = 0.08$, $T = 0.25$ and strike price, $X = \$100$. The normalised Fourier-Hermite expansion was derived for 4 time steps and 32 basis functions.

If we compare the figure 4.1 to the Fourier-Hermite figure 3.1, we see that the normalised method is a better representation for asset values further away from the strike price. If we take a closer look at the difference between the expansion approximation and the Black-Scholes formula.

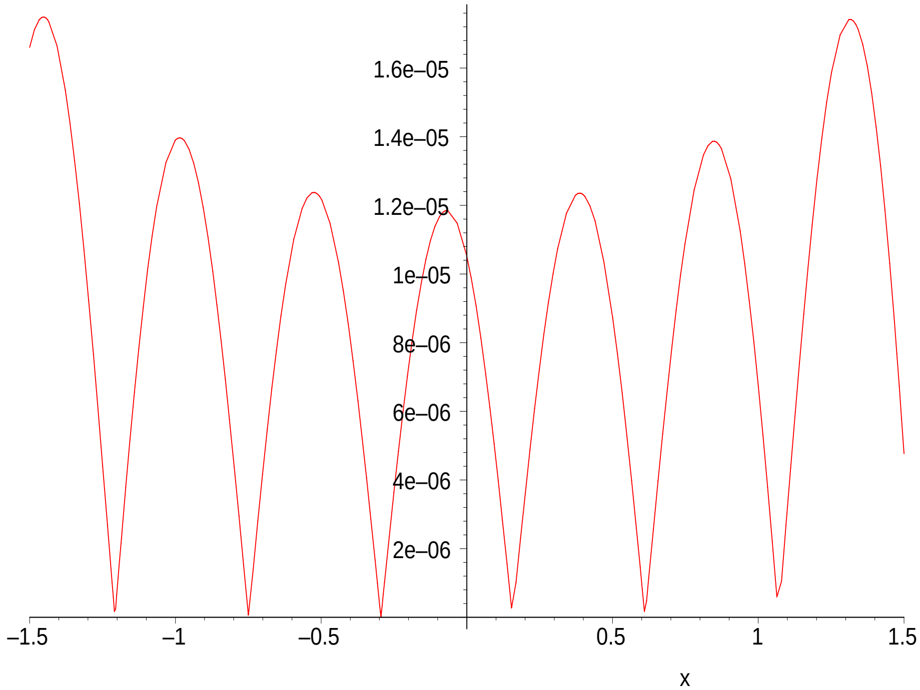

Figure 4.2: The absolute error of a normalised Fourier-Hermite expansion vs Black Scholes analytical solution for a European call $K = 4$, $N = 32$, $\sigma = 0.20$, $r = 0.08$, $T = 0.25$ and strike price of \$100.

Figures 4.1 and 4.2 shows the normalised method gives better results for asset values further from the strike price. However, options with this strike price (\$100) would not be written for these asset values. Closer to the strike price, the non-normalised method is better. Table 4.1 shows a numerical representation of the normalised Fourier-Hermite expansion method for a set of model parameters and 4 time steps. As with the non-normalised method, the absolute error is used to measure the accuracy of the prices.

Basis Functions	Asset Price ($)	Black-Scholes	Fourier-Hermite	Absolute Error
8	80	0.069017900	−0.006431766	7.54E−2
16			0.088458600	1.94E−2
32			0.069881862	8.64E−4
64			0.069009754	8.15E−6
128			0.069296396	2.78E−4
	90	1.0254530	0.97144092	5.40E−2
			1.0069471	1.85E−2
			1.0242163	1.24E−3
			1.0254483	4.70E−6
			1.0254531	1.00E−7
	100	5.0169820	5.1595170	1.43E−1
			5.0392292	2.22E−2
			5.0180316	1.05E−3
			5.0169829	9.00E−7
			5.0169781	3.90E−6
	110	12.620446	12.537716	8.27E−2
			12.597114	2.33E−2
			12.619442	1.00E−3
			12.620449	3.00E−6
			12.620442	4.00E−6
	120	22.066563	21.945115	1.21E−1
			22.076090	9.53E−3
			22.067815	1.25E−3
			22.066550	1.30E−5
			22.066718	1.55E−4

Table 4.1: Normalised Fourier-Hermite - European call option for 4 time steps, $\sigma = 0.20$, $r = 0.08$, $T = 0.25$, strike price of $100 and for various basis functions. Single precision was used to calculate the values.

Table 4.1 shows that for asset values of $90 to $110 and the number of basis functions used is approximately 64 to 128, the results for the normalised approach are as good as, if not better, than those evaluated for the non-normalised method.

Table 4.2 presents some numerical results using the same model parameters as table 4.1, but is performed with double precision.

Basis Functions	**Asset Price ($)**	**Black-Scholes**	**Fourier-Hermite**	**Absolute Error**
8	80	0.069017900		$7.54E{-2}$
16				$1.94E{-2}$
32				$8.64E{-4}$
64				$8.15E{-6}$
128				$2.78E{-4}$
	90	1.0254530		$5.40E{-2}$
				$1.85E{-2}$
				$1.24E{-3}$
				$4.70E{-6}$
				$1.00E{-7}$
	100	5.0169820		$1.43E{-1}$
				$2.22E{-2}$
				$1.05E{-3}$
				$9.00E{-7}$
				$3.90E{-6}$
	110	12.620446		$8.27E{-2}$
				$2.33E{-2}$
				$1.00E{-3}$
				$3.00E{-6}$
				$4.00E{-6}$
	120	22.066563		$1.21E{-1}$
				$9.53E{-3}$
				$1.25E{-3}$
				$1.30E{-5}$
				$1.55E{-4}$

Table 4.2: Normalised Fourier-Hermite - European call option for 4 time steps, $\sigma = 0.20$, $r = 0.08$, $T = 0.25$, strike price of $100 and for various basis functions. Double precision was used to calculate the values.

The Table 4.3 shows the prices and errors for expansions of 4 time steps and varying number of basis functions.

Basis Functions	**Asset Price ($)**	**Black-Scholes**	**Fourier-Hermite**	**Absolute Error**
8	80	18.08888506397669	18.01343751006435	7.54$E-2$
16			18.10832984562124	1.94$E-2$
32			18.08974907818617	8.64$E-4$
64			18.08887723039570	7.83$E-6$
128			18.08888506434852	3.72$E-10$
	90	9.045321064809460	8.991311526733935	5.40$E-2$
			9.026817995821769	1.85$E-2$
			9.044085364654334	1.24$E-3$
			9.045318182920060	2.88$E-6$
			9.045321065259884	4.50$E-10$
	100	3.036847936937940	3.179388674293241	1.43$E-1$
			3.059099678525065	2.23$E-2$
			3.037902053276338	1.05$E-3$
			3.036853028959601	5.09$E-6$
			3.036847937209272	2.71$E-10$
	110	0.6403158326585500	0.5575876168848387	8.27$E-2$
			0.6169841497201888	2.33$E-2$
			0.6393131629084357	1.00$E-3$
			0.6403188161068562	2.98$E-6$
			0.6403158329955574	3.37$E-10$
	120	0.08642753228261400	$-0.03501199899050 38$	1.21$E-1$
			0.09596066895079108	9.53$E-3$
			0.08768541887430010	1.26$E-3$
			0.08641956163094973	7.97$E-6$
			0.08642753286933688	5.87$E-10$

Table 4.3: Normalised Fourier-Hermite - European put option for 4 time steps, $\sigma = 0.20$, $r = 0.08$, $T = 0.25$, strike of $100 and for various basis functions. Double precision was used to calculate the values.

The parameters used for the call option (table 4.2) are identical to those used for the put option (table 4.3). For this parameter set, the normalised Fourier-Hermite expansion approximations work better for most of the put options compared to the call options. Varying the parameters have differing affects on the approximation using both types of Fourier-Hermite expansion techniques. An ability to find the optimal parameters would be advantageous.

4.3 American Put Options

The path integral (3.72) and early exercise point for the normalised method is the same, the difference being the Fourier set up which will incorporate normalised Hermite polynomials. Therefore, re-presenting the path integral

$$V^{k-1}(\xi_{k-1}) = \frac{e^{-r\Delta t}}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-(\xi_k - \mu(\xi_{k-1}))^2} V^k(\sqrt{2\Delta t}\,\xi_k)\, d\xi_k, \tag{4.45}$$

where V is the value of the American put option unexercised. If we denote F^{k-1} as the value of the option at time t_{k-1} and since an American option can be exercised at any time, ξ^*_{k-1} is introduced to denote the optimal exercise point, then

$$F^{k-1}(\xi_{k-1}) = \begin{cases} V^{k-1}(\xi_{k-1}), & \xi^*_{k-1} < \xi_{k-1} < \infty \\ 1 - e^{\sigma \xi_{k-1}}, & -\infty < \xi_{k-1} < \xi^*_{k-1} \end{cases}, \tag{4.46}$$

Since the American put option can be exercised at any time, the path integral (4.45) is split into two parts,

$$V^{k-1}(\xi_{k-1}) = h^{k-1}(\xi_{k-1}) + \frac{e^{-r\Delta t}}{\sqrt{\pi}} \int_{\frac{\xi^*_k}{\sqrt{2\Delta t}}}^{\infty} e^{-(\xi_k - \mu(\xi_{k-1}))^2} V^k(\sqrt{2\Delta t}\,\xi_k)\, d\xi_k, \tag{4.47}$$

where

$$h^{k-1}(\xi_{k-1}) = \frac{e^{-r\Delta t}}{\sqrt{\pi}} \int_{-\infty}^{\frac{\xi^*_k}{\sqrt{2\Delta t}}} e^{-(\xi_k - \mu(\xi_{k-1}))^2} (1 - e^{\sigma\sqrt{2\Delta t}\,\xi_k})\, d\xi_k. \tag{4.48}$$

The integral in (4.48) is the payoff or early exercise component of the path integral (4.47). The early exercise point ξ^*_{k-1} is the value of ξ which solves

$$V^{k-1}(\xi) = 1 - e^{\sigma\xi}. \tag{4.49}$$

So, the initial set up of the American Option is made, the following normalised Fourier-Hermite series expansions are introduced

$$V^{k-1}(\xi_{k-1}) \backsimeq \sum_{q=0}^{N} \alpha_q^{k-1} H_q^*(\xi_{k-1}), \tag{4.50}$$

and

$$h^{k-1}(\xi_{k-1}) \backsimeq \sum_{q=0}^{N} \gamma_q^{k-1} H_q^*(\xi_{k-1}). \tag{4.51}$$

As with the non normalised method, the most appropriate manner in evaluating V^0 is to treat the two integrals separately. Therefore, recurrence relations are formed for the coefficients of the normalised Fourier-Hermite series, the two parts are joined for final evaluation of the American put option price. Therefore, substituting (4.50) into (4.47) gives

$$\sum_{q=0}^{N} \alpha_q^{k-1} H_q^*(\xi_{k-1}) = \frac{e^{-r\Delta t}}{\sqrt{\pi}} \int_{\frac{\xi_k^*}{\sqrt{2\Delta t}}}^{\infty} e^{-(x_k-\mu(x_{k-1}))^2} V^k(\sqrt{2\Delta t}\,\xi_k)\, dx_k, \tag{4.52}$$

and substituting (3.78) into (3.75) gives

$$\sum_{q=0}^{N} \gamma_q^{k-1} H_q^*(\xi_{k-1}) = \frac{e^{-r\Delta t}}{\sqrt{\pi}} \int_{-\infty}^{\frac{\xi_k^*}{\sqrt{2\Delta t}}} e^{-(x_k-\mu(x_{k-1}))^2} (1 - e^{\sigma\sqrt{2\Delta t}\,\xi_k})\, dx_k. \tag{4.53}$$

Using the orthogonalisation property, (4.52) becomes

$$\begin{aligned}
&\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\xi_{k-1}^2} H_m^*(\xi_{k-1}) \sum_{q=0}^{N} \alpha_q^{k-1} H_q^*(\xi_{k-1})\, d\xi_{k-1} \\
&= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\frac{\xi_k^*}{\sqrt{2\Delta t}}} e^{-\xi_{k-1}^2} H_m^*(\xi_{k-1}) \frac{e^{-r\Delta t}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(\xi_k-\mu(\xi_{k-1}))^2} (1 - e^{\sigma\sqrt{2\Delta t}\,\xi_k})\, d\xi_k\, d\xi_{k-1} \\
&+ \frac{1}{\sqrt{\pi}} \int_{\frac{\xi_k^*}{\sqrt{2\Delta t}}}^{\infty} e^{-\frac{1}{2}\xi_{k-1}^2} H_m^*(\xi_{k-1}) \frac{e^{-r\Delta t}}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-(\xi_k-\mu(\xi_{k-1}))^2} V^k(\sqrt{2\Delta t}\,\xi_k)\, d\xi_k\, d\xi_{k-1}.
\end{aligned} \tag{4.54}$$

Prior to forming the evaluation of $\boldsymbol{\alpha}^{k-1}$, the coefficients of $\boldsymbol{\gamma}^k$ require generation. The values of $\boldsymbol{\gamma}^k$ are found recursively, with $\boldsymbol{\gamma}^{K-1} = 0$. To assist in the evaluation of the elements in $\boldsymbol{\gamma}^k$, the following Hermite polynomial and mathematical properties are used,

$$H_0^*(x) = 1, \quad H_1^*(x) = x, \tag{4.55}$$

$$H_n^*(ax+b) = (ax+b)H_{n-1}^*(ax+b) - (n-1)H_{n-2}^*(ax+b), \tag{4.56}$$

$$\frac{d}{dt}H_n^*(ax+b) = a\ n\ H_{n-1}^*(ax+b), \tag{4.57}$$

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} 2te^{-\frac{t^2}{2}}\ dt = 0, \tag{4.58}$$

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{t^2}{2}}\ dt = 1, \tag{4.59}$$

$$\frac{1}{\sqrt{2\pi}}\int_{x}^{\infty} H_1^*(t)e^{-\frac{t^2}{2}}\ dt = \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}, \tag{4.60}$$

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} H_1^*(t)e^{-\frac{t^2}{2}}\ dt = -\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}, \tag{4.61}$$

and

$$erfc(x) = \frac{2}{\sqrt{\pi}}\int_{x}^{\infty} e^{-t^2}\ dt = 1 - erf(x). \tag{4.62}$$

So,

$$\begin{aligned}
&\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}\xi_{k-1}^2} H_m^*(\xi_{k-1}) \sum_{q=0}^{N} \gamma_q^{k-1} H_q^*(\xi_{k-1})\ d\xi_{k-1} \\
&= \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\frac{\xi_k^*}{\sqrt{2\Delta t}}} e^{-\xi_{k-1}^2} H_m^*(\xi_{k-1}) \frac{e^{-r\Delta t}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(\xi_k-\mu(\xi_{k-1}))^2}(1-e^{\sigma\sqrt{2\Delta t}\,\xi_k})\ d\xi_k\ d\xi_{k-1},
\end{aligned}$$

which simplifies to

$$\begin{aligned}
\gamma_m^{k-1} =& \frac{e^{-r\Delta t}}{\sqrt{\pi}}\int_{-\infty}^{\frac{\xi_k^*}{\sqrt{2\Delta t}}} \\
&\left[\frac{1}{m!\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-[(\xi_k-\mu(\xi_{k-1}))^2+\xi_{k-1}^2]} H_m^*(\xi_{k-1})\ d\xi_{k-1}\right](1-e^{\sigma\sqrt{2\Delta t}\,\xi_k})\ d\xi_k, \\
=& \frac{e^{-r\Delta t}}{m!\sqrt{\pi}}\int_{-\infty}^{\frac{\xi_k^*}{\sqrt{2\Delta t}}} I_m^*(\xi_k)(1-e^{\sigma\sqrt{2\Delta t}\,\xi_k})\ d\xi_k,
\end{aligned} \tag{4.63}$$

where,

$$I_m^*(\xi_k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-[(\xi_k - \mu(\xi_{k-1}))^2 + \xi_{k-1}^2]} H_m^*(\xi_{k-1})\, d\xi_{k-1}. \tag{4.64}$$

Using the analytical solution to $I_m(\xi_k)$ as presented in sub-section 3.2 gives,

$$\gamma_m^{k-1} = \frac{e^{-r\Delta t}}{m!\sqrt{\pi}} \int_{-\infty}^{\frac{\xi_k^*}{\sqrt{2\Delta t}}} \frac{\sqrt{2\Delta t}\, e^{-\frac{1}{2}(\frac{\sqrt{2\Delta t}\,\xi_k - b}{\tau})^2} H_m^*(\frac{\sqrt{2\Delta t}\,\xi_k - b}{\tau})}{\tau^{m+1}} (1 - e^{\sigma\sqrt{2\Delta t}\,\xi_k})\, d\xi_k, \tag{4.65}$$

and by performing a further substitution,

$$z = \frac{\sqrt{2\Delta t}\,\xi_k - b}{\tau},$$

(4.65) is simplified to,

$$\begin{aligned} \gamma_m^{k-1} &= \frac{e^{-r\Delta t}}{m!\tau^m\sqrt{2\pi}} \int_{-\infty}^{\frac{\xi_k^* - b}{\tau}} e^{-\frac{z^2}{2}} H_m^*(z)(1 - e^{\sigma\tau z + \sigma b})\, dz \\ &= \frac{e^{-r\Delta t}}{m!\tau^m\sqrt{2\pi}} \left[\int_{-\infty}^{z_k} e^{-\frac{z^2}{2}} H_m^*(z)\, dz - e^{\sigma b} \int_{-\infty}^{z_k} e^{-\frac{z^2}{2} + \sigma\tau z} H_m^*(z)\, dz \right], \end{aligned} \tag{4.66}$$

and

$$z_k = \frac{\xi_k^* - b}{\tau}. \tag{4.67}$$

Given (4.66), the values of vector $\boldsymbol{\gamma}^{k-1}$ can be evaluated. Beginning with $m = 0$,

$$\begin{aligned} \gamma_0^{k-1} &= \frac{e^{-r\Delta t}}{0!\tau^0\sqrt{2\pi}} \left[\int_{-\infty}^{z_k} e^{-\frac{z^2}{2}} H_0^*(z)\, dz - e^{\sigma b} \int_{-\infty}^{z_k} e^{-\frac{z^2}{2} + \sigma\tau z} H_0^*(z)\, dz \right], \\ &= \frac{e^{-r\Delta t}}{\sqrt{2\pi}} \left[\int_{-\infty}^{z_k} e^{-\frac{z^2}{2}}\, dz - e^{\sigma b} \int_{-\infty}^{z_k} e^{-\frac{z^2}{2} + \sigma\tau z}\, dz \right]. \end{aligned} \tag{4.68}$$

Using property (4.24) and completing the square of the Gaussian in the right integral, (4.68) becomes

$$\gamma_0^{k-1} = e^{-r\Delta t}\Bigg[\frac{1}{2}erfc(-\frac{z_k}{\sqrt{2}}) - \frac{e^{\sigma b+\frac{\sigma^2\tau^2}{2}}}{\sqrt{2\pi}}\int_{-\infty}^{z_k} e^{-\frac{1}{2}(z-\sigma\tau)^2}\,dz\Bigg],$$

$$= \frac{e^{-r\Delta t}}{2}\Bigg[erfc(-\frac{z_k}{\sqrt{2}}) - e^{\sigma b+\frac{\sigma^2\tau^2}{2}}erfc\bigg(-\frac{\sigma\tau}{\sqrt{2}}+\frac{z_k}{\sqrt{2}}\bigg)\Bigg]. \tag{4.69}$$

For $m = 1$,

$$\gamma_1^{k-1} = \frac{e^{-r\Delta t}}{1!\tau^1\sqrt{2\pi}}\Bigg[\int_{-\infty}^{z_k} e^{-\frac{z^2}{2}}H_1^*(z)\,dz - e^{\sigma b}\int_{-\infty}^{z_k} e^{-\frac{z^2}{2}+\sigma\tau z}H_1^*(z)\,dz\Bigg],$$

$$= \frac{e^{-r\Delta t}}{\tau\sqrt{2\pi}}\Bigg[\int_{-\infty}^{z_k} ze^{-\frac{z^2}{2}}\,dz - e^{\sigma b+\frac{\sigma^2\tau^2}{2}}\int_{-\infty}^{z_k} ze^{-\frac{1}{2}(z-\sigma\tau)^2}\,dz\Bigg]. \tag{4.70}$$

Using properties (4.61) and (4.24), (4.70) becomes

$$\gamma_1^{k-1} = \frac{e^{-r\Delta t}}{\tau}\Bigg[-\frac{1}{\sqrt{2\pi}}e^{-\frac{z_k^2}{2}} + \frac{e^{\sigma b-\frac{z_k^2}{2}+\sigma\tau z_k}}{\sqrt{2\pi}} + \frac{\sigma\tau e^{\sigma b+\frac{\sigma^2\tau^2}{2}}}{2}erfc\bigg(-\frac{\sigma\tau}{\sqrt{2}}-\frac{z_k}{\sqrt{2}}\bigg)\Bigg]. \tag{4.71}$$

A proof of (4.71), can be found in appendix B.2.1.

For $m = 2, 3, \ldots, N$,

$$\gamma_m^{k-1} = \frac{e^{-r\Delta t}}{\tau^m}\Bigg[\Theta_m^{k-1} - \Phi_m^{k-1}\Bigg], \tag{4.72}$$

where

$$\Theta_m^{k-1} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z_k} e^{-\frac{z^2}{2}}H_m^*(z)\,dz, \tag{4.73}$$

and

$$\Phi_m^{k-1} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z_k} e^{-\frac{z^2}{2}+\sigma\tau z}H_m^*(z)\,dz. \tag{4.74}$$

and using,

$$H_n^*(ax+b) = (ax+b)H_{n-1}^*(ax+b) - (n-1)H_{n-2}^*(ax+b), \quad forn > 1,$$

becomes

$$\Theta_m^{k-1} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z_k} e^{-\frac{z^2}{2}} (zH_{m-1}^*(z) - (m-1)H_{m-2}^*(z))\, dz, \tag{4.75}$$

and

$$\Phi_m^{k-1} = \frac{e^{\sigma b + \frac{\sigma^2\tau^2}{2}}}{\sqrt{2\pi}} \int_{-\infty}^{z_k} e^{-\frac{1}{2}(z-\sigma\tau)^2} (zH_{m-1}^*(z) - (m-1)H_{m-2}^*(z))\, dz. \tag{4.76}$$

An analytical form for (4.75) is determined using properties, (4.21) and (4.60), and along with integration by parts gives

$$\Theta_m^{k-1} = -\frac{1}{\sqrt{2\pi}} e^{-\frac{z_k^2}{2}} H_{m-1}^*(z_k). \tag{4.77}$$

where z_k is given by (4.67). The proof for (4.77) can be found in Appendix B.2.2. For Φ, a recurrence relation is built using properties, (4.21) and (4.60), and along with integration by parts gives

$$\Phi_m^{k-1} = -\frac{e^{\sigma b + \frac{\sigma^2\tau^2}{2}}}{\sqrt{2\pi}} e^{-\frac{1}{2}(z_k-\sigma\tau)^2} H_{m-1}^*(z_k) + \sigma\tau\Phi_{m-1}^{k-1}. \tag{4.78}$$

The proof for (4.78) can also be found in Appendix B.2.3 leading to the evaluation (B.28). Therefore,

$$\begin{aligned} \gamma_m^{k-1} =& \frac{e^{-r\Delta t}}{m!\tau^m} \Bigg[-\frac{1}{\sqrt{2\pi}} e^{-\frac{z_k^2}{2}} H_{m-1}^*(z_k) + \frac{e^{\sigma b - \frac{z_k^2}{2} + \sigma\tau z_k}}{\sqrt{2\pi}} H_{m-1}^*(z_k) \\ & - \sigma\tau e^{\sigma b + \frac{\sigma^2\tau^2}{2}} \Phi_{m-1}^{k-1} \Bigg], \end{aligned} \tag{4.79}$$

To obtain a recurrence relation for γ_m^{k-1}, Φ_{m-1}^{k-1} is replaced with γ_{m-1}^{k-1} by rearranging

$$\gamma_{m-1}^{k-1} = \frac{e^{-r\Delta t}}{(m-1)!\tau^{m-1}} \Big[\Theta_{m-1}^{k-1} - \Phi_{m-1}^{k-1} \Big], \tag{4.80}$$

for Φ_{m-1}^{k-1} and substituting into (4.79). Therefore, (4.79) becomes

$$\gamma_m^{k-1} = \frac{\sigma}{m}\gamma_{m-1}^{k-1} + \frac{e^{-r\Delta t - \frac{z_k^2}{2}}}{m!\tau^m\sqrt{2\pi}} \Big[H_{m-1}^*(z_k)(e^{\sigma b + \sigma\tau z_k} - 1) + \sigma\tau H_{m-2}^*(z_k) \Big]. \tag{4.81}$$

The proof for (4.81) can be found in Appendix B.2.4.

Therefore, given (4.81), (4.54) becomes

$$\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}\xi_{k-1}^2} H_m^*(\xi_{k-1}) \sum_{q=0}^{N} \alpha_q^{k-1} H_q^*(\xi_{k-1})\, d\xi_{k-1} = \gamma_m^{k-1}$$
$$+\frac{1}{\sqrt{\pi}}\int_{\frac{\xi_k^*}{\sqrt{2\Delta t}}}^{\infty} e^{-\frac{1}{2}\xi_{k-1}^2} H_m^*(\xi_{k-1}) \frac{e^{-r\Delta t}}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-(\xi_k-\mu(\xi_{k-1}))^2} V^k(\sqrt{2\Delta t}\,\xi_k)\, d\xi_k\, d\xi_{k-1},$$

which simplifies to,

$$\alpha_m^{k-1} = \gamma_m^{k-1} + \frac{e^{-r\Delta t}}{m!\sqrt{\pi}} \int_{\frac{\xi_k^*}{\sqrt{2\Delta t}}}^{\infty}$$
$$\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}[2(\xi_k-\mu(\xi_{k-1}))^2+\xi_{k-1}^2]} H_m^*(\xi_{k-1}) d\xi_{k-1}\right] V^k(\sqrt{2\Delta t}\,\xi_k)\, d\xi_k,$$
$$= \frac{e^{-r\Delta t}}{m!\sqrt{\pi}} \int_{\frac{\xi_k^*}{\sqrt{2\Delta t}}}^{\infty} V^k(\sqrt{2\Delta t}\,\xi_k) I_m^*(\xi_k)\, d\xi_k, \tag{4.82}$$

where,

$$I_m^*(\xi_k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}[2(\xi_k-\mu(\xi_{k-1}))^2+\xi_{k-1}^2]} H_m^*(\xi_{k-1})\, d\xi_{k-1}. \tag{4.83}$$

Using the analytical solution to $I_m^*(\xi_k)$ as presented in sub-section 3.2 gives,

$$\alpha_m^{k-1} = \gamma_m^{k-1} + \frac{e^{-r\Delta t}}{m!\sqrt{\pi}} \int_{\frac{\xi_k^*}{\sqrt{2\Delta t}}}^{\infty} \frac{\sqrt{\Delta t}\, e^{-\frac{1}{2}(\frac{\sqrt{2\Delta t}\,\xi_k-b}{\tau})^2} H_m^*(\frac{\sqrt{2\Delta t}\,\xi_k-b}{\tau})}{\tau^{m+1}} V^k(\sqrt{2\Delta t}\,\xi_k)\, d\xi_k, \tag{4.84}$$

and by performing a further substitution,

$$z = \frac{\sqrt{2\Delta t}\,\xi_k - b}{\tau},$$

(4.84) is simplified to,

$$\alpha_m^{k-1} = \gamma_m^{k-1} + \frac{e^{-r\Delta t}}{m!\tau^m\sqrt{2\pi}} \int_{\frac{\xi_k^*-b}{v}}^{\infty} e^{-\frac{z^2}{2}} H_m^*(z) V^k(\tau z + b)\, dz. \tag{4.85}$$

Finally, a normalised Fourier-Hermite series is introduce for V^k to complete the relationship between $\boldsymbol{\alpha}^{k-1}$ and $\boldsymbol{\alpha}^k$,

$$V^k(\xi_k) \backsimeq \sum_{n=0}^{N} \alpha_n^k H_n^*(\xi_k), \tag{4.86}$$

and substituting (4.86) into (4.84) gives

$$\begin{aligned}\alpha_m^{k-1} &= \gamma_m^{k-1} + \frac{e^{-r\Delta t}}{m!\tau^m\sqrt{2\pi}} \int_{\frac{\xi_k^*-b}{\upsilon}}^{\infty} e^{-\frac{z^2}{2}} H_m^*(z) \sum_{n=0}^{N} \alpha_n^k H_n^*(\tau z + b)\, dz, \\ &= \gamma_m^{k-1} + \frac{e^{-r\Delta t}}{m!\tau^m\sqrt{2\pi}} \sum_{n=0}^{N} \alpha_n^k \int_{\frac{\xi_k^*-b}{\upsilon}}^{\infty} e^{-\frac{z^2}{2}} H_m^*(z) H_n^*(\tau z + b)\, dz. \end{aligned} \tag{4.87}$$

The expression (4.87) can be rewritten into

$$\alpha_m^{k-1} = \gamma_m^{k-1} + \sum_{n=0}^{N} \alpha_n^k A_{m,n}^k, \tag{4.88}$$

where

$$A_{m,n}^k = \frac{e^{-r\Delta t}}{m!\tau^m\sqrt{2\pi}} \int_{z_k}^{\infty} e^{-\frac{z^2}{2}} H_m^*(z) H_n^*(\tau z + b)\, dz. \tag{4.89}$$

and

$$z_k = \frac{\xi_k^* - b}{\tau}. \tag{4.90}$$

The elements of matrix $\mathbf{A}^k$ require evaluation with matrix $\mathbf{A}^k$ these elements changing for each time step because the optimal early exercise point, ξ_k^*, will differ. Therefore, for each time step, the coefficients, $\boldsymbol{\alpha}^{k-1}$ are found.

So element $A_{0,0}^k$ is given by

$$\begin{aligned} A_{0,0}^k &= \frac{e^{-r\Delta t}}{0!\tau^0\sqrt{2\pi}} \int_{z_k}^{\infty} e^{-\frac{z^2}{2}} H_0^*(z) H_0^*(\tau z + b)\, dz \\ &= \frac{e^{-r\Delta t}}{\sqrt{2\pi}} \int_{z_k}^{\infty} e^{-\frac{z^2}{2}}\, dz, \end{aligned} \tag{4.91}$$

therefore using (4.62),

$$A^k_{0,0} = \frac{e^{-r\Delta t}}{2} erfc(\frac{z_k}{\sqrt{2}}). \tag{4.92}$$

The next element $A^k_{0,1}$ is given by,

$$\begin{aligned} A^k_{0,1} &= \frac{e^{-r\Delta t}}{0!\tau^0\sqrt{2\pi}} \int_{z_k}^{\infty} e^{-\frac{z^2}{2}} H^*_0(z) H^*_1(\tau z + b)\, dz \\ &= \frac{e^{-r\Delta t}}{\sqrt{2\pi}} \Bigg[\tau \int_{z_k}^{\infty} z e^{-\frac{z^2}{2}} dz + b \int_{z_k}^{\infty} e^{-\frac{z^2}{2}}\, dz \Bigg]. \end{aligned} \tag{4.93}$$

Using properties (4.60) and (4.62), (4.93) becomes,

$$A^k_{0,1} = e^{-r\Delta t} \Bigg[\frac{b}{2}\, erfc(\frac{z_k}{\sqrt{2}}) + \frac{\tau}{\sqrt{2\pi}} e^{-\frac{z_k^2}{2}} \Bigg]. \tag{4.94}$$

For $m = 0$ and $n = 2, 3, \ldots, N$,

$$\begin{aligned} A^k_{0,n} &= \frac{e^{-r\Delta t}}{0!\tau^0\sqrt{2\pi}} \int_{z_k}^{\infty} e^{-\frac{z^2}{2}} H^*_0(z) H^*_n(\tau z + b)\, dz \\ &= \frac{e^{-r\Delta t}}{\sqrt{2\pi}} \int_{z_k}^{\infty} e^{-\frac{z^2}{2}} H^*_n(\tau z + b)\, dz, \end{aligned} \tag{4.95}$$

and using property (4.56), (4.95) can be expressed as,

$$\begin{aligned} A^k_{0,n} &= \frac{e^{-r\Delta t}}{\sqrt{2\pi}} \int_{z_k}^{\infty} e^{-\frac{z^2}{2}} \Bigg[(\tau z + b) H^*_{n-1}(\tau z + b) - (n-1) H^*_{n-2}(\tau z + b) \Bigg]\, dz, \\ &= \frac{e^{-r\Delta t}}{\sqrt{2\pi}} \int_{z_k}^{\infty} \tau z e^{-\frac{z^2}{2}}\, dz + b A^k_{0,n-1} - (n-1) A^k_{0,n-2}, \\ &= \frac{\tau e^{-r\Delta t}}{\sqrt{2\pi}} e^{-\frac{z_k^2}{2}} H^*_{n-1}(\tau z_k + b) + b A^k_{0,n-1} + (\tau^2 - 1)(n-1) A^k_{0,n-2}. \end{aligned} \tag{4.96}$$

For the value of $m = 1$ and $n = 0$,

$$A^k_{1,0} = \frac{e^{-r\Delta t}}{1!\tau^1\sqrt{2\pi}} \int_{z_k}^{\infty} e^{-\frac{z^2}{2}} H^*_1(z) H^*_0(\tau z + b)\, dz,$$

$$= \frac{e^{-r\Delta t}}{\tau\sqrt{2\pi}} \int_{z_k}^{\infty} z e^{-\frac{z^2}{2}}\, dz, \tag{4.97}$$

and using property (4.60), (4.97) becomes

$$A^k_{1,0} = \frac{e^{-r\Delta t}}{\tau\sqrt{2\pi}} e^{-\frac{z_k^2}{2}}. \tag{4.98}$$

For $m > 1$ and $n = 0$ and using property (4.56) gives,

$$A^k_{m,0} = \frac{e^{-r\Delta t}}{m!\tau^m\sqrt{\pi}} \int_{z_k}^{\infty} e^{-\frac{z^2}{2}} H^*_m(z) H^*_0(\tau z + b)\, dz,$$

$$= \frac{e^{-r\Delta t}}{m!\tau^m\sqrt{2\pi}} \int_{z_k}^{\infty} e^{-\frac{z^2}{2}} \left[z H^*_{m-1}(z) - (m-1) H^*_{m-2}(z)\right] dz,$$

$$= \frac{e^{-r\Delta t}}{m!\tau^m\sqrt{2\pi}} \int_{z_k}^{\infty} z e^{-\frac{z^2}{2}} H^*_{m-1}(z)\, dz - (m-1) A^k_{m-2,0}. \tag{4.99}$$

Using integration by parts, (4.99) reduces to,

$$A^k_{m,0} = \frac{e^{-r\Delta t}}{m!\tau^m\sqrt{2\pi}} e^{-\frac{z_k^2}{2}} H^*_{m-1}(z_k). \tag{4.100}$$

For general m and n and using property (4.56) for $H^*_m(z)$ gives,

$$A^k_{m,n} = \frac{e^{-r\Delta t}}{m!\tau^m\sqrt{2\pi}} \int_{z_k}^{\infty} e^{-\frac{z^2}{2}} H^*_m(z) H^*_n(\tau z + b)\, dz,$$

$$= \frac{e^{-r\Delta t}}{m!\tau^m\sqrt{2\pi}} \int_{z_k}^{\infty} e^{-\frac{z^2}{2}} H^*_n(\tau z + b) \left[z H^*_{m-1}(z) - (m-1) H^*_{m-2}(z)\right] dz,$$

$$= \frac{e^{-r\Delta t}}{m!\tau^m\sqrt{2\pi}} \int_{z_k}^{\infty} z e^{-\frac{z^2}{2}} H^*_{m-1}(z) H^*_n(\tau z + b)\, dz - (m-1) A^k_{m-2,n}. \tag{4.101}$$

Using integration by parts, (4.101) reduces to,

$$A^k_{m,n} = \frac{n}{m} A^k_{m-1,n-1} + \frac{e^{-r\Delta t}}{m!\tau^m \sqrt{2\pi}} e^{-\frac{z_k^2}{2}} H^*_{m-1}(z_k) H^*_n(\tau z_k + b). \tag{4.102}$$

Prior to finding all coefficients, as with the European options, the coefficients for the first time step $\boldsymbol{\alpha}^{K-1}$ are evaluated. Since the American put and European put at the first time step are equivalent, the expression (4.42) and (4.43) are used. Namely,

$$\alpha^{K-1}_m = \frac{\sigma}{\sqrt{m}} \left[\alpha^{K-1}_{m-1} + \frac{e^{-r\Delta t}}{\tau^{m-1}\sqrt{2\pi(m-1)}} e^{-\frac{1}{2}(\frac{b}{\tau})^2} H^*_{m-2}(-\frac{b}{\tau}) \right]. \tag{4.103}$$

for $m = 2, 3, \ldots N$, with initial conditions,

$$\begin{aligned} \alpha^{K-1}_0 =& \frac{e^{-r\Delta t}}{2} \left[erfc(\frac{b}{\sqrt{2}\tau}) + e^{\sigma b + \frac{1}{2}\sigma^2\tau^2} erfc(-\frac{b}{\sqrt{2}\tau} - \frac{\sigma\tau}{\sqrt{2}}) \right], \\ \alpha^{K-1}_1 =& -\frac{\sigma}{2} e^{-r\Delta t + \sigma b + \frac{1}{2}\sigma^2\tau^2} erfc(\frac{b}{\sqrt{2}\tau} + \frac{\sigma\tau}{\sqrt{2}}). \end{aligned} \tag{4.104}$$

Therefore, we are able to evaluate various American put options using the expressions and recurrence relations evaluated throughout this section.

4.3.1 Results and Analysis

Considering the normalised method realised similar recurrence relations and expressions, it is fair to say that we would expect similar results to those presented for the non-normalised examples. However, due to the oscillating nature of the two methods, a different parameter set N (the number of basis functions) and K (the number of time steps) may be required to achieve identical (or similar) prices. Table 4.4 shows some results for American put options when the number of basis functions used is 40.

Asset Price ($)	**Binomial Method**	**Norm F-H 60 steps**	**Norm F-H 80 Steps**	**Norm F-H 100 Steps**
80	20.000000	20.000000	20.000000	20.000000
90	10.037663	10.098764	10.112576	10.122039
100	3.224898	3.184179	3.202324	3.213752
110	0.665410	0.654379	0.666875	0.674574
120	0.088795	0.064938	0.067110	0.068702

Table 4.4: Normalised Fourier-Hermite - American put option for various time steps and 40 basis functions with $\sigma = 0.20$, $r = 0.08$, $T = 0.25$, strike of \$100. Double precision was used to calculate the values. The Binomial Method prices are optimal approximations (using various N) and were calculated based on Cox et al. (1979).

Table 4.4 are prices for various time steps given 40 basis functions. Table 4.5 presents accurate prices for 40 basis functions and the optimal number of time steps used to evaluate option price.

Asset Price ($)	**Binomial Method**	**F-H 40 Basis Functions**	**Norm F-H 40 Basis Functions**
90	10.037663	10.037439(43)	10.036482(25)
100	3.224898	3.224875(172)	3.225033(130)
110	0.665410	0.665417(363)	0.665395(77)
120	0.088795	0.088800(83)	0.079935(4)

Table 4.5: Comparison of the Fourier-Hermite expansion methods for various American put option prices for 40 basis functions and the best number of time steps with $\sigma = 0.20$, $r = 0.08$, $T = 0.25$, strike of \$100. Double precision was used to calculate the values. The number of time steps are in brackets after the price. The Binomial Method prices are optimal approximations (using various N) and were calculated based on Cox et al. (1979).

Table 4.6 presents American put option prices for 40 time steps using basis functions which evaluate these prices accurately.

Asset Price ($)	**Binomial Method**	**F-H 40 steps**	**Norm F-H 40 steps**
90	10.037663	10.037604(35)	10.047204(60)
100	3.224898	3.254261(12)	3.223757(23)
110	0.665410	0.631602(20)	0.630337(40)
120	0.088795	0.089520(13)	0.088798(30)

Table 4.6: Comparison of the Fourier-Hermite expansion methods for various American put option prices for 40 time steps and the best basis functions with $\sigma = 0.20$, $r = 0.08$, $T = 0.25$, strike of $100. Double precision was used to calculate the values. The number of basis functions are in brackets after the price. The Binomial Method prices are optimal approximations (using various N) and were calculated based on Cox et al. (1979).

Tables 4.5 and 4.6 show that in some cases the normalised method evaluated better prices than the non-normalised method. Also, some of the prices presented in these tables were comparable to the Binomial method. However, in other cases, the prices evaluated were not so accurate (refer to table 4.6, asset price $110).

4.4 Conclusion

The normalised Fourier-Hermite expansion presented in this chapter involves the same approach offered in Chapter 3 with the difference being the use of normalised Hermite orthogonal polynomials. These normalised polynomials have different properties to those of the non-normalised type. The Delta function for instance has only a factorial coefficient whereas the non-normalised has a factorial and exponential term.

The recurrence relations and expressions formed to evaluate the prices of the European and American put options had similarities to those presented in Chapter 3. The differences occurring due to the nature of the properties associated with the normalised Hermite polynomials (refer to (4.19) and (4.21)).

It is clear that both types of polynomials used lead to similar results, with one not better than the other in most cases. Computation times are relatively the same. In both types of Hermite polynomials, the optimal approximation would vary for differing N (basis functions) and K (number of time steps). So, it would be advantageous if some *a-prior* knowledge of these parameters were known to give an optimal approximation.

Chapter 5

Interpolation Polynomials, Quadrature Rules and European Options

The approaches to be offered in this chapter involves the use of interpolation polynomials and quadrature rules. This numerical method is an alternative to the spectral method covered in Chapter 3 and the normalised Fourier expansion version as presented in Chapter 4 and those traditionally used such as Monte Carlo simulation, finite differences and trees. The reason for the use of these methods for the path integral framework is due to the fact that a closed form solution is not possible at every time step. The approach being presented converts the path integral into a sum of "closed interval" integrals, which accurately prices options by utilising interpolation polynomials and various quadrature (Newton-Cotes) rules.

5.1 Introduction

The formulation of the path integral framework, as presented in Chapter 2, has no closed form solution at each time step. Therefore, alternative (numerical) methods are required to find an approximate solution, in this case, an option price. As elaborated in Chapter 1, the most common methods used in solving path integrals involves the use of Monte Carlo simulation and spectral methods like those presented in Chapters 3 and 4.

Interpolation polynomials have been used in many fields of mathematics and science. The polynomials, created from a set of data points (nodes), are used to represent a function that when manipulated in the context of the problem can give a closed form solution. Issues which affect this method of interpolation include, the types of polynomials to be used and grid allocations (discretization schemes).

Section 5.2 transforms the path integral into a form which allows for an efficient approximation to be found. An investigation of the weight function (kernel) used in the path integral framework is made. The weight function is used to transform the interval of integration from an infinite to closed form.

Section 5.3 investigates the formulation and implementation of the interpolation. Section 5.4 presents a thorough analysis of interpolating $f^k(x_k)$ for European options. The analysis covers the effects of the model parameters on the method. That is, how does changes in the Volatility, Interest Rates and Time to Expiry affect the method and the results. Various node allocations are presented.

An alternative to using interpolation polynomials to solve the modified path integral are quadrature rules. Section 5.5 presents results for European options utilizing various Newton-Cotes quadrature rules (endpoint, midpoint, trapezoidal and Simpson's). It will shown that the Newton-Cotes rules are very accurate and fast to compute (especially for the European options). Section 5.6 concludes the chapter, summarising the most important aspects of the approaches offered.

5.2 The Path Integral Framework

The path integral, as in Chapter 2 (equation (2.43)), is given by the following expression for $k = K, K-1, \dots, 1$,

$$f^{k-1}(x_{k-1}) = \upsilon \int_{-\infty}^{\infty} e^{-(x_k - \mu(x_{k-1}))^2} f^k(\sqrt{2\Delta t}\, x_k)\, dx_k, \tag{5.1}$$

where

$$\mu(x_{k-1}) = \frac{1}{\sqrt{2\Delta t}}(x_{k-1} + \frac{1}{\sigma}(r - \frac{1}{2}\sigma^2)\Delta t), \qquad \upsilon = \frac{e^{-r\Delta t}}{\sqrt{\pi}}$$

and

$$x_j = \frac{1}{\sigma}\ln(S), \qquad j = 0, 1, 2, \dots, K.$$

To assist in the implementation of this approach, a transformation of (5.1) is required by replacing $\sqrt{2\Delta t}\, x_k$ with x_k^* and neglecting the $*$ for convenience, so that

$$f^{k-1}(x_{k-1}) = \Omega \int_{-\infty}^{\infty} e^{-\left(\frac{x_k}{\sqrt{2\Delta t}} - \mu(x_{k-1})\right)^2} f^k(x_k)\, dx_k \tag{5.2}$$

where

$$\Omega = \frac{e^{-r\Delta t}}{\sqrt{2\Delta t \pi}}.$$

The reason for the change of variable is to simplify the interpolation of f^k and also improve the efficiency of the interpolation process. There is no closed form solution to the path integral (5.1) or (5.2), with the only exception being when $k = K$ namely,

$$f^{\mathrm{K}-1}(x_{\mathrm{K}-1}) = \Omega \int_{-\infty}^{\infty} e^{-(\frac{x_{\mathrm{K}}}{\sqrt{2\Delta t}} - \mu(x_{\mathrm{K}-1}))^2} f^{\mathrm{K}}(x_{\mathrm{K}})\, dx_{\mathrm{K}}, \tag{5.3}$$

where $f^{\mathrm{K}}(x_{\mathrm{K}})$ is the payoff function. For a **call** option,

$$f^{\mathrm{K}}(x_{\mathrm{K}}) = \begin{cases} e^{\sigma x_{\mathrm{K}}} - 1, & x_{\mathrm{K}} > 0 \\ 0, & x_{\mathrm{K}} \leq 0 \end{cases} \tag{5.4}$$

and for a **put** option,

$$f^{\mathrm{K}}(x_{\mathrm{K}}) = \begin{cases} e^{-\sigma x_{\mathrm{K}}} - 1, & x_{\mathrm{K}} < 0 \\ 0, & x_{\mathrm{K}} \geq 0. \end{cases} \tag{5.5}$$

Therefore, for a call option, the path integral $f^{\mathrm{K}-1}(x_{\mathrm{K}-1})$ has a closed form, which is derived by first substituting (5.4) into (5.3),

$$f^{\mathrm{K}-1}(x_{\mathrm{K}-1}) = \Omega \int_0^{\infty} e^{-(\frac{x_{\mathrm{K}}}{\sqrt{2\Delta t}} - \mu(x_{\mathrm{K}-1}))^2} (e^{\sigma x_{\mathrm{K}}} - 1) \, dx_{\mathrm{K}}. \tag{5.6}$$

The integral (5.6) can be split into two parts and using basic index laws, transforms (5.6) to,

$$f^{\mathrm{K}-1}(x_{\mathrm{K}-1}) = \Omega \left[\int_0^{\infty} e^{-(\frac{x_{\mathrm{K}}}{\sqrt{2\Delta t}} - \mu(x_{\mathrm{K}-1}))^2 + \sigma x_{\mathrm{K}}} \, dx_{\mathrm{K}} - \int_0^{\infty} e^{-(\frac{x_{\mathrm{K}}}{\sqrt{2\Delta t}} - \mu(x_{\mathrm{K}-1}))^2} \, dx_{\mathrm{K}} \right], \tag{5.7}$$

and by completing the square in the first integral of (5.7) gives,

$$f^{\mathrm{K}-1}(x_{\mathrm{K}-1}) = \Omega \left[\omega \int_0^{\infty} e^{-(x_{\mathrm{K}} - \mu(x_{\mathrm{K}-1}) - \sqrt{\tau})^2} \, dx_{\mathrm{K}} - \int_0^{\infty} e^{-(x_{\mathrm{K}} - \mu(x_{\mathrm{K}-1}))^2} \, dx_{\mathrm{K}} \right], \tag{5.8}$$

where

$$\omega = e^{-r\Delta t + \sigma\mu(x_{\mathrm{K}-1})\sqrt{2\Delta t} + \tau}, \tag{5.9}$$

and

$$\tau = \frac{\Delta t}{2}\sigma^2. \tag{5.10}$$

Therefore, the integrands in (5.8) take the form of a Gaussian function. With a change of variables,

$$\begin{aligned} u_1 &= x_{\mathrm{K}} - \mu(x_{\mathrm{K}-1}) - \sqrt{\tau} \\ u_2 &= x_{\mathrm{K}} - \mu(x_{\mathrm{K}-1}) \end{aligned} \tag{5.11}$$

equation (5.8) becomes

$$f^{\mathrm{K}-1}(x_{\mathrm{K}-1}) = \Omega\Big[\omega\int_{-\mu(x_{\mathrm{K}-1})-\sqrt{\tau}}^{\infty} e^{-u_1^2}\,du_1 - \int_{-\mu(x_{\mathrm{K}-1})}^{\infty} e^{-u_2^2}\,du_2\Big]. \tag{5.12}$$

Recalling

$$erfc(x) = \frac{2}{\sqrt{\pi}}\int_x^{\infty} e^{-t^2}\,dt = 1 - erf(x) \tag{5.13}$$

(5.12) can be defined in terms of the error function,

$$f^{\mathrm{K}-1}(x_{\mathrm{K}-1}) = e^{-r\Delta t}\Big[\omega\Big(\frac{1}{2}+\frac{1}{2}erf(\mu(x_{\mathrm{K}-1})+\sqrt{\tau})\Big) - \frac{1}{2} - \frac{1}{2}erf(\mu(x_{\mathrm{K}-1}))\Big], \tag{5.14}$$

with ω and τ given by (5.9) and (5.10) respectively. For the put option,

$$f^{\mathrm{K}-1}(x_{\mathrm{K}-1}) = \Omega\int_{-\infty}^{0} e^{-(\frac{x_{\mathrm{K}}}{\sqrt{2\Delta t}}-\mu(x_{\mathrm{K}-1}))^2}(e^{-\sigma x_{\mathrm{K}}}-1)\,dx_{\mathrm{K}}. \tag{5.15}$$

The integral (5.15) can be split into two parts as with the call option,

$$f^{\mathrm{K}-1}(x_{\mathrm{K}-1}) = \Omega\Big[\int_{-\infty}^{0} e^{-(\frac{x_{\mathrm{K}}}{\sqrt{2\Delta t}}-\mu(x_{\mathrm{K}-1}))^2-\sigma x_{\mathrm{K}}}\,dx_{\mathrm{K}} - \int_{-\infty}^{0} e^{-\frac{(x_{\mathrm{K}}}{\sqrt{2\Delta t}}-\mu(x_{\mathrm{K}-1}))^2}\,dx_{\mathrm{K}}\Big]. \tag{5.16}$$

Again, completing the square in the first integral of (5.16) gives,

$$f^{\mathrm{K}-1}(x_{\mathrm{K}-1}) = \Omega\Big[\omega\int_{-\infty}^{0} e^{-(x_{\mathrm{K}}-\mu(x_{\mathrm{K}-1})+\sqrt{\tau})^2}\,dx_{\mathrm{K}} - \int_{-\infty}^{0} e^{-(x_{\mathrm{K}}-\mu(x_{\mathrm{K}-1}))^2}\,dx_{\mathrm{K}}\Big]. \tag{5.17}$$

Using the change of variable (5.11), (5.17) becomes

$$f^{\mathrm{K}-1}(x_{\mathrm{K}-1}) = \Omega\Big[\omega\int_{-\infty}^{-\mu(x_{\mathrm{K}-1})+\sqrt{\tau}} e^{-u_1^2}\,du_1 - \int_{-\infty}^{-\mu(x_{\mathrm{K}-1})} e^{-u_2^2}\,du_2\Big]. \tag{5.18}$$

The equation (5.18) can be expressed in terms of the error function,

$$f^{\mathrm{K}-1}(x_{\mathrm{K}-1}) = e^{-r\Delta t}\Big[\frac{1}{2} - \frac{1}{2}erf(\mu(x_{\mathrm{K}-1})) - \omega(\frac{1}{2} - \frac{1}{2}erf(\mu(x_{\mathrm{K}-1})+\sqrt{\tau}))\Big], \tag{5.19}$$

with ω and τ given by (5.9) and (5.10) respectively. Since we know $f^{\mathrm{K}-1}(x_{\mathrm{K}-1})$ for both a put and call, it can be used for both a European or American style option.

At this point it is worth noting that it is evident that a closed form solution for all the subsequent integrals cannot be found due to the form of $f^{\mathrm{K}-1}(x_{\mathrm{K}-1})$.

Therefore, $f^{\mathrm{K}-1}(x_{\mathrm{K}-1})$ can be transformed/approximated which allows the path integral to have a closed form. Interpolating $f^{\mathrm{K}-1}(x_{\mathrm{K}-1})$ into many polynomials will have the desired impact. Each subsequent $f^k(x_k)$ is in turn interpolated until $f^0(x_0)$ is found.

Due to the nature of path integrals, the errors associated with interpolation are always carried forward to each subsequent time step. Therefore, the interpolation method and the discretisation of nodes are very important. Minimising errors in early time steps can only assist in achieving an accurate approximation.

A major issue that needs to be addressed prior to investigating the interpolation method, is obtaining a closed interval for each integral step. An investigation of the weight function is needed such that the integrals have a closed interval.

5.2.1 The Weight Function

In determining the most appropriate closed interval to interpolate over each time step, a thorough investigation of the weight function is required. The weight function in (5.2) is,

$$w(x_k, x_{k-1}) = \frac{1}{\sqrt{2\Delta t\pi}} e^{-\left(\frac{x_k}{\sqrt{2\Delta t}} - \mu(x_{k-1})\right)^2}. \tag{5.20}$$

Figure 5.1 is a graphical representation of (5.20).

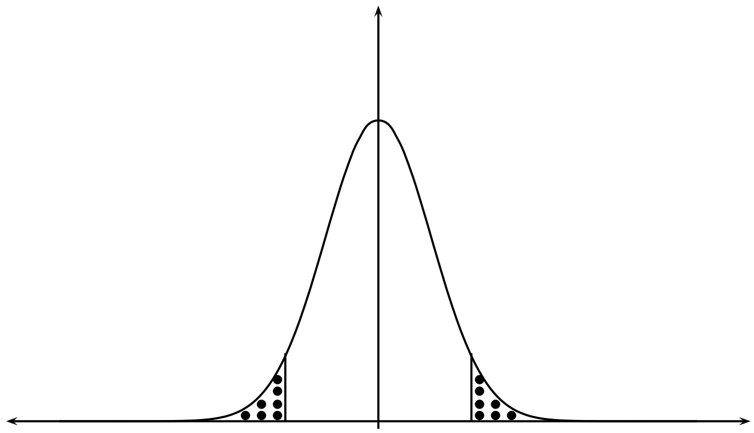

Figure 5.1: A graphical representation of the weight function

Given the nature of the weight in (5.2), and by setting the area under the tails to some *a-priori* bound, the infinite interval will convert to a closed interval with an associated error in doing so. Since the weight is symmetrical around the mean, the interval should take the form, $(\sqrt{2\Delta t}\,(L_{k-1}+\mu(x_{k-1})), \sqrt{2\Delta t}\,(R_{k-1}+\mu(x_{k-1})))$, where L_{k-1} is the left and R_{k-1} is the right side of the interval. The intervals formed were derived based on the Gaussian in the path integral being of the standard form, with a mean of 0 and a standard deviation of 1. Given the formulation of the closed interval, the path integral (5.2) will take the form,

$$\begin{aligned} f^{k-1}(x_{k-1}) =& \Psi(x_{k-1}, -\infty, \beta_1(x_{k-1})) \\ &+ \Psi(x_{k-1}, \beta_1(x_{k-1}), \beta_2(x_{k-1})) \\ &+ \Psi(x_{k-1}, \beta_2(x_{k-1}), \infty) \end{aligned} \tag{5.21}$$

where

$$\Psi(x_{k-1}, a, b) = \Omega \int_a^b e^{-(\frac{x_k}{\sqrt{2\Delta t}} - \mu(x_{k-1}))^2} f^k(x_k)\, dx_k \tag{5.22}$$

and

$$\beta_1(x_{k-1}) = \sqrt{2\Delta t}\,(L_{k-1} + \mu(x_{k-1})),$$
$$\beta_2(x_{k-1}) = \sqrt{2\Delta t}\,(R_{k-1} + \mu(x_{k-1})). \tag{5.23}$$

The path integral, $\Psi(x_{k-1}, \beta_1(x_{k-1}), \beta_2(x_{k-1}))$, has a closed interval so that interpolation is possible. Since the interval consists of the variable x_{k-1}, consideration must be given to the allocation of the intervals of integration for each time step. The path integral is backward recursive in nature, however, the optimal interval allocation must occur in a forward manner (i.e. for $k = 1, 2, \ldots, K-1$). All intervals are based on the value of the underlying x_0 (i.e the value of the underlying at the beginning of an options life). The value of the option is based on the payoff function (i.e. the boundary condition) and is used as the starting point in the path integral framework, with the option price calculated in a backward direction to $f^0(x_0)$.

To evaluate the option price f^0 for a certain underlying value, x_0, the interval of its integral and $f^1(x_1)$ is required. The interval of integration for $f^0(x_0)$, is dependent on the value of x_0. That is,

$$\Psi(x_0, \beta_1(x_0), \beta_2(x_0)) = \Omega \int_{\beta_1(x_0)}^{\beta_2(x_0)} e^{-(\frac{x_1}{\sqrt{2\Delta t}} - \mu(x_0))^2} f^1(x_1).dx_1. \tag{5.24}$$

The intervals of integration for $f^j(x_j)$, where $j = 1, 2, \ldots, K-2$, and $L_0 = R_0 = x_0$ being the value of the underlying, is determined by the pair of recursive equations,

$$\beta_1(L_j) = \sqrt{2\Delta t}\,(L_j + \mu(\beta_1(L_{j-1})))$$
$$\beta_2(R_j) = \sqrt{2\Delta t}\,(R_j + \mu(\beta_2(R_{j-1}))). \tag{5.25}$$

So, for each subsequent interval of integration, the previous interval values are used to determine the next. Table 5.1 is an example of the intervals of integration required when $K = 4$, with the value of the underlying, x_0, the intervals for each time step are given.

k	Function	Left Side	Right Side
3	$f^2(x_2)$	$L_3 = \beta_1(L_2)$	$R_3 = \beta_2(R_2)$
2	$f^1(x_1)$	$L_2 = \beta_1(L_1)$	$R_2 = \beta_2(R_1)$
1	$f^0(x_0)$	$L_1 = \beta_1(L_0)$	$R_1 = \beta_2(R_0)$

Table 5.1: An example of the intervals of integration used for pricing an option using 4 time steps.

Figure 5.2 is a graphical representation of the process involved in allocating the intervals of integration at each time step.

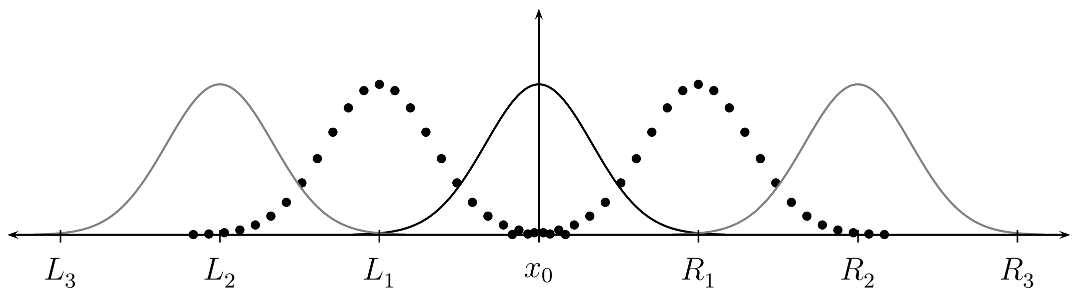

Figure 5.2: A graphical view point of the interval allocation for $K = 4$

So, starting with the underlying value x_0, the upper and lower bounds (R_1 and L_1) for the next time step are determined. The value of the upper bound (R_1) is then used to find the subsequent upper bound (R_2) and this process continues for the remaining time steps. The process is also performed for the lower bounds in the same manner.

Prior to using the recursive equations (5.25), determination of each L_j and R_j is required.

5.2.2 Closed Interval Allocation

Given the recursive equations (5.25), the optimal closed interval for each time step is found satisfying the following conditions,

$$\begin{aligned}\Psi(x_{k-1}, -\infty, \beta_1(x_{k-1})) &\leq \frac{\eta}{2} \\ \Psi(x_{k-1}, \beta_2(x_{k-1}), \infty) &\leq \frac{\eta}{2}.\end{aligned} \tag{5.26}$$

where η is an *a-priori* error set to a value close to zero.

Theoretically, the best option price possible is capped to the value of η. That is, if $\eta = 10^{-8}$ then the smallest error (difference between the approximated price and the so-called exact price) possible is 10^{-8}. However, this would only occur if the path integral had a closed form solution at each time step. Since this is not possible, errors associated with approximating the option price may vary from η.

The integrals (5.26), as (5.2), do not have a closed form solution. An approximation of $f^k(x^k)$ is required so that, firstly the integrals have a closed form solution and secondly, that the approximation is an upper bound to the exact $f^k(x^k)$.

From Black & Scholes (1973), any security price cannot be greater than the value of the asset or underlying itself. The upper bound for a call option therefore takes the following form,

$$f^k(x_k) < e^{\sigma x_k}, \tag{5.27}$$

and the upper bound for a put is

$$f^k(x_k) < e^{-\sigma x_k}. \tag{5.28}$$

Recalling (5.6) and (5.15) in Section 5.2, the integrals formed by using the upper bounds (5.27) and (5.28) are very similar.

$$\Psi_c(x_{k-1}, -\infty, \beta_1(x_{k-1})) < \frac{e^{-r\Delta t}}{\sqrt{\pi}} \int_0^\infty e^{-(\frac{x_k}{\sqrt{2\Delta t}} - \mu(x_{k-1}))^2} e^{\sigma x_k}\, dx_k \tag{5.29}$$

and

$$\Psi_p(x_{k-1}, \beta_2(x_{k-1}), \infty) < \frac{e^{-r\Delta t}}{\sqrt{\pi}} \int_0^{\infty} e^{-(\frac{x_k}{\sqrt{2\Delta t}} - \mu(x_{k-1}))^2} e^{-\sigma x_k}\, dx_k. \tag{5.30}$$

By referring to the first integral in (5.6) and (5.15), we complete the square so, a call takes the following form for the outer integrals,

$$\Psi_c(x_{k-1}, -\infty, \beta_1(x_{k-1})) < \frac{\omega}{\sqrt{2\Delta t \pi}} \int_{-\infty}^{\beta_1(x_{k-1})} e^{-(\frac{x_k}{\sqrt{2\Delta t}} - \mu(x_{k-1}) - \sqrt{\tau})^2}\, dx_k, \tag{5.31}$$

and

$$\Psi_c(x_{k-1}, \beta_2(x_{k-1}), \infty) < \frac{\omega}{\sqrt{2\Delta t \pi}} \int_{\beta_2(x_{k-1})}^{\infty} e^{-(\frac{x_k}{\sqrt{2\Delta t}} - \mu(x_{k-1}) - \sqrt{\tau})^2}\, dx_k. \tag{5.32}$$

For a put option,

$$\Psi_p(x_{k-1}, -\infty, \beta_1(x_{k-1})) < \frac{\omega}{\sqrt{2\Delta t \pi}} \int_{-\infty}^{\beta_1(x_{k-1})} e^{-(\frac{x_k}{\sqrt{2\Delta t}} - \mu(x_{k-1}) + \sqrt{\tau})^2}\, dx_k, \tag{5.33}$$

and

$$\Psi_p(x_{k-1}, \beta_2(x_{k-1}), \infty) < \frac{\omega}{\sqrt{2\Delta t \pi}} \int_{\beta_2(x_{k-1})}^{\infty} e^{-(\frac{x_k}{\sqrt{2\Delta t}} - \mu(x_{k-1}) + \sqrt{\tau})^2}\, dx_k, \tag{5.34}$$

with ω and τ given by (5.9) and (5.10) respectively. By performing a change of variable as in section 5.2 and recalling (5.13), firstly (5.31) and (5.32) become,

$$\Psi_c(x_{k-1}, -\infty, \beta_1(x_{k-1})) < \frac{\omega}{2}(1 - erf(L_{k-1} + \sqrt{\tau})) \tag{5.35}$$

and

$$\Psi_c(x_{k-1}, \beta_2(x_{k-1}), \infty) < \frac{\omega}{2}(1 - erf(R_{k-1} - \sqrt{\tau})). \tag{5.36}$$

For the put option, (5.33) and (5.34) become,

$$\Psi_p(x_{k-1}, -\infty, \beta_1(x_{k-1})) < \frac{\omega}{2}(1 - erf(L_{k-1} - \sqrt{\tau})) \tag{5.37}$$

and

$$\Psi_p(x_{k-1}, \beta_2(x_{k-1}), \infty) < \frac{\omega}{2}(1 - erf(R_{k-1} + \sqrt{\tau})). \tag{5.38}$$

Setting the outer integrals, (5.35), (5.36), (5.37) and (5.38) to an *a-priori* error margin, η, that is, the area under the tails, then values for L_{k-1} and R_{k-1} can be determined.

Using an asymptotic expansion given in Abramowitz & Stegun (1970),

$$1 - erf(x) \approx \frac{e^{-x^2}}{x\sqrt{\pi}}\left[1 - \frac{1}{2x^2} + O(x^{-4})\right]. \tag{5.39}$$

A general form for (5.35), (5.36),(5.37) and (5.38) is

$$\frac{\omega}{2}(1 - erf(X)) = \eta. \tag{5.40}$$

Since X is assumed to be large, substituting the first term in (5.39) into (5.40) is sufficient,

$$\frac{e^{-X^2}}{X\sqrt{\pi}} = \frac{2\eta}{\omega}, \tag{5.41}$$

and with some simple algebra,

$$\ln\left(\frac{2\eta\sqrt{\pi}}{\omega}\right) = -X^2 - \ln(X). \tag{5.42}$$

Since X is assumed to be large, (5.42) becomes,

$$\ln\left(\frac{2\eta\sqrt{\pi}}{\omega}\right) \approx -X^2, \tag{5.43}$$

and therefore

$$X \approx \sqrt{-\ln\left(\frac{2\eta\sqrt{\pi}}{\omega}\right)}. \tag{5.44}$$

Substituting (5.44) into (5.42) and solving for X gives,

$$X = \sqrt{-\ln\left(\frac{2\eta\sqrt{\pi}}{\omega}\right) - \frac{1}{2}\ln\left(\ln\left(\frac{\omega}{2\eta\sqrt{\pi}}\right)\right)}, \tag{5.45}$$

with ω is given by (5.9). Table 5.2 is an example of the intervals used for a particular European call option.

Asset Price ($)	η	Step	L_{k-1}	R_{k-1}
80	10^{-8}	3	-5.25402180800974	3.28294451560886
		2	-3.88650792296428	1.80435969849094
		1	-2.50707574146733	0.338140259899832
	10^{-16}	3	-7.19311250452033	5.22312967391776
		2	-5.17948762893245	3.09770361816799
		1	-3.15369033280151	0.984754851234016
	10^{-32}	3	-9.90611842381887	7.93666498266998
		2	-6.98829302802270	4.90668518241841
		1	-4.05815984534679	1.88922436377929

Table 5.2: European call option intervals of integration for $K = 4$, $\sigma = 0.20$, $r = 0.08$, $T = 0.25$ and strike of $100. Here η is the total error for the tails.

It is clear from Table 5.2, that the value of η has a significant influence on the intervals of integration. A balance is required between the value of η and the interpolation parameters. The wider the interval, the lower the accuracy of the interpolation. To improve the interpolation, a better grid allocation is required. However, to achieve this will lead to inefficiencies and greater computational effort.

5.3 Interpolation Polynomials

The objective of the interpolation is to convert the path integral (5.2), at each time step, to a form for which a closed form solution can be obtained. For a review of interpolation in general, we refer the reader to Atkinson (1989). de Boor (1978) gives a more detailed account on interpolation and the use of splines.

Issues which influence the interpolation include the number of partitions to be used (N) and the placement of nodes, the type of polynomials to be used and the sample data (including end points). Also, the values of the model parameters σ, r, T and K (time slicing) have their part to play. A change in any of these variables, invariably changes the final price. For example, a change in K, will require either a change in the value of N or the positioning of the node points to obtain a similar price.

The most commonly used functions for interpolating are polynomials of order q, in the form,

$$p(x) = \boldsymbol{\alpha}_0 + \alpha_1 x + \alpha_2 x^2 + \ldots + \alpha_{q-1} x^{q-1} + \alpha_q x^q. \tag{5.46}$$

To find the appropriate polynomials, discretization of the interval into N partitions is required. For example, Figure 5.3 is the grid for equally spaced nodes, given $n = 1, 2, \ldots, N-1, N$, takes the form (the grid allocation is an issue which is pursued later in this chapter),

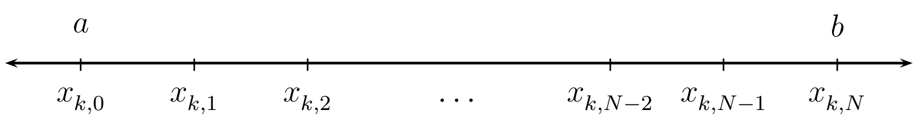

Figure 5.3: The discretization of x_k

From this point, results and analysis presented in this chapter are for Hermite interpolation polynomials (of order 4 (cubics)).

$$c^k(x, n) = \alpha_{0,n} + \alpha_{1,n} x + \alpha_{2,n} x^2 + \alpha_{3,n} x^3, \qquad n = 0, \ldots, N. \tag{5.47}$$

This method involves the interpolation of the European option, $f^k(x_k)$ (the American Put and a barrier option will be examined in the next chapter). By replacing the $f^k(x_k)$ with a series of polynomials (recalling that a closed form solution of $f^{\mathrm{K}}(x_{\mathrm{K}})$ is obtained by using the payoff functions (2.17) and (2.24)), the path integral (5.2) then becomes,

$$\Psi(x_{k-1}, \beta_1(x_{k-1}), \beta_2(x_{k-1})) = \Omega \sum_{n=1}^{N} \int_{x_{k,n-1}}^{x_{k,n}} e^{-\left(\frac{x_k}{\sqrt{2\Delta t}} - \mu(x_{k-1})\right)^2} c^k(x_k, n)\, dx_k \tag{5.48}$$

where Ω is given in (5.2).

Each component of the sum is an integral. Given the use of a Hermite interpolation polynomials, the components of the sum take the form,

$$I_q^{k-1}(x_{k-1}; a, b) = \sum_{q=0}^{3} \alpha_q \int_a^b e^{-(\frac{x_k}{\sqrt{2\Delta t}} - \mu(x_{k-1}))^2} x_k^q \, dx_k, \tag{5.49}$$

and from (5.48)

$$\Psi(x_{k-1}, \beta_1(x_{k-1}), \beta_2(x_{k-1})) = \Omega \sum_{n=1}^{N} \sum_{q=0}^{3} I_q^{k-1}(x_{k-1}; x_{k,n-1}, x_{k,n}). \tag{5.50}$$

Section 5.4 will present and analyse results with particular emphasis on the effects of node allocation and the various closed intervals used.

5.4 Interpolation and European Options

European style options are one of the simplest financial instruments to solve. It is wise to analyse thoroughly the affects of the method parameters such as N (number of partitions) and η (the *a-prior* bound used to close the path integral interval) for European options. Therefore, any findings from the analysis can easily be applied to more complex financial instruments, such as American or barrier options. Changes in the model parameters, such as K (time steps), σ (volatility), T (time to expiry) and r (interest rates), also have an affect on the option price.

Table 5.3 is a summary of European option prices using the Black-Scholes formula. These results are used when comparing the various approximations presented in later sections of this chapter. The errors presented are an absolute difference between the interpolation method (IPM) and Black-Scholes price.

Asset Price ($)	European Call Price ($)	European Put Price ($)
80	0.06901773330119	18.0888850639767
90	1.02545373413394	9.04532094721139
100	5.01698060626241	3.03684781734310
110	12.6204485019830	0.64031578148717
120	22.0665602016071	0.08642752091237

Table 5.3: Black-Scholes - European option prices with, $\sigma = 0.20$, $r = 0.08$, $T = 0.25$ and strike of $100.

The first part of the analysis involves the use of a fixed number of partitions, that are equally spaced at each time step. Since this node allocation is simple to implement, a thorough analysis of the model and method parameters are made.

5.4.1 Fixed Number of Partitions

In this node allocation, the number of nodes allocated at each time step are the same and are equally spaced over the interval of integration. Therefore, as the time step gets closer to $k = 0$, the distribution becomes denser (i.e the space between nodes (partition length) is decreasing). This is due to the fact that the interval of integration at the first time step is the widest and the last is the smallest (refer to table 5.2). With the intermediate interval lengths gradually decreasing.

Table 5.4 contains some numerical approximations for European call options with 128 partitions used at each time step.

Asset Price ($)	η	**IPM Result**	**Error**
80	10^{-8}	0.0690076240325474	$1.0109268638739191E-05$
	10^{-16}	0.0689726550385943	$4.5078262591853688E-05$
	10^{-32}	0.0688259946315314	$1.9173866965471135E-04$
90	10^{-8}	1.0254550797608930	$1.3456269492770345E-06$
	10^{-16}	1.0254601167095607	$6.3825756169838543E-06$
	10^{-32}	1.0254811418688603	$2.7407734916502779E-05$
100	10^{-8}	5.0170051428601026	$2.4536597691832140E-05$
	10^{-16}	5.0170889304529478	$1.0832419053696807E-04$
	10^{-32}	5.0174346307412199	$4.5402447880898977E-04$
110	10^{-8}	12.6204467004580998	$1.8015249403369182E-06$
	10^{-16}	12.6204414403296745	$7.0616533648371060E-06$
	10^{-32}	12.6204199493163536	$2.8552666686554318E-05$
120	10^{-8}	22.0665498972233856	$1.0304383724069055E-05$
	10^{-16}	22.0665161653561519	$4.4036250959611500E-05$
	10^{-32}	22.0663757561983438	$1.8444540876572102E-04$

Table 5.4: Interpolation method - European call option with 8 time steps, 128 partitions, $\sigma = 0.20$, $r = 0.08$, $T = 0.25$ and strike of $100. Here η is the total error for the tails (refer to (5.26)). Double precision was used to calculate the values.

Table 5.5 presents European put option prices for the same settings as table 5.4.

Asset Price (\$)	η	**IPM Result**	**Error**
80	10^{-8}	18.08629222663164	$2.592837345063992E{-3}$
	10^{-16}	18.08337631887139	$5.508745105312074E{-3}$
	10^{-32}	18.07740932173431	$1.147574224239223E{-2}$
90	10^{-8}	9.042396439929282	$2.924624880182303E{-3}$
	10^{-16}	9.039159958455835	$6.161106353629042E{-3}$
	10^{-32}	9.032634269541020	$1.268679526844341E{-2}$
100	10^{-8}	3.033601059225259	$3.246877712677690E{-3}$
	10^{-16}	3.030082741798865	$6.765195139071117E{-3}$
	10^{-32}	3.023154541128493	$1.369339580944290E{-2}$
110	10^{-8}	0.6366952507974828	$3.620581861078428E{-3}$
	10^{-16}	0.6327273319993394	$7.588500659221759E{-3}$
	10^{-32}	0.6247034663400692	$1.561236631849195E{-2}$
120	10^{-8}	0.08244929289049295	$3.978239392127833E{-3}$
	10^{-16}	0.07809231901426740	$8.335213268353394E{-3}$
	10^{-32}	0.06922013491199029	$1.720739737063049E{-2}$

Table 5.5: Interpolation method - European put option with 8 time steps, 128 partitions, $\sigma = 0.20$, $r = 0.08$, $T = 0.25$ and strike of \$100. Here η is the total error for the tails (refer to (5.26)). Double precision was used to calculate the values.

It is evident with this node distribution that as η decreases, the errors increase. This trend is due to the fact that as η becomes smaller, the interval lengths for the interpolation increase in size. The increase in interval lengths cause the partitions to be less dense. With a sparser distribution, the interpolation errors increase, which are then carried forward to proceeding time steps.

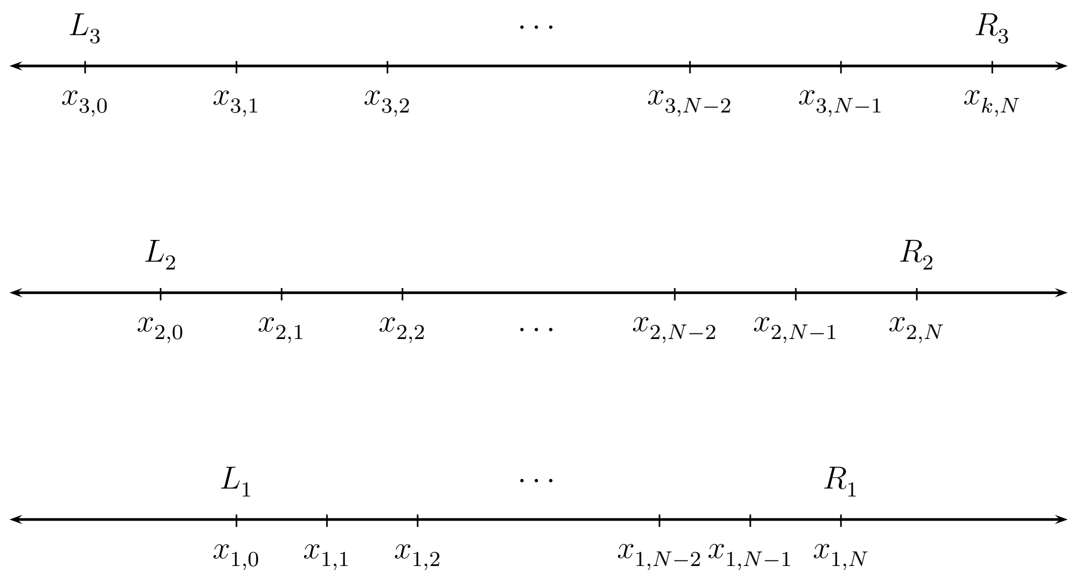

Figure 5.4: The discretization of x for $K = 4$ with a fixed number of partitions, equally spaced.

Figure 5.4 shows the discretization for each time step, given a fixed number of partitions that are equally spaced. As is illustrated in this figure, the densities of the distribution of nodes change at each time step. It is clear that even though a smaller η value theoretically gives a better approximation, this is countered by the decrease in density of the distribution of nodes for a wider interval of integration.

It is also clear that the closer two nodes are together, the better the interpolation becomes. However, too many interpolations can increase the error. Therefore, a compromise is required between the value of η used and the number of nodes being distributed throughout the interval of integration. In later sections, other distributions are used to alleviate the compromise between η and interpolation accuracy.

Prior to analysing these other distributions, an investigation of the effects of model parameters on the option price using the IPM is required. Since this node distribution is quite simple to implement, it is worthwhile investing time in understanding the effects of changing model parameter values have on the parameters of the method. That is, how do changes in K, σ, T and r affect the approximate option price, given certain values of N and η?

5.4.2 Parameter Analysis

Before investigating the method of interpolation in further detail, an in-depth analysis of model and method parameters is required. How do K, σ, T and r influence N and η, and vice-versa?

The first parameter to be investigated is the value of η. Since η controls the interval of integration, attempting to find an optimal η value is required. The plots presented were produced for a European call option.

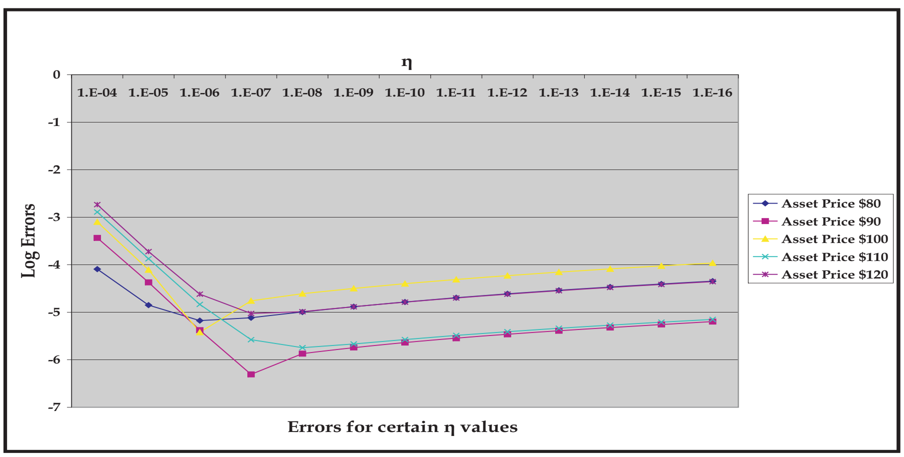

Figure 5.5: The effects of a changing η with $K = 8$, $N = 128$, $\sigma = 0.20$, $r = 0.08$, $T = 0.25$ and strike of \$100.

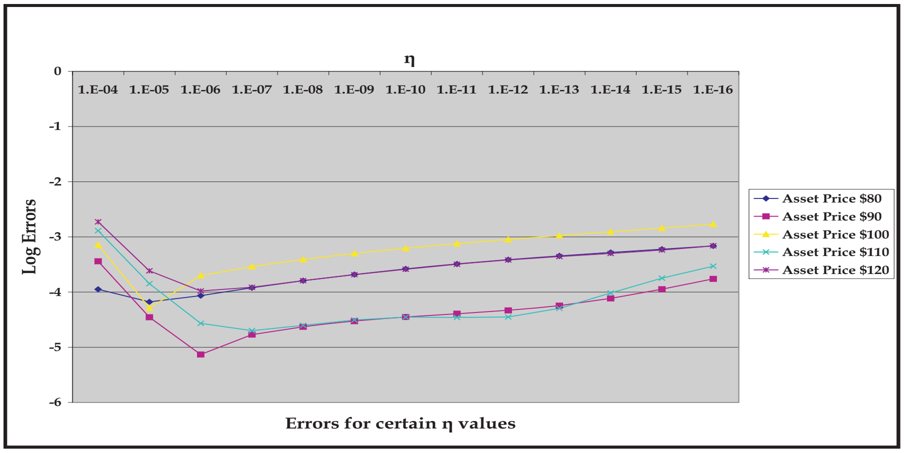

Figure 5.6: The effects of a changing η with $K = 8$, $N = 64$, $\sigma = 0.20$, $r = 0.08$, $T = 0.25$ and strike of \$100.

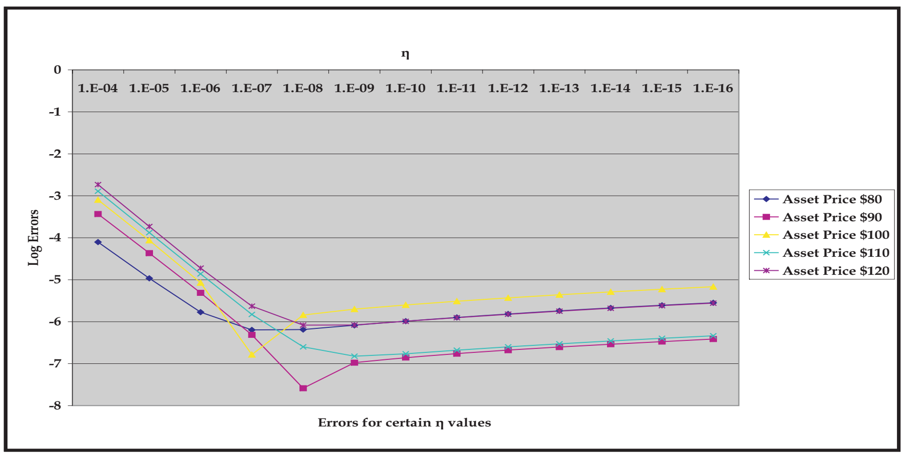

Figure 5.7: The effects of a changing η with $K = 8$, $N = 256$, $\sigma = 0.20$, $r = 0.08$, $T = 0.25$ and strike of \$100.

From figure 5.5, the optimal value of η is in the range $(10^{-6}, 10^{-8})$ for the parameter set $K = 8$, $N = 128$, $\sigma = 0.20$, $r = 0.08$, $T = 0.25$ and strike of \$100. It is worth noting that changes in this parameter set may change the optimal value of η.

Figure 5.6 shows an optimal η value in the range $(10^{-5}, 10^{-7})$. Figure 5.7 shows an optimal η value in the range $(10^{-7}, 10^{-9})$. The change in the optimal value is due to a change in the value of N. The change in the optimal range occurs due to the change in the number of partitions being used in the allocation of nodes. This means, for the case when $N = 64$, the interpolation is not as accurate. To compensate for the lack of accuracy, the value of η is increased. So, the node allocation will become denser. Figure 5.8 shows the difference in allocations, where interval (L_a, R_a) is for a larger η value compared to interval (L_b, R_b).

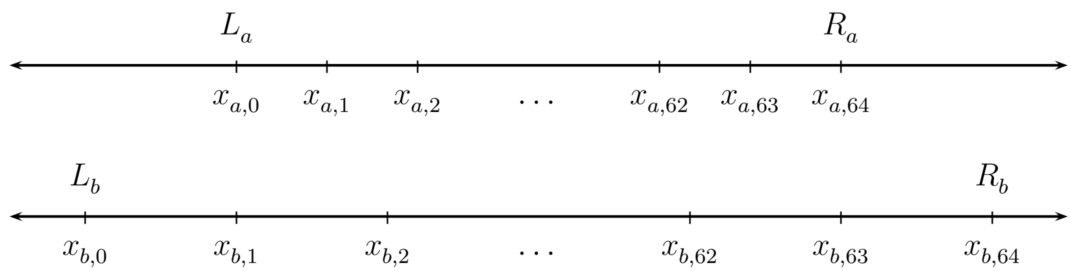

Figure 5.8: An example of the fixed number of nodes (equally spaced) discretization for $N = 64$ for varying interval of integration.

Increasing the value of η also restricts the best approximation achieved. Remembering the value of η is the point where the tails of the weight function are removed. Therefore, a compromise between η and N is needed. As Figures 5.5, 5.6 and 5.7 show, decreasing N invariably requires an increase in η.

Figures 5.9 - 5.11 show the effect on η when changing K from 8 to 6.

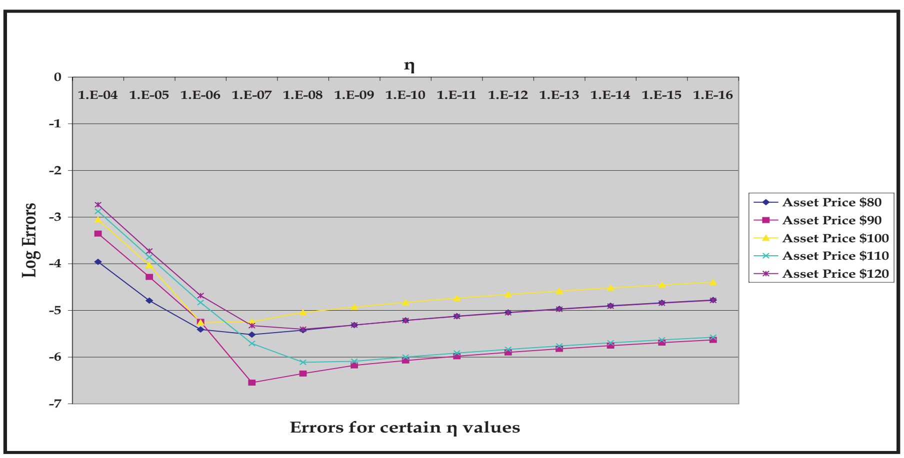

Figure 5.9: The effects of a changing η with $K = 6$, $N = 128$, $\sigma = 0.20$, $r = 0.08$, $T = 0.25$ and strike of \$100.

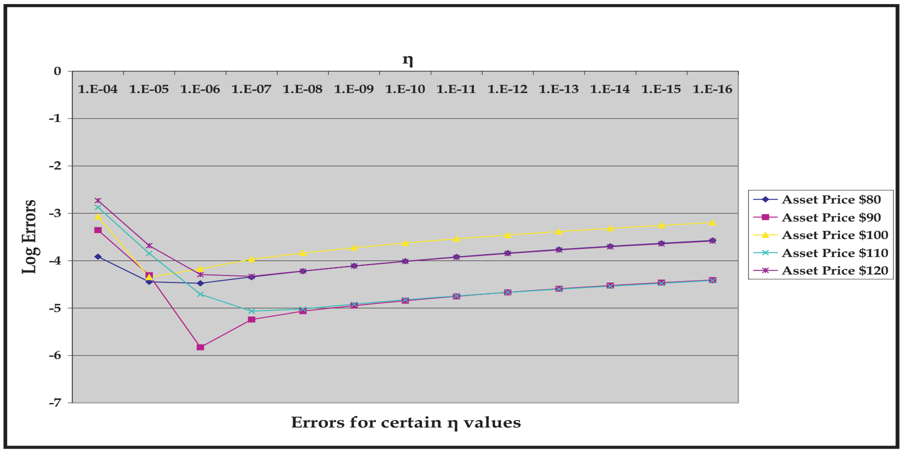

Figure 5.10: The effects of a changing η with $K = 6$, $N = 64$, $\sigma = 0.20$, $r = 0.08$, $T = 0.25$ and strike of \$100.

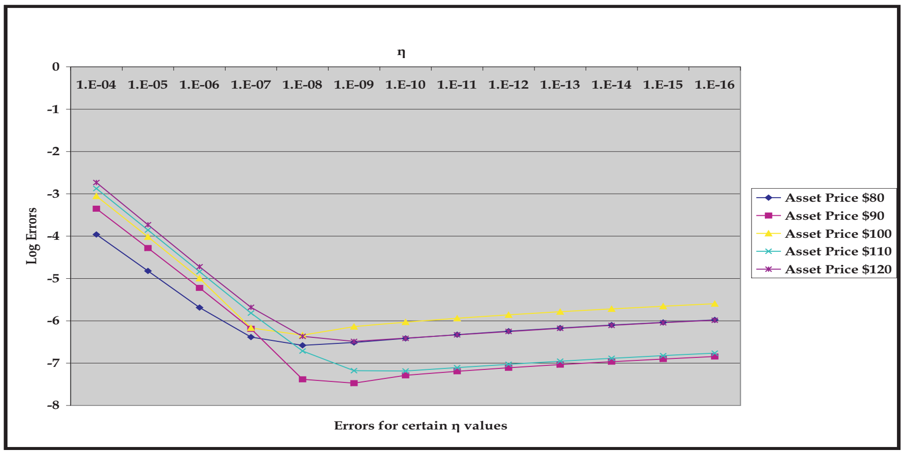

Figure 5.11: The effects of a changing η with $K = 6$, $N = 256$, $\sigma = 0.20$, $r = 0.08$, $T = 0.25$ and strike of \$100.

Comparing Figures 5.5 - 5.7 with Figures 5.9 - 5.11 shows a similar result in the optimal values for η. The only difference is the error obtained for each of the approximations. Appendix C.1.1 show results for optimal η for various K values. In analysing all these figures, it is clear that N has a major influence on the optimal η. The model parameter K, in combination with N and η, influences the final price. That is, given a certain N and η, there is a K which will give an improved approximation.

The interest rate, r, is the discounting factor on the value of money. In the analysis of the interest rate, various interest rates between 2% and 40% were used.

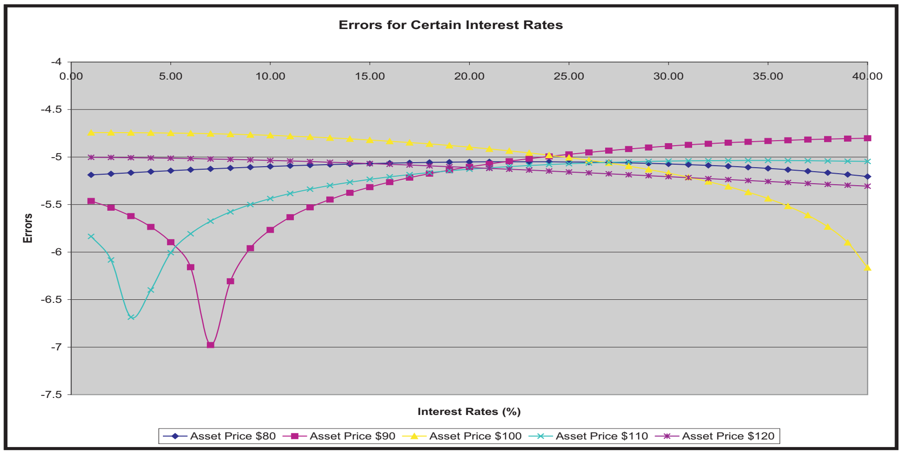

Figure 5.12: The effects of changing the Interest Rate with $K = 8$, $N = 128$, $\sigma = 0.20$, $T = 0.25$, $\eta = 10^{-7}$ and strike of $100.

From Figure 5.12, there exists an interest rate, in combination with a certain K, N and η, such that the approximation is optimal. However, it must be stated that the difference in approximations between all interest rates are similar.

The next parameter for analysis is σ, the volatility of the underlying. The volatilities used are between 5% and 50%.

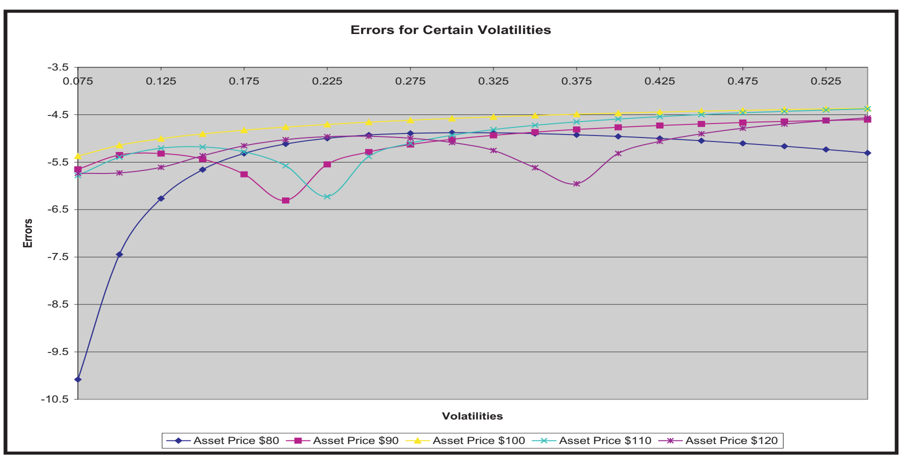

Figure 5.13: The effects of changing the Volatility with $K = 8$, $N = 128$, $\sigma = 0.20$, $T = 0.25$, $\eta = 10^{-7}$ and strike of $100.

The approximations, with volatility changes in Figure 5.13, behave similarly to those with interest rate changes. The only difference is that there may be more than one volatility value which gives a better approximation for a set of values for K, N and η. Therefore, it is possible for various local minimums to occur for a particular set of K, N and η, with one of these minima being the global minimum.

The final parameter to analyse is the Time to Expiry, T. The times used in this analysis include values between 0.1 year to 2 years.

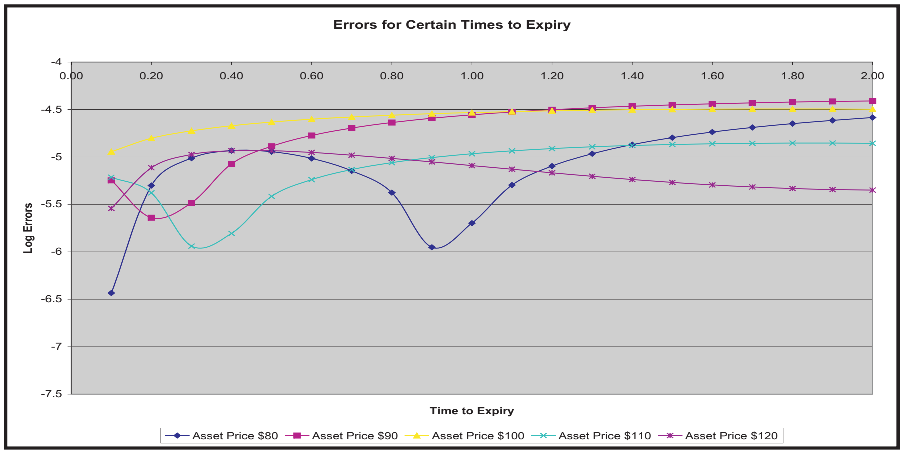

Figure 5.14: The effects of changing the Time to Expiry with $K = 8$, $N = 128$, $\sigma = 0.20$, $T = 0.25$, $\eta = 10^{-7}$ and strike of \$100.

The approximations, with Time to Expiry changes in Figure 5.14, behave similarly to those with interest rate and volatility changes.

It is obvious to find the optimal approximation is a multi-dimensional problem. Figure 5.15 shows the nature of the problem at hand.

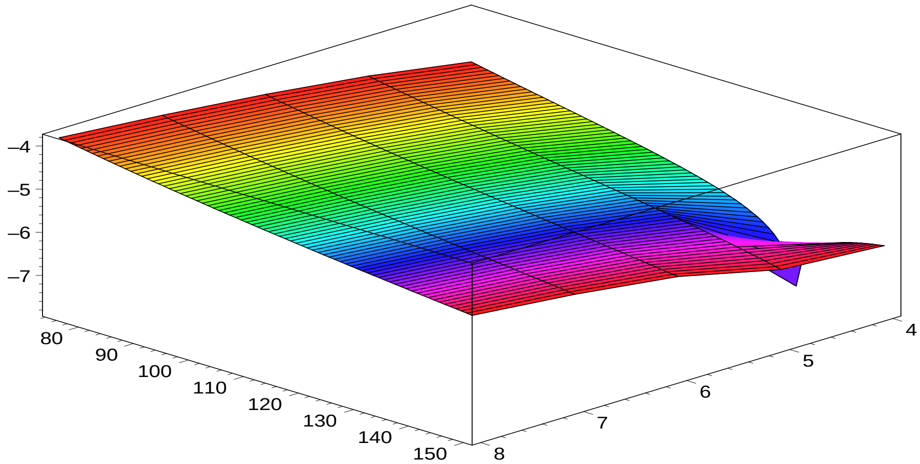

Figure 5.15: Approximations for various K (right axis) and N (left axis) with $\eta = 10^{-7}$, $\sigma = 0.20$, $T = 0.25$, asset price = \$100 and strike of \$100.

From figure 5.15 it is evident that the optimal approximation occurs when $K = 4$ and $N = 140$. This plot also shows other patterns such as when the number of time steps, K increases, to improve the approximation an increase in the number of partitions used is also required.

5.4.3 Fixed Spaced Partitions

The allocation of equally spaced partitions is an alternative to a fixed number of partitions. This distribution was not used with the intention to improve the results. The distribution, having identical densities, could lead to simpler analysis of the errors obtained. An obvious extension to this method would be to predict the best approximation for a certain set of parameters (σ, T, r, the asset and strike price). This extension could be achieved with most node distributions but should be easily implemented if the partitions were equally spaced. The previous allocation type, fixed number of partitions, which has varying densities from one time step to the next, requires analysis of errors for each interpolation. These calculations would require a greater computational effort.

As mentioned previously, theoretical error analysis is outside the scope of this thesis.

However, this analysis may assist in predicting the optimal method parameters (N and η) prior to approximating the option price. Table 5.6 contains European call option prices for various node spacing between 0.01 and 0.1.

η	Space (Total Partitions)	IPM Result	Error
10^{-8}	0.10 (173)	5.0170056697858438	$2.5063523432372614E-05$
	0.09 (192)	5.0169970139984361	$1.6407736024709463E-05$
	0.08 (216)	5.0169908039512894	$1.0197688878055278E-05$
	0.07 (246)	5.0169865302421375	$5.9239797262078397E-06$
	0.06 (288)	5.0169837421334647	$3.1358710537998657E-06$
	0.05 (345)	5.0169820480417426	$1.4417793313403759E-06$
	0.04 (431)	5.0169811159409443	$5.0967853340577740E-07$
	0.03 (574)	5.0169806737174278	$6.7455016572459670E-08$
	0.02 (861)	5.0169805093572686	$9.6905142937808719E-08$
	0.01 (1720)	5.0169804715224018	$1.3474000942759190E-07$
10^{-16}	0.10 (251)	5.0170058070948258	$2.5200832414612462E-05$
	0.09 (279)	5.0169971513073213	$1.6545044909804796E-05$
	0.08 (312)	5.0169909412589941	$1.0334996582844758E-05$
	0.07 (357)	5.0169866675485633	$6.0612861521591732E-06$
	0.06 (417)	5.0169838794404242	$3.2731780133521404E-06$
	0.05 (501)	5.0169821853506589	$1.5790882476607315E-06$
	0.04 (624)	5.0169812532524150	$6.4699000393297901E-07$
	0.03 (832)	5.0169808110206686	$2.0475825757149124E-07$
	0.02 (1247)	5.0169806466934990	$4.0431087644510200E-08$
	0.01 (2493)	5.0169806085350679	$2.2726570625408726E-09$
10^{-32}	0.10 (359)	5.0170058070942467	$2.5200831835214821E-05$
	0.09 (399)	5.0169971513067786	$1.6545044367183293E-05$
	0.08 (448)	5.0169909412581415	$1.0334995730748586E-05$
	0.07 (513)	5.0169866675484354	$6.0612860244835254E-06$
	0.06 (597)	5.0169838794412467	$3.2731788356110680E-06$
	0.05 (717)	5.0169821853609813	$1.5790985699593030E-06$
	0.04 (894)	5.0169812532574136	$6.4699500271214738E-07$
	0.03 (1193)	5.0169808110248049	$2.0476239384614736E-07$
	0.02 (1788)	5.0169806466714393	$4.0409028206900288E-08$
	0.01 (3576)	5.0169806090475753	$2.7851644601728509E-09$

Table 5.6: Interpolation method - European call option price using fixed spaced partitions for an asset price of \$100 with 4 time steps, $\sigma = 0.20$, $r = 0.08$, $T = 0.25$ and strike of \$100. Here η is the total error for the tails. The value in brackets, represents the number of interpolations made to obtain the option price. Double precision was used to calculate the values.

The errors are very similar for all three η values used. However, when the spacing is 0.01 and 0.02, the errors are significantly better for η values of 10^{-16} and 10^{-32}.

A valid comparison between equally spaced partitions and fixed number of partitions should be made when the total number of interpolations are identical (or approximately the same). In the results from the fixed number of partitions section (refer to 5.4), the total number of partitions used were 384.

η	Space	Interpolations Made	IPM Result	Absolute Relative Error
10^{-8}		384	5.0170051428601026	$2.4536597691832140E-05$
	0.05	345	5.0169820480417426	$1.4417793313403759E-06$
10^{-16}		384	5.0170889304529478	$1.0832419053696807E-04$
	0.07	357	5.0169866675485633	$6.0612861521591732E-06$
10^{-32}		384	5.0174346307412199	$4.5402447880898977E-04$
	0.09	399	5.0169971513067786	$1.6545044367183293E-05$

Table 5.7: Comparison of fixed number and fixed spaced partitions for a European call option prices for an asset price of \$100 with 4 time steps, $\sigma = 0.20$, $r = 0.08$, $T = 0.25$ and strike of \$100. Here η is the total error for the tails. The first line represents the price using fixed number of nodes and the second being equally spaced partitions.

From Table 5.7, it is clear that the equally spaced nodes will give better results. The magnitude of improvement is emphasised when $\eta = 10^{-16}$. Under the fixed number of partitions, at each time step, $f^k(x_k)$ is interpolated, and realises errors for each interpolation. The interpolation errors from previous time steps are carried to the subsequent time steps remaining, as previously mentioned. With the fixed number of partition distribution, the density of the nodes is less in the early steps, which incur greater interpolation errors. These errors carry forward to the final time steps when the distribution is denser. That is a greater number of interpolations are made, which propagates the errors of the earlier interpolation.

Using the equally spaced nodes has the advantage of using the nodes early. That is, the errors from interpolation are less than those incurred in the fixed number of nodes distribution. In the final steps, the density is the same and actually less interpolations are made. For example, when $\eta = 10^{-32}$, the nodes used are $(199, 133, 67)$. For the fixed number of partitions, 128 partitions are used at each time step. So, the first two time steps give a better interpolation for equally spaced nodes and less error being propagated to the final time step.

5.4.4 Adaptive Node Allocation

The adaptive node distribution is formed by controlling the errors of interpolation. That is, the nodes are found which give a fixed interpolating error. The error bounds used can vary as required. The results presented in this thesis, using the adaptive node allocation, uses the error bound Φ, such that,

$$0.8\epsilon \leq \Phi \leq 1.2\epsilon \tag{5.51}$$

where ϵ is of the L_1 (Lebesgue Norm) form, namely,

$$\epsilon = \int_{x_{k,n}}^{x_{k,n+1}} |f^k(x_k) - c^k(x_k, n)| \, dx_k. \tag{5.52}$$

This alternative to the previous distributions is important in minimising (as well as controlling) the error of interpolation to achieve better results. The minor disadvantage to this node distribution is the computational effort required to find the optimal nodes.

Table 5.8 presents some results for $\epsilon = 10^{-9}$. Even though requiring a greater computational effort to determine the node distribution, the effort is compensated by the fact the number of interpolations made is much less than the previous two distributions. A comparison of the results in table 5.7 with those in table 5.8, the adaptive nodes approximation is comparable if not better than the previous node distributions. Appendix C.1.3 contains further results for varying ϵ.

Asset Price ($)	η	**Partitions Used**	**IPM Result**	**Error**
80	10^{-8}	122	0.0690146803463858	$3.0529548003467750E-06$
	10^{-16}	150	0.0690146879852485	$3.0453159376598454E-06$
	10^{-32}	184	0.0690147040815196	$3.0292196664797634E-06$
90	10^{-8}	135	1.0254525910199330	$1.1431140107404092E-06$
	10^{-16}	163	1.0254529422904397	$7.9184350396110936E-07$
	10^{-32}	196	1.0254528224019526	$9.1173199111976855E-07$
100	10^{-8}	142	5.0169825138271680	$1.9075647568478349E-06$
	10^{-16}	169	5.0169824382954689	$1.8320330578969202E-06$
	10^{-32}	204	5.0169824313320062	$1.8250695947574780E-06$
110	10^{-8}	147	12.6204468274334065	$1.6745496322911890E-06$
	10^{-16}	176	12.6204468951690991	$1.6068139402358739E-06$
	10^{-32}	213	12.6204469508081285	$1.5511749107410822E-06$
120	10^{-8}	146	22.0665570766488592	$3.1249582504999651E-06$
	10^{-16}	178	22.0665570880606516	$3.1135464595433149E-06$
	10^{-32}	221	22.0665570674505034	$3.1341566064657655E-06$

Table 5.8: Interpolation method - European call option with adaptive node distribution with an interpolating error $\epsilon = 10^{-9}$ and with 4 time steps, $\sigma = 0.20$, $r = 0.08$, $T = 0.25$ and strike of \$100. Here η is the total error for the tails. Double precision was used to calculate the values.

After analysing the node distributions for each of the time steps, it is evident that the distributions are very similar. Figure 5.16 shows the densities of the node distribution for a specific European call option price for the first four time steps (for a case where $K = 8$). It is clear from these bar charts that the node distribution in each time step is very similar.

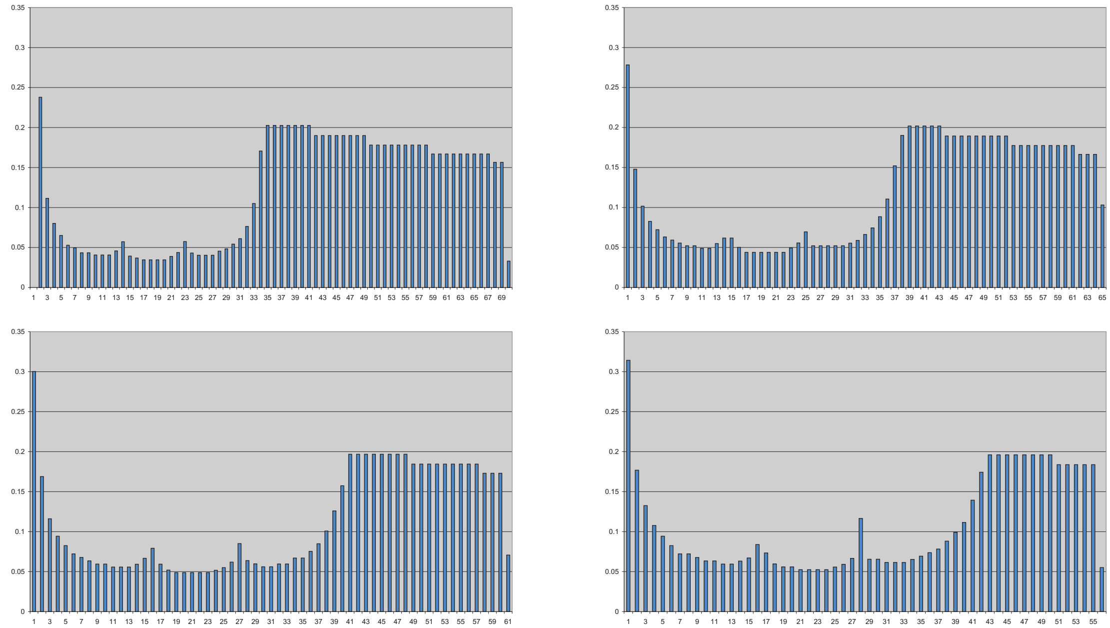

Figure 5.16: Adaptive Node Distribution for the first 4 time steps when $K = 8$

To combat the computational effort required for allocating nodes at each time step, nodes are distributed in the first time step only. This distribution is used in the subsequent time steps to follow. Since the intervals decrease in size, the nodes that are outside the interval are discarded and the endpoints are added (if not already included in the original distribution). Table 5.9 are results for the alternative adaptive node distribution.

Asset Price ($)	η	Partitions Used	IPM Result	Error
80	10^{-8}	116	0.0689375959379025	$8.0137363283629397E-05$
	10^{-16}	143	0.0689328220463222	$8.4911254863971088E-05$
	10^{-32}	174	0.0689226935416872	$9.5039759498958837E-05$
90	10^{-8}	133	1.0254499221687041	$3.8119652394710823E-06$
	10^{-16}	155	1.0254501539858176	$3.5801481260491763E-06$
	10^{-32}	186	1.0254502190435195	$3.5150904242886583E-06$
100	10^{-8}	140	5.0169804761446732	$1.3011773811189009E-07$
	10^{-16}	163	5.0169815587802802	$9.5251786944028360E-07$
	10^{-32}	195	5.0169816476442266	$1.0413818155030619E-06$
110	10^{-8}	148	12.6204385958538357	$9.9061292030411252E-06$
	10^{-16}	171	12.6204458862049833	$2.6157780552704679E-06$
	10^{-32}	205	12.6204459271643668	$2.5748186721630262E-06$
120	10^{-8}	157	22.0665483741688604	$1.1827438251010847E-05$
	10^{-16}	179	22.0665487395526547	$1.1462054455568804E-05$
	10^{-32}	215	22.0665486133647271	$1.1588242385118797E-05$

Table 5.9: Interpolation method - European call option with single adaptive node distribution at the first time step with an interpolating error $\epsilon = 10^{-9}$ and with 4 time steps, $\sigma = 0.20$, $r = 0.08$, $T = 0.25$ and strike of \$100. Here η is the total error for the tails. Double precision was used to calculate the values.

Generally, this method of distribution, gives similar results to the other adaptive distributions and in some cases, an improved result (refer to an asset price of \$100). The advantage of this method is, the reduced computational effort required to obtain very accurate results. To obtain more accurate results than those presented in Tables 5.8 and 5.9, it is advised that ϵ becomes smaller. However, a trade off for accuracy, is the computational effort required and hence time. Though it must be said that the method of allocating nodes at the fist time step alleviates this problem. As stated previously, further results can be found in Appendix C.1.3.

5.5 Traditional Quadrature Rules

An alternative to finding the European option price is the use of quadrature rules. The use of quadrature is very common for integral equations which have no closed form solution(s). Recalling the path integral,

$$f^{k-1}(x_{k-1}) = \Omega \int_{-\infty}^{\infty} e^{-\left(\frac{x_k}{\sqrt{2\Delta t}} - \mu(x_{k-1})\right)^2} f^k(x_k)\, dx_k \tag{5.53}$$

where

$$\Omega = \frac{e^{-r\Delta t}}{\sqrt{2\Delta t \pi}}.$$

For convenience, we denote that

$$g(x_k, x_{k-1}) = e^{-\left(\frac{x_k}{\sqrt{2\Delta t}} - \mu(x_{k-1})\right)^2} f^k(x_k). \tag{5.54}$$

As with the interpolation approach in the previous sections, the path integral interval has to be converted. We employ the same approach as previously so that,

$$\begin{aligned} f^{k-1}(x_{k-1}) =& \Psi(x_{k-1}, -\infty, \beta_1(x_{k-1})) \\ &+ \Psi(x_{k-1}, \beta_1(x_{k-1}), \beta_2(x_{k-1})) \\ &+ \Psi(x_{k-1}, \beta_2(x_{k-1}), \infty) \end{aligned} \tag{5.55}$$

where

$$\Psi(x_{k-1}, a, b) = \Omega \int_a^b g(x_k, x_{k-1})\, dx_k \tag{5.56}$$

and

$$\begin{aligned} \beta_1(x_{k-1}) &= \sqrt{2\Delta t}\, (L_{k-1} + \mu(x_{k-1})) \\ \beta_2(x_{k-1}) &= \sqrt{2\Delta t}\, (R_{k-1} + \mu(x_{k-1})). \end{aligned}$$

Section 5.2.2 has an explanation on how the intervals are set.

It is assumed in the quadrature rules that there are $N + 1$ sample points and that the partitions are equally spaced. Figure 5.17 is an example of a grid for equally spaced nodes, given $n = 0, 1, 2, \ldots, N-1, N$,

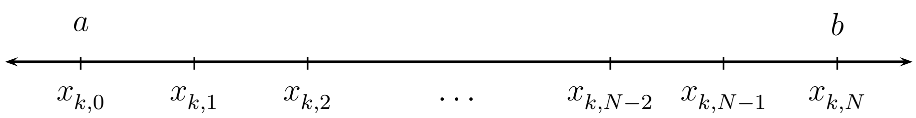

Figure 5.17: The discretization of x_k

The rest of this section will now investigate the most common quadrature rules (including endpoint, midpoint, trapezoidal and a composite Simpson rule).

5.5.1 Left and Right Endpoint Approximation

The Endpoint Approximation is a method of approximating rectangular areas. The sum of multiple rectangle areas (a Riemann Sum) are used to approximate the area under a particular curve. The left or right side of the rectangles are used for the height of the rectangle and the change in the x value is the width.

Clearly, this approximation is not overly accurate unless many rectangles are used. As described in the previous sections, it is not necessarily a given to use as many rectangles (or interpolations) as possible. Time constraints must be considered when using this approximation. Mathematically, the Left Endpoint approximation for (5.53) is given by,

$$\Psi(x_{k-1}, a, b) = \Delta x \sum_{i=0}^{N-1} g(x_{k,i}, x_{k-1}). \tag{5.57}$$

Therefore, (5.57) breaks the area under (5.54) into N rectangles. Table 5.10 shows results for 32 rectangles.

Asset Price ($)	η	Left End QR	Error
80	10^{-8}	0.06901764939587995	8.390530617519154$E{-}8$
	10^{-16}	0.06903104945227437	1.331615108823847$E{-}5$
	10^{-32}	0.06757412997851460	1.443603322671530$E{-}3$
90	10^{-8}	1.025453345956308	3.881776360348255$E{-}7$
	10^{-16}	1.025687042770189	2.333086362453773$E{-}4$
	10^{-32}	1.044118566638236	1.866483250429263$E{-}2$
100	10^{-8}	5.016980560499655	4.576275625112203$E{-}8$
	10^{-16}	5.017259560237605	2.789539751937298$E{-}4$
	10^{-32}	5.036957233365779	1.997662710336812$E{-}2$
110	10^{-8}	12.62044807790149	4.240815449918500$E{-}7$
	10^{-16}	12.62046538268648	1.688070343952575$E{-}5$
	10^{-32}	12.68516174817051	6.471324618746621$E{-}2$
120	10^{-8}	22.06655968365653	5.179505819796759$E{-}7$
	10^{-16}	22.06671664815568	1.564465485728306$E{-}4$
	10^{-32}	22.18405828892942	1.174980873223141$E{-}1$

Table 5.10: Left Endpoint - European call options with 4 time steps, 32 partitions, $\sigma = 0.20$, $r = 0.08$, $T = 0.25$ and strike of \$100. Here η is the total error for the tails (refer to (5.26)).

The results from table 5.10 show that for small values of η, the approximation performs poorly. The reason for this is quite logical; as η gets smaller, the intervals of integration increase, meaning that the 32 rectangles must cover a larger area. Table 5.11 shows the optimal η for each asset price and same parameters as those used in table 5.10.

Asset Price ($)	Optimal η	Left End QR	Error
80	10^{-10}	0.06901775932364641	2.602246063077018$E{-}8$
90	10^{-8}	1.025453457577973	2.765559710127263$E{-}7$
100	10^{-8}	5.016980560499655	4.576275625112203$E{-}8$
110	10^{-8}	12.62044807790149	4.240815449918500$E{-}7$
120	10^{-8}	22.06656015698260	4.462449609832220$E{-}8$

Table 5.11: Left Endpoint - European call options with 4 time steps, 32 partitions, $\sigma = 0.20$, $r = 0.08$, $T = 0.25$ and strike of \$100. Here η is the total error for the tails (refer to (5.26)).

Recalling that η controls the overall approximation of the option price, increasing the number of rectangles should improve the approximation. Table 5.12 are some results for 256 rectangles.

Asset Price (\$)	η	**Left End QR**	**Error**
80	10^{-8}	0.06901768008684646	$5.321433967087064E{-8}$
	10^{-16}	0.06901773330119115	$5.370703881624195E{-15}$
	10^{-32}	0.06901773330119265	$6.869504964868156E{-15}$
90	10^{-8}	1.025453625523113	$1.086108305514699E{-7}$
	10^{-16}	1.025453734133945	$1.554312234475219E{-15}$
	10^{-32}	1.025453734133948	$4.218847493575595E{-15}$
100	10^{-8}	5.016980450514724	$1.557476876357100E{-7}$
	10^{-16}	5.016980606262407	$4.440892098500626E{-15}$
	10^{-32}	5.016980606262412	$8.881784197001252E{-16}$
110	10^{-8}	12.62044830726072	$1.947223218223826E{-7}$
	10^{-16}	12.62044850198304	$3.552713678800501E{-15}$
	10^{-32}	12.62044850198304	$3.552713678800501E{-15}$
120	10^{-8}	22.06655997202600	$2.295811114549906E{-7}$
	10^{-16}	22.06656020160712	$1.776356839400250E{-14}$
	10^{-32}	22.06656020160712	$2.486899575160351E{-14}$

Table 5.12: Left Endpoint - European call options with 4 time steps, 256 partitions, $\sigma = 0.20$, $r = 0.08$, $T = 0.25$ and strike of \$100. Here η is the total error for the tails (refer to (5.26)).

The results from 5.12 show that with an increase in the number of rectangles, smaller values of η can achieve better approximations. By increasing the number of rectangles and decreasing η should improve the approximate option price. Table 5.13 shows the optimal η for each asset price for the same parameters as those used in table 5.12.

Asset Price ($)	**Optimal η**	**Left End QR**	**Error**
80	10^{-35}	0.06901773330119051	$4.732325642464730E{-15}$
90	10^{-22}	1.025453734133946	$1.998401444325282E{-15}$
100	10^{-23}	5.016980606262410	$8.881784197001252E{-16}$
110	10^{-35}	12.62044850198304	$0.000000000000000E0$
120	10^{-33}	22.06656020160710	$0.000000000000000E0$

Table 5.13: Left Endpoint - European call options with 4 time steps, 256 partitions, $\sigma = 0.20$, $r = 0.08$, $T = 0.25$ and strike of \$100. Here η is the total error for the tails (refer to (5.26)).

The increase in the number of rectangles in combination with a smaller η has produced excellent approximations. The method of Left Endpoint approximation is easy to implement, with simple function calls made. The results achieved are very accurate and are very fast to compute (less than 1 second).

The Right Endpoint approximation uses the right side of a rectangle to approximate area. As with the Left Endpoint approximation, the height is taken from the right side of the rectangle and width is the change in x. The Right Endpoint approximation for (5.53) is given by,

$$\Psi(x_{k-1}, a, b) = \Delta x \sum_{i=1}^{N} g(x_{k,i}, x_{k-1}) \tag{5.58}$$

Table 5.14 shows results for 32 rectangles.

Asset Price ($)	η	Right End QR	Error
80	10^{-8}	0.06901774109313175	$7.791945616508755E{-9}$
	10^{-16}	0.06903104945227856	$1.331615109242956E{-5}$
	10^{-32}	0.06757412997851460	$1.443603322671530E{-3}$
90	10^{-8}	1.025453540167024	$1.939669198591787E{-7}$
	10^{-16}	1.025687042770195	$2.333086362513725E{-4}$
	10^{-32}	1.044118566638236	$1.866483250429263E{-2}$
100	10^{-8}	5.016980843731448	$2374690373230237E{-7}$
	10^{-16}	5.017259560237612	$2789539752008352E{-4}$
	10^{-32}	5.036957233365779	$1997662710336812E{-2}$
110	10^{-8}	12.62044843424046	$6.774257599317934E{-8}$
	10^{-16}	12.62046538268649	$1.688070344840753E{-5}$
	10^{-32}	12.68516174817051	$6.471324618746621E{-2}$
120	10^{-8}	22.06656009480615	$1.068009609639375E{-7}$
	10^{-16}	22.06671664815569	$1.564465485834887E{-4}$
	10^{-32}	22.18405828892942	$1.174980873223141E{-1}$

Table 5.14: Right Endpoint - European call options with 4 time steps, 32 partitions, $\sigma = 0.20$, $r = 0.08$, $T = 0.25$ and strike of \$100. Here η is the total error for the tails (refer to (5.26)).

Similarly to the Left Endpoint approximation, the Right End Point performs poorly as η gets smaller for $N = 32$. The optimal approximations occur for $\eta = 10^{-8}$ except for an asset price of 120. For this asset value, $\eta = 10^{-9}$ gives the optimal approximation. So, table 5.15 uses 256 rectangles and presents the best approximations.

Asset Price ($)	Optimal η	Right End QR	Error
80	10^{-35}	0.069017733330119051	$4.732325642464730E{-15}$
90	10^{-29}	1.025453734133945	$1.776356839400250E{-15}$
100	10^{-23}	5.016980606262410	$8.881784197001252E{-16}$
110	10^{-16}	12.62044850198304	$0.000000000000000E0$
120	10^{-33}	22.06656020160710	$0.000000000000000E0$

Table 5.15: Right Endpoint - European call options with 4 time steps, 256 partitions, $\sigma = 0.20$, $r = 0.08$, $T = 0.25$ and strike of \$100. Here η is the total error for the tails (refer to (5.26)).

The approximations in table 5.15 are very similar to those presented in table 5.13. The only differences being for asset value of \$90 (a slightly better approximation) and some of the η values differ.

Since the option price function for a call $(f^{k-1}(x_{k-1}))$ is convex and increasing in nature for the interval of integration, the left endpoint approximation is a lower bound for the function and the right endpoint approximation is an upper bound. For the put option the situation is reversed.

5.5.2 Midpoint Approximation

An alternative to the Left and Right Endpoint approximation is to use the mid point of the rectangles, this rule is commonly known as the midpoint quadrature rule. Since the Left Endpoint underestimates the area and Right Endpoint overestimates, the Midpoint attempts to strike a balance. Therefore, in theory, it is a better approximation. The midpoint quadrature rule in terms of $g(x_k, x_{k-1})$ is

$$\Psi(x_{k-1}, a, b) = \Delta x \sum_{i=0}^{N-1} g(\frac{x_{k,i+1} + x_{k,i}}{2}, x_{k-1}). \tag{5.59}$$

Table 5.16 presents European call option prices with the number of partitions fixed to 256.

Asset Price ($)	η	**Mid Point QR**	**Error**
80	10^{-8}	0.06901768615240528	$4.71E{-}8$
	10^{-16}	0.06901773330119071	$4.58E{-}15$
	10^{-32}	0.06901773330119207	$5.94E{-}15$
90	10^{-8}	1.025453638403598	$9.57E{-}8$
	10^{-16}	1.025453734133945	$1.78E{-}15$
	10^{-32}	1.025453734133946	$2.00E{-}15$
100	10^{-8}	5.016980469322782	$1.37E{-}7$
	10^{-16}	5.016980606262410	$8.88E{-}16$
	10^{-32}	5.016980606262409	$1.78E{-}15$
110	10^{-8}	12.62044833094894	$1.71E{-}7$
	10^{-16}	12.62044850198305	$5.33E{-}15$
	10^{-32}	12.62044850198304	$1.78E{-}15$
120	10^{-8}	22.06655999943769	$2.02E{-}7$
	10^{-16}	22.06656020160712	$1.07E{-}14$
	10^{-32}	22.06656020160711	$3.55E{-}15$

Table 5.16: Midpoint - European call options with 4 time steps, 256 partitions, $\sigma = 0.20$, $r = 0.08$, $T = 0.25$ and strike of \$100. Here η is the total error for the tails (refer to (5.26)).

In most cases in table 5.16 the mid point rule approximation is better than the end point rules (the midpoint rule being an average of the left and right endpoint method). In the cases where the mid point rule is worse off, the differences are minimal and considering the accuracy of the approximations, these differences are negligible.

Table 5.17 presents some accurate results for similar options presented in table 5.16.

Asset Price ($)	**Optimal** η	**Mid Point QR**	**Error**
80	10^{-15}	0.06901773330118409	$2.04E{-}15$
90	10^{-15}	1.025453734133945	$1.78E{-}15$
100	10^{-15}	5.016980606262410	$8.88E{-}16$
110	10^{-16}	12.62044850198304	$0.00E0$
120	10^{-28}	22.06656020160711	$3.55E{-}15$

Table 5.17: Midpoint - European call options with 4 time steps, 256 partitions, $\sigma = 0.20$, $r = 0.08$, $T = 0.25$ and strike of \$100. Here η is the total error for the tails (refer to (5.26)).

The prices in table 5.17 are just as accurate as the left and right end point approximation. The only difference is in the mid point approximation the optimal η is not as small as those in the other rectangular quadrature rules.

5.5.3 Trapezoidal Rule

The trapezoidal rule is similar to the rectangular rules described previously, with the major difference being that instead of using rectangles, the areas of trapeziums are used. Since $f^{k-1}(x_{k-1})$ is convex in nature (increasing for a call option and decreasing for a put option), the approximation will be an upper bound of the analytic solution.

The trapezoidal rule for the path integral takes the form,

$$\Psi(x_{k-1}, a, b) = \Delta x \left[\frac{g(x_{k,0}, x_{k-1})}{2} + \sum_{i=1}^{N-1} g(x_{k,i}, x_{k-1}) + \frac{g(x_{k,N}, x_{k-1})}{2} \right] \qquad (5.60)$$

Table 5.18 presents the same options as those applied with the other quadrature rules (with the use of 256 trapeziums).

Asset Price ($)	η	**Trapezoidal QR**	**Error**
80	10^{-8}	0.06901768581792513	$4.75E{-}8$
	10^{-16}	0.06901773330119140	$5.27E{-}15$
	10^{-32}	0.06901773330119265	$6.52E{-}15$
90	10^{-8}	1.025453637661283	$9.65E{-}8$
	10^{-16}	1.025453734133945	$1.33E{-}15$
	10^{-32}	1.025453734133946	$2.44E{-}15$
100	10^{-8}	5.016980468216713	$1.38E{-}7$
	10^{-16}	5.016980606262407	$4.44E{-}15$
	10^{-32}	5.016980606262413	$1.78E{-}15$
110	10^{-8}	12.62044832953190	$1.72E{-}7$
	10^{-16}	12.62044850198304	$3.55E{-}15$
	10^{-32}	12.62044850198303	$5.33E{-}15$
120	10^{-8}	22.06655999772285	$2.04E{-}7$
	10^{-16}	22.06656020160712	$1.42E{-}14$
	10^{-32}	22.06656020160712	$1.07E{-}14$

Table 5.18: Trapezoidal - European call options with 4 time steps, 256 partitions, $\sigma = 0.20$, $r = 0.08$, $T = 0.25$ and strike of \$100. Here η is the total error for the tails (refer to (5.26)).

Table 5.19 presents some accurate results for similar options presented in 5.18.

Asset Price ($)	**Optimal** η	**Trapezoidal QR**	**Error**
80	10^{-15}	0.06901773330118310	$3.03E{-}15$
90	10^{-29}	1.025453734133945	$1.11E{-}15$
100	10^{-22}	5.016980606262410	$8.88E{-}16$
110	10^{-21}	12.62044850198304	$0.00E0$
120	10^{-25}	22.06656020160711	$3.55E{-}15$

Table 5.19: Trapezoidal - European call options with 4 time steps, 256 partitions, $\sigma = 0.20$, $r = 0.08$, $T = 0.25$ and strike of \$100. Here η is the total error for the tails (refer to (5.26)).

5.5.4 Composite Simpson Rule

The Simpson quadrature rule approximates the area under the curve using parabolic functions. The composite Simpson rule, derived from the Simpson's quadrature rule, is similar in form to the endpoint rules, the major difference being the weights used. The composite rule for the path integral takes the form,

$$\Psi(x_{k-1},a,b) = \frac{\Delta x}{3}\Bigg[g(x_{k,0},x_{k-1}) + 2\sum_{i=1}^{N/2-1} g(x_{k,2i},x_{k-1}) + 4\sum_{j=1}^{N/2} g(x_{k,2j-1},x_{k-1}) + g(x_{k,N},x_{k-1})\Bigg]. \tag{5.61}$$

Table 5.20 and 5.21 presents option prices using the composite Simpson rule with the number of partitions fixed to 256.

Asset Price ($)	η	**Composite Simpson's QR**	**Error**
80	10^{-8}	0.06901768177488442	$5.15E{-8}$
	10^{-16}	0.06901773330119121	$5.08E{-15}$
	10^{-32}	0.06901773330119260	$6.47E{-15}$
90	10^{-8}	1.025453629075925	$1.05E{-7}$
	10^{-16}	1.025453734133945	$1.33E{-15}$
	10^{-32}	1.025453734133946	$2.66E{-15}$
100	10^{-8}	5.016980455677901	$1.51E{-7}$
	10^{-16}	5.016980606262408	$2.66E{-15}$
	10^{-32}	5.016980606262408	$2.66E{-15}$
110	10^{-8}	12.62044831366562	$1.88E{-7}$
	10^{-16}	12.62044850198305	$7.11E{-15}$
	10^{-32}	12.62044850198305	$7.11E{-15}$
120	10^{-8}	22.06655997874159	$2.23E{-7}$
	10^{-16}	22.06656020160712	$1.07E{-14}$
	10^{-32}	22.06656020160712	$1.42E{-14}$

Table 5.20: Composite Simpson's Rule - European call options with 4 time steps, 256 partitions, $\sigma = 0.20$, $r = 0.08$, $T = 0.25$ and strike of $100. Here η is the total error for the tails (refer to (5.26)).

Table 5.21 presents some accurate results for similar options presented in 5.20.

Asset Price (\$)	**Optimal** η	**Composite Simpson's QR**	**Error**
80	10^{-35}	0.06901773330119064	$4.51E{-}15$
90	10^{-29}	1.025453734133944	$8.88E{-}16$
100	10^{-18}	5.016980606262412	$8.88E{-}16$
110	10^{-20}	12.62044850198304	$0.00E0$
120	10^{-35}	22.06656020160711	$3.55E{-}15$

Table 5.21: Composite Simpson's Rule - European call options with 4 time steps, 256 partitions, $\sigma = 0.20$, $r = 0.08$, $T = 0.25$ and strike of \$100. Here η is the total error for the tails (refer to (5.26)).

The results presented for the composite Simpson's rule are very similar to the results presented for the other rules. Computationally, the time taken to achieve these results are very similar to the other quadrature rules. This was expected for the composite Simpson's rule.

5.6 Conclusion

The approaches in this chapter are commonly used mathematical methods. The method of Mathematical interpolation and quadrature were applied to the pricing of European options. Firstly, the path integral was modified so that the interval became finite. This was achieved by using an upper bound of the underlying and the form of the Gaussian in the integrand. Using the modified path integral, an interpolation method was implemented to analyse the model parameters (r, the interest rate, σ, the volatility and T, the time to expiry). It showed that for a particular K (discretization of time) and η (the parameter that controls the interval of integration), there existed an accurate option price.

Various discretization schemes of the underlying were used. These schemes were formed to improve results and others used to improve computational effort and efficiency. A fixed number of nodes (equally spaced) were used since it was easy to implement and quite fast to obtain results. Equally spaced nodes were used as a scheme for future analysis of finding the most accurate result. One of the issues in the methods implemented in this chapter is knowing which parameters gives the most accurate price. By having equivalent interval integrals may assist in achieving this. The results for this scheme were better than those obtained in the fixed number allocation.

The final scheme used was an adaptive approach. Nodes were allocated based on a fixed interpolation error. By controlling the error of interpolation, nodes were distributed in a scheme most suitable. Nodes are placed in positions which allow the interpolation error to be fixed to a particular band of values. It was found that the distribution of nodes at each time step were very similar. So, to improve computational efficiency and speed, nodes were distributed at the first time step and then the scheme was used in the remaining time steps, with unused nodes being eliminated.

Various quadrature (Newton-Cotes) rules were also used to obtain the option price. The results obtained were highly accurate when compared to the Black-Scholes formula. The results obtained using these rules were more precise than those obtained using the interpolation method. A simple discretization scheme (fixed number of equally spaced nodes) were used for each rule.

In both approaches, one of the main issues arising is knowing when the best result can be obtained. Given a particular set of values for r, σ and T, what N (discretization scheme), K (time steps) and η (interval length) will give the most accurate result. A simple search technique, like a bi-section, was used in the data obtained in this chapter. Other, more sophisticated, techniques would also improve the approaches presented in this chapter.

The next chapter uses the approaches of this chapter and applies them to more complex options (American Put and Barrier down and out call options). One of the advantages of the modification made to the path integral is, the form of the finite interval is easily adaptable for options with barriers. Since the American put has a barrier for early exercise and the down and out call option ceases to exist at a particular barrier value, one side of the interval can be fixed to the barrier point. All discretization schemes are implemented using upper and lower interval values (whichever way they are found). Option values in these cases are zero when outside the barrier.

Chapter 6

American Put and Barrier Options

This chapter applies the approaches of Chapter 5 to an American put option and a Down and Out (European style) call option. Similar discretization schemes and parameters are utilized as in Chapter 5. A particular emphasis on performance and accuracy of the Interpolation and Quadrature methods are made. For these types of options, consideration must be given to the barriers required and those already formed by the finite interval evaluated.

6.1 Introduction

In this chapter we will apply the methods presented in Chapter 5 to an American put option and a Down and Out call option. Given the path integral has a finite interval, this is equivalent to having two barriers. These barriers are evaluated so that numerical methods like those in the previous chapter can be utilised to accurately approximate the option price.

For the American put and Down and Out options, the barriers will vary depending on the price required. If the option barrier is inside the finite interval, then the path integral interval will change to cater for the option barrier. Therefore, these numerical methods will require different parameters to those used for the European options to achieve accurate results. The same discretization schemes will be applied for these options.

Section 6.2 and 6.3 applies the interpolation method to the American put and Down and Out call option respectively. Various discretization schemes, as described in Chapter 5 are applied to the two options. Section 6.4 and 6.5 applies the various Newton-Cotes rules for the American put and Down and Out call option. Section 6.6 concludes the chapter.

6.2 Interpolation Polynomials and American Put Options

The interpolation techniques employed for the European option can also be applied to the American put option. However, since the American put option contains a barrier (denoted by $B(x_{k-1})$), the interval of integration L_j is fixed to this barrier and R_j is obtained by solving (5.38).

Therefore, the option price at each time step is determined by the following expression for $k = 1, 2, \ldots, K-1$,

$$\begin{aligned} f^{k-1}(x_{k-1}) =& \Psi(x_{k-1}, -\infty, B(x_{k-1})) \\ &+ \Psi(x_{k-1}, B(x_{k-1}), \beta_2(x_{k-1})) \\ &+ \Psi(x_{k-1}, \beta_2(x_{k-1}), \infty) \end{aligned} \tag{6.1}$$

where

$$\begin{aligned} \Psi(x_{k-1}, -\infty, B(x_{k-1})) &= \Omega \int_{-\infty}^{B(x_{k-1})} e^{-(\frac{x_k}{\sqrt{2\Delta t}} - \mu(x_{k-1}))^2} (1 - e^{\sigma x_k})\, dx_k, \\ \Psi(x_{k-1}, \beta_2(x_{k-1}), \infty) &\leq \frac{\eta}{2}. \end{aligned} \tag{6.2}$$

The barrier point $B(x_k) = x_k^*$, is found such that, x_k^* is the solution to the following expression

$$f^k(x_k) = 1 - e^{\sigma x_k} \tag{6.3}$$

and the middle integral in (6.1) is determined by using the interpolation method presented earlier in Chapter 5.

6.2.1 Fixed Number of Partitions

We firstly apply the interpolation method to an American put option using a fixed number of partitions (equally spaced) at each time step. Table 6.1 presents results for a varying number of time steps and the number of partitions (N) used is fixed to 100.

Asset Price (\$)	**Binomial Method**	**IPM 4 Steps**	**IPM 8 Steps**	**IPM 16 Steps**
80	20.000000	20.000000	20.000000	20.000000
90	10.037663	9.824579	9.976936	10.406923
100	3.224899	3.182479	3.291575	3.638102
110	0.665410	0.654543	0.684210	0.778183
120	0.088796	0.084440	0.084569	0.092523

Table 6.1: Interpolation method - American put option for 100 partitions and various time steps with $\sigma = 0.20$, $r = 0.08$, $T = 0.25$, $\eta = 10^{-32}$ and strike of \$100. The values are calculated in this table are performed in double precision. The Binomial Method prices are optimal approximations (using various N) and were calculated based on Cox et al. (1979).

The results in table 6.1 show similar patterns to those presented in table 3.6. Therefore, there should be an optimal number of time steps for 100 partitions.

Asset Price (\$)	**Time Steps**	**Binomial Method**	**IPM**
90	10	10.037663	10.036710
100	6	3.224899	3.229790
110	6	0.665410	0.666986
120	13	0.088796	0.088617

Table 6.2: Interpolation method - American put option for 100 partitions and optimal time steps with $\sigma = 0.20$, $r = 0.08$, $T = 0.25$, $\eta = 10^{-32}$ and strike of \$100. The values are calculated in this table are performed in double precision. The Binomial Method prices are optimal approximations (using various N) and were calculated based on Cox et al. (1979).

The results in table 6.2 are not as precise as those in table 3.6. However, if we increase the number of interpolations per time step to 200, we would envisage that we should be able to obtain better results, especially for optimal K.

Asset Price ($)	Binomial Method	IPM 4 Steps	IPM 8 Steps	IPM 16 Steps
80	20.000000	20.000000	20.000000	20.000000
90	10.037663	9.821295	9.948753	10.051384
100	3.224899	3.174616	3.220669	3.342221
110	0.665410	0.653157	0.664813	0.698396
120	0.088796	0.086430	0.086948	0.089930

Table 6.3: Interpolation method - American put option for 200 partitions and various time steps with $\sigma = 0.20$, $r = 0.08$, $T = 0.25$, $\eta = 10^{-32}$ and strike of $100. The values are calculated in this table are performed in double precision. The Binomial Method prices are optimal approximations (using various N) and were calculated based on Cox et al. (1979).

Table 6.3 presents results for varying time steps and the number of partitions fixed to 200. Table 6.4 presents results for the optimal time steps for 200 partitions.

Asset Price ($)	Time Steps	Binomial Method	IPM
90	15	10.037663	10.038215
100	8	3.224899	3.220669
110	8	0.665410	0.664813
120	14	0.088796	0.088955

Table 6.4: Interpolation method - American put option for 200 partitions and optimal time steps with $\sigma = 0.20$, $r = 0.08$, $T = 0.25$, $\eta = 10^{-32}$ and strike of $100. The values are calculated in this table are performed in double precision. The Binomial Method prices are optimal approximations (using various N) and were calculated based on Cox et al. (1979).

Table 6.4 shows an improvement in the American put option price to those present in table 6.2. We can also investigate the effect of varying the number of partitions for a fixed number of time steps. Table 6.5 presents approximate prices for an American put option with the number of time steps fixed to 8.

Asset Price ($)	Binomial Method	IPM $N = 32$	IPM $N = 64$	IPM $N = 128$	IPM $N = 256$
80	20.000000	20.000000	20.000000	20.000000	20.000000
90	10.037663	10.704074	10.045959	9.9617033	9.945430
100	3.224899	4.006660	3.426438	3.254724	3.211479
110	0.665410	0.881138	0.719538	0.674214	0.662247
120	0.088796	0.090384	0.079813	0.085813	0.087252

Table 6.5: Interpolation method - American put option for 8 time steps and various node points with $\sigma = 0.20$, $r = 0.08$, $T = 0.25$, $\eta = 10^{-32}$ and strike of \$100. The values are calculated in this table are performed in double precision. The Binomial Method prices are optimal approximations (using various N) and were calculated based on Cox et al. (1979).

As with the fixed number of partitions, we can find the optimal number of partitions for a fixed number of time steps. Table 6.6 presents prices for the optimal number of partitions when the number of time steps are fixed to 8.

Asset Price ($)	Nodes	Binomial Method	IPM
90	66	10.0376631	10.037994
100	184	3.224899	3.224953
110	191	0.665410	0.665446
120	32	0.088796	0.090384

Table 6.6: Interpolation method - American put option for 8 time steps and optimal partitions with $\sigma = 0.20$, $r = 0.08$, $T = 0.25$, $\eta = 10^{-32}$ and strike of \$100. The values are calculated in this table are performed in double precision. The Binomial Method prices are optimal approximations (using various N) and were calculated based on Cox et al. (1979).

As we can see in table 6.6, the results are quite accurate, except for an asset price of \$120. This could be due to the fact that when $K = 8$, there is no N which allows for an accurate result (like those for the other asset prices). For this interpolation method it may be advantageous to fix the number of time steps and vary the number of partitions to find a precise approximation.

6.2.2 Fixed Spaced Partitions

We can also apply the fixed spaced partitions distribution to the American put option. Table 5.6 contains European call option prices for various partition spacing between 0.01 and 0.1.

Binomial Method	**Space (Total Partitions)**	**IPM**	**Error**
3.224899	0.10 (619)	3.343134	$1.182362E-01$
	0.09 (686)	3.316435	$9.153747E-02$
	0.08 (770)	3.292111	$6.721259E-02$
	0.07 (879)	3.270316	$4.541829E-02$
	0.06 (1026)	3.251185	$2.628746E-02$
	0.05 (1228)	3.234828	$9.930157E-03$
	0.04 (1535)	3.221333	$3.564979E-03$
	0.03 (2046)	3.210769	$1.412878E-02$
	0.02 (3086)	3.203188	$2.170986E-02$
	0.01 (6129)	3.198623	$2.627530E-02$

Table 6.7: Interpolation method - American put option price using fixed spaced partitions for an asset price of \$100 with 8 time steps, $\sigma = 0.20$, $r = 0.08$, $T = 0.25$ and strike of \$100. Here η is the total error for the tails. The value in brackets, represents the number of interpolations made to obtain the option price. Double precision was used to calculate the values. The Binomial Method price is an optimal approximation and was calculated based on Cox et al. (1979).

Table 6.7 shows that a good approximation occurs when the fixed partition space is approximately 0.04. After this spacing the error increases again as the number of partitions increase. So, as with the European option, a balance between spacing and the number of interpolations needs to be achieved. Table 6.8 investigates the length of space around 0.04 by adding an additional decimal place.

Binomial Method	**Space (Total Partitions)**	**IPM**	**Error**
3.224899	0.049 (1254)	3.233349	$8.450544E-03$
	0.048 (1282)	3.231898	$6.999583E-03$
	0.047 (1308)	3.230476	$5.577595E-03$
	0.046 (1337)	3.229082	$4.184343E-03$
	0.045 (1365)	3.227718	$2.820075E-03$
	0.044 (1397)	3.226383	$1.484753E-03$
	0.043 (1429)	3.225058	$1.600883E-04$
	0.042 (1463)	3.223799	$1.098535E-03$
	0.041 (1499)	3.222552	$2.346364E-03$

Table 6.8: Interpolation method - American put option price using fixed spaced partitions (with an extra decimal place) for an asset price of \$100 with 8 time steps, $\sigma = 0.20$, $r = 0.08$, $T = 0.25$ and strike of \$100. Here $\eta = 10^{-32}$ is the total error for the tails. The value in brackets, represents the number of interpolations made to obtain the option price. Double precision was used to calculate the values. The Binomial Method price is an optimal approximation and was calculated based on Cox et al. (1979).

Table 6.8 presents improved results as expected. The optimal approximation occurs between 0.042 and 0.043. Table 6.9 investigates this spacing interval.

Binomial Method	**Space (Total Partitions)**	**IPM**	**Error**
3.224899	0.0429 (1431)	3.224948	$4.954533E-05$
	0.0428 (1436)	3.224819	$7.918370E-05$
	0.0427 (1438)	3.224690	$2.076232E-04$
	0.0426 (1443)	3.224562	$3.357855E-04$
	0.0425 (1445)	3.224434	$4.638370E-04$
	0.0424 (1449)	3.224307	$5.727084E-04$
	0.0423 (1452)	3.224180	$6.960137E-04$
	0.0422 (1456)	3.224053	$8.190258E-04$
	0.0421 (1459)	3.223926	$9.417471E-04$

Table 6.9: Interpolation method - American put option price using fixed spaced partitions (with an extra decimal place) for an asset price of \$100 with 8 time steps, $\sigma = 0.20$, $r = 0.08$, $T = 0.25$ and strike of \$100. Here $\eta = 10^{-32}$ is the total error for the tails. The value in brackets, represents the number of interpolations made to obtain the option price. Double precision was used to calculate the values. The Binomial Method price is an optimal approximation and was calculated based on Cox et al. (1979).

Again improved results are shown in table 6.9, with the best approximation in the spacing interval of 0.0428 and 0.0429. Table 6.10 investigates further.

Binomial Method	**Space (Total Partitions)**	**IPM**	**Error**
3.224899	0.04289 (1431)	3.224935	$3.552170E-05$
	0.04288 (1431)	3.224922	$2.303481E-05$
	0.04287 (1433)	3.224909	$1.029906E-05$
	0.04286 (1433)	3.224896	$1.942798E-06$
	0.04285 (1434)	3.224883	$1.439585E-05$
	0.04284 (1434)	3.224870	$2.685903E-05$
	0.04283 (1434)	3.224857	$3.933016E-05$
	0.04282 (1434)	3.224845	$5.178347E-05$
	0.04281 (1436)	3.224832	$6.425259E-05$

Table 6.10: Interpolation method - American put option price using fixed spaced partitions (with an extra decimal place) for an asset price of \$100 with 8 time steps, $\sigma = 0.20$, $r = 0.08$, $T = 0.25$ and strike of \$100. Here $\eta = 10^{-32}$ is the total error for the tails. The value in brackets, represents the number of interpolations made to obtain the option price. Double precision was used to calculate the values. The Binomial Method price is an optimal approximation and was calculated based on Cox et al. (1979).

Table 6.10 shows that a spacing of 0.04286 is an excellent approximation of a American put option price. It must be said that further investigation is possible by obtaining better spacing precision. However, considering that the Binomial method obtained is to 6 digit accuracy, the price obtained for a spacing of 0.04286 in table 6.10 is very accurate compared to the Binomial method. Table 6.11 are optimal results for various asset values.

Asset Price (\$)	**Space (Total Partitions)**	**IPM**	**Error**
90	0.08876 (654)	10.037655	$8.042291393950407E-06$
100	0.04286 (1433)	3.224896	$1.942798209561975E-06$
110	0.05371 (1206)	0.665410	$4.356533792569672E-07$
120	0.007 (9654)	0.087797	$9.993420516646702E-06$

Table 6.11: Interpolation method - precise American put option price for an asset price of \$100 with $K = 8$, $\sigma = 0.20$, $r = 0.08$, $T = 0.25$ and strike of \$100. Here $\eta = 10^{-32}$ is the total error for the tails. The value in brackets, represents the number of interpolations made to obtain the option price. Double precision was used to calculate the values.

Table 6.11 presents some highly accurate results for various asset prices. For asset price \$120, many partitions were required to achieve the evaluated result compared to the other asset values.

6.2.3 Adaptive Nodes

For the American put option, the adaptive nodes will need to be calculated at each time step as the interval of integration will expand on the left hand side. With the European option, the intervals of integration move inward from both sides, therefore the allocation could be sliced for either side. The American put option interval differs because the barrier (or left side of the interval) tends to move out, while the right side moves inward. Therefore, allocations at each time step is performed.

Table 6.12 shows some American put option prices for an adaptive node distribution when the asset value is \$100.

ϵ	**Binomial Method**	4 **time steps**	8 **time steps**
10^{-1}	3.224899	15.628738	24.023563
10^{-2}		5.021539	8.322211
10^{-3}		3.601882	4.234894
10^{-4}		3.244270	3.415439
10^{-5}		3.183102	3.232615
10^{-6}		3.173328	3.203636

Table 6.12: Interpolation method - American put option (asset value of \$100) for adaptive node points and 4 time steps with $\eta = 10^{-32}$, $\sigma = 0.20$, $r = 0.08$, $T = 0.25$ and strike of \$100. The values are calculated in this table are performed in double precision. The Binomial Method price is an optimal approximation and was calculated based on Cox et al. (1979).

When the number of time steps used is 4, the best ϵ is somewhere between 10^{-4} and 10^{-5}. For 8 time steps the best ϵ is between 10^{-5} and 10^{-6}. Table 6.13 will expand on the results found in Table 6.12 by using a more precise ϵ (additional decimal places).

ϵ	**Binomial Method**	4 **time steps**
$9E{-}5$	3.224899	3.240004
$8E{-}5$		3.236440
$7E{-}5$		3.228739
$6E{-}5$		3.221673
$5E{-}5$		3.216960
$4E{-}5$		3.209042
$3E{-}5$		3.201310
$2E{-}5$		3.191629

Table 6.13: Interpolation method - American put option (asset value of \$100) for adaptive node points and 4 time steps with $\eta = 10^{-32}$, $\sigma = 0.20$, $r = 0.08$, $T = 0.25$ and strike of \$100. The values are calculated in this table are performed in double precision. The Binomial Method price is an optimal approximation and was calculated based on Cox et al. (1979).

Table 6.13 shows with a better ϵ value (more precision), an improved approximation can be found. Table 6.14 adds another decimal place to ϵ.

ϵ	**Binomial Method**	4 **time steps**
$6.9E{-}5$	3.224899	3.228715
$6.8E{-}5$		3.226489
$6.7E{-}5$		3.227012
$6.6E{-}5$		3.227030
$6.5E{-}5$		3.226771
$6.4E{-}5$		3.225582
$6.3E{-}5$		3.224506
$6.2E{-}5$		3.222634
$6.1E{-}5$		3.221698

Table 6.14: Interpolation method - American put option (asset value of \$100) for adaptive node points and 4 time steps with $\eta = 10^{-32}$, $\sigma = 0.20$, $r = 0.08$, $T = 0.25$ and strike of \$100. The values are calculated in this table are performed in double precision. The Binomial Method price is an optimal approximation and was calculated based on Cox et al. (1979).

Clearly the best result is around $\epsilon = 6.3E{-}5$. So, with further investigation in table 6.15, the best approximation was found to be at $\epsilon = 6.21E{-}5$.

ϵ	**Binomial Method**	4 **time steps**
$6.29E{-}5$	3.224899	3.224506
$6.28E{-}5$		3.224506
$6.27E{-}5$		3.224498
$6.26E{-}5$		3.224498
$6.25E{-}5$		3.224498
$6.24E{-}5$		3.224498
$6.23E{-}5$		3.224574
$6.22E{-}5$		3.224574
$6.21E{-}5$		3.224574

Table 6.15: Interpolation method - American put option (asset value of \$100) for adaptive node points and 4 time steps with $\eta = 10^{-32}$, $\sigma = 0.20$, $r = 0.08$, $T = 0.25$ and strike of \$100. The values are calculated in this table are performed in double precision. The Binomial Method price is an optimal approximation and was calculated based on Cox et al. (1979).

The results in table 6.15 can be similar for differing ϵ values because the node distributions are the same. Therefore, when $\epsilon = 6.21E{-}5$, $6.22E{-}5$ and $6.23E{-}5$, the number of nodes and the distribution of such are identical, so the approximations are the same. Table 6.16, 6.17 and 6.18 are results for 8 time steps with varying values of ϵ.

ϵ	**Binomial Method**	8 **time steps**
$9E{-}6$	3.224899	3.229158
$8E{-}6$		3.227386
$7E{-}6$		3.223595
$6E{-}6$		3.221340
$5E{-}6$		3.218585
$4E{-}6$		3.214595
$3E{-}6$		3.211248
$2E{-}6$		3.207711

Table 6.16: Interpolation method - American put option (asset value of \$100) for adaptive node points and 8 time steps with $\eta = 10^{-32}$, $\sigma = 0.20$, $r = 0.08$, $T = 0.25$ and strike of \$100. The values are calculated in this table are performed in double precision. The Binomial Method price is an optimal approximation and was calculated based on Cox et al. (1979).

ϵ	**Binomial Method**	8 **time steps**
$7.9E{-}6$	3.224899	3.226953
$7.8E{-}6$		3.226481
$7.7E{-}6$		3.226008
$7.6E{-}6$		3.226185
$7.5E{-}6$		3.225254
$7.4E{-}6$		3.225483
$7.3E{-}6$		3.225154
$7.2E{-}6$		3.223834
$7.1E{-}6$		3.224639

Table 6.17: Interpolation method - American put option (asset value of \$100) for adaptive node points and 8 time steps with $\eta = 10^{-32}$, $\sigma = 0.20$, $r = 0.08$, $T = 0.25$ and strike of \$100. The values are calculated in this table are performed in double precision. The Binomial Method price is an optimal approximation and was calculated based on Cox et al. (1979).

ϵ	**Binomial Method**	8 **time steps**
$7.29E{-}6$	3.224899	3.224884
$7.28E{-}6$		3.224410
$7.27E{-}6$		3.224448
$7.26E{-}6$		3.224448
$7.25E{-}6$		3.224448
$7.24E{-}6$		3.224448
$7.23E{-}6$		3.224267
$7.22E{-}6$		3.224347
$7.21E{-}6$		3.223810

Table 6.18: Interpolation method - American put option (asset value of \$100) for adaptive node points and 8 time steps with $\eta = 10^{-32}$, $\sigma = 0.20$, $r = 0.08$, $T = 0.25$ and strike of \$100. The values are calculated in this table are performed in double precision. The Binomial Method price is an optimal approximation and was calculated based on Cox et al. (1979).

For the case of 8 time steps the value of ϵ needs to be smaller since there are more time steps and therefore more interpolations required. In this case, the optimal ϵ value is $7.29E{-}6$ which gives an excellent approximation compared to the Binomial method.

6.3 Interpolation Polynomials and Barrier Options

Barrier options are options which are dependent on whether the underlying asset price reaches a pre-determined level within a certain time period. The payoff of the option will vary depending on the level the asset price achieves. The are two specific types of barrier options, they are *knock-out options* or *knock-in options*. A knock-out option ceases to exists when the underlying asset price reaches a barrier. Whereas, a knock-in option will come into existence if the underlying asset reaches a barrier.

In this section we will apply the Interpolation method to a knock-out option called a *down and out call*. The down and out call option is similar to a normal European call option that ceases to exist when it reaches a barrier H. The barrier level for a down and out call is set below the initial asset price. Hull (2006) gives a closed form for the down and out call option, c_{do} in terms of the European call, c, as given in (2.21) and the corresponding down and in call, c_{di}. Namely,

$$c_{do} = c - c_{di} \tag{6.4}$$

where

$$c_{di} = xe^{-rT}(\frac{H}{x})^{2\lambda} erfc(-y) - ce^{-rT}(\frac{H}{x})^{2\lambda-2} erfc(\sigma\sqrt{T} - y), \tag{6.5}$$

$$\lambda = \frac{r + \frac{\sigma^2}{2}}{\sigma^2},$$

$$y = \frac{\ln\left(\frac{H^2}{xc}\right)}{\sigma\sqrt{T}} + \lambda\sigma\sqrt{T}$$

and x is the underlying asset price and with c the strike price.

One of the advantages of this method is that the transformation of the path integral (5.1) is in a barrier form. For options like the down and out barrier option, $\beta_1(x_{k-1})$ is fixed to H unless the underlying asset value determines that $\beta_1(x_{k-1})$ is inside the barrier H.

Asset Price ($)	η	**Step**	L_{k-1}	R_{k-1}
100	10^{-8}	9	-1.43841036225890	8.44118710000083
		8	-1.43841036225890	7.48359933286901
		7	-1.43841036225890	6.53093355998789
		6	-1.43841036225890	5.58318874745056
		5	-1.43841036225890	4.64036384470732
		4	-1.43841036225890	3.70245778422486
		3	-1.43841036225890	2.76946948113699
		2	-1.43841036225890	1.84139783288643
		1	-0.893241718857600	0.918241718857600

Table 6.19: Down and Out call option intervals of integration for $K = 10$, $\sigma = 0.20$, $r = 0.08$, $T = 0.25$, $H = \$75$ and strike of \$100. Here η is the total error for the tails.

In table 6.19 the left interval point is fixed for all time steps except for the final time step, when the method dictates that the barrier is different. It should be noted that the barrier of $H = \$75$ is converted using $\frac{1}{\sigma}\ln(H)$.

6.3.1 Fixed Number of Partitions

We firstly apply a down and out call option to the interpolation method with fixed number of partitions. The errors calculated in the tables are the absolute difference between the prices evaluated from the analytical form presented in Hull (2006) against those computed using the interpolation method.

Asset Price ($)	Optimal η	IPM	Error
80	10^{-3}	0.0683055896929662	$8.8606713925206467E-04$
90	10^{-6}	1.0254500127567925	$2.5591529289811787E-06$
100	10^{-6}	5.0170110377369452	$3.0446349936347206E-05$
110	10^{-8}	12.6204428872442822	$5.6147134247508390E-06$
120	10^{-7}	22.0665241762180813	$3.6025388994165297E-05$

Table 6.20: Interpolation method - Down and Out call option (asset value of \$100) for fixed number of node points ($N = 64$) and 8 time steps with $\sigma = 0.20$, $r = 0.08$, $T = 0.25$, strike of \$100 and barrier of \$75. The values are calculated in this table are performed in double precision.

Table 6.20 presents results for a various down and out call options with varying η such that the best approximation is evaluated. Table 6.21 is for 128 partitions.

Asset Price ($)	Optimal η	IPM	Error
80	10^{-3}	0.0683072619700805	$8.8773941636632056E-04$
90	10^{-6}	1.0254484846409615	$1.0310370980592953E-06$
100	10^{-7}	5.0169828478871548	$2.2565001457641731E-06$
110	10^{-9}	12.6204480608633283	$4.4109437769002113E-07$
120	10^{-8}	22.0665573941933566	$2.8074137209399552E-06$

Table 6.21: Interpolation method - Down and Out call option (asset value of \$100) for fixed number of node points ($N = 128$) and 8 time steps with $\sigma = 0.20$, $r = 0.08$, $T = 0.25$, strike of \$100 and barrier of \$75. The values are calculated in this table are performed in double precision.

An increase in the number of partitions (that is, number of interpolations made) means that η can be decreased. A decrease in η allows for the potential of better approximations, as is the case in table 6.21. Appendix C.2.1 presents prices for the 128 partitions and varying η (where the optimal prices in table 6.21 were derived from). Table 6.22 uses 256 partitions at each time step.

Asset Price ($)	Optimal η	IPM	Error
80	10^{-3}	0.0683073667812694	$8.87844227555521517E-04$
90	10^{-6}	1.0254483906967153	$9.3709285192700165E-07$
100	10^{-8}	5.0169807549790546	$1.6359204546567696E-07$
110	10^{-10}	12.6204484686408858	$3.3316821523854401E-08$
120	10^{-9}	22.0665599904518572	$2.1115521831038819E-07$

Table 6.22: Interpolation method - Down and Out call option for fixed number of node points ($N = 256$) and 8 time steps with $\sigma = 0.20$, $r = 0.08$, $T = 0.25$, strike of $100 and barrier of $75. The values are calculated in this table are performed in double precision.

The prices in tables 6.20 - 6.22 shows with an increase in the number of partitions, the best η for asset values decreased for asset values $100, $110 and $120. For these asset values the approximations improved. For an asset value of $90, the best η remained the same at 10^{-6} and the approximation improved. Given the asset value of $80, which is near the barrier value, the best η remain the same and the approximation did not improve.

Tables 6.23 and 6.24 are down and out call option prices for an increasing number of time steps.

Asset Price ($)	Optimal η	Approximation	Error
80	10^{-3}	0.0684750295337522	$1.0555069800379977E-03$
90	10^{-6}	1.0254506693384327	$3.2157345690800310E-06$
100	10^{-6}	5.0169913171112990	$1.0725724289978311E-05$
110	10^{-8}	12.6204459132041311	$2.5887535753943425E-06$
120	10^{-7}	22.0665443382339959	$1.5863373081792531E-05$

Table 6.23: Interpolation method - Down and Out call option (asset value of $100) for fixed number of node points ($N = 128$) and 16 time steps with $\sigma = 0.20$, $r = 0.08$, $T = 0.25$, strike of $100 and barrier of $75. The values are calculated in this table are performed in double precision.

Table 6.23 shows an improvement in the approximate prices when the asset value is $100, $110 and $120 compared to the corresponding values in table 6.21. For asset values of $80 and $90, the approximations in table 6.21 are slightly better. In all cases, the best η's remained the same, despite the change in the number of time steps. Table 6.24 are prices for down and out call options with 32 time steps.

Asset Price ($)	Optimal η	IPM	Error
80	10^{-3}	0.0684788368401993	$1.0593142864851065E-03$
90	10^{-7}	1.0254608988946279	$1.3445290764427242E-05$
100	10^{-5}	5.0170147735455517	$3.4182158542561680E-05$
110	10^{-9}	12.6204430800773935	$5.4218803141603544E-06$
120	10^{-6}	22.0664688777809666	$9.1323826109657169E-05$

Table 6.24: Interpolation method - Down and Out call option (asset value of $100) for fixed number of node points ($N = 128$) and 32 time steps with $\sigma = 0.20$, $r = 0.08$, $T = 0.25$, strike of $100 and barrier of $75. The values are calculated in this table are performed in double precision.

The approximate prices in table 6.24 did not improve with a further increase in the number of time steps. Also, the best η changed for each asset value except for an asset value of $80.

In appendix C.2.1 a range of prices are presented for varying η values for the parameter set used in tables 6.20 - 6.24. These are where the prices in the tables were derived from.

6.3.2 Fixed Spaced Partitions

We now apply the down and out call option with the use of fixed spaced partitions. Table 6.25 are prices for varying spaced partitions and η's.

η	**Space (Partitions)**	**IPM**	**Error**
10^{-6}	10^{-1} (348)	5.0170300926179250	$4.9501230916376704E-05$
	10^{-2} (386)	5.0170098978847584	$2.9306497749403970E-05$
	10^{-3} (435)	5.0169954087381203	$1.4817351111362509E-05$
	10^{-4} (496)	5.0169854371542311	$4.8457672222057546E-06$
	10^{-5} (578)	5.0169789317218401	$1.6596651684908093E-06$
	10^{-6} (692)	5.0169749788864193	$5.6125005899743119E-06$
	10^{-7} (866)	5.0169728040157544	$7.7873712547860130E-06$
	10^{-8} (1151)	5.0169717721315097	$8.8192554988097971E-06$
	10^{-9} (1726)	5.0169713887179688	$9.2026690400037747E-06$
	10^{-10} (3448)	5.0169713007819920	$9.2906050165597609E-06$
10^{-7}	10^{-1} (370)	5.0170384148661213	$5.7823479112512066E-05$
	10^{-2} (410)	5.0170182201453981	$3.7628758389057770E-05$
	10^{-3} (462)	5.0170037310055946	$2.3139618585826804E-05$
	10^{-4} (526)	5.0169937594270468	$1.3168040038230577E-05$
	10^{-5} (615)	5.0169872540022604	$6.6626152518678428E-06$
	10^{-6} (738)	5.0169833011640650	$2.7097770562145573E-06$
	10^{-7} (921)	5.0169811262921549	$5.3490514587140048E-07$
	10^{-8} (1226)	5.0169800944138707	$4.9697313833640955E-07$
	10^{-9} (1839)	5.0169797110474379	$8.8033957068578417E-07$
	10^{-10} (3673)	5.0169796230779342	$9.6830907458933524E-07$
10^{-8}	10^{-1} (392)	5.0170392994777373	$5.8708090728415430E-05$
	10^{-2} (434)	5.0170191047581509	$3.8513371142245845E-05$
	10^{-3} (488)	5.0170046156198866	$2.4024232878061547E-05$
	10^{-4} (557)	5.0169946440423354	$1.4052655326196595E-05$
	10^{-5} (650)	5.0169881386167852	$7.5472297758616413E-06$
	10^{-6} (779)	5.0169841857825546	$3.5943955457862220E-06$
	10^{-7} (973)	5.0169820109170020	$1.4195299935515493E-06$
	10^{-8} (1296)	5.0169809790307083	$3.8764369977850599E-07$
	10^{-9} (1944)	5.0169805955836511	$4.1966423391937013E-09$
	10^{-10} (3881)	5.0169805076564362	$8.3730572475460008E-08$
10^{-9}	10^{-1} (411)	5.0170393928197772	$5.8801432768068596E-05$
	10^{-2} (456)	5.0170191981008427	$3.8606713834155038E-05$
	10^{-3} (513)	5.0170047089619541	$2.4117574944776399E-05$
	10^{-4} (583)	5.0169947373830777	$1.4145996068970490E-05$
	10^{-5} (682)	5.0169882319565895	$7.6405695804970808E-06$
	10^{-6} (818)	5.0169842791279953	$3.6877409861912902E-06$
	10^{-7} (1020)	5.0169821042577594	$1.5128707508971218E-06$
	10^{-8} (1361)	5.0169810723648807	$4.8097787141987425E-07$
	10^{-9} (2038)	5.0169806888920911	$9.7505082480298100E-08$
	10^{-10} (4074)	5.0169806010175773	$9.6305685648445660E-09$

Table 6.25: Interpolation method - Down and Out call option (asset price of \$100) for fixed spaced partitions and 8 time steps with $\sigma = 0.20$, $r = 0.08$, $T = 0.25$, strike of \$100 and barrier of \$75. The values are calculated in this table are performed in double precision.

Table 6.25 shows the same patterns as the equivalent European and American options with the same partition distribution. As the distance of the partitions decrease, so does the error. However, as also stated previously, eventually too many interpolations occur which causes the error to rise. If we refer to $\eta = 10^{-8}$ in table 6.25, when the partition space is 10^{-10}, the error rises compared to a space of 10^{-9}. The number of interpolations made for 10^{-10} is 3881, which is an increase of 1937 compared to the number of interpolations made for a partition space of 10^{-9}.

6.3.3 Adaptive Nodes

The down and out option is now applied using an adaptive node allocation. Table 6.26 are prices with $\epsilon = 10^{-6}$ and the number of time steps used fixed at 8.

Asset Price ($)	η	Partitions Used	IPM	Error
80	10^{-3}	58	0.0667076971555293	$2.3100361456567875E-03$
	10^{-4}	65	0.0674275336015360	$1.5901996996501005E-03$
	10^{-5}	68	0.0672370933556727	$1.7806399455134320E-03$
	10^{-6}	72	0.0672915745025081	$1.7261587986780111E-03$
	10^{-7}	75	0.0672845568851327	$1.7331764160534316E-03$
	10^{-8}	80	0.0672923905737283	$1.7253427274578517E-03$
	10^{-9}	84	0.0673127309525815	$1.7050023486045907E-03$
90	10^{-3}	67	1.0222277060305562	$3.2260281033874225E-03$
	10^{-4}	72	1.0250202921474112	$4.3344198653247346E-04$
	10^{-5}	73	1.1458590413285348	$1.2040530719459111E-01$
	10^{-6}	81	1.0251509390672204	$3.0279506672333673E-04$
	10^{-7}	83	1.0252210711434961	$2.3266299044744781E-04$
	10^{-8}	82	1.4620555381593765	$4.3660180402543286E-01$
	10^{-9}	91	1.0251726792727522	$2.8105486119152367E-04$
100	10^{-3}	74	5.0194331391387381	$2.4525328763271015E-03$
	10^{-4}	77	5.0175642069767097	$5.8360071429897076E-04$
	10^{-5}	82	5.0179963399931822	$1.0157337307706293E-03$
	10^{-6}	85	5.0180193978603107	$1.0387915978994633E-03$
	10^{-7}	89	5.0180505947074918	$1.0699884450807162E-03$
	10^{-8}	92	5.0179777799471799	$9.9717368476845425E-04$
	10^{-9}	95	5.0180132019509127	$1.0325956885014198E-03$
110	10^{-3}	76	12.6071276354699311	$1.3320866513108487E-02$
	10^{-4}	80	12.6180020418139698	$2.4464601690699084E-03$
	10^{-5}	85	12.6191418142999243	$1.3066876831141494E-03$
	10^{-6}	89	12.6193887652008971	$1.0597367821413295E-03$
	10^{-7}	94	12.6193160043933528	$1.1324975896870315E-03$
	10^{-8}	96	12.6192126954519068	$1.2358065311329769E-03$
	10^{-9}	99	12.6192355201199309	$1.2129818631079203E-03$
120	10^{-3}	74	22.0639763674506710	$2.5838341564388134E-03$
	10^{-4}	82	22.0628994217529844	$3.6607798541254954E-03$
	10^{-5}	86	22.0645480731361268	$2.0121284709850862E-03$
	10^{-6}	90	22.0647955148665176	$1.7646867405934907E-03$
	10^{-7}	94	22.0647950020762238	$1.7651995308870516E-03$
	10^{-8}	98	22.0647691902863059	$1.7910113208063905E-03$
	10^{-9}	102	22.0648056145526148	$1.7545870544949382E-03$

Table 6.26: Interpolation method - Down and Out call option for Adaptive node points and 8 time steps with $\epsilon = 10^{-6}$, $\sigma = 0.20$, $r = 0.08$, $T = 0.25$, strike of \$100 and barrier of \$75. The values are calculated in this table are performed in double precision.

To improve the results presented in table 6.26 we can evaluate prices with a smaller ϵ value. Table 6.27 are down and out call option prices with ϵ set to 10^{-11}.

Asset Price ($)	η	Partitions Used	IPM	Error
80	10^{-3}	587	0.0683072146821381	$7.1051861904801146E-04$
	10^{-4}	642	0.0687790910800854	$2.3864222110076385E-04$
	10^{-5}	682	0.0688473789731971	$1.7035432798904017E-04$
	10^{-6}	716	0.0688566150165243	$1.6111828466187006E-04$
	10^{-7}	752	0.0688578031005921	$1.5993020059402443E-04$
	10^{-8}	783	0.0688579502199794	$1.5978308120669213E-04$
	10^{-9}	812	0.0688579672710672	$1.5976603011890432E-04$
90	10^{-3}	679	1.0224396345886602	$3.0140995452834426E-03$
	10^{-4}	719	1.0250854707610884	$3.6826337285536526E-04$
	10^{-5}	706	1.1459434877436201	$1.2048975360967641E-01$
	10^{-6}	793	1.0254498837255983	$3.8504083454612981E-06$
	10^{-7}	830	1.0254530234446562	$7.1068928762796357E-07$
	10^{-8}	802	1.4629639165870467	$4.3751018245310297E-01$
	10^{-9}	896	1.0254532527596778	$4.8137426596900346E-07$
100	10^{-3}	719	5.0097134777741434	$7.2671284882676324E-03$
	10^{-4}	766	5.0161797967156616	$8.0080954674988103E-04$
	10^{-5}	809	5.0169295251541817	$5.1081108229378280E-05$
	10^{-6}	845	5.0169764105428980	$4.1957195134889602E-06$
	10^{-7}	878	5.0169797237830682	$8.8247934307283948E-07$
	10^{-8}	910	5.0169806051998531	$1.0625576807310466E-09$
	10^{-9}	947	5.0169807015054770	$9.5243065517669478E-08$
110	10^{-3}	746	12.6103791197460211	$1.0069382237018476E-02$
	10^{-4}	797	12.6191599217570669	$1.2885802259721135E-03$
	10^{-5}	842	12.6203153900204281	$1.3311196261078795E-04$
	10^{-6}	886	12.6204408845995211	$7.6173835183990235E-06$
	10^{-7}	924	12.6204469987161083	$1.5032669303804980E-06$
	10^{-8}	956	12.6204482797488637	$2.2223417550648605E-07$
	10^{-9}	1025	12.6204483926597995	$1.0932323935008981E-07$
120	10^{-3}	737	22.0482537285707458	$1.8306473036364124E-02$
	10^{-4}	815	22.0647170331095133	$1.8431684975978024E-03$
	10^{-5}	857	22.0663747563650574	$1.8544524205410529E-04$
	10^{-6}	907	22.0665413973320668	$1.8804275042638707E-05$
	10^{-7}	939	22.0665586636416684	$1.5379654411118437E-06$
	10^{-8}	982	22.0665599361222640	$2.6548484521882187E-07$
	10^{-9}	1024	22.0665600027465594	$1.9886055024898042E-07$

Table 6.27: Interpolation method - Down and Out call option for Adaptive node points and 8 time steps with $\epsilon = 10^{-11}$, $\sigma = 0.20$, $r = 0.08$, $T = 0.25$, strike of $100 and barrier of $75. The values are calculated in this table are performed in double precision.

The errors in table 6.27 are greatly improved compared to those presented in table 6.26. However, as presented in both tables 6.26 and 6.27, the number of nodes differ immensely. So, when $\epsilon = 10^{-6}$, the number of nodes used are in the order of 30. In the case of $\epsilon = 10^{-11}$, the number of nodes used vary between approximately 600 to 1000. Therefore, improvements should be expected.

Appendix C.2.3 presents further results for differing ϵ. These tables emphasise the improvements in accuracy for the changes in ϵ. It must be noted that depending on the requirements, a balance in accuracy and computational effort may be needed. Smaller values of ϵ requires greater computational effort to derive an accurate price.

6.4 Quadrature Rules and American Put Options

Using the Quadrature (Newton-Cotes) rules employed for the European options, table 6.28 presents results for American put options with the 200 partitions and various number of time steps.

Asset Price ($)	**Binomial Method**	**Left End 4 Steps**	**Left End 8 Steps**	**Left End 16 Steps**
80	20.000000	20.000000	20.000000	20.000000
90	10.037663	9.419727	10.233520	10.673077
100	3.224899	3.248504	3.435497	3.615112
110	0.665410	0.677737	0.714048	0.760814
120	0.088796	0.089300	0.094368	0.101612

Table 6.28: Left Endpoint Quadrature - American put option for 200 partitions and various time steps with $\sigma = 0.20$, $r = 0.08$, $T = 0.25$, $\eta = 10^{-32}$ and strike of \$100. The values are calculated in this table are performed in double precision. The Binomial Method prices are optimal approximations (using various N) and were calculated based on Cox et al. (1979).

The option prices in table 6.28 shows similar variations as in the interpolation method. It can be shown that when not enough partitions are being used, the early exercise barrier is miscalculated. Even though some of the results in table 6.28 are close to the Binomial price, there are imprecisions in the calculations of the price using the Left endpoint quadrature approximation.

Table 6.29 contains American put prices for an optimal amount of partitions given 8 time steps.

Asset Price ($)	**Partitions**	**Binomial Method**	**Left End QR**
90	825	10.037663	10.038135
100	2229	3.224899	3.224905
110	2150	0.665410	0.665411
120	1561	0.088796	0.088796

Table 6.29: Left Endpoint Quadrature - American put option for 8 time steps and optimal partitions with $\sigma = 0.20$, $r = 0.08$, $T = 0.25$, $\eta = 10^{-32}$ and strike of $100. The values are calculated in this table are performed in double precision. The Binomial Method prices are optimal approximations (using various N) and were calculated based on Cox et al. (1979).

Even though the prices in table 6.29 are excellent approximations for the American put option price, the computational effort was great. As mentioned in the previous paragraph, the calculation of the Exercise Barrier at each time step is not as precise as the interpolation and Fourier methods described earlier.

Table 6.30 are some further American put option prices evaluated using various (Newton-Cotes) quadrature rules.

Asset Price ($)	**Binomial Method**	**Right End**	**Mid Point**	**Trapezoidal**	**Composite Simpson's**
90	10.037663	10.000000	10.000000	10.000000	10.000000
100	3.224899	2.599630	3.187915	3.186625	3.070412
110	0.665410	2.516689	0.658364	0.657891	0.635625
120	0.088796	2.492115	0.087753	0.087676	0.085229

Table 6.30: Various Quadrature Rules - American put option for 8 time steps and 512 partitions with $\sigma = 0.20$, $r = 0.08$, $T = 0.25$, $\eta = 10^{-32}$ and strike of $100. The values are calculated in this table are performed in double precision. The Binomial Method prices are optimal approximations (using various N) and were calculated based on Cox et al. (1979).

Clearly the results in table 6.30 are poor compared to the Binomial method and those presented throughout the thesis. The poor accuracy of these prices are attributed to the inaccuracy of the barrier points (early exercise boundary) at each time step. The barriers calculated by the quadrature methods compared to those evaluated by the Interpolation method are different. Given the higher accuracy achieved by the latter method, the deficiency of the quadrature methods can be attributed to the evaluation of the barrier points.

The inaccuracy in the barrier calculation can be attributed to the nature of the interval. For the American put option the barrier moves outward at each time step (that is, the left interval). Therefore, to calculate the next barrier point, the method, whether IPM or Quadrature, requires an estimate of function values that are outside the domain of the current time step. This is a form of *extrapolation*, which the Quadrature methods do not perform well. To show the inaccuracy occurs, due to the calculation of inferior barrier points, table 6.31 are American put option prices with the barrier points obtained by an accurate Interpolation method approximation. For each time step, the barrier point obtained by the interpolation method is used within the Quadrature methods.

Quadrature Method	**Binomial Method**	**Optimal Nodes**	**Approximation**
	3.224899		
Left End		138	3.225513
Mid Point		57	3.224436

Table 6.31: Left End and Mid point Quadrature Rules - American put option for an Asset Price of \$100, 8 time steps and an optimal amount of partitions with $\sigma = 0.20$, $r = 0.08$, $T = 0.25$, $\eta = 10^{-32}$ and strike of \$100. The values are calculated in this table are performed in double precision. The Binomial Method prices are optimal approximations (using various N) and were calculated based on Cox et al. (1979).

As presented in table 6.31, the evaluated prices for the left end and mid point quadrature rules have improved greatly compared to those in table 6.30. It must be stated that for the other rules (right end point, trapezoidal and the composite Simpson's) did not improve enough. In excess of 4096 node points were used in these other rules and prices were in the range of \$3.16 and \$3.18. So, an improvement is required in the evaluation of the barrier points at each time step. This is an issue for all American put option prices and requires further investigation. It is envisaged that the Quadrature methods will perform well for the down and out call option as the barrier is fixed at each time step.

6.5 Quadrature Rules and Barrier Options

Using the Quadrature (Newton-Cotes) rules employed for the European and American put options, table 6.32 presents results for the down and out call option with various partitions and 8 time steps.

Asset Price (\$)	Optimal η	Left Endpoint	Error
$N = 64$			
80	10^{-3}	0.068265704858581	$8.461823048671463E-04$
90	10^{-6}	1.025446584949125	$8.686547385838850E-07$
100	10^{-9}	5.016980587577223	$3.809786264241666E-09$
110	10^{-11}	12.62044850128009	$6.776161853849771E-10$
120	10^{-11}	22.06656019939149	$2.215585936937714E-09$
$N = 128$			
80	10^{-3}	0.068287400942693	$8.678783889792541E-04$
90	10^{-6}	1.025447547820377	$9.421651370189466E-08$
100	10^{-9}	5.016980591447074	$6.006484198906037E-11$
110	10^{-12}	12.62044850195964	$1.936228954946273E-12$
120	10^{-15}	22.06656020160710	$2.131628207280301E-13$
$N = 256$			
80	10^{-3}	0.068297602247680	$8.780796939656527E-04$
90	10^{-6}	1.025447982063936	$5.284600721466859E-07$
100	10^{-9}	5.016980593114734	$1.727725518207990E-09$
110	10^{-12}	12.62044850196316	$5.451639140119369E-12$
120	10^{-18}	22.06656020160708	$0.000000000000000E-00$

Table 6.32: Left Endpoint Quadrature - Down and Out call option for 8 time steps with $\sigma = 0.20$, $r = 0.08$, $T = 0.25$, strike of \$100 and barrier of \$75. The values are calculated in this table are performed in double precision.

Table 6.32 shows an improvement in the approximation as the number of partitions increase for all asset values other than \$80. The justification for less accurate results and a lack of improvement at this asset value is its proximity to the barrier. A possible solution to this problem is to consider other discretization schemes, where extra nodes are used near the barrier. Refer to the adaptive discretization for the interpolation method (Section 6.3.3) where results improved for the asset value of \$80.

Table 6.33 are prices using Right Endpoint Quadrature.

Asset Price (\$)	**Optimal η**	**Right Endpoint**	**Error**
$N = 64$			
80	10^{-3}	0.068342191541811	$9.226689880968697E-04$
90	10^{-6}	1.025449676530992	$2.222927128814334E-06$
100	10^{-9}	5.016980598662323	$7.275314573007563E-09$
110	10^{-11}	12.62044850149923	$4.584759238923652E-10$
120	10^{-10}	22.06656020049667	$1.110404213022775E-09$
$N = 128$			
80	10^{-3}	0.068325632015505	$9.061094617912047E-04$
90	10^{-6}	1.025449093148066	$1.639544201958998E-06$
100	10^{-9}	5.016980596987800	$5.600790942139611E-09$
110	10^{-12}	12.62044850197052	$1.281463823943341E-11$
120	10^{-15}	22.06656020160710	$1.776356839400250E-14$
$N = 256$			
80	10^{-3}	0.068316716249908	$8.971936961940374E-04$
90	10^{-6}	1.025448754669658	$1.301065794656608E-06$
100	10^{-9}	5.016980595884865	$4.497856309626513E-09$
110	10^{-12}	12.62044850196860	$1.089084378236294E-11$
120	10^{-18}	22.06656020160708	$0.000000000000000E-00$

Table 6.33: Right Endpoint Quadrature - Down and Out call option for 8 time steps with $\sigma = 0.20$, $r = 0.08$, $T = 0.25$, strike of \$100 and barrier of \$75. The values are calculated in this table are performed in double precision.

As expected the right endpoint approximations were similar to those evaluated by the left endpoint rule. It should also be noted that the prices evaluated by the right endpoint are larger than those approximated by the left endpoint.

Table 6.34 are prices using Midpoint Quadrature.

Asset Price ($)	**Optimal η**	**Midpoint**	**Error**
$N = 64$			
80	10^{-3}	0.068309240511536	$8.897179578215703E - 04$
90	10^{-6}	1.025448511612557	$1.058008693766155E - 06$
100	10^{-9}	5.016980595318302	$3.931292624770322E - 09$
110	10^{-11}	12.62044850150694	$4.507683115662076E - 10$
120	10^{-10}	22.06656019977550	$1.831576668109847E - 09$
$N = 128$			
80	10^{-3}	0.068307840878315	$8.883183246010951E - 04$
90	10^{-6}	1.025448416586935	$9.629830712842846E - 07$
100	10^{-9}	5.016980594782908	$3.395898673375086E - 09$
110	10^{-12}	12.62044850196667	$8.967049325292464E - 12$
120	10^{-16}	22.06656020160709	$1.065814103640150E - 14$
$N = 256$			
80	10^{-3}	0.068307490431320	$8.879678776056721E - 04$
90	10^{-6}	1.025448392448670	$9.388448065639210E - 07$
100	10^{-9}	5.016980594642222	$3.255212988051426E - 09$
110	10^{-12}	12.62044850196628	$8.576250820624409E - 12$
120	10^{-16}	22.06656020160708	$0.000000000000000E - 00$

Table 6.34: Midpoint Quadrature - Down and Out call option for 8 time steps with $\sigma = 0.20$, $r = 0.08$, $T = 0.25$, strike of \$100 and barrier of \$75. The values are calculated in this table are performed in double precision.

In most cases, the approximated down and out call option price using the midpoint rule was between the values evaluated for the left and right endpoint.

Table 6.35 are prices using the Trapezoidal Quadrature rule.

Asset Price ($)	**Optimal η**	**Trapezoidal**	**Error**
$N = 64$			
80	10^{-3}	0.068303637736538	$8.841151828233723E-04$
90	10^{-6}	1.025448128067019	$6.744631551036662E-07$
100	10^{-9}	5.016980593113967	$1.726958132053369E-09$
110	10^{-11}	12.62044850138965	$5.680540482444485E-10$
120	10^{-11}	22.06656019951860	$2.088476946937590E-09$
$N = 128$			
80	10^{-3}	0.068306438824764	$8.869162710496564E-04$
90	10^{-6}	1.025448319815123	$8.662112598312177E-07$
100	10^{-9}	5.016980594215982	$2.828972611723657E-09$
110	10^{-12}	12.62044850196508	$7.373657240350440E-12$
120	10^{-17}	22.06656020160710	$1.776356839400250E-14$
$N = 256$			
80	10^{-3}	0.068307139832806	$8.876172790918402E-04$
90	10^{-6}	1.025448368199469	$9.145956059164462E-07$
100	10^{-9}	5.016980594499438	$3.112429425300434E-09$
110	10^{-12}	12.62044850196587	$8.167688747562352E-12$
120	10^{-18}	22.06656020160708	$0.000000000000000E-00$

Table 6.35: Trapezoidal Quadrature - Down and Out call option for 8 time steps with $\sigma = 0.20$, $r = 0.08$, $T = 0.25$, strike of \$100 and barrier of \$75. The values are calculated in this table are performed in double precision.

In some cases the Trapezoidal rule achieved better results than the rectangular rules.

Table 6.36 are prices using the Composite Simpson's Quadrature rule.

Asset Price ($)	**Optimal η**	**Simpson's**	**Error**
$N = 64$			
80	10^{-3}	0.068240065246154	$8.205426924401821E-04$
90	10^{-6}	1.025446090099999	$1.363503864482141E-06$
100	10^{-7}	5.016975630482336	$4.960904672834943E-06$
110	10^{-6}	12.62050955140471	$6.104944700346948E-05$
120	10^{-5}	22.06647082185757	$8.937974951095384E-05$
$N = 128$			
80	10^{-3}	0.068275532665032	$8.560101113182089E-04$
90	10^{-6}	1.025447574142025	$1.205381610080991E-07$
100	10^{-9}	5.016980591439553	$5.254374713103971E-11$
110	10^{-12}	12.62044850173511	$2.226006046157636E-10$
120	10^{-11}	22.06656020112946	$4.776161688369029E-10$
$N = 256$			
80	10^{-3}	0.068291909854050	$8.723873003353422E-04$
90	10^{-6}	1.025448011081464	$5.574776003047788E-07$
100	10^{-9}	5.016980593200113	$1.813104333336923E-09$
110	10^{-12}	12.62044850196314	$5.432099214885966E-12$
120	10^{-19}	22.06656020160709	$1.421085471520200E-14$

Table 6.36: Composite Simpson's Quadrature - Down and Out call option for 8 time steps with $\sigma = 0.20$, $r = 0.08$, $T = 0.25$, strike of \$100 and barrier of \$75. The values are calculated in this table are performed in double precision.

When 256 partitions were used for the Composite Simpson's rule, the option prices evaluated were as accurate, if not better than the other rules. Table 6.37 are the errors associated with each Quadrature rule for a specific type of Down and Out call option.

As table 6.37 shows, there are minor differences between the Newton-Cotes rules. All prices are very precise compared to the analytical solution. It is difficult to compare the performance of the quadrature methods due to the propagating errors from time step to time step. In most cases, a single integral is evaluated using a particular rule and comparison is made by comparing the results. In these cases, propagation of errors has also an influence on the performance. Some rules work better with different discretization schemes also.

Asset Price ($)	**Left Error**	**Right Error**	**Mid Error**	**Trapezoidal Error**	**Simpson's Error**
80	$1.448386347E-03$	$1.428410997E-03$	$1.438701810E-03$	$1.438357809E-03$	$1.431487249E-03$
90	$5.857078917E-06$	$5.731699053E-06$	$5.800528943E-06$	$5.794178022E-06$	$5.738509854E-05$
100	$1.038337150E-07$	$7.759018938E-08$	$8.984506561E-08$	$9.071228657E-08$	$1.003878749E-07$
110	$1.633102080E-07$	$1.278341930E-07$	$1.443648117E-07$	$1.455721996E-07$	$1.590288523E-07$
120	$2.098224208E-07$	$1.696865439E-07$	$1.881378076E-07$	$1.897544841E-07$	$2.075375924E-07$

Table 6.37: Comparing Quadrature - Down and Out call option (asset value of \$100) for 8 time steps and 256 partitions with $\sigma = 0.20$, $r = 0.08$, $T = 0.25$, strike of \$100, $\eta = 10^{-8}$ and barrier of \$75. The values are calculated in this table are performed in double precision.

6.6 Conclusion

The methods used in the previous chapter were applied to an American put option and a Barrier down and out call option. The interpolation method performed well when applied to these different options. Similar behavioural issues arose in these cases compared to the European options. The option prices evaluated were quite precise compared to the binomial and analytical solutions obtained from the literature. The results compared favourably to those achieved by the Fourier series. As with the Fourier series, *a-priori* knowledge of the method parameters would allow optimal evaluations to be gained for less computational effort.

The quadrature methods worked quite well for the barrier option but performed poorly for the American put option. The poor performance occurred due to the miscalculation of the early exercise boundary at each time step. Even using the correct boundary values (evaluated by the interpolation method) did not assist. Some methods (right end point, trapezoidal and the composite Simpson's rules) did not improve significantly; with a high number of node points required to obtain an average result.

A possible improvement for this issue is the use of more sophisticated quadrature rules. Weighted rules which take into account the weight within the integrand may assist. Many of the weighted rules however are associated with single integrals, integrated over one variable. These rules would need to be extended to take into account a second variable. Since our path integral has two variables, the underlying and time.

Chapter 7

Conclusions and Recommendations

The main aim of the thesis was to present a numerical investigation of the path integral framework. The framework developed and presented by Chiarella et al. (1999) follows the Black Scholes paradigm and was summarised in Chapter 2. The path integral is an alternative representation of an option price than the traditional partial differential equations, namely

$$f^{k-1}(\xi_{k-1}) = \frac{e^{-r\Delta t}}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-(\xi_k - \mu(\xi_{k-1},\Delta t))^2} f^k(\sqrt{2\Delta t}\xi_k) d\xi_k. \tag{7.1}$$

In Chapter 3, we represent the approach of Chiarella et al. (1999). The Fourier-Hermite series is used to represent the underlying, $f^k(\sqrt{2\Delta t}\xi_k)$, within the path integral framework. The main advantage of this spectral method is the continuous representation of the option price as a polynomial. This allows multiple option prices to be evaluated from the same polynomial (that is, an option price for a particular underlying value).

One of the major disadvantages with this technique is the computational effort required to obtain accurate prices. This can be attributed to the exponential and factorial terms found in the recurrence relations, namely for a European and American put option,

$$\alpha_m^{K-1} = \frac{\sigma}{2m}\left[\frac{e^{-r\Delta t}}{2^{m-1}(m-1)!\upsilon^{m-1}\sqrt{\pi}} e^{-(\frac{b}{\upsilon})^2} H_{m-2}(-\frac{b}{\upsilon}) + \alpha_{m-1}^{K-1}\right]. \tag{7.2}$$

Also, the matrix multiplication(s) required to find the coefficients of the option price polynomial can be time consuming, especially when a large number of time steps (K) and/or basis functions (N) are to be used,

$$\boldsymbol{\alpha}^0 = e^{-r(K-1)\Delta t}\mathbf{A}^{K-1}\boldsymbol{\alpha}^{K-1}. \tag{7.3}$$

The coefficients determined (for a European option) in (7.3) requires matrix multiplication, with the dimensions of $\mathbf{A}$ and $\boldsymbol{\alpha}$ determined by the number of basis functions.

In an effort to combat the computational effort, in chapter 4 we offer a Normalised version of the same technique. The recurrence relations are formed using similar methods to the alternative, with differing Hermite orthogonal polynomial properties, forming different relations. For example,

$$\alpha_m^{K-1} = \frac{\sigma}{m}\left[\alpha_{m-1}^{K-1} + \frac{e^{-r\Delta t}}{(m-1)!\tau^{m-1}\sqrt{2\pi}} e^{-\frac{1}{2}(\frac{b}{\tau})^2} H_{m-2}^{*}(-\frac{b}{\tau})\right], \tag{7.4}$$

evaluates the coefficients α_m^{K-1} for the normalised version. This is equivalent to (7.2) in the non-normalised method. The major difference being the 2^{m-1} term does not exist in the normalised version. Investigations showed that the computational time did not improve by any great amount, in most cases, less than five percent.

Therefore, one of the most time consuming parts of this type of method is the matrix multiplication. If this matrix multiplication could be eliminated or the effort required to calculate was drastically reduced, the computational time required to obtain an accurate result could be reduced. Diagonalisation or other efficient methods to evaluate $\mathbf{A}^k$ would be worthwhile.

Another issue that arises in this method is the oscillating nature of the Fourier series. That is, for a given K (the number of time steps) and N (the number of basis functions), there are underlying values that will give more accurate results than others.

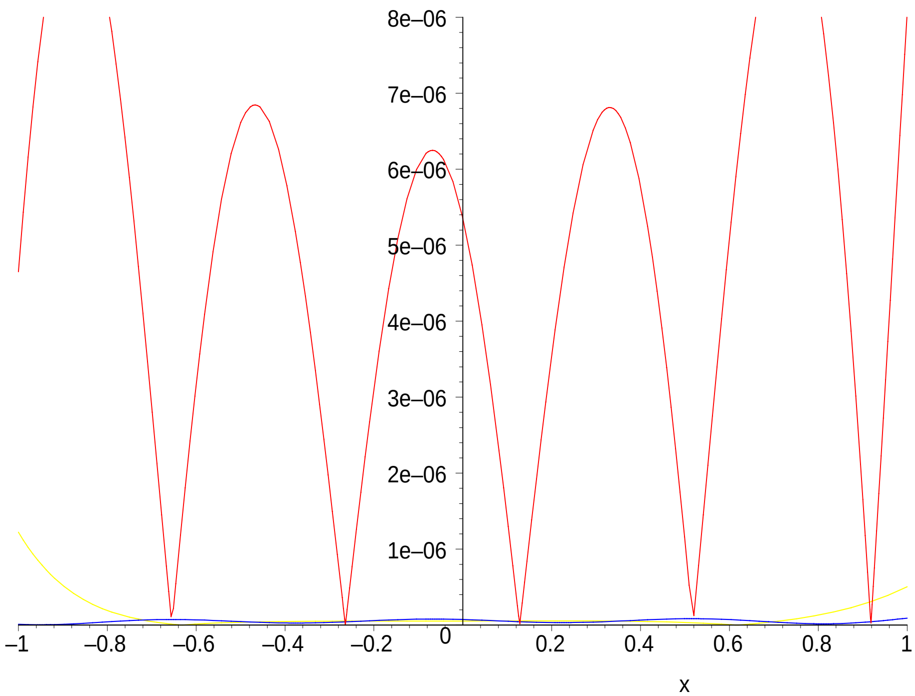

Figure 7.1: The absolute error of a Fourier-Hermite expansion vs Black-Scholes for a European call with $K = 4$, $N = 16$ (red), $N = 32$ (blue), $N = 64$ (yellow), $\sigma = 0.20$, $r = 0.08$, $T = 0.25$ and strike of \$100.

Figure 7.1 shows which prices were better than other prices. Therefore, an *a-priori* knowledge of what K and N is required, given a set of model parameters (volatility, interest rates and time to expiry), to give a better result would be advantageous.

The methods presented in Chapters 5 and 6 uses the same path integral framework, (7.1), but is modified so that interpolation polynomials and Newton-Cotes quadrature can be applied to find accurate results. The path integral is modified due to the infinite interval in (7.1). An upper bound (the value of the underlying) is used to approximate $f^k(\xi_k)$ at each time step, which allows a finite interval to be formed.

Using a better upper bound would be an obvious improvement in determining the finite intervals. Better intervals will lead to higher accuracy in the option price evaluation. In evaluating the intervals, a Taylor series was used for the error function. Use of a better approximation for the error function could also improve the determination of the interval at each time step.

In the thesis, the interpolation method used Hermite interpolation polynomials (of order 4) to represent $f^k(\xi_k)$ in (7.1). These commonly used polynomials, representing the underlying, achieved precise option prices. The prices evaluated for American put and the Barrier (down and out call) options were comparable to, the Binomial and analytical solutions found in the literature, respectively.

One of the major advantages of this interpolation method is the ease of implementation. The implementation will also allow for the use of other Hermite interpolation polynomials (different orders). Accurate results are evaluated with minimal computational effort. As with the Fourier method, *a-priori* knowledge of method parameters given a set of model parameters, would be advantageous.

For the interpolation method, various discretization schemes of the underlying were offered. Each scheme having its advantages over the others. The fixed schemes allowed for fast and precise results and the adaptive scheme traded computational effort for higher accuracy. Other discretization schemes, Gauss types for instance, could be used to improve the evaluation of the option price.

Various Newton-Cotes rules were applied to the path integral (7.1) to obtain accurate option prices in Chapter 6. These rules achieve fast results and in the case of the European options with high precision. Inaccuracies arose for the American put option, specifically the calculation of the exercise boundary. To compensate for this problem, the barriers were manually placed (obtained via the interpolation technique) to investigate the merits of the technique. This issue requires further attention and may be a flaw in using these types of rules.

Only one type of discretization scheme was used (fixed number of nodes). Other schemes, including Gauss type should be investigated and may assist in the issues arising with the American put option. Other types of quadrature, including weighted rules for multi-variable and multi-dimensional integrals, would be worthwhile investigating. Many of these rules may assist with obtaining *a-priori* knowledge of the method parameters. Some of these weighted rules have associated errors. By bounding the errors, appropriate and optimal K and N can be determined. The single issue associated with all the techniques present in the thesis is the lack of knowledge in regards to the number of time steps and basis functions/nodes prior to evaluation of the option price.

Given the differing methods offered in the thesis, a further extension worth exploring is different discretization of time. The implementations offered in the thesis, has fixed spaced time steps. The use of an alternative scheme for time may lead to improved precision, it is envisaged though that a greater computational effort would be required. Other types of options could also be applied to the path integral framework. Various barrier type options may be applied to the interpolation and quadrature methods. Asian options, where payoff is determined by an average value of the underlying, could be implemented, though may require further manipulation of the path integral to compensate for the unusual payoff scheme.

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Appendix A

Fourier-Hermite Series Expansion

These appendices contain many proofs and analysis of the methods and techniques described in Chapter 3.

A.1 European Options

A.1.1 Completing the Square

This is a step by step evaluation of the power of the exponential in the path integral (3.14) being converted to a complete square.

$$(\xi_k - \mu(\xi_{k-1}))^2 + \xi_{k-1}^2 = (\xi_k - \frac{1}{\sqrt{2\Delta t}}(\xi_{k-1} + b))^2 + \xi_{k-1}^2$$

$$= \xi_k^2 + \frac{1}{2\Delta t}(\xi_{k-1} + b)^2 - \frac{2\xi_k}{\sqrt{2\Delta t}}(\xi_{k-1} + b) + \xi_{k-1}^2$$

$$= \xi_k^2 + \frac{\xi_{k-1}^2}{2\Delta t} + \frac{b^2}{2\Delta t} + \frac{2\xi_{k-1}b}{2\Delta t} - \frac{2\xi_{k-1}\xi_k}{\sqrt{2\Delta t}} - \frac{2\xi_k b}{\sqrt{2\Delta t}} + \xi_{k-1}^2$$

$$= \xi_{k-1}^2\left(\frac{2\Delta t + 1}{2\Delta t}\right) - \frac{2\xi_{k-1}}{\sqrt{2\Delta t}}\left(\frac{\sqrt{2\Delta t}\xi_k - b}{\sqrt{2\Delta t}}\right) + \left(\xi_k - \frac{b}{\sqrt{2\Delta t}}\right)^2$$

$$= \frac{\xi_{k-1}^2 \upsilon^2}{2\Delta t} - \frac{2\xi_{k-1}}{\sqrt{2\Delta t}}\left(\frac{\sqrt{2\Delta t}\xi_k - b}{\sqrt{2\Delta t}}\right) + \left(\frac{\sqrt{2\Delta t}\xi_k - b}{\sqrt{2\Delta t}}\right)^2$$

$$= \frac{\xi_{k-1}^2 \upsilon^2}{2\Delta t} - \frac{2\xi_{k-1}\upsilon}{\sqrt{2\Delta t}}\left(\frac{\sqrt{2\Delta t}\xi_k - b}{\upsilon\sqrt{2\Delta t}}\right) + \left(\frac{\sqrt{2\Delta t}\xi_k - b}{\sqrt{2\Delta t}}\right)^2. \tag{A.1}$$

The expression (A.1) is in a form such that we can complete the square.

$$= \left[\frac{\xi_{k-1}\upsilon}{\sqrt{2\Delta t}} - \frac{\sqrt{2\Delta t}\xi_k - b}{\upsilon\sqrt{2\Delta t}}\right]^2 + \left(\frac{\sqrt{2\Delta t}\xi_k - b}{\sqrt{2\Delta t}}\right)^2 - \left(\frac{\sqrt{2\Delta t}\xi_k - b}{\upsilon\sqrt{2\Delta t}}\right)^2$$

$$= \left[\frac{\xi_{k-1}\upsilon}{\sqrt{2\Delta t}} - \frac{\sqrt{2\Delta t}\xi_k - b}{\upsilon\sqrt{2\Delta t}}\right]^2 + \frac{(\sqrt{2\Delta t}\xi_k - b)^2}{2\Delta t}\left[1 - \frac{1}{\upsilon^2}\right]$$

$$= \left[\frac{\xi_{k-1}\upsilon}{\sqrt{2\Delta t}} - \frac{\sqrt{2\Delta t}\xi_k - b}{\upsilon\sqrt{2\Delta t}}\right]^2 + \frac{(\sqrt{2\Delta t}\xi_k - b)^2}{2\Delta t}\left[\frac{\upsilon^2 - 1}{\upsilon^2}\right]$$

$$= \left[\frac{\xi_{k-1}\upsilon}{\sqrt{2\Delta t}} - \frac{\sqrt{2\Delta t}\xi_k - b}{\upsilon\sqrt{2\Delta t}}\right]^2 + \frac{(\sqrt{2\Delta t}\xi_k - b)^2}{2\Delta t}\left[\frac{1 + 2\Delta t - 1}{\upsilon^2}\right]$$

$$= \left[\frac{\upsilon\xi_{k-1}}{\sqrt{2\Delta t}} - \frac{\sqrt{2\Delta t}\,\xi_k - b}{\upsilon\sqrt{2\Delta t}}\right]^2 + \left[\frac{\sqrt{2\Delta t}\,\xi_k - b}{\upsilon}\right]^2. \tag{A.2}$$

A.1.2 Evaluating $A_{m,n}$

This is the complete evaluation of elements $A_{m,n}$. The first step requires the transformation of (3.30) to a form so that integration by parts can be used.

$$A_{m,n} = \frac{1}{2^m m! \upsilon^m \sqrt{\pi}} \int_{-\infty}^{\infty} e^{-z^2} H_m(z) \left[\frac{d}{dz}\frac{1}{\upsilon}\frac{1}{2(n+1)} H_{n+1}(\upsilon z + b)\right] dz,$$

$$= \frac{1}{2^m m! \upsilon^m}\left(\frac{1}{\upsilon}\frac{1}{2(n+1)}\frac{1}{\sqrt{\pi}} e^{-z^2} H_m(z) H_{n+1}(\upsilon z + b)\right]_{\infty}^{\infty}$$

$$- \frac{1}{\upsilon}\frac{1}{2(n+1)}\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} H_{n+1}(\upsilon z + b)\left(\frac{d}{dz} e^{-z^2} H_m(z)\right) dz\Bigg),$$

$$= \frac{1}{2^m m! \upsilon^m}\left[-\frac{1}{\upsilon}\frac{1}{2(n+1)}\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} H_{n+1}(\upsilon z + b)\left(\frac{d}{dz} e^{-z^2} H_m(z)\right) dz\right]. \tag{A.3}$$

The derivative in (A.3) can be solved using property (3.6) and the product rule as described in the chapter.

$$\left(\frac{d}{dz}e^{-z^2}H_m(z)\right) = 2me^{-z^2}H_{m-1}(z) - 2ze^{-z^2}H_m(z),$$

$$= e^{-z^2}\big[2mH_{m-1}(z) - 2zH_m(z)\big],$$

$$= e^{-z^2}\big[-H_{m+1}(z)\big]. \tag{A.4}$$

So, to evaluate the element $A_{m,n}$, (A.4) is substituted into (A.3). Since (A.4) is expressed in a forward manner, rearrangement is required so that $A_{m,n}$ is expressed in terms of $A_{m-1,n-1}$. Therefore, as presented in the chapter

$$A_{m,n} = \frac{1}{2^m m! \upsilon^m}\left[-\frac{1}{\upsilon}\frac{1}{2(n+1)}\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} H_{n+1}(\upsilon z+b)e^{-z^2}\Big(-H_{m+1}(z)\Big)\,dz\right],$$

$$= \frac{1}{2^m m! \upsilon^m}\left[\frac{1}{\upsilon}\frac{1}{2(n+1)}\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-z^2}H_{m+1}(z)H_{n+1}(\upsilon z+b)\,dz\right],$$

$$= \frac{1}{2^{m+1}(m+1)! \upsilon^{m+1}}\left[\frac{m+1}{n+1}\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-z^2}H_{m+1}(z)H_{n+1}(\upsilon z+b)\,dz\right],$$

$$= \frac{m+1}{n+1}A_{m+1,n+1},$$

and so

$$A_{m+1,n+1} = \frac{n+1}{m+1}A_{m,n},$$

giving

$$A_{m,n} = \frac{n}{m}A_{m-1,n-1}. \tag{A.5}$$

It must be noted that when $m > n$ element $A_{m,n} = 0$.

A.1.3 Evaluating $\Psi^c_m(-\frac{b}{\upsilon})$

The following is a derivation of $\Psi^c_m(-\frac{b}{\upsilon})$ for a European call option. $\Psi^c_m(-\frac{b}{\upsilon})$ is transformed so that it can assist in the evaluation of $\boldsymbol{\alpha}^{K-1}$. The aim is to join the two exponential in the integrand of (3.37) into a single exponential. The exponential also has to be transformed so that Hermite polynomials and their properties can be used. This is achieved via simple algebra.

$$
\begin{aligned}
\Psi^c_m(-\frac{b}{\upsilon}) &= \frac{1}{\sqrt{\pi}} \int_{-\frac{b}{\upsilon}}^{\infty} e^{\sigma \upsilon z} e^{-z^2} H_m(z)\, dz, \\
&= \frac{1}{\sqrt{\pi}} \int_{-\frac{b}{\upsilon}}^{\infty} e^{-(z^2 - \sigma \upsilon z)} H_m(z)\, dz, \\
&= \frac{1}{\sqrt{\pi}} \int_{-\frac{b}{\upsilon}}^{\infty} e^{-(z^2 - \sigma \upsilon z + \frac{\sigma^2 \upsilon^2}{4})} e^{\frac{\sigma^2 \upsilon^2}{4}} H_m(z)\, dz, \\
&= \frac{e^{\frac{1}{4}\sigma^2 \upsilon^2}}{\sqrt{\pi}} \int_{-\frac{b}{\upsilon}}^{\infty} e^{-(z - \frac{\sigma \upsilon}{2})^2} H_m(z)\, dz, \qquad \text{(A.6)}
\end{aligned}
$$

The proof for Ψ^c_0, as presented in (3.41), is as follows

$$
\begin{aligned}
\Psi^c_0(-\frac{b}{\upsilon}) &= \frac{e^{\frac{1}{4}\sigma^2 \tau^2}}{\sqrt{\pi}} \int_{-\frac{b}{\upsilon}}^{\infty} e^{-(z - \frac{\sigma \upsilon}{2})^2} H_0(z) dz, \\
&= \frac{e^{\frac{1}{4}\sigma^2 \tau^2}}{\sqrt{\pi}} \int_{-\frac{b}{\upsilon}}^{\infty} e^{-(z - \frac{\sigma \upsilon}{2})^2} dz \qquad \text{(A.7)}
\end{aligned}
$$

let,

$$
u = z - \frac{\sigma \upsilon}{2} \qquad \text{(A.8)}
$$

and substituting (A.8) into (A.7) gives

$$\Psi_0^c(-\frac{b}{\upsilon}) = \frac{e^{\frac{1}{4}\sigma^2\upsilon^2}}{\sqrt{\pi}} \int_{-\frac{b}{\upsilon}-\frac{\sigma\upsilon}{2}}^{\infty} e^{-u^2}\frac{du}{dz}dz,$$

$$= \frac{e^{\frac{1}{4}\sigma^2\upsilon^2}}{2} + \frac{e^{\frac{1}{4}\sigma^2\upsilon^2}}{\sqrt{\pi}} \int_0^{\frac{b}{\upsilon}+\frac{\sigma\upsilon}{2}} e^{-u^2}du,$$

$$= e^{\frac{1}{4}\sigma^2\upsilon^2}\left[\frac{1}{2} + \frac{1}{2}erf\big(\frac{b}{\upsilon} + \frac{\sigma\upsilon}{2})\right],$$

$$= \frac{e^{\frac{1}{4}\sigma^2\upsilon^2}}{2}erfc\big(-\frac{b}{\upsilon} - \frac{\sigma\upsilon}{2}), \tag{A.9}$$

The proof for Ψ_1^c, as in (3.43), is

$$\Psi_1^c(-\frac{b}{\upsilon}) = \frac{e^{\frac{1}{4}\sigma^2\upsilon^2}}{\sqrt{\pi}} \int_{-\frac{b}{\upsilon}}^{\infty} e^{-(z-\frac{\sigma\upsilon}{2})^2}H_1(z)dz,$$

$$= \frac{e^{\frac{1}{4}\sigma^2\upsilon^2}}{\sqrt{\pi}} \int_{-\frac{b}{\upsilon}}^{\infty} 2ze^{-(z-\frac{\sigma\upsilon}{2})^2}dz \tag{A.10}$$

let,

$$u = z - \frac{\sigma\upsilon}{2}, \tag{A.11}$$

and substituting (A.11) into (A.10) gives

$$\Psi_1^c(-\frac{b}{\upsilon}) = \frac{e^{\frac{1}{4}\sigma^2\upsilon^2}}{\sqrt{\pi}} \int_{-\frac{b}{\upsilon}-\frac{\sigma\upsilon}{2}}^{\infty} 2(u+\frac{\sigma\upsilon}{2})e^{-u^2}.\frac{du}{dz}dz,$$

$$= e^{\frac{1}{4}\sigma^2\upsilon^2}\left[\frac{2}{\sqrt{\pi}}\int_{-\frac{b}{\upsilon}-\frac{\sigma\upsilon}{2}}^{\infty} ue^{-u^2}du + \frac{\sigma\upsilon}{\sqrt{\pi}}\int_{-\frac{b}{\upsilon}-\frac{\sigma\upsilon}{2}}^{\infty} e^{-u^2}du\right]. \tag{A.12}$$

Performing the substitution,

$$x = u^2,$$

into (A.12) gives

$$\Psi_1^c(-\frac{b}{\upsilon}) = e^{\frac{1}{4}\sigma^2\upsilon^2}\left[\frac{2}{\sqrt{\pi}}\int_{-\frac{b}{\upsilon}-\frac{\sigma\upsilon}{2}}^{\infty} ue^{-x}\frac{dx}{2udu}du + \frac{\sigma\upsilon}{2}erfc\big(-\frac{b}{\upsilon}-\frac{\sigma\upsilon}{2}\big)\right],$$

$$= e^{\frac{1}{4}\sigma^2\upsilon^2}\left[\frac{1}{\sqrt{\pi}}\int_{-\frac{b}{\upsilon}-\frac{\sigma\upsilon}{2}}^{\infty} e^{-x}dx + \frac{\sigma\upsilon}{2}erfc\big(-\frac{b}{\upsilon}-\frac{\sigma\upsilon}{2}\big)\right],$$

$$= e^{\frac{1}{4}\sigma^2\upsilon^2}\left[\frac{\sigma\upsilon}{2}erfc\big(-\frac{b}{\upsilon}-\frac{\sigma\upsilon}{2}\big) - \frac{1}{\sqrt{\pi}}e^{-x}\Big]_{-\frac{b}{\upsilon}-\frac{\sigma\upsilon}{2}}^{\infty}\right],$$

$$= e^{\frac{1}{4}\sigma^2\upsilon^2}\left[\frac{\sigma\upsilon}{2}erfc\big(-\frac{b}{\upsilon}-\frac{\sigma\upsilon}{2}\big) + \frac{1}{\sqrt{\pi}}e^{-(-\frac{b}{\upsilon}-\frac{\sigma\upsilon}{2})^2}\right],$$

$$= e^{\frac{1}{4}\sigma^2\upsilon^2}\left[\frac{\sigma\upsilon}{2}erfc\big(-\frac{b}{\upsilon}-\frac{\sigma\upsilon}{2}\big) + \frac{1}{\sqrt{\pi}}e^{-((\frac{b}{\upsilon})^2+\frac{\sigma^2\upsilon^2}{4}+\sigma b)}\right],$$

$$= e^{\frac{1}{4}\sigma^2\upsilon^2}\frac{\sigma\upsilon}{2}erfc\big(-\frac{b}{\upsilon}-\frac{\sigma\upsilon}{2}\big) + \frac{1}{\sqrt{\pi}}e^{-((\frac{b}{\upsilon})^2+\sigma b)}, \tag{A.13}$$

The proof for Ψ_m^c, as in (3.45), is

$$\begin{aligned}
\Psi_m^c(-\frac{b}{\upsilon}) =& \frac{e^{\frac{1}{4}\sigma^2\upsilon^2}}{\sqrt{\pi}} \int_{-\infty}^{-\frac{b}{\upsilon}} e^{-(z-\frac{\sigma\upsilon}{2})^2} \Big[2zH_{m-1}(z) - 2(m-1)H_{m-2}(z) \Big] dz, \\
=& e^{\frac{1}{4}\sigma^2\upsilon^2} \Big[\frac{1}{\sqrt{\pi}} \int_{-\infty}^{-\frac{b}{\upsilon}} 2ze^{-(z-\frac{\sigma\upsilon}{2})^2} H_{m-1}(z)dz \\
& - \frac{2(m-1)}{\sqrt{\pi}} \int_{-\infty}^{-\frac{b}{\upsilon}} e^{-(z-\frac{\sigma\upsilon}{2})^2} H_{m-2}(z)dz \Big], \\
=& e^{\frac{1}{4}\sigma^2\upsilon^2} \Big[\frac{1}{\sqrt{\pi}} \int_{-\infty}^{-\frac{b}{\upsilon}} 2ze^{-(z-\frac{\sigma\upsilon}{2})^2} H_{m-1}(z)dz \\
& - \frac{1}{\sqrt{\pi}} \int_{-\infty}^{-\frac{b}{\upsilon}} e^{-(z-\frac{\sigma\upsilon}{2})^2} \big(\frac{d}{dz} H_{m-1}(z)\big) dz \Big], \\
=& e^{\frac{1}{4}\sigma^2\upsilon^2} \Big[\frac{1}{\sqrt{\pi}} \int_{-\infty}^{-\frac{b}{\upsilon}} 2ze^{-(z-\frac{\sigma\upsilon}{2})^2} H_{m-1}(z)dz \\
& - \frac{1}{\sqrt{2\pi}} e^{-(z-\frac{\sigma\upsilon}{2})^2} H_{m-1}(z) \Big]_{-\infty}^{-\frac{b}{\upsilon}} - \frac{1}{\sqrt{\pi}} \int_{-\infty}^{-\frac{b}{\upsilon}} (z - \frac{\sigma\upsilon}{2}) e^{-(z-\frac{\sigma\upsilon}{2})^2} H_{m-1}(z)dz \Big], \\
=& e^{\frac{1}{4}\sigma^2\upsilon^2} \Big[\frac{\sigma\upsilon}{\sqrt{\pi}} \int_{-\infty}^{-\frac{b}{\upsilon}} e^{-(z-\frac{\sigma\upsilon}{2})^2} H_{m-1}(z)dz - \frac{1}{\sqrt{\pi}} e^{-((-\frac{b}{\upsilon}-\frac{\sigma\upsilon}{2})^2} H_{m-1}(-\frac{b}{\upsilon}) \Big], \\
=& \Big[\sigma\upsilon\Psi_{m-1}^c(-\frac{b}{\upsilon}) - \frac{e^{\frac{1}{4}\sigma^2\tau^2}}{\sqrt{\pi}} e^{-((\frac{b}{\upsilon})^2 + \frac{\sigma^2\upsilon^2}{4} + \sigma b)} H_{m-1}(-\frac{b}{\upsilon}) \Big], \\
=& \Big[\sigma\upsilon\Psi_{m-1}^c(-\frac{b}{\upsilon}) - \frac{1}{\sqrt{\pi}} e^{-((\frac{b}{\upsilon})^2 + \sigma b)} H_{m-1}(-\frac{b}{\upsilon}) \Big],
\end{aligned} \tag{A.14}$$

A.1.4 Evaluating $\Omega_m^c(-\frac{b}{\upsilon})$

The proof for Ω_0^c, as in (3.42), is given by

$$\Omega_0^c(-\frac{b}{\upsilon}) = \frac{1}{\sqrt{\pi}} \int_{-\frac{b}{\upsilon}}^{\infty} e^{-z^2} H_0(z) dz,$$

$$= \frac{1}{\sqrt{\pi}} \int_{-\frac{b}{\upsilon}}^{\infty} e^{-z^2} dz,$$

$$= \frac{1}{2} + \frac{1}{\sqrt{\pi}} \int_{0}^{\frac{b}{\upsilon}} e^{-z^2} dz,$$

$$= \frac{1}{2} + \frac{1}{2} erf\big(\frac{b}{\upsilon}\big),$$

$$= \frac{1}{2} erfc\big(-\frac{b}{\upsilon}\big). \tag{A.15}$$

The proof for Ω_1^c, as in (3.44), is

$$\begin{aligned} \Omega_1^c(-\frac{b}{\upsilon}) &= \frac{1}{\sqrt{\pi}} \int_{-\frac{b}{\upsilon}}^{\infty} e^{-z^2} H_1(z) dz, \\ &= \frac{1}{\sqrt{\pi}} \int_{-\frac{b}{\upsilon}}^{\infty} 2z e^{-z^2} dz. \end{aligned} \tag{A.16}$$

Performing the substitution,

$$u = z^2,$$

into (A.16) gives

$$\Omega_1^c(-\frac{b}{\upsilon}) = \frac{1}{\sqrt{\pi}}\int_{-\frac{b}{\upsilon}}^{\infty} 2ze^{-u}\frac{du}{2zdz}dz,$$

$$= \frac{1}{\sqrt{\pi}}\int_{-\frac{b}{\upsilon}}^{\infty} e^{-u}du,$$

$$= -\frac{1}{\sqrt{\pi}}e^{-u}\Big]_{-\frac{b}{\upsilon}}^{\infty},$$

$$= \frac{1}{\sqrt{\pi}}e^{-(\frac{b}{\upsilon})^2}. \tag{A.17}$$

The proof for Ω_m^c, as in (3.46), is

$$\Omega_m^c(-\frac{b}{\upsilon}) =\frac{1}{\sqrt{\pi}}\int_{-\frac{b}{\upsilon}}^{\infty} e^{-z^2}\Big[2zH_{m-1}(z) - 2(m-1)H_{m-2}(z)\Big]dz,$$

$$=\frac{1}{\sqrt{\pi}}\int_{-\frac{b}{\upsilon}}^{\infty} 2ze^{-z^2}H_{m-1}(z)dz - \frac{1}{\sqrt{\pi}}\int_{-\frac{b}{\upsilon}}^{\infty} 2(m-1)e^{-z^2}H_{m-2}(z)dz,$$

$$=\frac{1}{\sqrt{\pi}}\int_{-\frac{b}{\upsilon}}^{\infty} 2ze^{-z^2}H_{m-1}(z)dz - 2(m-1)\Omega_{m-2}^c(-\frac{b}{\upsilon}),$$

$$=-\frac{1}{\sqrt{\pi}}e^{-z^2}H_{m-1}(z)\Big]_{-\frac{b}{\upsilon}}^{\infty}$$
$$+ 2(m-1)\frac{1}{\sqrt{\pi}}\int_{-\frac{b}{\upsilon}}^{\infty} e^{-z^2}H_{m-2}(z)dz - 2(m-1)\Omega_{m-2}^c(-\frac{b}{\upsilon}),$$

$$=\frac{1}{\sqrt{\pi}}e^{-(\frac{b}{\upsilon})^2}H_{m-1}(-\frac{b}{\upsilon}). \tag{A.18}$$

A.1.5 Evaluating $\Psi_m^p(-\frac{b}{\upsilon})$

The proof for Ψ_0^p as in (3.63), is

$$\Psi_0^p(-\frac{b}{\upsilon}) = \frac{e^{\frac{1}{4}\sigma^2\upsilon^2}}{\sqrt{\pi}} \int_{-\infty}^{-\frac{b}{\upsilon}} e^{-(z-\frac{\sigma\upsilon}{2})^2} H_0(z)\, dz,$$

$$= \frac{e^{\frac{1}{4}\sigma^2\upsilon^2}}{\sqrt{\pi}} \int_{-\infty}^{-\frac{b}{\upsilon}} e^{-(z-\frac{\sigma\upsilon}{2})^2}\, dz,$$

$$= \frac{e^{\frac{1}{4}\sigma^2\upsilon^2}}{\sqrt{\pi}} \int_{-\infty}^{-\frac{b}{\upsilon}-\frac{\sigma\upsilon}{2}} e^{-u^2}\, du,$$

$$= \frac{e^{\frac{1}{4}\sigma^2\upsilon^2}}{2} erfc(\frac{b}{\upsilon} + \frac{\sigma\upsilon}{2}). \tag{A.19}$$

The proof for Ψ_1^p as in (3.63), is

$$\Psi_1^p(-\frac{b}{\upsilon}) = \frac{e^{\frac{1}{4}\sigma^2\upsilon^2}}{\sqrt{\pi}} \int_{-\infty}^{-\frac{b}{\upsilon}} e^{-(z-\frac{\sigma\upsilon}{2})^2} H_1(z)\, dz,$$

$$= \frac{e^{\frac{1}{4}\sigma^2\upsilon^2}}{\sqrt{\pi}} \int_{-\infty}^{-\frac{b}{\upsilon}} 2ze^{-(z-\frac{\sigma\upsilon}{2})^2}\, dz,$$

$$= \frac{e^{\frac{1}{4}\sigma^2\upsilon^2}}{\sqrt{\pi}} \int_{-\infty}^{-\frac{b}{\upsilon}-\frac{\sigma\upsilon}{2}} 2(u + \frac{\sigma\upsilon}{2})e^{-u^2}\, du,$$

$$= \frac{2e^{\frac{1}{4}\sigma^2\upsilon^2}}{\sqrt{\pi}} \left[\int_{-\infty}^{-\frac{b}{\upsilon}-\frac{\sigma\upsilon}{2}} ue^{-u^2}\, du + \frac{\sigma\upsilon}{2} \int_{-\infty}^{-\frac{b}{\upsilon}-\frac{\sigma\upsilon}{2}} e^{-u^2}\, du \right],$$

$$= e^{\frac{1}{4}\sigma^2\upsilon^2} \left[\frac{1}{\sqrt{\pi}} \int_{-\infty}^{-\frac{b}{\upsilon}-\frac{\sigma\upsilon}{2}} e^{-v}\, dv + \frac{\sigma\upsilon}{2} erfc(\frac{b}{\upsilon} + \frac{\sigma\upsilon}{2}) \right],$$

$$= \frac{\sigma\upsilon e^{\frac{1}{4}\sigma^2\upsilon^2}}{2} erfc(\frac{b}{\upsilon} + \frac{\sigma\upsilon}{2}) - \frac{1}{\sqrt{\pi}} e^{-((\frac{b}{\upsilon})^2 + \sigma b)}. \tag{A.20}$$

The proof for Ψ_m^p as in (3.63), is

$$\Psi_m^p(-\frac{b}{\upsilon}) = \frac{e^{\frac{1}{4}\sigma^2\upsilon^2}}{\sqrt{\pi}} \int_{-\infty}^{-\frac{b}{\upsilon}} e^{-(z-\frac{\sigma\upsilon}{2})^2} H_m(z)\, dz,$$

$$= \frac{e^{\frac{1}{4}\sigma^2\upsilon^2}}{\sqrt{\pi}} \int_{-\infty}^{-\frac{b}{\upsilon}} e^{-(z-\frac{\sigma\upsilon}{2})^2} \Big[2zH_{m-1}(z) - 2(m-1)H_{m-2}(z)\Big]\, dz,$$

$$= e^{\frac{1}{4}\sigma^2\upsilon^2}\Bigg[\frac{1}{\sqrt{\pi}} \int_{-\infty}^{-\frac{b}{\upsilon}} 2ze^{-(z-\frac{\sigma\upsilon}{2})^2} H_{m-1}(z)\, dz$$
$$- \frac{1}{\sqrt{\pi}} \int_{-\infty}^{-\frac{b}{\upsilon}} 2(m-1)e^{-(z-\frac{\sigma\upsilon}{2})^2} H_{m-2}(z)\Bigg]\, dz,$$

$$= e^{\frac{1}{4}\sigma^2\upsilon^2}\Bigg[\frac{1}{\sqrt{\pi}} \int_{-\infty}^{-\frac{b}{\upsilon}} 2ze^{-(z-\frac{\sigma\upsilon}{2})^2} H_{m-1}(z)\, dz$$
$$- \frac{1}{\sqrt{\pi}} \int_{-\infty}^{-\frac{b}{\upsilon}} 2(m-1)e^{-(z-\frac{\sigma\upsilon}{2})^2} \left(\frac{d}{dz}\frac{1}{2(m-1)}H_{m-1}(z)\right)\Bigg]\, dz,$$

$$= e^{\frac{1}{4}\sigma^2\upsilon^2}\Bigg[\frac{1}{\sqrt{\pi}} \int_{-\infty}^{-\frac{b}{\upsilon}} 2ze^{-(z-\frac{\sigma\upsilon}{2})^2} H_{m-1}(z)\, dz - \frac{1}{\sqrt{\pi}}e^{-(z-\frac{\sigma\upsilon}{2})^2} H_{m-1}(z)\Bigg]_{-\infty}^{-\frac{b}{\upsilon}}$$
$$- \frac{1}{\sqrt{\pi}} \int_{-\infty}^{-\frac{b}{\upsilon}} 2(z-\frac{\sigma\upsilon}{2})e^{-(z-\frac{\sigma\upsilon}{2})^2} H_{m-1}(z)\Bigg]\, dz,$$

$$= e^{\frac{1}{4}\sigma^2\upsilon^2}\Bigg[-\frac{1}{\sqrt{\pi}}e^{-(-\frac{b}{\upsilon}-\frac{\sigma\upsilon}{2})^2} H_{m-1}(-\frac{b}{\upsilon}) + \frac{\sigma\upsilon}{\sqrt{\pi}} \int_{-\infty}^{-\frac{b}{\upsilon}} e^{-(z-\frac{\sigma\upsilon}{2})^2} H_{m-1}(z)\, dz\Bigg],$$

$$= \sigma\upsilon\Psi_{m-1}^p(-\frac{b}{\upsilon}) - \frac{1}{\sqrt{\pi}}e^{-((\frac{b}{\upsilon})^2+\sigma b)} H_{m-1}(-\frac{b}{\upsilon}). \tag{A.21}$$

A.1.6 Evaluating $\Omega_m^p(-\frac{b}{\upsilon})$

The proof for Ω_0^p as in (3.66), is

$$\Omega_0^p(-\frac{b}{v}) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{-\frac{b}{v}} e^{-z^2} H_0(z)\, dz$$

$$= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{-\frac{b}{v}} e^{-z^2}\, dz,$$

$$= \frac{1}{2} erfc(\frac{b}{v}). \tag{A.22}$$

The proof for Ω_1^p as in (3.66), is

$$\Omega_1^p(-\frac{b}{v}) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{-\frac{b}{v}} e^{-z^2} H_1(z)\, dz,$$

$$= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{-\frac{b}{v}} 2ze^{-z^2}\, dz,$$

$$= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{-\frac{b}{v}} e^{-u}\, du,$$

$$= -\frac{1}{\sqrt{\pi}} e^{-(\frac{b}{v})^2}. \tag{A.23}$$

The proof for Ω_m^p as in (3.66), is

$$\Omega_m^p(-\frac{b}{\upsilon}) = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{-\frac{b}{\upsilon}} e^{-z^2} H_m(z)\,dz,$$

$$= \frac{1}{\sqrt{\pi}}\int_{-\infty}^{-\frac{b}{\upsilon}} e^{-z^2}\Big[2zH_{m-1}(z) - 2(m-1)H_{m-2}(z)\,dz\Big],$$

$$= \frac{1}{\sqrt{\pi}}\int_{-\infty}^{-\frac{b}{\upsilon}} e^{-z^2} 2zH_{m-1}(z) - 2(m-1)\Omega_{m-2}^p(-\frac{b}{\upsilon}),$$

$$= -\frac{1}{\sqrt{\pi}} e^{-z^2} H_{m-1}(z)\Big]_{-\infty}^{-\frac{b}{\upsilon}} + 2(m-1)\Omega_{m-2}^p(-\frac{b}{\upsilon}) - 2(m-1)\Omega_{m-2}^p(-\frac{b}{\upsilon}),$$

$$= -\frac{1}{\sqrt{\pi}} e^{-(\frac{b}{\upsilon})^2} H_{m-1}(-\frac{b}{\upsilon}). \tag{A.24}$$

A.2 American Put Option

A.2.1 Evaluating γ_1^{k-1}

The proof to γ_1^{k-1} as in (3.97) can be formed by using properties (3.86) and (3.88) and integration.

$$\gamma_1^{k-1} = \frac{e^{-r\Delta t}}{2^1 1! \upsilon^1 \sqrt{\pi}}\Big[\int_{-\infty}^{z_k} e^{-z^2} H_1(z)\,dz - e^{\sigma b}\int_{-\infty}^{z_k} e^{-z^2+\sigma\upsilon z} H_1(z)\,dz\Big],$$

$$= \frac{e^{-r\Delta t}}{2\upsilon\sqrt{\pi}}\Big[\int_{-\infty}^{z_k} 2ze^{-z^2}\,dz - e^{\sigma b + \frac{\sigma^2\upsilon^2}{4}}\int_{-\infty}^{z_k} 2ze^{-(z-\frac{\sigma\upsilon}{2})^2}\,dz\Big],$$

$$= \frac{e^{-r\Delta t}}{2\upsilon}\Big[-\frac{1}{\sqrt{\pi}}e^{-z_k^2} - \frac{1}{\sqrt{\pi}}e^{\sigma b + \frac{\sigma^2\upsilon^2}{4}}\int_{-\infty}^{z_k-\sigma\upsilon}(2u+\sigma\upsilon)e^{-u^2}\,du\Big],$$

$$= \frac{e^{-r\Delta t}}{2\upsilon}\Big[-\frac{1}{\sqrt{\pi}}e^{-z_k^2} - \frac{1}{\sqrt{\pi}}e^{\sigma b + \frac{\sigma^2\upsilon^2}{4}}\Big[\int_{-\infty}^{z_k-\sigma\upsilon} 2ue^{-u^2}\,du + \sigma\upsilon\int_{-\infty}^{z_k-\sigma\upsilon} e^{-u^2}\,du\Big]\Big],$$

$$= \frac{e^{-r\Delta t}}{2\upsilon}\Big[-\frac{1}{\sqrt{\pi}}e^{-z_k^2} + \frac{e^{\sigma b - z_k^2 + \sigma\upsilon z_k}}{\sqrt{\pi}} - \frac{\sigma\upsilon e^{\sigma b + \frac{\sigma^2\upsilon^2}{4}}}{2}\, erfc\Big(\frac{\sigma\upsilon}{2} - z_k\Big)\Big]. \tag{A.25}$$

A.2.2 Evaluating Θ_m^{k-1}

The proof for Θ_m^{k-1} as in (3.103), is.

$$
\begin{aligned}
\Theta_m^{k-1} &= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{z_k} e^{-z^2} H_m(z)\, dz, \\
&= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{z_k} e^{-z^2} (2zH_{m-1}(z) - 2(m-1)H_{m-2}(z))\, dz, \\
&= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{z_k} 2ze^{-z^2} H_{m-1}(z)\, dz - \frac{1}{\sqrt{\pi}} \int_{-\infty}^{z_k} 2(m-1)e^{-z^2} H_{m-2}(z)\, dz, \\
&= -\frac{1}{\sqrt{\pi}} e^{-z_k^2} H_{m-1}(z_k) \\
&+ \frac{1}{\sqrt{\pi}} \int_{-\infty}^{z_k} 2(m-1)e^{-z^2} H_{m-2}(z)\, dz - \frac{1}{\sqrt{\pi}} \int_{-\infty}^{z_k} 2(m-1)e^{-z^2} H_{m-2}(z)\, dz, \\
&= -\frac{1}{\sqrt{\pi}} e^{-z_k^2} H_{m-1}(z_k).
\end{aligned} \tag{A.26}
$$

A.2.3 Evaluating Φ_m^{k-1}

The proof for Φ_m^{k-1} as in (3.104), is.

$$\Phi_m^{k-1} = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{z_k} e^{-z^2+\sigma\upsilon z} H_m(z)\, dz,$$

$$= \frac{e^{\sigma b + \frac{\sigma^2\upsilon^2}{4}}}{\sqrt{\pi}} \int_{-\infty}^{z_k} e^{-(z-\frac{\sigma\upsilon}{2})^2} (2zH_{m-1}(z) - 2(m-1)H_{m-2}(z))\, dz,$$

$$= \frac{e^{\sigma b + \frac{\sigma^2\upsilon^2}{4}}}{\sqrt{\pi}} \left[\int_{-\infty}^{z_k} 2ze^{-(z-\frac{\sigma\upsilon}{2})^2} H_{m-1}(z)\, dz \right.$$

$$\left. - \frac{e^{\sigma b + \frac{\sigma^2\upsilon^2}{4}}}{\sqrt{\pi}} \int_{-\infty}^{z_k} 2(m-1) e^{-(z-\frac{\sigma\upsilon}{2})^2} H_{m-2}(z)\, dz \right],$$

$$= \frac{e^{\sigma b + \frac{\sigma^2\upsilon^2}{4}}}{\sqrt{\pi}} \left[\int_{-\infty}^{z_k - \frac{\sigma\upsilon}{2}} (2u+\sigma\upsilon) e^{-u^2} H_{m-1}(u + \frac{\sigma\upsilon}{2})\, du \right.$$
$$\left. - \int_{-\infty}^{z_k} 2(m-1) e^{-(z-\frac{\sigma\upsilon}{2})^2} H_{m-2}(z)\, dz \right],$$

$$= \frac{e^{\sigma b + \frac{\sigma^2\upsilon^2}{4}}}{\sqrt{\pi}} \left[\int_{-\infty}^{z_k - \frac{\sigma\upsilon}{2}} 2ue^{-u^2} H_{m-1}(u + \frac{\sigma\upsilon}{2})\, du \right.$$

$$\left. + \sigma\upsilon \int_{-\infty}^{z_k} e^{-(z-\frac{\sigma\upsilon}{2})^2} H_{m-2}(z)\, dz - \int_{-\infty}^{z_k} 2(m-1) e^{-(z-\frac{\sigma\upsilon}{2})^2} H_{m-2}(z)\, dz \right],$$

$$= \frac{e^{\sigma b + \frac{\sigma^2\upsilon^2}{4}}}{\sqrt{\pi}} \left[e^{-(z_k-\frac{\sigma\upsilon}{2})^2} H_{m-1}(z_k) + \int_{-\infty}^{z_k} 2(m-1) e^{-(z-\frac{\sigma\upsilon}{2})^2} H_{m-2}(z)\, dz \right.$$

$$\left. + \sigma\upsilon \int_{-\infty}^{z_k} e^{-(z-\frac{\sigma\upsilon}{2})^2} H_{m-2}(z)\, dz - \int_{-\infty}^{z_k} 2(m-1) e^{-(z-\frac{\sigma\upsilon}{2})^2} H_{m-2}(z)\, dz \right],$$

$$= -\frac{e^{\sigma b + \frac{\sigma^2\upsilon^2}{4}}}{\sqrt{\pi}} e^{-(z_k-\frac{\sigma\upsilon}{2})^2} H_{m-1}(z_k) + \sigma\upsilon e^{\sigma b + \frac{\sigma^2\upsilon^2}{4}} \Phi_{m-1}^{k-1}. \tag{A.27}$$

A.2.4 Evaluating γ_m^{k-1}

The proof for γ_m^{k-1} as in (3.107), is

$$
\begin{aligned}
\gamma_m^{k-1} =& \frac{e^{-r\Delta t}}{2^m m! \upsilon^m}\Bigg[-\frac{1}{\sqrt{\pi}} e^{-z_k^2} H_{m-1}(z_k) + \frac{e^{\sigma b - z_k^2 + \sigma \upsilon z_k}}{\sqrt{\pi}} H_{m-1}(z_k) \\
&- \sigma \upsilon e^{\sigma b + \frac{\sigma^2 \upsilon^2}{4}} \Phi_{m-1}^{k-1}\Bigg],
\end{aligned}
$$

$$
\begin{aligned}
=& \Bigg[-\frac{1}{2^m m! \upsilon^m \sqrt{\pi}} e^{-r\Delta t - z_k^2} H_{m-1}(z_k) + \frac{e^{-r\Delta t + \sigma b - z_k^2 + \sigma \upsilon z_k}}{2^m m! \upsilon^m \sqrt{\pi}} H_{m-1}(z_k) \\
&- \frac{\sigma e^{-r\Delta t} e^{\sigma b + \frac{\sigma^2 \upsilon^2}{4}}}{2^m m! \upsilon^{m-1}} \Phi_{m-1}^{k-1}\Bigg],
\end{aligned}
\tag{A.28}
$$

To obtain a recurrence relation for γ_m^{k-1}, Φ_{m-1}^{k-1} is replaced with γ_{m-1}^{k-1} by rearranging

$$
\gamma_{m-1}^{k-1} = \frac{e^{-r\Delta t}}{2^{m-1}(m-1)! \upsilon^{m-1}}\Bigg[\Theta_{m-1}^{k-1} - \Phi_{m-1}^{k-1}\Bigg],
$$

$$
\frac{\sigma e^{-r\Delta t}}{2^{m-1}(m-1)! \upsilon^{m-1}} \Phi_{m-1}^{k-1} = \frac{e^{-r\Delta t}}{2^{m-1}(m-1)! \upsilon^{m-1}} \Theta_{m-1}^{k-1} - \gamma_{m-1}^{k-1},
$$

$$
\frac{\sigma e^{-r\Delta t}}{2^m m! \upsilon^{m-1}} \Phi_{m-1}^{k-1} = \frac{\sigma e^{-r\Delta t}}{2^m m! \upsilon^{m-1}} \Theta_{m-1}^{k-1} - \frac{\sigma}{2m} \gamma_{m-1}^{k-1},
$$

$$
\frac{\sigma e^{-r\Delta t}}{2^m m! \upsilon^{m-1}} \Phi_{m-1}^{k-1} = \frac{\sigma e^{-r\Delta t}}{2^m m! \upsilon^{m-1} \sqrt{\pi}} e^{-z_k^2} H_{m-2}(z_k) - \frac{\sigma}{2m} \gamma_{m-1}^{k-1}, \tag{A.29}
$$

substituting (A.29) into (A.28). Therefore, (A.28) becomes

$$
\gamma_m^{k-1} = \frac{\sigma}{2m} \gamma_{m-1}^{k-1} + \frac{e^{-r\Delta t - z_k^2}}{2^m m! \upsilon^m \sqrt{\pi}} \Bigg[H_{m-1}(z_k)(e^{\sigma b + \sigma \upsilon z_k} - 1) + \sigma \upsilon H_{m-2}(z_k)\Bigg]. \tag{A.30}
$$

A.2.5 Evaluating $A_{0,n}^k$

The proof to the solution of $A_{0,n}^k$ as in (3.120), is.

$$
\mathbf{A}_{0,n}^k = \frac{e^{-r\Delta t}}{2^0 0! \upsilon^0 \sqrt{\pi}} \int_{z_k}^{\infty} e^{-z^2} H_0(z) H_n(\upsilon z + b)\, dz
$$

$$
= \frac{e^{-r\Delta t}}{\sqrt{\pi}} \int_{z_k}^{\infty} e^{-z^2} H_n(\upsilon z + b)\, dz, \tag{A.31}
$$

and using property (3.83), (A.31) can be expressed as,

$$\mathbf{A}_{0,n}^{k} = \frac{e^{-r\Delta t}}{\sqrt{\pi}} \int_{z_k}^{\infty} e^{-z^2} \Big[2(\upsilon z + b) H_{n-1}(\upsilon z + b) - 2(n-1) H_{n-2}(\upsilon z + b) \Big] \, dz,$$

$$= \frac{e^{-r\Delta t}}{\sqrt{\pi}} \int_{z_k}^{\infty} 2\upsilon z e^{-z^2} H_{n-1}(\upsilon z_k + b) \, dz + 2b\mathbf{A}_{0,n-1}^{k} - 2(n-1)\mathbf{A}_{0,n-2}^{k}, \quad \text{(A.32)}$$

and finally the integral in (A.32) is evaluated using (3.84) and integration by parts, so

$$\mathbf{A}_{0,n}^{k} = \frac{e^{-r\Delta t}}{\sqrt{\pi}} \left[\upsilon e^{-z^2} H_{n-1}(\upsilon z_k + b) + 2\upsilon^2 (n-1) \int_{z_k}^{\infty} e^{-z^2} H_{n-2}(\upsilon z_k + b) \, dz \right]$$

$$+ 2b\mathbf{A}_{0,n-1}^{k} - 2(n-1)\mathbf{A}_{0,n-2}^{k},$$

$$= \frac{\upsilon e^{-r\Delta t}}{\sqrt{\pi}} e^{-z_k^2} H_{n-1}(\upsilon z_k + b) + 2b\mathbf{A}_{0,n-1}^{k} + 2(\upsilon^2 - 1)(n-1)\mathbf{A}_{0,n-2}^{k}. \quad \text{(A.33)}$$

Appendix B

Normalised Fourier-Hermite Series Expansion

These appendices contain many proofs and analysis of the methods and techniques described in Chapter 4.

B.1 European Options

B.1.1 Completing the Square

This is a step by step evaluation of the power of the exponential in the path integral (4.8) being converted to a complete square.

$$
\begin{aligned}
2(\xi_k - \mu(\xi_{k-1}))^2 + \xi_{k-1}^2 &= 2(\xi_k - \frac{1}{\sqrt{2\Delta t}}(\xi_{k-1} + b))^2 + \xi_{k-1}^2 \\
&= 2\xi_k^2 + \frac{2}{2\Delta t}(\xi_{k-1} + b)^2 - \frac{4\xi_k}{\sqrt{2\Delta t}}(\xi_{k-1} + b) + \xi_{k-1}^2 \\
&= 2\xi_k^2 + \frac{2\xi_{k-1}^2}{2\Delta t} + \frac{2b^2}{2\Delta t} + \frac{4\xi_{k-1}b}{2\Delta t} - \frac{4\xi_{k-1}\xi_k}{\sqrt{2\Delta t}} - \frac{4\xi_k}{\sqrt{2\Delta t}} + \xi_{k-1}^2 \\
&= \xi_{k-1}^2\left(\frac{\Delta t + 1}{\Delta t}\right) + \frac{4\xi_{k-1}}{\sqrt{2\Delta t}}\left(\frac{b - \sqrt{2\Delta t}\xi_k}{\sqrt{2\Delta t}}\right) + \left(\xi_k - \frac{b}{\sqrt{2\Delta t}}\right)^2 \\
&= \frac{\xi_{k-1}^2\tau^2}{\Delta t} + \frac{4\xi_{k-1}}{\sqrt{2\Delta t}}\left(\frac{b - \sqrt{2\Delta t}\xi_k}{\sqrt{2\Delta t}}\right) + \left(\frac{\sqrt{2\Delta t}\xi_k - b}{\sqrt{2\Delta t}}\right)^2 \\
&= \frac{\xi_{k-1}^2\tau^2}{\Delta t} + \frac{4\xi_{k-1}\tau}{\sqrt{2\Delta t}}\left(\frac{b - \sqrt{2\Delta t}\xi_k}{\tau\sqrt{2\Delta t}}\right) + \left(\frac{\sqrt{2\Delta t}\xi_k - b}{\sqrt{2\Delta t}}\right)^2 \qquad \text{(B.1)}
\end{aligned}
$$

The expression (B.1) is in a form such that we can complete the square.

$$= \left[\frac{\xi_{k-1}\tau}{\sqrt{2\Delta t}} - \frac{\sqrt{2\Delta t}\,\xi_k - b}{\tau\sqrt{2\Delta t}}\right]^2 + \left(\frac{\sqrt{2\Delta t}\xi_k - b}{\sqrt{2\Delta t}}\right)^2 + \left(\frac{\sqrt{2\Delta t}\xi_k - b}{\tau\sqrt{2\Delta t}}\right)^2$$

$$= \left[\frac{\xi_{k-1}\tau}{\sqrt{2\Delta t}} - \frac{\sqrt{2\Delta t}\,\xi_k - b}{\tau\sqrt{2\Delta t}}\right]^2 + \frac{(\sqrt{2\Delta t}\xi_k - b)^2}{\Delta t}\left[1 - \frac{1}{\tau^2}\right]^2$$

$$= \left[\frac{\xi_{k-1}\tau}{\sqrt{2\Delta t}} - \frac{\sqrt{2\Delta t}\,\xi_k - b}{\tau\sqrt{2\Delta t}}\right]^2 + \frac{(\sqrt{2\Delta t}\xi_k - b)^2}{\Delta t}\left[\frac{\tau^2 - 1}{\tau^2}\right]^2$$

$$= \left[\frac{\xi_{k-1}\tau}{\sqrt{2\Delta t}} - \frac{\sqrt{2\Delta t}\,\xi_k - b}{\tau\sqrt{2\Delta t}}\right]^2 + \frac{(\sqrt{2\Delta t}\xi_k - b)^2}{\Delta t}\left[\frac{\Delta t + 1 - 1}{\Delta t + 1}\right]^2$$

$$= \left(\frac{\xi_{k-1}\tau}{\sqrt{\Delta t}} - \frac{\sqrt{2\Delta t}\,\xi_k - b}{\tau\sqrt{\Delta t}}\right)^2 + \left(\frac{\sqrt{2\Delta t}\,\xi_k - b}{\tau}\right)^2. \tag{B.2}$$

B.1.2 Evaluating $\Psi_m^*(-\frac{b}{\tau})$

The following is a derivation of $\Psi_m^*(-\frac{b}{\tau})$ for a European call option. $\Psi_m^*(-\frac{b}{\tau})$ is transformed so that it can assist in the evaluation of $\boldsymbol{\alpha}^{K-1}$. The aim is to join the two exponential in the integrand of (4.35) into a single exponential. The exponential also has to be transformed so that normalised Hermite polynomials and their properties can be used. This is achieved via simple algebra.

$$\Psi_m^*(-\frac{b}{\tau}) = \frac{1}{\sqrt{2\pi}}\int_{-\frac{b}{\tau}}^{\infty} e^{\sigma\tau z} e^{-\frac{1}{2}z^2} H_m(z)\,dz,$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\frac{b}{\tau}}^{\infty} e^{-\frac{1}{2}(z^2 - 2\sigma\tau z)} H_m(z)\,dz,$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\frac{b}{\tau}}^{\infty} e^{-\frac{1}{2}(z^2 - 2\sigma\tau z + \sigma^2\tau^2)} e^{\frac{\sigma^2\tau^2}{2}} H_m(z)\,dz,$$

$$= \frac{e^{\frac{1}{2}\sigma^2\tau^2}}{\sqrt{2\pi}}\int_{-\frac{b}{\tau}}^{\infty} e^{-\frac{1}{2}(z-\sigma\tau)^2} H_m(z)\,dz, \tag{B.3}$$

The proof for Ψ_0^*, as presented in (4.36), is as follows

$$\Psi_0^*(-\frac{b}{\tau}) = \frac{e^{\frac{1}{2}\sigma^2\tau^2}}{\sqrt{2\pi}} \int_{-\frac{b}{\tau}}^{\infty} e^{-\frac{1}{2}(z-\sigma v)^2} H_0(z)\, dz,$$

$$= \frac{e^{\frac{1}{2}\sigma^2\tau^2}}{\sqrt{2\pi}} \int_{-\frac{b}{\tau}}^{\infty} e^{-\frac{1}{2}(z-\sigma\tau)^2}\, dz \tag{B.4}$$

let,

$$u = \frac{z-\sigma\tau}{\sqrt{2}} \tag{B.5}$$

and substituting (B.5) into (B.4) gives

$$\Psi_0^*(-\frac{b}{\tau}) = \frac{e^{\frac{1}{2}\sigma^2\tau^2}}{\sqrt{2\pi}} \int_{-\frac{b}{\sqrt{2}\,\tau}-\frac{\sigma\tau}{\sqrt{2}}}^{\infty} e^{-u^2} \frac{\sqrt{2}\,du}{dz} dz,$$

$$= \frac{e^{\frac{1}{2}\sigma^2\tau^2}}{2} + \frac{e^{\frac{1}{2}\sigma^2\tau^2}}{\sqrt{\pi}} \int_0^{\frac{b}{\sqrt{2}\,\tau}+\frac{\sigma\tau}{\sqrt{2}}} e^{-u^2} du,$$

$$= e^{\frac{1}{2}\sigma^2\tau^2}\left[\frac{1}{2} + \frac{1}{2} erf\big(\frac{b}{\sqrt{2}\,\tau} + \frac{\sigma\tau}{\sqrt{2}}\big)\right],$$

$$= \frac{e^{\frac{1}{2}\sigma^2\tau^2}}{2} erfc\big(-\frac{b}{\sqrt{2}\,\tau} - \frac{\sigma\tau}{\sqrt{2}}\big), \tag{B.6}$$

The proof for Ψ_1^*, as in (4.36), is

$$\Psi_1^*(-\frac{b}{\tau}) = \frac{e^{\frac{1}{2}\sigma^2\tau^2}}{\sqrt{2\pi}} \int_{-\frac{b}{\tau}}^{\infty} e^{-\frac{1}{2}(z-\sigma\tau)^2} H_1(z)\, dz,$$

$$= \frac{e^{\frac{1}{2}\sigma^2\tau^2}}{\sqrt{2\pi}} \int_{-\frac{b}{\tau}}^{\infty} z e^{-\frac{1}{2}(z-\sigma\tau)^2}\, dz \tag{B.7}$$

let,

$$u = \frac{z-\sigma\tau}{\sqrt{2}}, \tag{B.8}$$

and substituting (B.8) into (B.7) gives

$$\Psi_1^*(-\frac{b}{\tau}) = \frac{e^{\frac{1}{2}\sigma^2\tau^2}}{\sqrt{2\pi}} \int_{-\frac{b}{\sqrt{2}\,\tau}-\frac{\sigma\tau}{\sqrt{2}}}^{\infty} (\sqrt{2}\,u + \sigma\tau)e^{-u^2}\ \frac{\sqrt{2}\,du}{dz}dz,$$

$$= e^{\frac{1}{2}\sigma^2\tau^2}\left[\frac{1}{\sqrt{\pi}} \int_{-\frac{b}{\sqrt{2}\,\tau}-\frac{\sigma\tau}{\sqrt{2}}}^{\infty} \sqrt{2}\,ue^{-u^2}\ du + \frac{\sigma\tau}{\sqrt{\pi}} \int_{-\frac{b}{\sqrt{2}\,\tau}-\frac{\sigma\tau}{\sqrt{2}}}^{\infty} e^{-u^2}du\right]. \qquad \text{(B.9)}$$

Performing the substitution,

$$x = u^2,$$

into (B.9) gives

$$\Psi_1^*(-\frac{b}{\tau}) = e^{\frac{1}{2}\sigma^2\tau^2}\left[\frac{1}{\sqrt{\pi}} \int_{-\frac{b}{\sqrt{2}\,\tau}-\frac{\sigma\tau}{\sqrt{2}}}^{\infty} \sqrt{2}\,ue^{-x}\frac{dx}{2u\,du}du + \frac{\sigma\tau}{2}erfc\big(-\frac{b}{\sqrt{2}\,\tau} - \frac{\sigma\tau}{\sqrt{2}}\big)\right],$$

$$= e^{\frac{1}{2}\sigma^2\tau^2}\left[\frac{1}{\sqrt{2\pi}} \int_{-\frac{b}{\sqrt{2}\,\tau}-\frac{\sigma\tau}{\sqrt{2}}}^{\infty} e^{-x}dx + \frac{\sigma\tau}{2}erfc\big(-\frac{b}{\tau} - \frac{\sigma\tau}{\sqrt{2}}\big)\right],$$

$$= e^{\frac{1}{2}\sigma^2\tau^2}\left[\frac{\sigma\tau}{2}erfc\big(-\frac{b}{\sqrt{2}\,\tau} - \frac{\sigma\tau}{\sqrt{2}}\big) - \frac{1}{\sqrt{2\pi}}e^{-x}\Big]_{-\frac{b}{\sqrt{2}\,\tau}-\frac{\sigma\tau}{\sqrt{2}}}^{\infty}\right],$$

$$= e^{\frac{1}{2}\sigma^2\tau^2}\left[\frac{\sigma\tau}{2}erfc\big(-\frac{b}{\sqrt{2}\,\tau} - \frac{\sigma\tau}{\sqrt{2}}\big) + \frac{1}{\sqrt{2\pi}}e^{-(-\frac{b}{\sqrt{2}\,\tau}-\frac{\sigma\tau}{\sqrt{2}})^2}\right],$$

$$= e^{\frac{1}{2}\sigma^2\tau^2}\left[\frac{\sigma\tau}{2}erfc\big(-\frac{b}{\sqrt{2}\,\tau} - \frac{\sigma\tau}{\sqrt{2}}\big) + \frac{1}{\sqrt{2\pi}}e^{-((\frac{b}{\sqrt{2}\,\tau})^2+\frac{\sigma^2\upsilon^2}{4}+\sigma b)}\right],$$

$$= e^{\frac{1}{2}\sigma^2\tau^2}\frac{\sigma\tau}{2}erfc\big(-\frac{b}{\sqrt{2}\,\tau} - \frac{\sigma\tau}{\sqrt{2}}\big) + \frac{1}{\sqrt{2\pi}}e^{-((\frac{b}{\tau})^2+2\sigma b)}, \qquad \text{(B.10)}$$

The proof for Ψ_m^*, as in (4.36), is

$$\begin{aligned}
\Psi_m^*(-\frac{b}{\tau}) =&\frac{e^{\frac{1}{2}\sigma^2\tau^2}}{\sqrt{2\pi}} \int_{-\infty}^{-\frac{b}{\tau}} e^{-\frac{1}{2}(z-\sigma\tau)^2} \Big[zH_{m-1}(z) - (m-1)H_{m-2}(z)\Big]\, dz, \\
=&e^{\frac{1}{2}\sigma^2\tau^2}\Bigg[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-\frac{b}{\tau}} ze^{-\frac{1}{2}(z-\sigma\tau)^2} H_{m-1}(z)\, dz \\
&- \frac{(m-1)}{\sqrt{2\pi}} \int_{-\infty}^{-\frac{b}{\tau}} e^{-\frac{1}{2}(z-\sigma\tau)^2} H_{m-2}(z)\, dz\Bigg], \\
=&e^{\frac{1}{2}\sigma^2\tau^2}\Bigg[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-\frac{b}{\tau}} ze^{-\frac{1}{2}(z-\sigma\tau)^2} H_{m-1}(z)\, dz \\
&- \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-\frac{b}{\tau}} e^{-\frac{1}{2}(z-\sigma\tau)^2} \big(\frac{d}{dz} H_{m-1}(z)\big)\, dz\Bigg], \\
=&e^{\frac{1}{2}\sigma^2\tau^2}\Bigg[\frac{1}{2\sqrt{\pi}} \int_{-\infty}^{-\frac{b}{\tau}} ze^{-\frac{1}{2}(z-\sigma\tau)^2} H_{m-1}(z)\, dz \\
&- \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(z-\sigma\tau)^2} H_{m-1}(z)\Bigg]_{-\infty}^{-\frac{b}{\tau}} - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-\frac{b}{\tau}} (z-\sigma\tau) e^{-\frac{1}{2}(z-\sigma\tau)^2} H_{m-1}(z)\, dz\Bigg], \\
=&e^{\frac{1}{2}\sigma^2\tau^2}\Bigg[\frac{\sigma\tau}{\sqrt{2\pi}} \int_{-\infty}^{-\frac{b}{\tau}} e^{-\frac{1}{2}(z-\sigma\tau)^2} H_{m-1}(z)\, dz - \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}((-\frac{b}{\tau}-\sigma\tau)^2} H_{m-1}(-\frac{b}{\tau})\Bigg], \\
=&\Bigg[\sigma\tau\Psi_{m-1}^*(-\frac{b}{\tau}) - \frac{e^{\frac{1}{2}\sigma^2\tau^2}}{\sqrt{2\pi}} e^{-\frac{1}{2}((\frac{b}{\tau})^2+\sigma^2\tau^2+2\sigma b)} H_{m-1}(-\frac{b}{\tau})\Bigg], \\
=&\Bigg[\sigma\tau\Psi_{m-1}^*(-\frac{b}{\tau}) - \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}((\frac{b}{\tau})^2+\sigma b)} H_{m-1}(-\frac{b}{\tau})\Bigg], \qquad \text{(B.11)}
\end{aligned}$$

B.1.3 Evaluating $\Omega_m^*(-\frac{b}{\tau})$

The proof for Ω_0^*, as in (4.36), is given by

$$\Omega_0^*(-\frac{b}{\tau}) = \frac{1}{\sqrt{2\pi}} \int_{-\frac{b}{\tau}}^{\infty} e^{-\frac{1}{2}z^2} H_0(z)\, dz,$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\frac{b}{\tau}}^{\infty} e^{-\frac{1}{2}z^2}\, dz,$$

$$= \frac{1}{2} + \frac{1}{\sqrt{\pi}} \int_{0}^{\frac{b}{\sqrt{2}\,\tau}} e^{-u^2}\, du,$$

$$= \frac{1}{2} + \frac{1}{2} erf(\frac{b}{\sqrt{2}\,\tau}),$$

$$= \frac{1}{2} erfc(-\frac{b}{\sqrt{2}\,\tau}). \tag{B.12}$$

The proof for Ω_1^*, as in (4.36), is

$$\Omega_1^*(-\frac{b}{\tau}) = \frac{1}{\sqrt{2\pi}} \int_{-\frac{b}{\tau}}^{\infty} e^{-\frac{1}{2}z^2} H_1(z)\, dz,$$
$$= \frac{1}{\sqrt{2\pi}} \int_{-\frac{b}{\tau}}^{\infty} z e^{-\frac{1}{2}z^2}\, dz. \tag{B.13}$$

Performing the substitution,

$$u = z^2,$$

into (B.13) gives

$$\begin{aligned}\Omega_1^*(-\frac{b}{\tau}) &= \frac{1}{\sqrt{2\pi}}\int_{-\frac{b}{\sqrt{2}\,\tau}}^{\infty} ze^{-u}\frac{du}{zdz}\,dz,\\ &= \frac{1}{\sqrt{2\pi}}\int_{-\frac{b}{\sqrt{2}\,\tau}}^{\infty} e^{-u}du,\\ &= -\frac{1}{\sqrt{2\pi}}e^{-u}\Big]_{-\frac{b}{\sqrt{2}\,\tau}}^{\infty},\\ &= \frac{1}{\sqrt{2\pi}}e^{-(\frac{b}{\sqrt{2}\,\tau})^2}. \end{aligned} \tag{B.14}$$

The proof for Ω_m^*, as in (4.36), is

$$\begin{aligned}\Omega_m^*(-\frac{b}{\tau}) =&\frac{1}{\sqrt{2\pi}}\int_{-\frac{b}{\tau}}^{\infty} e^{-\frac{1}{2}z^2}\Big[zH_{m-1}(z)-(m-1)H_{m-2}(z)\Big]\,dz,\\ =&\frac{1}{\sqrt{2\pi}}\int_{-\frac{b}{\tau}}^{\infty} ze^{-\frac{1}{2}z^2}H_{m-1}(z)\,dz-\frac{1}{\sqrt{2\pi}}\int_{-\frac{b}{\tau}}^{\infty}(m-1)e^{-\frac{1}{2}z^2}H_{m-2}(z)\,dz,\\ =&\frac{1}{\sqrt{2\pi}}\int_{-\frac{b}{\tau}}^{\infty} ze^{-\frac{1}{2}z^2}H_{m-1}(z)\,dz-(m-1)\Omega_{m-2}^*(-\frac{b}{\tau}),\\ =&-\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}H_{m-1}(z)\Big]_{-\frac{b}{\tau}}^{\infty}\\ &+(m-1)\frac{1}{\sqrt{2\pi}}\int_{-\frac{b}{\tau}}^{\infty}e^{-\frac{1}{2}z^2}H_{m-2}(z)\,dz-(m-1)\Omega_{m-2}^*(-\frac{b}{\tau}),\\ =&\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(\frac{b}{\tau})^2}H_{m-1}(-\frac{b}{\tau}). \end{aligned} \tag{B.15}$$

B.1.4 Evaluating α^{K-1} for a European Call Option

Since we have solved the initial and general cases for Ψ^* and Ω^*, a recurrence relation for α_m^{K-1} with $m = 1, 2, \ldots, N$ can be formed from (4.37) and using Ψ_m^* and Ω_m^*, gives

$$\begin{aligned}\alpha_m^{K-1} =& \frac{e^{-r\Delta t}}{m!\tau^m}\Bigg[e^{\sigma b}\Bigg(\sigma\tau\Psi_{m-1}^*(-\frac{b}{\tau}) + \frac{1}{\sqrt{2\pi}}e^{-(\frac{1}{2}(\frac{b}{\tau})^2+\sigma b)}H_{m-1}(-\frac{b}{\tau})\Bigg) \\ &- \frac{1}{\sqrt{2\pi}}e^{-(\frac{b}{\tau})^2}H_{m-1}(-\frac{b}{\tau})\Bigg],\end{aligned} \tag{B.16}$$

and so

$$\alpha_m^{K-1} = \frac{e^{-r\Delta t}}{m!\tau^m}\Bigg[\sigma\tau e^{\sigma b}\Psi_{m-1}^*(-\frac{b}{\tau})\Bigg]. \tag{B.17}$$

The next step is to find a relationship between α_m^{K-1} and α_{m-1}^{K-1}. Given (4.37) for coefficient $m-1$,

$$\alpha_{m-1}^{K-1} = \frac{e^{-r\Delta t}}{(m-1)!\tau^{m-1}}\Bigg[e^{\sigma b}\Psi_{m-1}^*(-\frac{b}{\tau}) - \Omega_{m-1}^*(-\frac{b}{\tau})\Bigg],$$

therefore, rearranging α_{m-1}^{K-1} for,

$$e^{\sigma b}\Psi_{m-1}^*(-\frac{b}{\tau}) = \Bigg[\Omega_{m-1}^*(-\frac{b}{\tau}) + e^{r\Delta t}(m-1)!\tau^{m-1}\alpha_{m-1}^{K-1}\Bigg], \tag{B.18}$$

and substituting (B.18) into (B.16) gives

$$\begin{aligned}\alpha_m^{K-1} =& \frac{e^{-r\Delta t}}{m!\tau^m}\Bigg[\sigma\tau\Bigg(\Omega_{m-1}^*(-\frac{b}{\tau}) + e^{r\Delta t}\tau^{m-1}(m-1)!\alpha_{m-1}^{K-1}\Bigg)\Bigg], \\ =& \frac{e^{-r\Delta t}}{m!\tau^m}\Bigg[\sigma\tau\Omega_{m-1}^*(-\frac{b}{\tau}) + \sigma\tau e^{r\Delta t}(m-1)!\tau^{m-1}\alpha_{m-1}^{K-1}\Bigg], \\ =& \sigma\Bigg[\frac{e^{-r\Delta t}}{m!\tau^{m-1}}\Omega_{m-1}^*(-\frac{b}{\tau}) + \frac{\alpha_{m-1}^{K-1}}{m}\Bigg], \\ =& \frac{\sigma}{m}\Bigg[\frac{e^{-r\Delta t}}{(m-1)!\tau^{m-1}\sqrt{2\pi}}e^{-\frac{1}{2}(\frac{b}{\tau})^2}H_{m-2}^*(-\frac{b}{\tau}) + \alpha_{m-1}^{K-1}\Bigg].\end{aligned} \tag{B.19}$$

B.1.5 Evaluating $\hat{\Psi}^*_m(-\frac{b}{\tau})$

The proof for $\hat{\Psi}^*_0$ as in (4.41), is

$$\begin{aligned}
\hat{\Psi}^*_0(-\frac{b}{\tau}) &= \frac{e^{\frac{1}{2}\sigma^2\tau^2}}{\sqrt{2\pi}} \int_{-\infty}^{-\frac{b}{\tau}} e^{-\frac{1}{2}(z-\sigma\tau)^2} H_0(z)\, dz, \\
&= \frac{e^{\frac{1}{2}\sigma^2\tau^2}}{\sqrt{2\pi}} \int_{-\infty}^{-\frac{b}{\tau}} e^{-\frac{1}{2}(z-\sigma\tau)^2}\, dz, \\
&= \frac{e^{\frac{1}{2}\sigma^2\tau^2}}{\sqrt{\pi}} \int_{-\infty}^{-\frac{b}{\sqrt{2}\,\tau}-\frac{\sigma\tau}{\sqrt{2}}} e^{-u^2}\, du, \\
&= \frac{e^{\frac{1}{2}\sigma^2\tau^2}}{2} erfc(\frac{b}{\sqrt{2}\,\tau} + \frac{\sigma\tau}{\sqrt{2}}).
\end{aligned} \tag{B.20}$$

The proof for $\hat{\Psi}^*_1$ as in (4.41), is

$$\begin{aligned}
\hat{\Psi}^*_1(-\frac{b}{\tau}) &= \frac{e^{\frac{1}{2}\sigma^2\tau^2}}{\sqrt{2\pi}} \int_{-\infty}^{-\frac{b}{\tau}} e^{-\frac{1}{2}(z-\sigma\tau)^2} H_1(z)\, dz, \\
&= \frac{e^{\frac{1}{2}\sigma^2\tau^2}}{\sqrt{2\pi}} \int_{-\infty}^{-\frac{b}{\tau}} z e^{-\frac{1}{2}(z-\sigma\tau)^2}\, dz, \\
&= \frac{e^{\frac{1}{2}\sigma^2\tau^2}}{\sqrt{\pi}} \int_{-\infty}^{-\frac{b}{\sqrt{2}\,\tau}-\frac{\sigma\tau}{\sqrt{2}}} (\sqrt{2}\,u + \sigma\tau) e^{-u^2}\, du, \\
&= \frac{e^{\frac{1}{2}\sigma^2\tau^2}}{\sqrt{\pi}} \left[\sqrt{2} \int_{-\infty}^{-\frac{b}{\sqrt{2}\,\tau}-\frac{\sigma\tau}{\sqrt{2}}} u e^{-u^2}\, du + \frac{\sigma\tau}{2} \int_{-\infty}^{-\frac{b}{\sqrt{2}\,\tau}-\frac{\sigma\tau}{\sqrt{2}}} e^{-u^2}\, du\right], \\
&= e^{\frac{1}{2}\sigma^2\tau^2}\left[\frac{1}{\sqrt{\pi}} \int_{-\infty}^{-\frac{b}{\sqrt{2}\,\tau}-\frac{\sigma\tau}{\sqrt{2}}} e^{-v}\, dv + \frac{\sigma\tau}{2} erfc(\frac{b}{\sqrt{2}\tau} + \frac{\sigma\tau}{\sqrt{2}})\right], \\
&= \frac{\sigma\tau e^{\frac{1}{2}\sigma^2\tau^2}}{2} erfc(\frac{b}{\sqrt{2}\tau} + \frac{\sigma\tau}{\sqrt{2}}) - \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}((\frac{b}{\tau})^2 + 2\sigma b)}.
\end{aligned} \tag{B.21}$$

The proof for $\hat{\Psi}^*_m$ as in (4.41), is

$$\hat{\Psi}^*_m(-\frac{b}{\tau}) = \frac{e^{\frac{1}{2}\sigma^2\tau^2}}{\sqrt{2\pi}} \int_{-\infty}^{-\frac{b}{\tau}} e^{-\frac{1}{2}(z-\sigma\tau)^2} H_m(z)\, dz,$$

$$= \frac{e^{\frac{1}{2}\sigma^2\tau^2}}{\sqrt{2\pi}} \int_{-\infty}^{-\frac{b}{\tau}} e^{-\frac{1}{2}(z-\sigma\tau)^2} \Big[zH_{m-1}(z) - (m-1)H_{m-2}(z) \Big]\, dz,$$

$$= e^{\frac{1}{2}\sigma^2\tau^2} \Bigg[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-\frac{b}{\tau}} ze^{-\frac{1}{2}(z-\sigma\tau)^2} H_{m-1}(z)\, dz$$
$$- \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-\frac{b}{\tau}} (m-1)e^{-\frac{1}{2}(z-\sigma\tau)^2} H_{m-2}(z) \Bigg]\, dz,$$

$$= e^{\frac{1}{2}\sigma^2\tau^2} \Bigg[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-\frac{b}{\tau}} ze^{-\frac{1}{2}(z-\sigma\tau)^2} H_{m-1}(z)\, dz$$
$$- \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-\frac{b}{\tau}} (m-1)e^{-\frac{1}{2}(z-\sigma\tau)^2} \left(\frac{d}{dz} \frac{1}{(m-1)} H_{m-1}(z) \right) \Bigg]\, dz,$$

$$= e^{\frac{1}{2}\sigma^2\tau^2} \Bigg[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-\frac{b}{\tau}} ze^{-\frac{1}{2}(z-\sigma\tau)^2} H_{m-1}(z)\, dz - \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(z-\sigma\tau)^2} H_{m-1}(z) \Bigg]_{-\infty}^{-\frac{b}{\tau}}$$
$$- \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-\frac{b}{\tau}} (z-\sigma\tau)e^{-\frac{1}{2}(z-\sigma\tau)^2} H_{m-1}(z) \Bigg]\, dz,$$

$$= e^{\frac{1}{2}\sigma^2\tau^2} \Bigg[-\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(\frac{b}{\tau}-\sigma\tau)^2} H_{m-1}(-\frac{b}{\tau}) + \frac{\sigma\tau}{\sqrt{2\pi}} \int_{-\infty}^{-\frac{b}{\tau}} e^{-\frac{1}{2}(z-\sigma\tau)^2} H_{m-1}(z)\, dz \Bigg],$$

$$= \sigma\tau\hat{\Psi}^*_{m-1}(-\frac{b}{\tau}) - \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}((\frac{b}{\tau})^2 - 2\sigma b)} H_{m-1}(-\frac{b}{\tau}). \tag{B.22}$$

B.1.6 Evaluating $\hat{\Omega}^*_m(-\frac{b}{\tau})$

The proof for $\hat{\Omega}^*_0$ as in (4.41), is

$$\hat{\Omega}_0^*(-\frac{b}{\tau}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-\frac{b}{\tau}} e^{-\frac{1}{2}z^2} H_0(z)\, dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-\frac{b}{\tau}} e^{-\frac{1}{2}z^2}\, dz,$$

$$= \frac{1}{2} erfc(\frac{b}{\sqrt{2}\,\tau}). \tag{B.23}$$

The proof for $\hat{\Omega}_1^*$ as in (4.41), is

$$\hat{\Omega}_1^*(-\frac{b}{\tau}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-\frac{b}{\tau}} e^{-\frac{1}{2}z^2} H_1(z)\, dz,$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-\frac{b}{\tau}} z e^{-\frac{1}{2}z^2}\, dz,$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-\frac{b}{\tau}} e^{-u}\, du,$$

$$= -\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(\frac{b}{\tau})^2}. \tag{B.24}$$

The proof for $\hat{\Omega}_m^*$ as in (4.41), is

$$\hat{\Omega}^*_m(-\frac{b}{\tau}) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-\frac{b}{\tau}} e^{-\frac{1}{2}z^2} H_m(z)\,dz,$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-\frac{b}{\tau}} e^{-\frac{1}{2}z^2}\Big[zH_{m-1}(z) - (m-1)H_{m-2}(z)\,dz\Big],$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-\frac{b}{\tau}} e^{-\frac{1}{2}z^2} zH_{m-1}(z) - (m-1)\hat{\Omega}^*_{m-2}(-\frac{b}{\tau}),$$

$$= -\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}H_{m-1}(z)\Big]_{-\infty}^{-\frac{b}{\tau}} + (m-1)\hat{\Omega}^*_{m-2}(-\frac{b}{\tau}) - (m-1)\hat{\Omega}^*_{m-2}(-\frac{b}{\tau}),$$

$$= -\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(\frac{b}{\tau})^2}H_{m-1}(-\frac{b}{\tau}). \tag{B.25}$$

B.2 American Put Option

B.2.1 Evaluating γ_1^{k-1}

The proof to γ_1^{k-1} as in (4.71) can be formed by using properties (4.61) and (4.24) and integration.

$$\gamma_1^{k-1} = \frac{e^{-r\Delta t}}{1!\tau^1\sqrt{2\pi}}\Big[\int_{-\infty}^{z_k} e^{-\frac{1}{2}z^2}H_1(z)\,dz - e^{\sigma b}\int_{-\infty}^{z_k} e^{-\frac{1}{2}z^2+\sigma\tau z}H_1(z)\,dz\Big],$$

$$= \frac{e^{-r\Delta t}}{\tau\sqrt{2\pi}}\Big[\int_{-\infty}^{z_k} ze^{-\frac{1}{2}z^2}\,dz - e^{\sigma b+\frac{\sigma^2\tau^2}{2}}\int_{-\infty}^{z_k} ze^{-\frac{1}{2}(z-\sigma\tau)^2}\,dz\Big],$$

$$= \frac{e^{-r\Delta t}}{\tau}\Big[-\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z_k^2} - \frac{1}{2\sqrt{\pi}}e^{\sigma b+\frac{\sigma^2\tau^2}{2}}\int_{-\infty}^{z_k-\sigma\tau}(u+\sigma\tau)e^{-\frac{1}{2}u^2}\,du\Big],$$

$$= \frac{e^{-r\Delta t}}{\tau}\Big[-\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z_k^2} - \frac{1}{2\sqrt{\pi}}e^{\sigma b+\frac{\sigma^2\tau^2}{2}}\Big[\int_{-\infty}^{z_k-\sigma\tau} ue^{-\frac{1}{2}u^2}\,du + \sigma\tau\int_{-\infty}^{z_k-\sigma\tau} e^{-\frac{1}{2}u^2}\,du\Big]\Big],$$

$$= \frac{e^{-r\Delta t}}{\tau}\Big[-\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z_k^2} + \frac{e^{\sigma b-\frac{1}{2}z_k^2+\sigma\tau z_k}}{\sqrt{2\pi}} - \frac{\sigma\tau e^{\sigma b+\frac{\sigma^2\tau^2}{2}}}{2}\,erfc\Big(\frac{-\sigma\tau}{2}+\frac{z_k}{\sqrt{2}}\Big)\Big]. \tag{B.26}$$

B.2.2 Evaluating Θ_m^{k-1}

The proof for Θ_m^{k-1} as in (4.77), is.

$$
\begin{aligned}
\Theta_m^{k-1} &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z_k} e^{-\frac{1}{2}z^2} H_m(z)\, dz, \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z_k} e^{-\frac{1}{2}z^2} (zH_{m-1}(z) - (m-1)H_{m-2}(z))\, dz, \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z_k} ze^{-\frac{1}{2}z^2} H_{m-1}(z)\, dz - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z_k} (m-1)e^{-\frac{1}{2}z^2} H_{m-2}(z)\, dz, \\
&= -\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z_k^2} H_{m-1}(z_k) \\
&+ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z_k} (m-1)e^{-\frac{1}{2}z^2} H_{m-2}(z)\, dz - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z_k} (m-1)e^{-\frac{1}{2}z^2} H_{m-2}(z)\, dz, \\
&= -\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z_k^2} H_{m-1}(z_k). \qquad \text{(B.27)}
\end{aligned}
$$

B.2.3 Evaluating Φ_m^{k-1}

The proof for Φ_m^{k-1} as in (4.78), is.

$$
\begin{aligned}
\Phi_m^{k-1} &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z_k} e^{-\frac{1}{2}z^2+\sigma\tau z} H_m(z)\, dz, \\
&= \frac{e^{\sigma b+\frac{\sigma^2\tau^2}{2}}}{\sqrt{2\pi}} \int_{-\infty}^{z_k} e^{-\frac{1}{2}(z-\sigma\tau)^2} \left(z H_{m-1}(z) - (m-1) H_{m-2}(z)\right) dz, \\
&= \frac{e^{\sigma b+\frac{\sigma^2\tau^2}{2}}}{\sqrt{2\pi}} \left[\int_{-\infty}^{z_k} z e^{-\frac{1}{2}(z-\sigma\tau)^2} H_{m-1}(z)\, dz \right. \\
&\left. - \frac{e^{\sigma b+\frac{\sigma^2\tau^2}{2}}}{\sqrt{2\pi}} \int_{-\infty}^{z_k} (m-1) e^{-\frac{1}{2}(z-\sigma\tau)^2} H_{m-2}(z)\, dz \right], \\
&= \frac{e^{\sigma b+\frac{\sigma^2\tau^2}{2}}}{\sqrt{2\pi}} \left[\int_{-\infty}^{z_k-\sigma\tau} (u+\sigma\tau) e^{-\frac{1}{2}u^2} H_{m-1}(u+\sigma\tau)\, du \right. \\
&\left. - \int_{-\infty}^{z_k} (m-1) e^{-\frac{1}{2}(z-\sigma\tau)^2} H_{m-2}(z)\, dz \right], \\
&= \frac{e^{\sigma b+\frac{\sigma^2\tau^2}{2}}}{\sqrt{2\pi}} \left[\int_{-\infty}^{z_k-\sigma\tau} u e^{-\frac{1}{2}u^2} H_{m-1}(u+\sigma\tau)\, du \right. \\
&\left. + \sigma\tau \int_{-\infty}^{z_k} e^{-\frac{1}{2}(z-\sigma\tau)^2} H_{m-2}(z)\, dz - \int_{-\infty}^{z_k} (m-1) e^{-\frac{1}{2}(z-\sigma\tau)^2} H_{m-2}(z)\, dz \right], \\
&= \frac{e^{\sigma b+\frac{\sigma^2\tau^2}{2}}}{\sqrt{2\pi}} \left[e^{-\frac{1}{2}(z_k-\sigma\tau)^2} H_{m-1}(z_k) + \int_{-\infty}^{z_k} (m-1) e^{-\frac{1}{2}(z-\sigma\tau)^2} H_{m-2}(z)\, dz \right. \\
&\left. + \sigma\tau \int_{-\infty}^{z_k} e^{-\frac{1}{2}(z-\sigma\tau)^2} H_{m-2}(z)\, dz - \int_{-\infty}^{z_k} (m-1) e^{-\frac{1}{2}(z-\sigma\tau)^2} H_{m-2}(z)\, dz \right], \\
&= -\frac{e^{\sigma b+\frac{\sigma^2\tau^2}{2}}}{\sqrt{2\pi}} e^{-\frac{1}{2}(z_k-\sigma\tau)^2} H_{m-1}(z_k) + \sigma\tau e^{\sigma b+\frac{\sigma^2\tau^2}{2}} \Phi_{m-1}^{k-1}. \qquad \text{(B.28)}
\end{aligned}
$$

B.2.4 Evaluating γ_m^{k-1}

The proof for γ_m^{k-1} as in (4.81), is

$$
\begin{aligned}
\gamma_m^{k-1} =& \frac{e^{-r\Delta t}}{m!\tau^m}\Bigg[-\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z_k^2}H_{m-1}(z_k) + \frac{e^{\sigma b-\frac{1}{2}z_k^2+\sigma\tau z_k}}{\sqrt{2\pi}}H_{m-1}(z_k) \\
&- \sigma\tau e^{\sigma b+\frac{\sigma^2\tau^2}{2}}\Phi_{m-1}^{k-1}\Bigg],
\end{aligned}
$$

$$
\begin{aligned}
=& \Bigg[-\frac{1}{m!\tau^m\sqrt{2\pi}}e^{-r\Delta t-\frac{1}{2}z_k^2}H_{m-1}(z_k) + \frac{e^{-r\Delta t+\sigma b-\frac{1}{2}z_k^2+\sigma\tau z_k}}{m!\tau^m\sqrt{2\pi}}H_{m-1}(z_k) \\
&- \frac{\sigma e^{-r\Delta t}e^{\sigma b+\frac{\sigma^2\tau^2}{2}}}{m!\tau^{m-1}}\Phi_{m-1}^{k-1}\Bigg], \qquad \text{(B.29)}
\end{aligned}
$$

To obtain a recurrence relation for γ_m^{k-1}, Φ_{m-1}^{k-1} is replaced with γ_{m-1}^{k-1} by rearranging

$$
\gamma_{m-1}^{k-1} = \frac{e^{-r\Delta t}}{(m-1)!\tau^{m-1}}\Bigg[\Theta_{m-1}^{k-1} - \Phi_{m-1}^{k-1}\Bigg],
$$

$$
\frac{\sigma e^{-r\Delta t}}{(m-1)!\tau^{m-1}}\Phi_{m-1}^{k-1} = \frac{e^{-r\Delta t}}{(m-1)!\tau^{m-1}}\Theta_{m-1}^{k-1} - \gamma_{m-1}^{k-1},
$$

$$
\frac{\sigma e^{-r\Delta t}}{m!\tau^{m-1}}\Phi_{m-1}^{k-1} = \frac{\sigma e^{-r\Delta t}}{m!\tau^{m-1}}\Theta_{m-1}^{k-1} - \frac{\sigma}{m}\gamma_{m-1}^{k-1},
$$

$$
\frac{\sigma e^{-r\Delta t}}{m!\tau^{m-1}}\Phi_{m-1}^{k-1} = \frac{\sigma e^{-r\Delta t}}{m!\tau^{m-1}\sqrt{2\pi}}e^{-\frac{1}{2}z_k^2}H_{m-2}(z_k) - \frac{\sigma}{m}\gamma_{m-1}^{k-1}, \qquad \text{(B.30)}
$$

substituting (B.30) into (B.29). Therefore, (B.29) becomes

$$
\gamma_m^{k-1} = \frac{\sigma}{m}\gamma_{m-1}^{k-1} + \frac{e^{-r\Delta t-\frac{1}{2}z_k^2}}{m!\tau^m\sqrt{2\pi}}\Bigg[H_{m-1}(z_k)(e^{\sigma b+\sigma\tau z_k} - 1) + \sigma\tau H_{m-2}(z_k)\Bigg]. \qquad \text{(B.31)}
$$

Appendix C

Interpolation Polynomials

This appendix contains further data analysis for the InterPolation Method (IPM) described in Chapter 5. For convenience the approximate results using this method are in the column labeled IPM.

C.1 European Options

This section contains results for various European options using the fixed number of nodes (per time step), fixed spaced partitions and adaptive node distributions.

C.1.1 Fixed Number of Partitions

An analysis of the parameters were made in section 5.4.2. The results are numerical prices of the data graphed.

Varying η and Partitions (N)

Tables C.1 - C.5 are European call options prices for varying η and number of partitions (N) fixed at 64 at each time step.

Asset Price (\$)	η	**IPM**	**Error**
80	10^{-4}	0.0689055676694494	$1.1216563173675103E-04$
	10^{-5}	0.0689512512544143	$6.6482046771855138E-05$
	10^{-6}	0.0689317130989882	$8.6020202197910345E-05$
	10^{-7}	0.0688979303334288	$1.1980296775736558E-04$
	10^{-8}	0.0688569109541093	$1.6082234707682054E-04$
	10^{-9}	0.0688093126214839	$2.0842067970220680E-04$
	10^{-10}	0.0687548495225413	$2.6288377864486841E-04$
	10^{-11}	0.0686943512286288	$3.2338207255733871E-04$
	10^{-12}	0.0686302706720580	$3.8746262912817044E-04$
	10^{-13}	0.0685646166087813	$4.5311669240483696E-04$
	10^{-14}	0.0684962694270779	$5.2146387410820748E-04$
	10^{-15}	0.0684201655575928	$5.9756774359330566E-04$
	10^{-16}	0.0683288557435221	$6.8887755766399448E-04$

Table C.1: European call option with $K = 8$, $N = 64$, $\sigma = 0.20$, $r = 0.08$, $T = 0.25$, asset value of \$80 and strike of \$100. Here η is the total error for the tails (refer to (5.26)). The values are calculated in this table are performed in double precision.

Asset Price ($)	η	IPM	Error
90	10^{-4}	1.0250907293696512	$3.6300476429253214E-04$
	10^{-5}	1.0254186444924345	$3.5089641509285230E-05$
	10^{-6}	1.0254611444368389	$7.4103028952637873E-06$
	10^{-7}	1.0254707252843509	$1.6991150407312527E-05$
	10^{-8}	1.0254771780032472	$2.3443869303502174E-05$
	10^{-9}	1.0254834100119699	$2.9675878026116931E-05$
	10^{-10}	1.0254891455639026	$3.5411429958817631E-05$
	10^{-11}	1.0254943604339977	$4.0626300053941966E-05$
	10^{-12}	1.0255004294501593	$4.6695316215615568E-05$
	10^{-13}	1.0255106884899348	$5.6954355991123418E-05$
	10^{-14}	1.0255304691906100	$7.6735056666298040E-05$
	10^{-15}	1.0255667049156083	$1.1297078166463276E-04$
	10^{-16}	1.0256272991139672	$1.7356498002354254E-04$

Table C.2: European call option with $K = 8$, $N = 64$, $\sigma = 0.20$, $r = 0.08$, $T = 0.25$, asset value of \$90 and strike of \$100. Here η is the total error for the tails (refer to (5.26)). The values are calculated in this table are performed in double precision.

Asset Price ($)	η	IPM	Error
100	10^{-4}	5.0162632406806198	$7.1736558179130716E-04$
	10^{-5}	5.0170322174573707	$5.1611194959444440E-05$
	10^{-6}	5.0171791741258476	$1.9856786343599997E-04$
	10^{-7}	5.0172713387960828	$2.9073253367192797E-04$
	10^{-8}	5.0173701613216330	$3.8955505922214084E-04$
	10^{-9}	5.0174812387290979	$5.0063246668646810E-04$
	10^{-10}	5.0176036943619984	$6.2308809958694367E-04$
	10^{-11}	5.0177364876208452	$7.5588135843435156E-04$
	10^{-12}	5.0178803792558169	$8.9977299340543437E-04$
	10^{-13}	5.0180390615210957	$1.0584552586845319E-03$
	10^{-14}	5.0182194891752729	$1.2388829128621992E-03$
	10^{-15}	5.0184314664499077	$1.4508601874962945E-03$
	10^{-16}	5.0186867485302180	$1.7061422678071869E-03$

Table C.3: European call option with $K = 8$, $N = 64$, $\sigma = 0.20$, $r = 0.08$, $T = 0.25$, asset value of \$100 and strike of \$100. Here η is the total error for the tails (refer to (5.26)). The values are calculated in this table are performed in double precision.

Asset Price ($)	η	IPM	Error
110	10^{-4}	12.6191544371908044	$1.2940647922343729E-03$
	10^{-5}	12.6203063995743552	$1.4210240868450796E-04$
	10^{-6}	12.6204213847275479	$2.7117255491093673E-05$
	10^{-7}	12.6204285276262400	$1.9974356799123960E-05$
	10^{-8}	12.6204236143870254	$2.4887596014600000E-05$
	10^{-9}	12.6204172337181451	$3.1268264893968833E-05$
	10^{-10}	12.6204133528399645	$3.5149143073898159E-05$
	10^{-11}	12.6204138257078373	$3.4676275201261930E-05$
	10^{-12}	12.6204131606021477	$3.5341380891229512E-05$
	10^{-13}	12.6203977568857795	$5.0745097260329608E-05$
	10^{-14}	12.6203523736852699	$9.6128297769060289E-05$
	10^{-15}	12.6202691572634187	$1.7934471961966736E-04$
	10^{-16}	12.6201536957545688	$2.9480622847077864E-04$

Table C.4: European call option with $K = 8$, $N = 64$, $\sigma = 0.20$, $r = 0.08$, $T = 0.25$, asset value of \$110 and strike of \$100. Here η is the total error for the tails (refer to (5.26)). The values are calculated in this table are performed in double precision.

Asset Price ($)	η	IPM	Error
120	10^{-4}	22.0646820659120060	$1.8781356951047545E-03$
	10^{-5}	22.0663171950582573	$2.4300654885289763E-04$
	10^{-6}	22.0664553901652560	$1.0481144185614522E-04$
	10^{-7}	22.0664377909500047	$1.2241065710516530E-04$
	10^{-8}	22.0663993040532951	$1.6089755381643034E-04$
	10^{-9}	22.0663536010475880	$2.0660055952137490E-04$
	10^{-10}	22.0663005261894050	$2.5967541770477442E-04$
	10^{-11}	22.0662402158038269	$3.1998580328485016E-04$
	10^{-12}	22.0661783306269328	$3.8187098017894439E-04$
	10^{-13}	22.0661196904855252	$4.4051112158549621E-04$
	10^{-14}	22.0660586099259390	$5.0159168117069886E-04$
	10^{-15}	22.0659795863288402	$5.8061527827002468E-04$
	10^{-16}	22.0658681773066263	$6.9202430048320718E-04$

Table C.5: European call option with $K = 8$, $N = 64$, $\sigma = 0.20$, $r = 0.08$, $T = 0.25$, asset value of \$120 and strike of \$100. Here η is the total error for the tails (refer to (5.26)). The values are calculated in this table are performed in double precision.

Tables C.6 - C.10 are European call option prices for varying η and number of partitions (N) fixed at 128 at each time step.

Asset Price ($)	η	IPM	Error
80	10^{-4}	0.0689367552870456	$8.0978014140216121E-05$
	10^{-5}	0.0690036402171109	$1.4093084074902600E-05$
	10^{-6}	0.0690110725171198	$6.6607840659487980E-06$
	10^{-7}	0.0690100652256005	$7.6680755852951386E-06$
	10^{-8}	0.0690076240513432	$1.0109249842587490E-05$
	10^{-9}	0.0690046587459066	$1.3074555279179913E-05$
	10^{-10}	0.0690012819366726	$1.6451364513183318E-05$
	10^{-11}	0.0689975064663605	$2.0226834825250953E-05$
	10^{-12}	0.0689933332174374	$2.4400083748388375E-05$
	10^{-13}	0.0689887618405712	$2.8971460614541332E-05$
	10^{-14}	0.0689837918932166	$3.3941407969145885E-05$
	10^{-15}	0.0689784230627737	$3.9310238412126278E-05$
	10^{-16}	0.0689726550373553	$4.5078263830446250E-05$

Table C.6: European call option with $K = 8$, $N = 128$, $\sigma = 0.20$, $r = 0.08$, $T = 0.25$, asset value of $80 and strike of $100. Here η is the total error for the tails (refer to (5.26)).

Asset Price ($)	η	IPM	Error
	10^{-4}	1.0250862290958813	$3.6750503806248780E-04$
	10^{-5}	1.0254110435852024	$4.2690548741376733E-05$
	10^{-6}	1.0254495443869083	$4.1897470354212984E-06$
	10^{-7}	1.0254542278377765	$4.9370383285035624E-07$
	10^{-8}	1.0254550797537640	$1.3456198202574376E-06$
	10^{-9}	1.0254555517806849	$1.8176467412797659E-06$
	10^{-10}	1.0254560327402125	$2.2986062688662434E-06$
	10^{-11}	1.0254565662080921	$2.8320741484594292E-06$
	10^{-12}	1.0254571574257660	$3.4232918222354414E-06$
	10^{-13}	1.0254578075044520	$4.0733705084297633E-06$
	10^{-14}	1.0254585170377859	$4.7829038422528281E-06$
	10^{-15}	1.0254592865874521	$5.5524535084311433E-06$
	10^{-16}	1.0254601167095867	$6.3825756430047065E-06$

Table C.7: European call option with $K = 8$, $N = 128$, $\sigma = 0.20$, $r = 0.08$, $T = 0.25$, asset value of $90 and strike of $100. Here η is the total error for the tails (refer to (5.26)).

Asset Price ($)	η	**IPM**	**Error**
	10^{-4}	5.0161849394953126	$7.9566676709844075E-04$
	10^{-5}	5.0169024397291455	$7.8166533265533467E-05$
	10^{-6}	5.0169843854911038	$3.7792286924909213E-06$
	10^{-7}	5.0169980171018702	$1.7410839459053307E-05$
	10^{-8}	5.0170051428605795	$2.4536598168534152E-05$
	10^{-9}	5.0170124121918205	$3.1805929409656208E-05$
	10^{-10}	5.0170205348414312	$3.9928579020215293E-05$
	10^{-11}	5.0170295868793682	$4.8980616956795231E-05$
	10^{-12}	5.0170395769473197	$5.8970684908843385E-05$
	10^{-13}	5.0170505062850035	$6.9900022592328526E-05$
	10^{-14}	5.0170623750761409	$8.1768813729732326E-05$
	10^{-15}	5.0170751832305012	$9.4576968089760127E-05$
	10^{-16}	5.0170889304522124	$1.0832418980144531E-04$

Table C.8: European call option with $K = 8$, $N = 128$, $\sigma = 0.20$, $r = 0.08$, $T = 0.25$, asset value of \$100 and strike of \$100. Here η is the total error for the tails (refer to (5.26)).

Asset Price ($)	η	**IPM**	**Error**
	10^{-4}	12.6191596559569792	$1.2888460260596091E-03$
	10^{-5}	12.6203148892311905	$1.3361275184864407E-04$
	10^{-6}	12.6204339094032747	$1.4592579763994529E-05$
	10^{-7}	12.6204458492171359	$2.6527659041652996E-06$
	10^{-8}	12.6204467005803540	$1.8014026853530041E-06$
	10^{-9}	12.6204463572961778	$2.1446868614383163E-06$
	10^{-10}	12.6204458386694647	$2.6633135752351933E-06$
	10^{-11}	12.6204452494020103	$3.2525810295114255E-06$
	10^{-12}	12.6204446010568372	$3.9009262015055057E-06$
	10^{-13}	12.6204438953959279	$4.6065871117662027E-06$
	10^{-14}	12.6204431328405455	$5.3691424944934951E-06$
	10^{-15}	12.6204423143259366	$6.1876571033625538E-06$
	10^{-16}	12.6204414403427752	$7.0616402642054155E-06$

Table C.9: European call option with $K = 8$, $N = 128$, $\sigma = 0.20$, $r = 0.08$, $T = 0.25$, asset value of \$110 and strike of \$100. Here η is the total error for the tails (refer to (5.26)).

Asset Price (\$)	η	**IPM**	**Error**
	10^{-4}	22.0647150059023964	$1.8451957047027623E-03$
	10^{-5}	22.0663713079930517	$1.8889361404761384E-04$
	10^{-6}	22.0665361629037093	$2.4038703388962190E-05$
	10^{-7}	22.0665507510834509	$9.4505236469810683E-06$
	10^{-8}	22.0665498984873985	$1.0303119701848829E-05$
	10^{-9}	22.0665471529241799	$1.3048682920291377E-05$
	10^{-10}	22.0665438770243512	$1.6324582749471794E-05$
	10^{-11}	22.0665402071496359	$1.9994457464700410E-05$
	10^{-12}	22.0665361583697397	$2.4043237359383518E-05$
	10^{-13}	22.0665317289970986	$2.8472610000274123E-05$
	10^{-14}	22.0665269206924073	$3.3280914693567532E-05$
	10^{-15}	22.0665217331912231	$3.8468415877734863E-05$
	10^{-16}	22.0665161654550772	$4.4036152022086661E-05$

Table C.10: European call option with $K = 8$, $N = 128$, $\sigma = 0.20$, $r = 0.08$, $T = 0.25$, asset value of \$120 and strike of \$100. Here η is the total error for the tails (refer to (5.26)).

Tables C.11 - C.15 are European call option prices for varying η and number of partitions (N) fixed at 256 at each time step.

Asset Price (\$)	η	**IPM**	**Error**
80	10^{-4}	0.0689387078581818	$7.9025443004362723E-05$
	10^{-5}	0.0690069218268951	$1.0811474290990924E-05$
	10^{-6}	0.0690160461211648	$1.6871800213001664E-06$
	10^{-7}	0.0690170968441743	$6.3645701182208844E-07$
	10^{-8}	0.0690170820091693	$6.5129201681302443E-07$
	10^{-9}	0.0690169130923457	$8.2020884045134329E-07$
	10^{-10}	0.0690167037042556	$1.0295969305753293E-06$
	10^{-11}	0.0690164674532151	$1.2658479710400336E-06$
	10^{-12}	0.0690162060942279	$1.5272069582268968E-06$
	10^{-13}	0.0690159197675642	$1.8135336219728126E-06$
	10^{-14}	0.0690156084374017	$2.1248637844476642E-06$
	10^{-15}	0.0690152719832055	$2.4613179806461973E-06$
	10^{-16}	0.0690149104757052	$2.8228254809049662E-06$

Table C.11: European call option with $K = 8$, $N = 256$, $\sigma = 0.20$, $r = 0.08$, $T = 0.25$, asset value of \$80 and strike of \$100. Here η is the total error for the tails (refer to (5.26)). The values are calculated in this table are performed in double precision.

Asset Price ($)	η	IPM	Error
90	10^{-4}	1.0250859556035634	$3.6777853038034297E-04$
	10^{-5}	1.0254105854941611	$4.3148639782691900E-05$
	10^{-6}	1.0254488510621849	$4.8830717589223860E-06$
	10^{-7}	1.0254532474503681	$4.8668357555819997E-07$
	10^{-8}	1.0254537601255047	$2.5991560986815543E-08$
	10^{-9}	1.0254538397174204	$1.0558347668387924E-07$
	10^{-10}	1.0254538744589849	$1.4032504134370294E-07$
	10^{-11}	1.0254539074482094	$1.7331426571676189E-07$
	10^{-12}	1.0254539432989465	$2.0916500267920135E-07$
	10^{-13}	1.0254539824909914	$2.4835704760112209E-07$
	10^{-14}	1.0254540251466810	$2.9101273734538635E-07$
	10^{-15}	1.0254540713075531	$3.3717360934520935E-07$
	10^{-16}	1.0254541209016621	$3.8676771837303781E-07$

Table C.12: European call option with $K = 8$, $N = 256$, $\sigma = 0.20$, $r = 0.08$, $T = 0.25$, asset value of $90 and strike of $100. Here η is the total error for the tails (refer to (5.26)). The values are calculated in this table are performed in double precision.

Asset Price ($)	η	IPM	Error
100	10^{-4}	5.0161800213911496	$8.0058487126194455E-04$
	10^{-5}	5.0168942765180953	$8.6329744315716272E-05$
	10^{-6}	5.0169721147311641	$8.4915312469069359E-06$
	10^{-7}	5.0169807698623208	$1.6359991000802765E-07$
	10^{-8}	5.0169820459554524	$1.4396930411719744E-06$
	10^{-9}	5.0169825896216302	$1.9833592192208815E-06$
	10^{-10}	5.0169831088098622	$2.5025474512108126E-06$
	10^{-11}	5.0169836784466115	$3.0721842000924759E-06$
	10^{-12}	5.0169843064752984	$3.7002128874807916E-06$
	10^{-13}	5.0169849938369833	$4.3875745724120119E-06$
	10^{-14}	5.0169857406521343	$5.1343897231048707E-06$
	10^{-15}	5.0169865470375985	$5.9407751870632630E-06$
	10^{-16}	5.0169874130037595	$6.8067413483419159E-06$

Table C.13: European call option with $K = 8$, $N = 256$, $\sigma = 0.20$, $r = 0.08$, $T = 0.25$, asset value of $100 and strike of $100. Here η is the total error for the tails (refer to (5.26)). The values are calculated in this table are performed in double precision.

Asset Price ($)	η	**IPM**	**Error**
110	10^{-4}	12.6191599959021907	$1.2885060808487392E-03$
	10^{-5}	12.6203154456118938	$1.3305637114602309E-04$
	10^{-6}	12.6204347405830664	$1.3761399972334054E-05$
	10^{-7}	12.6204470136595468	$1.4883234922269395E-06$
	10^{-8}	12.6204482510154126	$2.5096762723553212E-07$
	10^{-9}	12.6204483510113672	$1.5097167127553490E-07$
	10^{-10}	12.6204483299921879	$1.7199085211583309E-07$
	10^{-11}	12.6204482932302575	$2.0875278172738376E-07$
	10^{-12}	12.6204482507003952	$2.5128264469209682E-07$
	10^{-13}	12.6204482042078769	$2.9777516286078054E-07$
	10^{-14}	12.6204481546761080	$3.4730693043716343E-07$
	10^{-15}	12.6204481000043227	$4.0197871697955634E-07$
	10^{-16}	12.6204480418787988	$4.6010424081810442E-07$

Table C.14: European call option with $K = 8$, $N = 256$, $\sigma = 0.20$, $r = 0.08$, $T = 0.25$, asset value of \$110 and strike of \$100. Here η is the total error for the tails (refer to (5.26)). The values are calculated in this table are performed in double precision.

Asset Price ($)	η	**IPM**	**Error**
120	10^{-4}	22.0647170628561504	$1.8431387509593167E-03$
	10^{-5}	22.0663747124115588	$1.8548919555327892E-04$
	10^{-6}	22.0665412512933408	$1.8950313768195670E-05$
	10^{-7}	22.0665578560066891	$2.3456004222266458E-06$
	10^{-8}	22.0665593724022706	$8.2920484090198698E-07$
	10^{-9}	22.0665593655224157	$8.3608469592100221E-07$
	10^{-10}	22.0665591840155990	$1.0175915110544054E-06$
	10^{-11}	22.0665589565979374	$1.2450091735338731E-06$
	10^{-12}	22.0665586934555975	$1.5081515147841174E-06$
	10^{-13}	22.0665584174259628	$1.7841811472685265E-06$
	10^{-14}	22.0665581126993722	$2.0889077384511623E-06$
	10^{-15}	22.0665577884289199	$2.4131781906078231E-06$
	10^{-16}	22.0665574419953892	$2.7596117224693018E-06$

Table C.15: European call option with $K = 8$, $N = 256$, $\sigma = 0.20$, $r = 0.08$, $T = 0.25$, asset value of \$120 and strike of \$100. Here η is the total error for the tails (refer to (5.26)). The values are calculated in this table are performed in double precision.

Varying η and Time Steps (K)

Tables C.16 - C.20 are European call prices for varying η and number of partitions (N) fixed at 128 at each time step and the number of time steps fixed at 4.

Asset Price ($)	η	IPM	Error
80	10^{-4}	0.0688383390069953	$1.7939429419047215 \times 10^{-4}$
	10^{-5}	0.0689925677644777	$2.5165536708053170 \times 10^{-5}$
	10^{-6}	0.0690140651674954	$3.6681336904346648 \times 10^{-6}$
	10^{-7}	0.0690167067450898	$1.0265560959556337 \times 10^{-6}$
	10^{-8}	0.0690168398575387	$8.9344364703845131 \times 10^{-7}$
	10^{-9}	0.0690166313256694	$1.1019755164040808 \times 10^{-6}$
	10^{-10}	0.0690163525693472	$1.3807318385633191 \times 10^{-6}$
	10^{-11}	0.0690160361673992	$1.6971337865644490 \times 10^{-6}$
	10^{-12}	0.0690156858456771	$2.0474555086948121 \times 10^{-6}$
	10^{-13}	0.0690153019864084	$2.4313147773606485 \times 10^{-6}$
	10^{-14}	0.0690148845928768	$2.8487083089472585 \times 10^{-6}$
	10^{-15}	0.0690144336276348	$3.2996735509303182 \times 10^{-6}$
	10^{-16}	0.0690139490618014	$3.7842393843848764 \times 10^{-6}$

Table C.16: European call option with $K = 4$, $N = 128$, $\sigma = 0.20$, $r = 0.08$, $T = 0.25$, asset vale of $80 and strike of $100. Here η is the total error for the tails (refer to (5.26)). The values are calculated in this table are performed in double precision.

Asset Price ($)	η	IPM	Error
90	10^{-4}	1.0248647586828115	$5.8897545113217181 \times 10^{-4}$
	10^{-5}	1.0253843055081435	$6.9428625800067867 \times 10^{-5}$
	10^{-6}	1.0254458861939419	$7.8479400016789880 \times 10^{-6}$
	10^{-7}	1.0254529419206306	$7.9221331300560749 \times 10^{-7}$
	10^{-8}	1.0254537546582394	$2.0524295804569270 \times 10^{-8}$
	10^{-9}	1.0254538746841340	$1.4055019023628823 \times 10^{-7}$
	10^{-10}	1.0254539229216315	$1.8878768773961108 \times 10^{-7}$
	10^{-11}	1.0254539675401335	$2.3340618986178452 \times 10^{-7}$
	10^{-12}	1.0254540159389953	$2.8180505168978742 \times 10^{-7}$
	10^{-13}	1.0254540689281852	$3.3479424163540550 \times 10^{-7}$
	10^{-14}	1.0254541266163428	$3.9248239918493466 \times 10^{-7}$
	10^{-15}	1.0254541890303279	$4.5489638427625989 \times 10^{-7}$
	10^{-16}	1.0254542561916775	$5.2205773384828014 \times 10^{-7}$

Table C.17: European call option with $K = 4$, $N = 128$, $\sigma = 0.20$, $r = 0.08$, $T = 0.25$, asset vale of $90 and strike of $100. Here η is the total error for the tails (refer to (5.26)). The values are calculated in this table are performed in double precision.

Asset Price ($)	η	IPM	Error
100	10^{-4}	5.0159521235252402	$1.0284827371712102 \times 10^{-3}$
	10^{-5}	5.0168683465368709	$1.1225972554040897 \times 10^{-4}$
	10^{-6}	5.0169695112331354	$1.1095029275876023 \times 10^{-5}$
	10^{-7}	5.0169808511259220	$2.4486351071772994 \times 10^{-7}$
	10^{-8}	5.0169825382459337	$1.9319835224684834 \times 10^{-6}$
	10^{-9}	5.0169832646039971	$2.6583415857484027 \times 10^{-6}$
	10^{-10}	5.0169839600465584	$3.3537841469255536 \times 10^{-6}$
	10^{-11}	5.0169847231833717	$4.1169209603897361 \times 10^{-6}$
	10^{-12}	5.0169855645086203	$4.9582462087471857 \times 10^{-6}$
	10^{-13}	5.0169864852090669	$5.8789466558795134 \times 10^{-6}$
	10^{-14}	5.0169874854752265	$6.8792128153638554 \times 10^{-6}$
	10^{-15}	5.0169885653782496	$7.9591158386183025 \times 10^{-6}$
	10^{-16}	5.0169897249702746	$9.1187078637977592 \times 10^{-6}$

Table C.18: European call option with $K = 4$, $N = 128$, $\sigma = 0.20$, $r = 0.08$, $T = 0.25$, asset vale of \$100 and strike of \$100. Here η is the total error for the tails (refer to (5.26)). The values are calculated in this table are performed in double precision.

Asset Price ($)	η	IPM	Error
110	10^{-4}	12.6190176889321766	$1.4308130508622341 \times 10^{-3}$
	10^{-5}	12.6202974977079112	$1.5100427512848569 \times 10^{-4}$
	10^{-6}	12.6204325970149238	$1.5904968114766049 \times 10^{-5}$
	10^{-7}	12.6204467451297599	$1.7568532795220193 \times 10^{-6}$
	10^{-8}	12.6204481895572425	$3.1242579634493950 \times 10^{-7}$
	10^{-9}	12.6204483027103329	$1.9927270666908470 \times 10^{-7}$
	10^{-10}	12.6204482727472005	$2.2923583864464803 \times 10^{-7}$
	10^{-11}	12.6204482231697881	$2.7881325026513082 \times 10^{-7}$
	10^{-12}	12.6204481671445130	$3.3483852635018962 \times 10^{-7}$
	10^{-13}	12.6204481057453357	$3.9623770375918355 \times 10^{-7}$
	10^{-14}	12.6204480391416460	$4.6284139298968796 \times 10^{-7}$
	10^{-15}	12.6204479673705841	$5.3461245430597160 \times 10^{-7}$
	10^{-16}	12.6204478904231117	$6.1155992769901246 \times 10^{-7}$

Table C.19: European call option with $K = 4$, $N = 128$, $\sigma = 0.20$, $r = 0.08$, $T = 0.25$, asset vale of \$110 and strike of \$100. Here η is the total error for the tails (refer to (5.26)). The values are calculated in this table are performed in double precision.

Asset Price ($)	η	IPM	Error
120	10^{-4}	22.0646922057009292	$1.8679959061712603 \times 10^{-3}$
	10^{-5}	22.0663695920392975	$1.9060956780170280 \times 10^{-4}$
	10^{-6}	22.0665403269740032	$1.9874633097605177 \times 10^{-5}$
	10^{-7}	22.0665575821960616	$2.6194110380739488 \times 10^{-6}$
	10^{-8}	22.0665591499322957	$1.0516748027988854 \times 10^{-6}$
	10^{-9}	22.06655908737521010	$1.1142318873114121 \times 10^{-6}$
	10^{-10}	22.0665588287621048	$1.3728449960170863 \times 10^{-6}$
	10^{-11}	22.0665585223048666	$1.6793022333816410 \times 10^{-6}$
	10^{-12}	22.0665581821209393	$2.0194861616307236 \times 10^{-6}$
	10^{-13}	22.0665578098858539	$2.3917212460533577 \times 10^{-6}$
	10^{-14}	22.0665574055449163	$2.7960621845624090 \times 10^{-6}$
	10^{-15}	22.0665569688028569	$3.2328042437468341 \times 10^{-6}$
	10^{-16}	22.0665565000875041	$3.7015195963885361 \times 10^{-6}$

Table C.20: European call option with $K = 4$, $N = 128$, $\sigma = 0.20$, $r = 0.08$, $T = 0.25$, asset vale of \$120 and strike of \$100. Here η is the total error for the tails (refer to (5.26)). The values are calculated in this table are performed in double precision.

Tables C.21 - C.25 are European call prices for varying η and number of partitions (N) fixed at 128 at each time step and the number of time steps fixed at 8.

Asset Price ($)	η	IPM	Error
80	10^{-4}	0.0689367552870456	$8.0978014140216121 \times 10^{-5}$
	10^{-5}	0.06900364021711010	$1.4093084074902600 \times 10^{-5}$
	10^{-6}	0.0690110725171198	$6.6607840659487980 \times 10^{-6}$
	10^{-7}	0.0690100652256005	$7.6680755852951386 \times 10^{-6}$
	10^{-8}	0.0690076240513432	$1.0109249842587490 \times 10^{-5}$
	10^{-9}	0.0690046587459066	$1.3074555279179913 \times 10^{-5}$
	10^{-10}	0.0690012819366726	$1.6451364513183318 \times 10^{-5}$
	10^{-11}	0.0689975064663605	$2.0226834825250953 \times 10^{-5}$
	10^{-12}	0.0689933332174374	$2.4400083748388375 \times 10^{-5}$
	10^{-13}	0.0689887618405712	$2.8971460614541332 \times 10^{-5}$
	10^{-14}	0.0689837918932166	$3.3941407969145885 \times 10^{-5}$
	10^{-15}	0.0689784230627737	$3.9310238412126278 \times 10^{-5}$
	10^{-16}	0.0689726550373553	$4.5078263830446250 \times 10^{-5}$

Table C.21: European call option with $K = 8$, $N = 128$, $\sigma = 0.20$, $r = 0.08$, $T = 0.25$, asset value of \$80 and strike of \$100. Here η is the total error for the tails (refer to (5.26)). The values are calculated in this table are performed in double precision.

Asset Price ($)	η	IPM	Error
90	10^{-4}	1.0250862290958813	$3.6750503806248780E-04$
	10^{-5}	1.0254110435852024	$4.2690548741376733E-05$
	10^{-6}	1.0254495443869083	$4.1897470354212984E-06$
	10^{-7}	1.0254542278377765	$4.9370383285035624E-07$
	10^{-8}	1.0254550797537640	$1.3456198202574376E-06$
	10^{-9}	1.0254555517806849	$1.8176467412797659E-06$
	10^{-10}	1.0254560327402125	$2.2986062688662434E-06$
	10^{-11}	1.0254565662080921	$2.8320741484594292E-06$
	10^{-12}	1.0254571574257660	$3.4232918222354414E-06$
	10^{-13}	1.0254578075044520	$4.0733705084297633E-06$
	10^{-14}	1.0254585170377859	$4.7829038422528281E-06$
	10^{-15}	1.0254592865874521	$5.5524535084311433E-06$
	10^{-16}	1.0254601167095867	$6.3825756430047065E-06$

Table C.22: European call option with $K = 8$, $N = 128$, $\sigma = 0.20$, $r = 0.08$, $T = 0.25$, asset value of \$90 and strike of \$100. Here η is the total error for the tails (refer to (5.26)). The values are calculated in this table are performed in double precision.

Asset Price ($)	η	IPM	Error
100	10^{-4}	5.0161849394953126	$7.9566676709844075E-04$
	10^{-5}	5.0169024397291455	$7.8166533265533467E-05$
	10^{-6}	5.0169843854911038	$3.7792286924909213E-06$
	10^{-7}	5.0169980171018702	$1.7410839459053307E-05$
	10^{-8}	5.0170051428605795	$2.4536598168534152E-05$
	10^{-9}	5.0170124121918205	$3.1805929409656208E-05$
	10^{-10}	5.0170205348414312	$3.9928579020215293E-05$
	10^{-11}	5.0170295868793682	$4.8980616956795231E-05$
	10^{-12}	5.0170395769473197	$5.8970684908843385E-05$
	10^{-13}	5.0170505062850035	$6.9900022592328526E-05$
	10^{-14}	5.0170623750761409	$8.1768813729732326E-05$
	10^{-15}	5.0170751832305012	$9.4576968089760127E-05$
	10^{-16}	5.0170889304522124	$1.0832418980144531E-04$

Table C.23: European call option with $K = 8$, $N = 128$, $\sigma = 0.20$, $r = 0.08$, $T = 0.25$, asset value of \$100 and strike of \$100. Here η is the total error for the tails (refer to (5.26)).

Asset Price ($)	η	IPM	Error
110	10^{-4}	12.6191596559569792	$1.2888460260596091E-03$
	10^{-5}	12.6203148892311905	$1.3361275184864407E-04$
	10^{-6}	12.6204339094032747	$1.4592579763994529E-05$
	10^{-7}	12.6204458492171359	$2.6527659041652996E-06$
	10^{-8}	12.6204467005803540	$1.8014026853530041E-06$
	10^{-9}	12.6204463572961778	$2.1446868614383163E-06$
	10^{-10}	12.6204458386694647	$2.6633135752351933E-06$
	10^{-11}	12.6204452494020103	$3.2525810295114255E-06$
	10^{-12}	12.6204446010568372	$3.9009262015055057E-06$
	10^{-13}	12.6204438953959279	$4.6065871117662027E-06$
	10^{-14}	12.6204431328405455	$5.3691424944934951E-06$
	10^{-15}	12.6204423143259366	$6.1876571033625538E-06$
	10^{-16}	12.6204414403427752	$7.0616402642054155E-06$

Table C.24: European call option with $K = 8$, $N = 128$, $\sigma = 0.20$, $r = 0.08$, $T = 0.25$, asset value of \$110 and strike of \$100. Here η is the total error for the tails (refer to (5.26)).

Asset Price ($)	η	IPM	Error
120	10^{-4}	22.0647150059023964	$1.8451957047027623E-03$
	10^{-5}	22.0663713079930517	$1.8889361404761384E-04$
	10^{-6}	22.0665361629037093	$2.4038703388962190E-05$
	10^{-7}	22.0665507510834509	$9.4505236469810683E-06$
	10^{-8}	22.0665498984873985	$1.0303119701848829E-05$
	10^{-9}	22.0665471529241799	$1.3048682920291377E-05$
	10^{-10}	22.0665438770243512	$1.6324582749471794E-05$
	10^{-11}	22.0665402071496359	$1.9994457464700410E-05$
	10^{-12}	22.0665361583697397	$2.4043237359383518E-05$
	10^{-13}	22.0665317289970986	$2.8472610000274123E-05$
	10^{-14}	22.0665269206924073	$3.3280914693567532E-05$
	10^{-15}	22.0665217331912231	$3.8468415877734863E-05$
	10^{-16}	22.0665161654550772	$4.4036152022086661E-05$

Table C.25: European call option with $K = 8$, $N = 128$, $\sigma = 0.20$, $r = 0.08$, $T = 0.25$, asset value of \$120 and strike of \$100. Here η is the total error for the tails (refer to (5.26)).

Tables C.26 - C.30 are European call prices for varying η and number of partitions (N) fixed at 128 at each time step and the number of time steps fixed at 16.

Asset Price ($)	η	IPM	Error
80	10^{-4}	0.0689582771692606	$5.9456131925217544E-05$
	10^{-5}	0.0689793368560128	$3.8396445172970105E-05$
	10^{-6}	0.0689665708855032	$5.1162415682531361E-05$
	10^{-7}	0.0689462056729448	$7.1527628240936753E-05$
	10^{-8}	0.0689216843832525	$9.6048917933261140E-05$
	10^{-9}	0.0688935607390907	$1.2417256209509364E-04$
	10^{-10}	0.0688611168765195	$1.5661642466626487E-04$
	10^{-11}	0.0688232380201808	$1.9449528100494266E-04$
	10^{-12}	0.0687810515966192	$2.3668170456656729E-04$
	10^{-13}	0.0687384809806727	$2.7925232051312886E-04$
	10^{-14}	0.0686993995053716	$3.1833379581415336E-04$
	10^{-15}	0.0686641977283687	$3.5353557281709181E-04$
	10^{-16}	0.0686285034915563	$3.8922980962951756E-04$

Table C.26: European call option with $K = 16$, $N = 128$, $\sigma = 0.20$, $r = 0.08$, $T = 0.25$, asset value of \$80 and strike of \$100. Here η is the total error for the tails (refer to (5.26)).

Asset Price ($)	η	IPM	Error
90	10^{-4}	1.0252112526988435	$2.4248143510024328E-04$
	10^{-5}	1.0254304211527474	$2.3312981196378646E-05$
	10^{-6}	1.0254577677306331	$4.0335966894691766E-06$
	10^{-7}	1.0254636223942994	$9.8882603556671445E-06$
	10^{-8}	1.0254677244796533	$1.3990345709528895E-05$
	10^{-9}	1.0254724796790455	$1.8745545101936378E-05$
	10^{-10}	1.0254782416938892	$2.4507559945492752E-05$
	10^{-11}	1.0254848560957128	$3.1121961769145501E-05$
	10^{-12}	1.0254916916648129	$3.7957530869131562E-05$
	10^{-13}	1.0254975951859269	$4.3861051983104526E-05$
	10^{-14}	1.0255009251387170	$4.7191004773350365E-05$
	10^{-15}	1.0254996504810641	$4.5916347120311674E-05$
	10^{-16}	1.0254914799169168	$3.7745782973178099E-05$

Table C.27: European call option with $K = 16$, $N = 128$, $\sigma = 0.20$, $r = 0.08$, $T = 0.25$, asset value of \$90 and strike of \$100. Here η is the total error for the tails (refer to (5.26)).

Asset Price ($)	η	IPM	Error
100	10^{-4}	5.0163672497859464	$6.1335647646501568E-04$
	10^{-5}	5.0169930707709156	$1.2464508504506089E-05$
	10^{-6}	5.0170976275381935	$1.1702127578200328E-04$
	10^{-7}	5.0171546171944028	$1.7401093199143070E-04$
	10^{-8}	5.0172150865578313	$2.3448029542044724E-04$
	10^{-9}	5.0172851661250997	$3.0455986268843049E-04$
	10^{-10}	5.0173624377283215	$3.8183146591069805E-04$
	10^{-11}	5.0174384587526966	$4.5785249028554298E-04$
	10^{-12}	5.0175049287416051	$5.2432247919440012E-04$
	10^{-13}	5.0175626290813584	$5.8202281894709218E-04$
	10^{-14}	5.0176260795912890	$6.4547332887798792E-04$
	10^{-15}	5.0177211545220777	$7.4054825966660132E-04$
	10^{-16}	5.0178778465529241	$8.9724029051288512E-04$

Table C.28: European call option with $K = 16$, $N = 128$, $\sigma = 0.20$, $r = 0.08$, $T = 0.25$, asset value of \$100 and strike of \$100. Here η is the total error for the tails (refer to (5.26)).

Asset Price ($)	η	IPM	Error
110	10^{-4}	12.6192243540329905	$1.2241479500491037E-03$
	10^{-5}	12.6203187087153221	$1.2979326771689337E-04$
	10^{-6}	12.6204275920049565	$2.0909978082350733E-05$
	10^{-7}	12.6204357462574137	$1.2755725625757108E-05$
	10^{-8}	12.6204330975581236	$1.5404424916209969E-05$
	10^{-9}	12.6204274184788865	$2.1083504153640575E-05$
	10^{-10}	12.6204265898257120	$2.1912157327497184E-05$
	10^{-11}	12.6204299463401579	$1.8555642880557777E-05$
	10^{-12}	12.6204123097258307	$3.6192257207856571E-05$
	10^{-13}	12.6203658604232931	$8.2641559745466608E-05$
	10^{-14}	12.6203331480489922	$1.1535393404649152E-04$
	10^{-15}	12.6203715480197118	$7.6953963326631403E-05$
	10^{-16}	12.6204925573686229	$4.4055385584051976E-05$

Table C.29: European call option with $K = 16$, $N = 128$, $\sigma = 0.20$, $r = 0.08$, $T = 0.25$, asset value of \$110 and strike of \$100. Here η is the total error for the tails (refer to (5.26)).

Asset Price ($)	η	**IPM**	**Error**
120	10^{-4}	22.0646946915641315	$1.8655100429670890E-03$
	10^{-5}	22.0663408517722779	$2.1934983482074788E-04$
	10^{-6}	22.0664901703076524	$7.0031299448380935E-05$
	10^{-7}	22.0664862587156740	$7.3942891426481694E-05$
	10^{-8}	22.0664638811254008	$9.6320481698941585E-05$
	10^{-9}	22.0664361600788546	$1.2404152824330161E-04$
	10^{-10}	22.0664045675190117	$1.5563408808649815E-04$
	10^{-11}	22.0663735070014617	$1.8669460563891516E-04$
	10^{-12}	22.0663294149184424	$2.3078668865694763E-04$
	10^{-13}	22.0662782667328869	$2.8193487421335206E-04$
	10^{-14}	22.0662494584594882	$3.1074314761070365E-04$
	10^{-15}	22.0662299096733996	$3.3029193370059939E-04$
	10^{-16}	22.0661684686939346	$3.9173291316585424E-04$

Table C.30: European call option with $K = 16$, $N = 128$, $\sigma = 0.20$, $r = 0.08$, $T = 0.25$, asset value of \$120 and strike of \$100. Here η is the total error for the tails (refer to (5.26)).

C.1.2 Fixed Spaced Partitions

Table C.31 are European call option prices using Fixed Space Partitions and 8 time steps.

η	**Space (Partitions)**	**IPM**	**Error**
10^{-8}	0.10 (571)	5.0170393001964610	$5.8693934049808272E-05$
	0.09 (635)	5.0170191054794246	$3.8499217013682197E-05$
	0.08 (712)	5.0170046163516142	$2.4010089202941565E-05$
	0.07 (812)	5.0169946447725833	$1.4038510171704432E-05$
	0.06 (949)	5.0169881393371067	$7.5330746952817496E-06$
	0.05 (1139)	5.0169841865188891	$3.5802564780917923E-06$
	0.04 (1421)	5.0169820116252728	$1.4053628615007252E-06$
	0.03 (1895)	5.0169809797489462	$3.7348653544877486E-07$
	0.02 (2840)	5.0169805964410719	$9.8213388810552971E-09$
	0.01 (5676)	5.0169805081153545	$9.8147057148167960E-08$
10^{-16}	0.10 (826)	5.0170394044695792	$5.8798207167726213E-05$
	0.09 (917)	5.0170192097522701	$3.8603489858901607E-05$
	0.08 (1031)	5.0170047206242590	$2.4114361847626942E-05$
	0.07 (1176)	5.0169947490435307	$1.4142781119136361E-05$
	0.06 (1372)	5.0169882436167867	$7.6373543753116557E-06$
	0.05 (1648)	5.0169842907878239	$3.6845254125505988E-06$
	0.04 (2058)	5.0169821159164245	$1.5096540136039227E-06$
	0.03 (2744)	5.0169810840361189	$4.7777370740242908E-07$
	0.02 (4113)	5.0169807006890261	$9.4426615138143433E-08$
	0.01 (8224)	5.0169806119534828	$5.6910719192782011E-09$
10^{-32}	0.10 (1183)	5.0170394044699238	$5.8798207512589240E-05$
	0.09 (1314)	5.0170192097533661	$3.8603490955246844E-05$
	0.08 (1477)	5.0170047206247226	$2.4114362311145054E-05$
	0.07 (1689)	5.0169947490452049	$1.4142782793491460E-05$
	0.06 (1970)	5.0169882436156605	$7.6373532498230645E-06$
	0.05 (2362)	5.0169842907883959	$3.6845259850093459E-06$
	0.04 (2952)	5.0169821159098253	$1.5096474140219307E-06$
	0.03 (3935)	5.0169810840248106	$4.7776239978092327E-07$
	0.02 (5901)	5.0169807006477285	$9.4385316923295548E-08$
	0.01 (11798)	5.0169806119866811	$5.7242696693826645E-09$

Table C.31: European call option price for an asset price of \$100 with $K = 8$, $\sigma = 0.20$, $r = 0.08$, $T = 0.25$ and strike of \$100. Here η is the total error for the tails. The value in brackets, represents the number of partitions used to obtain the option price. Double precision was used to calculate the values.

C.1.3 Adaptive Node Allocation

The following are varying results for European Call options using the Adaptive Node Allocation for each time step.

Asset Price (\$)	η	Partitions Used	IPM	Error
80	10^{-8}	32	0.0683036148058220	7.1411849536408408$E-04$
	10^{-16}	38	0.0683161742817790	7.0155901940712135$E-04$
	10^{-32}	48	0.0682909719159381	7.2676138524808195$E-04$
90	10^{-8}	35	1.0253784795629517	7.5254570991965808$E-05$
	10^{-16}	42	1.0254737759957526	2.0041861808865746$E-05$
	10^{-32}	50	1.0252199185734290	2.3381556051478569$E-04$
100	10^{-8}	37	5.0174861712502077	5.0556498779677495$E-04$
	10^{-16}	45	5.0174971044474770	5.1649818506582790$E-04$
	10^{-32}	52	5.0174392446016380	4.5863833922701880$E-04$
110	10^{-8}	38	12.6200227904004230	4.2571158261617281$E-04$
	10^{-16}	45	12.6199792761028000	4.6922588023967648$E-04$
	10^{-32}	55	12.6199707960413612	4.7770594167884095$E-04$
120	10^{-8}	38	22.0657953634293591	7.6483817774974927$E-04$
	10^{-16}	46	22.0657571768756462	8.0302473146331987$E-04$
	10^{-32}	57	22.0658059171494401	7.5428445767133923$E-04$

Table C.32: European call option with adaptive node distribution with an interpolating error $\epsilon = 10^{-6}$ and with $K = 4$, $\sigma = 0.20$, $r = 0.08$, $T = 0.25$ and strike of \$100. Here η is the total error for the tails. Double precision was used to calculate the values.

Asset Price (\$)	η	Partitions Used	IPM	Error
80	10^{-8}	50	0.0688954102687032	1.2232303248293705$E-04$
	10^{-16}	61	0.0688940066247114	1.2372667647477312$E-04$
	10^{-32}	75	0.0689046674456321	1.1306585555407320$E-04$
90	10^{-8}	55	1.0254277685535811	2.5965580362566354$E-05$
	10^{-16}	66	1.0254169779481670	3.6756185776802563$E-05$
	10^{-32}	79	1.0254329508733471	2.0783260596540787$E-05$
100	10^{-8}	58	5.0170432852059683	6.2678943557603617$E-05$
	10^{-16}	68	5.0170593597916806	7.8753529269420808$E-05$
	10^{-32}	83	5.0170530272327625	7.2420970351205760$E-05$
110	10^{-8}	59	12.6203772670746748	7.1234908363693172$E-05$
	10^{-16}	70	12.6203810305626245	6.7471420414633165$E-05$
	10^{-32}	86	12.6203805761256920	6.7925857347161944$E-05$
120	10^{-8}	60	22.0664371647642739	1.2303684283621052$E-04$
	10^{-16}	70	22.0664348117661007	1.2538984101029271$E-04$
	10^{-32}	88	22.0664328279561737	1.2737365093773434$E-04$

Table C.33: European call option with adaptive node distribution with an interpolating error $\epsilon = 10^{-7}$ and with $K = 4$, $\sigma = 0.20$, $r = 0.08$, $T = 0.25$ and strike of \$100. Here η is the total error for the tails. Double precision was used to calculate the values.

Asset Price ($)	η	Partitions Used	IPM	Error
80	10^{-8}	78	0.0689989134074569	$1.8819893729201970E-05$
	10^{-16}	96	0.0689991677138930	$1.8565587293082123E-05$
	10^{-32}	117	0.0689988705545568	$1.8862746629304088E-05$
90	10^{-8}	88	1.0254467440320314	$6.9901019124005148E-06$
	10^{-16}	103	1.0254474372179319	$6.2969160118803158E-06$
	10^{-32}	124	1.0254479143915620	$5.8197423817574201E-06$
100	10^{-8}	89	5.0169924302855735	$1.1824023162659847E-05$
	10^{-16}	109	5.0169913953935454	$1.0789131134336660E-05$
	10^{-32}	130	5.0169899157673292	$9.3095049176483702E-06$
110	10^{-8}	92	12.6204399615872873	$8.5403957517105056E-06$
	10^{-16}	111	12.6204381116031996	$1.0390379839853026E-05$
	10^{-32}	136	12.6204367174553553	$1.1784527684244317E-05$
120	10^{-8}	92	22.0665399725005678	$2.0229106542646313E-05$
	10^{-16}	112	22.0665405299263355	$1.9671680773214462E-05$
	10^{-32}	139	22.0665405886243384	$1.9612982771199938E-05$

Table C.34: European call option with adaptive node distribution with an interpolating error $\epsilon = 10^{-8}$ and with $K = 4$, $\sigma = 0.20$, $r = 0.08$, $T = 0.25$ and strike of \$100. Here η is the total error for the tails. Double precision was used to calculate the values.

Asset Price ($)	η	Partitions Used	IPM	Error
80	10^{-8}	122	0.0690146803463858	$3.0529548003467750E-06$
	10^{-16}	150	0.0690146879852485	$3.0453159376598454E-06$
	10^{-32}	184	0.0690147040815196	$3.0292196664797634E-06$
90	10^{-8}	135	1.0254525910199330	$1.1431140107404092E-06$
	10^{-16}	163	1.0254529422904397	$7.9184350396110936E-07$
	10^{-32}	196	1.0254528224019526	$9.1173199111976855E-07$
100	10^{-8}	142	5.0169825138271680	$1.9075647568478349E-06$
	10^{-16}	169	5.0169824382954689	$1.8320330578969202E-06$
	10^{-32}	204	5.0169824313320062	$1.8250695947574780E-06$
110	10^{-8}	147	12.6204468274334065	$1.6745496322911890E-06$
	10^{-16}	176	12.6204468951690991	$1.6068139402358739E-06$
	10^{-32}	213	12.6204469508081285	$1.5511749107410822E-06$
120	10^{-8}	146	22.0665570766488592	$3.1249582504999651E-06$
	10^{-16}	178	22.0665570880606516	$3.1135464595433149E-06$
	10^{-32}	221	22.0665570674505034	$3.1341566064657655E-06$

Table C.35: European call option with adaptive node distribution with an interpolating error $\epsilon = 10^{-9}$ and with $K = 4$, $\sigma = 0.20$, $r = 0.08$, $T = 0.25$ and strike of \$100. Here η is the total error for the tails. Double precision was used to calculate the values.

Asset Price ($)	η	**Partitions Used**	**IPM**	**Error**
80	10^{-8}	193	0.0690172078013851	$5.2549980107414324E-07$
	10^{-16}	236	0.0690172582546654	$4.7504652075100809E-07$
	10^{-32}	296	0.0690172579420216	$4.7535916448786242E-07$
90	10^{-8}	214	1.0254534930640242	$2.4106991940858746E-07$
	10^{-16}	254	1.0254535991744871	$1.3495945663105635E-07$
	10^{-32}	313	1.0254536019086471	$1.3222529668399652E-07$
100	10^{-8}	225	5.0169808299027157	$2.2364030488608577E-07$
	10^{-16}	271	5.0169809155872507	$3.0932483935375288E-07$
	10^{-32}	333	5.0169809095824611	$3.0332005032662757E-07$
110	10^{-8}	229	12.6204482262521953	$2.7573084371290690E-07$
	10^{-16}	276	12.6204482376241689	$2.6435887090503485E-07$
	10^{-32}	344	12.6204482351633924	$2.6681964693242577E-07$
120	10^{-8}	231	22.0665595105130130	$6.9109409583933967E-07$
	10^{-16}	288	22.0665597140188332	$4.8758827808637051E-07$
	10^{-32}	366	22.0665596995283586	$5.0207875079699704E-07$

Table C.36: European call option with adaptive node distribution with an interpolating error $\epsilon = 10^{-10}$ and with $K = 4$, $\sigma = 0.20$, $r = 0.08$, $T = 0.25$ and strike of \$100. Here η is the total error for the tails. Double precision was used to calculate the values.

Asset Price ($)	η	**Partitions Used**	**IPM**	**Error**
80	10^{-8}	307	0.0690176085869599	$1.2471422625084244E-07$
	10^{-16}	377	0.0690176576342929	$7.5666893237236188E-08$
	10^{-32}	481	0.0690176577220457	$7.5579140450428295E-08$
90	10^{-8}	337	1.0254536147954754	$1.1933846827405503E-07$
	10^{-16}	403	1.0254537106797199	$2.3454223806174124E-08$
	10^{-32}	491	1.0254537106768995	$2.3457044119601367E-08$
100	10^{-8}	362	5.0169805370092924	$6.9253119211465375E-08$
	10^{-16}	435	5.0169806541162956	$4.7853884049819939E-08$
	10^{-32}	519	5.0169806519832498	$4.5720838920404461E-08$
110	10^{-8}	366	12.6204484257445184	$7.6238521051763541E-08$
	10^{-16}	459	12.6204484630551867	$3.8927852608061642E-08$
	10^{-32}	532	12.6204484645069286	$3.7476111103273979E-08$
120	10^{-8}	366	22.0665599222584845	$2.7934862478762312E-07$
	10^{-16}	441	22.0665601247959060	$7.6811204618998374E-08$
	10^{-32}	532	22.0665601256640791	$7.5943032418201994E-08$

Table C.37: European call option with adaptive node distribution with an interpolating error $\epsilon = 10^{-11}$ and with $K = 4$, $\sigma = 0.20$, $r = 0.08$, $T = 0.25$ and strike of \$100. Here η is the total error for the tails. Double precision was used to calculate the values.

The next results are for European Call options using the Adaptive Node Allocation at the first time step only.

Asset Price ($)	η	**Partitions Used**	**IPM**	**Error**
80	10^{-8}	30	0.0647530362187676	$4.2646970824184848E-03$
	10^{-16}	37	0.0652485905418126	$3.7691427593735337E-03$
	10^{-32}	46	0.0655335425615821	$3.4841907396040549E-03$
90	10^{-8}	34	1.0250303800479135	$4.2335408603020458E-04$
	10^{-16}	40	1.0249006559712861	$5.5307816265760812E-04$
	10^{-32}	47	1.0249922062839603	$4.6152784998337859E-04$
100	10^{-8}	37	5.0169731408602267	$7.4654021846309870E-06$
	10^{-16}	43	5.0173086543014094	$3.2804803899860735E-04$
	10^{-32}	50	5.0172921435895832	$3.1153732717240090E-04$
110	10^{-8}	38	12.4745040775131812	$1.4594442446985711E-01$
	10^{-16}	45	12.6200147834617464	$4.3371852129214794E-04$
	10^{-32}	53	12.6199203722459821	$5.2812973705684829E-04$
120	10^{-8}	40	22.0635446242469051	$3.0155773602047464E-03$
	10^{-16}	46	22.0647869190571448	$1.7732825499661153E-03$
	10^{-32}	55	22.0649245913760943	$1.6356102310155496E-03$

Table C.38: European call option with single adaptive node distribution at the first time step with an interpolating error $\epsilon = 10^{-6}$ and with $K = 4$, $\sigma = 0.20$, $r = 0.08$, $T = 0.25$ and strike of \$100. Here η is the total error for the tails. Double precision was used to calculate the values.

Asset Price ($)	η	**Partitions Used**	**IPM**	**Error**
80	10^{-8}	46	0.0678365793214134	$1.1811539797726943E-03$
	10^{-16}	57	0.0679305218950064	$1.0872114061796995E-03$
	10^{-32}	70	0.0680434099544715	$9.7432334671464314E-04$
90	10^{-8}	52	1.0253647729123416	$8.8961221602137930E-05$
	10^{-16}	62	1.0253546486926255	$9.9085441318189083E-05$
	10^{-32}	73	1.0253609300581759	$9.2804075767921601E-05$
100	10^{-8}	56	5.0160950281385528	$8.8557812385828272E-04$
	10^{-16}	65	5.0170287433638627	$4.8137101451434239E-05$
	10^{-32}	78	5.0170291647828407	$4.8558520429620167E-05$
110	10^{-8}	60	12.6181838452583648	$2.2646567246736860E-03$
	10^{-16}	68	12.6203476962336545	$1.0080574938398090E-04$
	10^{-32}	82	12.6203456755827563	$1.0282640028336232E-04$
120	10^{-8}	63	22.0648712217616776	$1.6889798454322591E-03$
	10^{-16}	71	22.0660650839243218	$4.9511768278887036E-04$
	10^{-32}	85	22.0661100951545244	$4.5010645258469761E-04$

Table C.39: European call option with single adaptive node distribution at the first time step with an interpolating error $\epsilon = 10^{-7}$ and with $K = 4$, $\sigma = 0.20$, $r = 0.08$, $T = 0.25$ and strike of $100. Here η is the total error for the tails. Double precision was used to calculate the values.

Asset Price ($)	η	**Partitions Used**	**IPM**	**Error**
80	10^{-8}	75	0.0687180173132312	$2.9971598795491871E-04$
	10^{-16}	91	0.0686897011165346	$3.2803218465150786E-04$
	10^{-32}	111	0.0686884080908636	$3.2932521032250387E-04$
90	10^{-8}	85	1.0254344795077599	$1.9254626183740076E-05$
	10^{-16}	97	1.0254362120841831	$1.7522049760611280E-05$
	10^{-32}	117	1.0254326921476347	$2.1041986309096317E-05$
100	10^{-8}	91	5.0169835301475638	$2.9238851527446652E-06$
	10^{-16}	104	5.0169856227568159	$5.0164944045971716E-06$
	10^{-32}	124	5.0169862259386448	$5.6196762340809947E-06$
110	10^{-8}	91	12.6203376167489605	$1.1088523407809525E-04$
	10^{-16}	109	12.6204320012411717	$1.6500741867320201E-05$
	10^{-32}	129	12.6204321899372314	$1.6312045808541953E-05$
120	10^{-8}	97	22.0664723399513498	$8.7861655762022650E-05$
	10^{-16}	113	22.0664889287525021	$7.1272854607062897E-05$
	10^{-32}	134	22.0664942622059996	$6.5939401111547724E-05$

Table C.40: European call option with single adaptive node distribution at the first time step with an interpolating error $\epsilon = 10^{-8}$ and with $K = 4$, $\sigma = 0.20$, $r = 0.08$, $T = 0.25$ and strike of $100. Here η is the total error for the tails. Double precision was used to calculate the values.

Asset Price ($)	η	**Partitions Used**	**IPM**	**Error**
80	10^{-8}	116	0.0689375959379025	$8.0137363283629397E-05$
	10^{-16}	143	0.0689328220463222	$8.4911254863971088E-05$
	10^{-32}	174	0.0689226935416872	$9.5039759498958837E-05$
90	10^{-8}	133	1.0254499221687041	$3.8119652394710823E-06$
	10^{-16}	155	1.0254501539858176	$3.5801481260491763E-06$
	10^{-32}	186	1.0254502190435195	$3.5150904242886583E-06$
100	10^{-8}	140	5.0169804761446732	$1.3011773811189009E-07$
	10^{-16}	163	5.0169815587802802	$9.5251786944028360E-07$
	10^{-32}	195	5.0169816476442266	$1.0413818155030619E-06$
110	10^{-8}	148	12.6204385958538357	$9.9061292030411252E-06$
	10^{-16}	171	12.6204458862049833	$2.6157780552704679E-06$
	10^{-32}	205	12.6204459271643668	$2.5748186721630262E-06$
120	10^{-8}	157	22.0665483741688604	$1.1827438251010847E-05$
	10^{-16}	179	22.0665487395526547	$1.1462054455568804E-05$
	10^{-32}	215	22.0665486133647271	$1.1588242385118797E-05$

Table C.41: European call option with single adaptive node distribution at the first time step with an interpolating error $\epsilon = 10^{-9}$ and with $K = 4$, $\sigma = 0.20$, $r = 0.08$, $T = 0.25$ and strike of \$100. Here η is the total error for the tails. Double precision was used to calculate the values.

Asset Price ($)	η	**Partitions Used**	**IPM**	**Error**
80	10^{-8}	182	0.0689941859085118	$2.3547392674334953E-05$
	10^{-16}	226	0.0689968093376596	$2.0923963526501155E-05$
	10^{-32}	274	0.0689946039240728	$2.3129377113291320E-05$
90	10^{-8}	207	1.0254527202812522	$1.0138526915573487E-06$
	10^{-16}	244	1.0254530669426853	$6.6719125831754544E-07$
	10^{-32}	308	1.0254531282419075	$6.0589203619565435E-07$
100	10^{-8}	221	5.0169789819791211	$1.6242832898138992E-06$
	10^{-16}	260	5.0169808246016006	$2.1833918978364508E-07$
	10^{-32}	306	5.0169807846794052	$1.7841699367915353E-07$
110	10^{-8}	236	12.6204479354339512	$5.6654908819897543E-07$
	10^{-16}	272	12.6204481078901640	$3.9409287500014045E-07$
	10^{-32}	330	12.6204481122970229	$3.8968601678845971E-07$
120	10^{-8}	246	22.0665579241181149	$2.2774889973931067E-06$
	10^{-16}	286	22.0665582758502872	$1.9257568234509748E-06$
	10^{-32}	351	22.0665582779249583	$1.9236821524604153E-06$

Table C.42: European call option with single adaptive node distribution at the first time step with an interpolating error $\epsilon = 10^{-10}$ and with $K = 4$, $\sigma = 0.20$, $r = 0.08$, $T = 0.25$ and strike of \$100. Here η is the total error for the tails. Double precision was used to calculate the values.

Asset Price ($)	η	Partitions Used	IPM	Error
80	10^{-8}	292	0.0690123703176366	$5.3629835495357277E-06$
	10^{-16}	369	0.0690118570575226	$5.8762436635248320E-06$
	10^{-32}	466	0.0690118143068898	$5.9189942963220818E-06$
90	10^{-8}	331	1.0254535143641064	$2.1976983739790956E-07$
	10^{-16}	381	1.0254536379048673	$9.6229076443354877E-08$
	10^{-32}	468	1.0254536268056598	$1.0732828393394955E-07$
100	10^{-8}	357	5.0169804793704280	$1.2689198303217353E-07$
	10^{-16}	408	5.0169806387585929	$3.2496182134078566E-08$
	10^{-32}	500	5.0169806386728908	$3.2410479855471408E-08$
110	10^{-8}	374	12.6204476899931777	$8.1198986146002738E-07$
	10^{-16}	458	12.6204484366120280	$6.5371011381820665E-08$
	10^{-32}	514	12.6204484348346639	$6.7148375748615763E-08$
120	10^{-8}	386	22.0665597106699458	$4.9093716314452251E-07$
	10^{-16}	454	22.0665599138313162	$2.8777579486050087E-07$
	10^{-32}	545	22.0665599109669692	$2.9064014250845815E-07$

Table C.43: European call option with single adaptive node distribution at the first time step with an interpolating error $\epsilon = 10^{-11}$ and with $K = 4$, $\sigma = 0.20$, $r = 0.08$, $T = 0.25$ and strike of $100. Here η is the total error for the tails. Double precision was used to calculate the values.

Asset Price ($)	η	Partitions Used	IPM	Error
80	10^{-8}	467	0.0690164011553810	$1.3321458051680085E-06$
	10^{-16}	657	0.0690164645751736	$1.2687260124975321E-06$
	10^{-32}	928	0.0690164148041142	$1.3184970719176334E-06$
90	10^{-8}	548	1.0254535519164236	$1.8221752006580827E-07$
	10^{-16}	631	1.0254537167652471	$1.7368696585140508E-08$
	10^{-32}	738	1.0254537179865473	$1.6147396440790995E-08$
100	10^{-8}	578	5.0169802295469976	$3.7671541372463757E-07$
	10^{-16}	725	5.0169806118828433	$5.6204318976682544E-09$
	10^{-32}	871	5.0169806141676014	$7.9051903911775412E-09$
110	10^{-8}	651	12.6204482528549295	$2.4912810980826094E-07$
	10^{-16}	797	12.6204484915003992	$1.0482639756226320E-08$
	10^{-32}	966	12.6204484932440018	$8.7390372716100728E-09$
120	10^{-8}	612	22.0665597225116343	$4.7909547440383449E-07$
	10^{-16}	801	22.0665601559853357	$4.5621773026027768E-08$
	10^{-32}	960	22.0665601581406321	$4.3466477639420020E-08$

Table C.44: European call option with single adaptive node distribution at the first time step with an interpolating error $\epsilon = 10^{-12}$ and with $K = 4$, $\sigma = 0.20$, $r = 0.08$, $T = 0.25$ and strike of $100. Here η is the total error for the tails. Double precision was used to calculate the values.

C.2 Barrier Option

C.2.1 Fixed Number of Partitions

The following are various down and out call option prices using the interpolation method. The first three tables C.45 - C.49 show the affects in changing the (fixed) number of partitions used at each time step for varying η values.

Asset Price ($)	Best η	IPM	Error
80	10^{-3}	0.0683055896929662	$8.8606713925206467E-04$
	10^{-4}	0.0687759760647901	$1.3564535110758853E-03$
	10^{-5}	0.0688423253431654	$1.4228027894512573E-03$
	10^{-6}	0.0688491774848888	$1.4296549311745843E-03$
	10^{-7}	0.0688475393165816	$1.4280167628674266E-03$
	10^{-8}	0.0688444210785239	$1.4248985248097425E-03$
	10^{-9}	0.0688407366243453	$1.4212140706310853E-03$
	10^{-10}	0.0688366018755622	$1.4170793218480464E-03$
	10^{-11}	0.0688320324871941	$1.4125099334799099E-03$
90	10^{-3}	1.0224401440197386	$3.0073095841248385E-03$
	10^{-4}	1.0250862935897256	$3.6116001413781518E-04$
	10^{-5}	1.0254112962308759	$3.6157372987517811E-05$
	10^{-6}	1.0254500127567925	$2.5591529289811787E-06$
	10^{-7}	1.0254549329916194	$7.4793877558390620E-06$
	10^{-8}	1.0254560401453998	$8.5865415362998410E-06$
	10^{-9}	1.0254567853829346	$9.3317790711769377E-06$
	10^{-10}	1.0254575565850177	$1.0102981154273827E-05$
	10^{-11}	1.0254583966701443	$1.0943066280737213E-05$
100	10^{-3}	5.0097261031306912	$7.2544882563176094E-03$
	10^{-4}	5.0161999919981266	$7.8059938888250247E-04$
	10^{-5}	5.0169230663740523	$5.7525012956555210E-05$
	10^{-6}	5.0170110377369452	$3.0446349936347206E-05$
	10^{-7}	5.0170311406572363	$5.0549270227268073E-05$
	10^{-8}	5.0170451456981162	$6.4554311107745121E-05$
	10^{-9}	5.0170596724315883	$7.9081044579432103E-05$
	10^{-10}	5.0170754061278133	$9.4814740804471942E-05$
	10^{-11}	5.0170924023187995	$1.1181093179032930E-04$
110	10^{-3}	12.6080060802346914	$1.2442421723016239E-02$
	10^{-4}	12.6191581209801011	$1.2903809776060315E-03$
	10^{-5}	12.6203128367279405	$1.3566522976682371E-04$
	10^{-6}	12.6204313002239772	$1.7201733729099722E-05$
	10^{-7}	12.6204426500682523	$5.8518894535675159E-06$
	10^{-8}	12.6204428872442822	$5.6147134247508390E-06$
	10^{-9}	12.6204419103890277	$6.5915686797701767E-06$
	10^{-10}	12.6204407360192867	$7.7659384201123061E-06$
	10^{-11}	12.6204394713437171	$9.0306139899354321E-06$
120	10^{-3}	22.0482436846082166	$1.8316516998859167E-02$
	10^{-4}	22.0647021565586066	$1.8580450484695366E-03$
	10^{-5}	22.0663540887229992	$2.0611288407967887E-04$
	10^{-6}	22.0665143847740346	$4.5816833041389948E-05$
	10^{-7}	22.0665241762180813	$3.6025388994165297E-05$
	10^{-8}	22.0665182612426989	$4.1940364378656447E-05$
	10^{-9}	22.0665102067487418	$4.9994858336566139E-05$
	10^{-10}	22.0665013888027026	$5.8812804373808980E-05$
	10^{-11}	22.0664919800444252	$6.8221562651848977E-05$

Table C.45: Interpolation method - Down and Out call option (asset value of \$100) for fixed number of node points ($N = 64$) and 8 time steps with $\sigma = 0.20$, $r = 0.08$, $T = 0.25$, strike of \$100 and barrier of \$75. The values are calculated in this table are performed in double precision.

Asset Price ($)	Best η	IPM	Error
80	10^{-3}	0.0683072619700805	$8.8773941636632056E-04$
	10^{-4}	0.0687790449978890	$1.3595224441748642E-03$
	10^{-5}	0.0688472118448349	$1.4276892911207128E-03$
	10^{-6}	0.0688562991647970	$1.4367766110828686E-03$
	10^{-7}	0.0688573103486822	$1.4377877949680500E-03$
	10^{-8}	0.0688572530953786	$1.4377305416644654E-03$
	10^{-9}	0.0688570389472511	$1.4375163935368709E-03$
	10^{-10}	0.0688567818923351	$1.4372593386209327E-03$
	10^{-11}	0.0688564958021260	$1.4369732484118144E-03$
90	10^{-3}	1.0224397192638275	$3.0077343400360113E-03$
	10^{-4}	1.0250855724302372	$3.6188117362636468E-04$
	10^{-5}	1.0254102072014823	$3.7246402381091448E-05$
	10^{-6}	1.0254484846409615	$1.0310370980592953E-06$
	10^{-7}	1.0254528951658006	$5.4415619371583901E-06$
	10^{-8}	1.0254534228871026	$5.9692832390587092E-06$
	10^{-9}	1.0254535187915284	$6.0651876647790925E-06$
	10^{-10}	1.0254535708590353	$6.1172551718557955E-06$
	10^{-11}	1.0254536219906023	$6.1683867387182789E-06$
100	10^{-3}	5.0097141673701895	$7.2664240168197891E-03$
	10^{-4}	5.0161809638127917	$7.9962757421753183E-04$
	10^{-5}	5.0168955688406109	$8.5022546397750531E-05$
	10^{-6}	5.0169737857093084	$6.8056777005609526E-06$
	10^{-7}	5.0169828478871548	$2.2565001457641731E-06$
	10^{-8}	5.0169845570965599	$3.9657095510237106E-06$
	10^{-9}	5.0169855581428315	$4.9667558225940933E-06$
	10^{-10}	5.0169865574496173	$5.9660626085256130E-06$
	10^{-11}	5.0169876286881063	$7.0373010974122963E-06$
110	10^{-3}	12.6080072437573278	$1.2441258200379668E-02$
	10^{-4}	12.6191598972215822	$1.2886047361243547E-03$
	10^{-5}	12.6203153152431966	$1.3318671450990305E-04$
	10^{-6}	12.6204345739939541	$1.3927963751769745E-05$
	10^{-7}	12.6204468065268198	$1.6954308873495805E-06$
	10^{-8}	12.6204480039080575	$4.9804964874500257E-07$
	10^{-9}	12.6204480608633283	$4.4109437769002113E-07$
	10^{-10}	12.6204479956123556	$5.0634535175841933E-07$
	10^{-11}	12.6204479125415716	$5.8941613534368997E-07$
120	10^{-3}	22.0482530542952233	$1.8307147311855121E-02$
	10^{-4}	22.0647162683832398	$1.8439332238356920E-03$
	10^{-5}	22.0663736286873480	$1.8657291972756784E-04$
	10^{-6}	22.0665398716459471	$2.0329961128440210E-05$
	10^{-7}	22.0665561888958024	$4.0127112743704174E-06$
	10^{-8}	22.0665573941933566	$2.8074137209399552E-06$
	10^{-9}	22.0665570491997904	$3.1524072852118223E-06$
	10^{-10}	22.0665565108357207	$3.6907713579781287E-06$
	10^{-11}	22.0665559208715365	$4.2807355410401371E-06$

Table C.46: Interpolation method - Down and Out call option (asset value of \$100) for fixed number of node points ($N = 128$) and 8 time steps with $\sigma = 0.20$, $r = 0.08$, $T = 0.25$, strike of \$100 and barrier of \$75. The values are calculated in this table are performed in double precision.

Asset Price ($)	Best η	IPM	Error
80	10^{-3}	0.0683073667812694	$8.8784422755521517E-04$
	10^{-4}	0.0687792371757303	$1.3597146220160937E-03$
	10^{-5}	0.0688475178867252	$1.4279953330110346E-03$
	10^{-6}	0.0688567448060473	$1.4372222523331376E-03$
	10^{-7}	0.0688579217104104	$1.4383991566962735E-03$
	10^{-8}	0.0688580562510999	$1.4385336973857677E-03$
	10^{-9}	0.0688580593096919	$1.4385367559777000E-03$
	10^{-10}	0.0688580452150260	$1.4385226613118358E-03$
	10^{-11}	0.0688580276916166	$1.4385051379024413E-03$
90	10^{-3}	1.0224396930333099	$3.0077605705535956E-03$
	10^{-4}	1.0250855277497526	$3.6192585411085054E-04$
	10^{-5}	1.0254101400634417	$3.7313540421878011E-05$
	10^{-6}	1.0254483906967153	$9.3709285192700165E-07$
	10^{-7}	1.0254527701057747	$5.3165019112816347E-06$
	10^{-8}	1.0254532626717492	$5.8090678857480871E-06$
	10^{-9}	1.0254533192929198	$5.8656890563474340E-06$
	10^{-10}	1.0254533280191160	$5.8744152525019855E-06$
	10^{-11}	1.0254533318310959	$5.8782272324003904E-06$
100	10^{-3}	5.0097134200483904	$7.2671713386186776E-03$
	10^{-4}	5.0161797719151666	$8.0081947184257318E-04$
	10^{-5}	5.0168938457101655	$8.6745676843025254E-05$
	10^{-6}	5.0169714501784908	$9.1412085179154445E-06$
	10^{-7}	5.0169798187731356	$7.7261387287919092E-07$
	10^{-8}	5.0169807549790546	$1.6359204546567696E-07$
	10^{-9}	5.0169809051381016	$3.1375109316167382E-07$
	10^{-10}	5.0169809768787967	$3.8549178779656579E-07$
	10^{-11}	5.0169810449513959	$4.5356438674315491E-07$
110	10^{-3}	12.6080073177338665	$1.2441184223840884E-02$
	10^{-4}	12.6191600107495923	$1.2884912081151390E-03$
	10^{-5}	12.6203154726816322	$1.3302927607539150E-04$
	10^{-6}	12.6204347840687063	$1.3717888999731365E-05$
	10^{-7}	12.6204470745996868	$1.4273580206669578E-06$
	10^{-8}	12.6204483356843866	$1.6627332011243112E-07$
	10^{-9}	12.6204484607604144	$4.1197292799388663E-08$
	10^{-10}	12.6204484686408858	$3.3316821523854401E-08$
	10^{-11}	12.6204484644797326	$3.7477973502397788E-08$
120	10^{-3}	22.0482536251306662	$1.8306576476409542E-02$
	10^{-4}	22.0647171354911116	$1.8430661159668515E-03$
	10^{-5}	22.0663748536137589	$1.8534799331726415E-04$
	10^{-6}	22.0665414615643449	$1.8740042731502093E-05$
	10^{-7}	22.0665581896313014	$2.0119757748737754E-06$
	10^{-8}	22.0665598451358989	$3.5647117757608981E-07$
	10^{-9}	22.0665599904518572	$2.1115521831038819E-07$
	10^{-10}	22.0665599668678283	$2.3473925025996323E-07$
	10^{-11}	22.0665599400823353	$2.6152474075225030E-07$

Table C.47: Interpolation method - Down and Out call option for fixed number of node points ($N = 256$) and 8 time steps with $\sigma = 0.20$, $r = 0.08$, $T = 0.25$, strike of $100 and barrier of $75. The values are calculated in this table are performed in double precision.

Asset Price ($)	Best η	IPM	Error
80	10^{-3}	0.0684750295337522	$1.0555069800379977E-03$
	10^{-4}	0.0687307337863579	$1.3112112326437342E-03$
	10^{-5}	0.0687640794551370	$1.3445569014227830E-03$
	10^{-6}	0.0687671383006468	$1.3476157469326211E-03$
	10^{-7}	0.0687660746061996	$1.3465520524854604E-03$
	10^{-8}	0.0687642758953696	$1.3447533416554615E-03$
	10^{-9}	0.0687621648461850	$1.3426422924708360E-03$
	10^{-10}	0.0687597936993950	$1.3402711456808429E-03$
	10^{-11}	0.0687571690260584	$1.3376464723441968E-03$
90	10^{-3}	1.0233568354873057	$2.0906181165576687E-03$
	10^{-4}	1.0252080882368706	$2.3936536699287886E-04$
	10^{-5}	1.0254255393210923	$2.1914282771241800E-05$
	10^{-6}	1.0254506693384327	$3.2157345690800310E-06$
	10^{-7}	1.0254537488210473	$6.2952171838229209E-06$
	10^{-8}	1.0254543806071188	$6.9270032553719885E-06$
	10^{-9}	1.0254547766813371	$7.3230774735866255E-06$
	10^{-10}	1.0254551827690328	$7.7291651693484065E-06$
	10^{-11}	1.0254556257670291	$8.1721631656206384E-06$
100	10^{-3}	5.0107865014472113	$6.1940899397976146E-03$
	10^{-4}	5.0163260928138591	$6.5449857314936466E-04$
	10^{-5}	5.0169232979320393	$5.7293454969881141E-05$
	10^{-6}	5.0169913171112990	$1.0725724289978311E-05$
	10^{-7}	5.0170037717753750	$2.3180388365739990E-05$
	10^{-8}	5.0170110222316202	$3.0430844611545327E-05$
	10^{-9}	5.0170183757780897	$3.7784391080514235E-05$
	10^{-10}	5.0170263831041142	$4.5791717104975982E-05$
	10^{-11}	5.0170350954736609	$5.4504086651613326E-05$
110	10^{-3}	12.6084866167184035	$1.1961885239303305E-02$
	10^{-4}	12.6192269843602123	$1.2215175974938930E-03$
	10^{-5}	12.6203231672093068	$1.2533474839926395E-04$
	10^{-6}	12.6204343701590691	$1.4131798636851656E-05$
	10^{-7}	12.6204452616290510	$3.2403286553961408E-06$
	10^{-8}	12.6204459132041311	$2.5887535753943425E-06$
	10^{-9}	12.6204454808093587	$3.0211483481590307E-06$
	10^{-10}	12.6204448953281236	$3.6066295838743656E-06$
	10^{-11}	12.6204442510343959	$4.2509233105780808E-06$
120	10^{-3}	22.0482012901957809	$1.8358911411295309E-02$
	10^{-4}	22.0647099550959105	$1.8502465111674971E-03$
	10^{-5}	22.0663672076409192	$1.9299396615823916E-04$
	10^{-6}	22.0665307799731067	$2.9421633970794225E-05$
	10^{-7}	22.0665443382339959	$1.5863373081792531E-05$
	10^{-8}	22.0665425671607061	$1.7634446372594148E-05$
	10^{-9}	22.0665389660413034	$2.1235565772492748E-05$
	10^{-10}	22.0665348926796412	$2.5308927434353201E-05$
	10^{-11}	22.0665304961700173	$2.9705437060156825E-05$

Table C.48: Interpolation method - Down and Out call option (asset value of \$100) for fixed number of node points ($N = 128$) and 16 time steps with $\sigma = 0.20$, $r = 0.08$, $T = 0.25$, strike of \$100 and barrier of \$75. The values are calculated in this table are performed in double precision.

Asset Price ($)	**Best η**	**IPM**	**Error**
80	10^{-3}	0.0684788368401993	$1.0593142864851065E-03$
	10^{-4}	0.0686338213621091	$1.2142988083949763E-03$
	10^{-5}	0.0686443300824745	$1.2248075287602835E-03$
	10^{-6}	0.0686351783551779	$1.2156558014637248E-03$
	10^{-7}	0.0686217445868527	$1.2022220331385611E-03$
	10^{-8}	0.0686057758141439	$1.1862532604296891E-03$
	10^{-9}	0.0685870452324950	$1.1675226787807771E-03$
	10^{-10}	0.0685689740662313	$1.1494515125171599E-03$
	10^{-11}	0.0685421262130522	$1.1226036593380375E-03$
90	10^{-3}	1.0238303171003096	$1.6171365035538843E-03$
	10^{-4}	1.0252744162133363	$1.7303739052731970E-04$
	10^{-5}	1.0254371765819830	$1.0277021880548964E-05$
	10^{-6}	1.0254568005622298	$9.3469583663241540E-06$
	10^{-7}	1.0254608988946279	$1.3445290764427242E-05$
	10^{-8}	1.0254669810358874	$1.9527432023984381E-05$
	10^{-9}	1.0254614192326361	$1.3965628772524830E-05$
	10^{-10}	1.0254837011014923	$3.6247497628774661E-05$
	10^{-11}	1.0254408154691956	$6.6381346678148767E-06$
100	10^{-3}	5.0113887800292716	$5.5918113577371753E-03$
	10^{-4}	5.0164559507026985	$5.2464068431082311E-04$
	10^{-5}	5.0170147735455517	$3.4182158542561680E-05$
	10^{-6}	5.0171082270027094	$1.2763561570081072E-04$
	10^{-7}	5.0171567070606340	$1.7611567362518787E-04$
	10^{-8}	5.0172102741930491	$2.2968280604038216E-04$
	10^{-9}	5.0172748855295328	$2.9429414252352126E-04$
	10^{-10}	5.0173205275582715	$3.3993617126262810E-04$
	10^{-11}	5.0173753581525844	$3.9476676557542723E-04$
110	10^{-3}	12.6087131035068634	$1.1735398450843371E-02$
	10^{-4}	12.6192547805435993	$1.1937214141083397E-03$
	10^{-5}	12.6203210075730823	$1.2749438462467122E-04$
	10^{-6}	12.6204265992730740	$2.1902684632646441E-05$
	10^{-7}	12.6204344136141948	$1.4088343511242662E-05$
	10^{-8}	12.6204246877480770	$2.3814209629269278E-05$
	10^{-9}	12.6204430800773935	$5.4218803141603544E-06$
	10^{-10}	12.6203934588103799	$5.5043147326183650E-05$
	10^{-11}	12.6204451597451470	$3.3422125594873009E-06$
120	10^{-3}	22.0481386543581515	$1.8421547248925751E-02$
	10^{-4}	22.0646746436143886	$1.8855579926896349E-03$
	10^{-5}	22.0663217006377472	$2.3850096932953591E-04$
	10^{-6}	22.0664688777809666	$9.1323826109657169E-05$
	10^{-7}	22.0664642714516432	$9.5930155433188169E-05$
	10^{-8}	22.0664378380520461	$1.2236355503292273E-04$
	10^{-9}	22.0664168518034600	$1.4334980361629945E-04$
	10^{-10}	22.0663714764300956	$1.8872517698054203E-04$
	10^{-11}	22.0663681987526772	$1.9200285439890941E-04$

Table C.49: Interpolation method - Down and Out call option (asset value of \$100) for fixed number of node points ($N = 128$) and 32 time steps with $\sigma = 0.20$, $r = 0.08$, $T = 0.25$, strike of \$100 and barrier of \$75. The values are calculated in this table are performed in double precision.

C.2.2 Fixed Spaced Partitions

Table C.50 are down and out call option prices for an asset value of \$100 and a barrier of \$75.

η	**Space (Partitions)**	**IPM**	**Error**
10^{-3}	10^{-1} (263)	5.0097725451313302	$7.2080462556786684E-03$
	10^{-2} (290)	5.0097522535224988	$7.2283378645100760E-03$
	10^{-3} (328)	5.0097376462119341	$7.2429451750746776E-03$
	10^{-4} (376)	5.0097276078056465	$7.2529835813621435E-03$
	10^{-5} (436)	5.0097210616502732	$7.2595297367354650E-03$
	10^{-6} (522)	5.0097170816403391	$7.2635097466695886E-03$
	10^{-7} (653)	5.0097148912581355	$7.2657001288736756E-03$
	10^{-8} (868)	5.0097138514636370	$7.2667399233714014E-03$
	10^{-9} (1302)	5.0097134653422977	$7.2671260447111641E-03$
	10^{-10} (2598)	5.0097133768095778	$7.2672145774312080E-03$
10^{-4}	10^{-1} (295)	5.0162385027547556	$7.4208863225302957E-04$
	10^{-2} (328)	5.0162183045148181	$7.6228687219057067E-04$
	10^{-3} (369)	5.0162038122478085	$7.7677913919993458E-04$
	10^{-4} (421)	5.0161938381544244	$7.8675323258448060E-04$
	10^{-5} (490)	5.0161873314695171	$7.9325991749132352E-04$
	10^{-6} (587)	5.0161833777435998	$7.9721364340881729E-04$
	10^{-7} (735)	5.0161812024217403	$7.9938896526826619E-04$
	10^{-8} (977)	5.0161801703202373	$8.0042106677113978E-04$
	10^{-9} (1465)	5.0161797868068314	$8.0080458017758871E-04$
	10^{-10} (2925)	5.0161796989618219	$8.0089242518732728E-04$
10^{-5}	10^{-1} (324)	5.0169525294902453	$2.8061896763936778E-05$
	10^{-2} (359)	5.0169323345752970	$4.8256811711888670E-05$
	10^{-3} (404)	5.0169178453012382	$6.2746085770226667E-05$
	10^{-4} (461)	5.0169078736315234	$7.2717755485174340E-05$
	10^{-5} (538)	5.0169013681475443	$7.9223239464465411E-05$
	10^{-6} (644)	5.0168974152741921	$8.3176112816718550E-05$
	10^{-7} (804)	5.0168952403875577	$8.5350999450700682E-05$
	10^{-8} (1072)	5.0168942084994228	$8.6382887586056167E-05$
	10^{-9} (1604)	5.0168938250672355	$8.6766319773445844E-05$
	10^{-10} (3205)	5.0168937372413733	$8.6854145635306690E-05$

Table C.50: Interpolation method - Down and Out call option (asset price of \$100) for fixed spaced node points and 8 time steps with $\sigma = 0.20$, $r = 0.08$, $T = 0.25$, strike of \$100 and barrier of \$75. The values are calculated in this table are performed in double precision.

C.2.3 Adaptive Node Allocation

The following tables are Down and Out option prices using the adaptive node allocation scheme.

Asset Price ($)	η	**Partitions Used**	**IPM**	**Error**
80	10^{-3}	93	0.0685075848117223	$5.1014848946381993E-04$
	10^{-4}	103	0.0686057314282196	$4.1200187296657484E-04$
	10^{-5}	106	0.0685984492775369	$4.1928402364919048E-04$
	10^{-6}	114	0.0686030816908914	$4.1465161029477157E-04$
	10^{-7}	120	0.0686093311489033	$4.0840215228278012E-04$
	10^{-8}	124	0.0686078655367144	$4.0986776447168296E-04$
	10^{-9}	130	0.0686078595662187	$4.0987373496747127E-04$
90	10^{-3}	108	1.0223730565834366	$3.0806775505071160E-03$
	10^{-4}	115	1.0250367575065114	$4.1697662743235553E-04$
	10^{-5}	115	1.1459312486363369	$1.2047751450239320E-01$
	10^{-6}	125	1.0253966828071532	$5.7051326790395762E-05$
	10^{-7}	133	1.0253705445670849	$8.3189566858765662E-05$
	10^{-8}	129	1.4628655386171763	$4.3741180448323252E-01$
	10^{-9}	142	1.0253695034687036	$8.4230665240055125E-05$
100	10^{-3}	114	5.0138820734270544	$3.0985328353563824E-03$
	10^{-4}	123	5.0163450488826369	$6.3555737977449844E-04$
	10^{-5}	128	5.0170592935254312	$7.8687263019988629E-05$
	10^{-6}	135	5.0171397171182806	$1.5911085586919893E-04$
	10^{-7}	141	5.0171511737617154	$1.7056749930452475E-04$
	10^{-8}	146	5.0171487064646545	$1.6810020224342082E-04$
	10^{-9}	150	5.0171528726403052	$1.7226637789441446E-04$
110	10^{-3}	119	12.6078589179667553	$1.2589584016284294E-02$
	10^{-4}	128	12.6189831392058114	$1.4653627772281963E-03$
	10^{-5}	133	12.6202608652958812	$1.8763668715726656E-04$
	10^{-6}	140	12.6202601728833805	$1.8832909965960098E-04$
	10^{-7}	147	12.6202859465386208	$1.6255544441901026E-04$
	10^{-8}	151	12.6202831221511076	$1.6537983193176675E-04$
	10^{-9}	157	12.6202858584042339	$1.6264357880491076E-04$
120	10^{-3}	120	22.0479818604872690	$1.8578341119840358E-02$
	10^{-4}	128	22.0653467635675966	$1.2134380395156796E-03$
	10^{-5}	137	22.0662530238758130	$3.0717773129851356E-04$
	10^{-6}	143	22.0662560718336991	$3.0412977340965597E-04$
	10^{-7}	148	22.0662726930162556	$2.8750859085435732E-04$
	10^{-8}	155	22.0662753321348966	$2.8486947221406655E-04$
	10^{-9}	162	22.0662809753278779	$2.7922627923171461E-04$

Table C.51: Interpolation Method - Down and Out call option for Adaptive node points and 8 time steps with $\epsilon = 10^{-7}$, $\sigma = 0.20$, $r = 0.08$, $T = 0.25$, strike of \$100 and barrier of \$75. The values are calculated in this table are performed in double precision.

Asset Price ($)	η	Partitions Used	IPM	Error
80	10^{-3}	155	0.0682681029425389	$7.4963035864719987E-04$
	10^{-4}	159	0.0688133107559223	$2.0442254526387453E-04$
	10^{-5}	169	0.0688105313096403	$2.0720199154581611E-04$
	10^{-6}	179	0.0688164761735872	$2.0125712759888831E-04$
	10^{-7}	191	0.0688180496788722	$1.9968362231388963E-04$
	10^{-8}	198	0.0688183518866204	$1.9938141456568624E-04$
	10^{-9}	205	0.0688177961746860	$1.9993712650015222E-04$
90	10^{-3}	173	1.0224227920094464	$3.0309421244972129E-03$
	10^{-4}	181	1.0250721271529495	$3.8160698099427848E-04$
	10^{-5}	178	1.1459237309539734	$1.2046999682002978E-01$
	10^{-6}	201	1.0254321387973033	$2.1595336640277407E-05$
	10^{-7}	210	1.0254389104161938	$1.4823717749828935E-05$
	10^{-8}	203	1.4629484595152973	$4.3749472538135348E-01$
	10^{-9}	224	1.0254385779112785	$1.5156222665270447E-05$
100	10^{-3}	187	5.0097391179938802	$7.2414882685313542E-03$
	10^{-4}	193	5.0162055297531118	$7.7507650929897753E-04$
	10^{-5}	204	5.0169182656975186	$6.2340564892238159E-05$
	10^{-6}	213	5.0169960512148535	$1.5444952441989734E-05$
	10^{-7}	223	5.0170043499404455	$2.3743678034121585E-05$
	10^{-8}	232	5.0170050607473788	$2.4454484967573187E-05$
	10^{-9}	238	5.0170060349250409	$2.5428662629722876E-05$
110	10^{-3}	192	12.6079839801466065	$1.2464521836433318E-02$
	10^{-4}	201	12.6196794430449231	$7.6905893811596293E-04$
	10^{-5}	211	12.6202898843292282	$1.5861765381153248E-04$
	10^{-6}	221	12.6204201860852265	$2.8315897812603019E-05$
	10^{-7}	231	12.6204237385471298	$2.4763435910180043E-05$
	10^{-8}	241	12.6204222389087377	$2.6263074301424716E-05$
	10^{-9}	248	12.6204216696886338	$2.6832294405809698E-05$
120	10^{-3}	193	22.0482087559703572	$1.8351445636755059E-02$
	10^{-4}	202	22.0646722191764582	$1.8879824306511361E-03$
	10^{-5}	216	22.0664604134588203	$9.9788148288926237E-05$
	10^{-6}	226	22.0665014574836249	$5.8744123485054978E-05$
	10^{-7}	238	22.0665131638000638	$4.7037807046623747E-05$
	10^{-8}	246	22.0665146076756500	$4.5593931460863324E-05$
	10^{-9}	255	22.0665149761169381	$4.5225490172273730E-05$

Table C.52: Interpolation Method - Down and Out call option for Adaptive node points and 8 time steps with $\epsilon = 10^{-8}$, $\sigma = 0.20$, $r = 0.08$, $T = 0.25$, strike of \$100 and barrier of \$75. The values are calculated in this table are performed in double precision.

Asset Price ($)	η	**Partitions Used**	**IPM**	**Error**
80	10^{-3}	243	0.0683012678226094	$7.1646547857673153E-04$
	10^{-4}	256	0.0687883682913956	$2.2936500979055209E-04$
	10^{-5}	269	0.0688413317325429	$1.7640156864325322E-04$
	10^{-6}	284	0.0688505856898115	$1.6714761137461573E-04$
	10^{-7}	299	0.0688517112947402	$1.6602200644597184E-04$
	10^{-8}	312	0.0688519157980230	$1.6581750316314176E-04$
	10^{-9}	325	0.0688519182439113	$1.6581505727484734E-04$
90	10^{-3}	283	1.0224378064467050	$3.0159276872386218E-03$
	10^{-4}	285	1.0250834161862925	$3.7031794765125109E-04$
	10^{-5}	280	1.1458934530507712	$1.2043971891682742E-01$
	10^{-6}	317	1.0254485095276962	$5.2246062474600730E-06$
	10^{-7}	330	1.0254510932790835	$2.6408548601392079E-06$
	10^{-8}	319	1.4629617358079796	$4.3750800167403592E-01$
	10^{-9}	354	1.0254514797949690	$2.2543389746956799E-06$
100	10^{-3}	304	5.0097173385506721	$7.2632677117392963E-03$
	10^{-4}	305	5.0164384356775296	$5.4217058488154057E-04$
	10^{-5}	321	5.0168976984638567	$8.2907798554243683E-05$
	10^{-6}	336	5.0169754244275229	$5.1818348882626264E-06$
	10^{-7}	350	5.0169837275339875	$3.1212715759232346E-06$
	10^{-8}	364	5.0169845201876893	$3.9139252783204626E-06$
	10^{-9}	378	5.0169845632036898	$3.9569412788043046E-06$
110	10^{-3}	316	12.6080034995456867	$1.2445002437352914E-02$
	10^{-4}	324	12.6191561968738597	$1.2923051091795479E-03$
	10^{-5}	333	12.6203114432710670	$1.3705871197144948E-04$
	10^{-6}	346	12.6204357780953451	$1.2723887693466551E-05$
	10^{-7}	366	12.6204440064880323	$4.4954950068065713E-06$
	10^{-8}	379	12.6204444400538023	$4.0619292374088900E-06$
	10^{-9}	393	12.6204447061330356	$3.7958500037627019E-06$
120	10^{-3}	320	22.0482466301654547	$1.8313571441655463E-02$
	10^{-4}	331	22.0647099954582728	$1.8502061488373522E-03$
	10^{-5}	338	22.0664406160850355	$1.1958552207647077E-04$
	10^{-6}	356	22.0665343491479859	$2.5852459126229910E-05$
	10^{-7}	372	22.0665523753337922	$7.8262733183009914E-06$
	10^{-8}	390	22.0665529843572266	$7.2172498838396493E-06$
	10^{-9}	403	22.0665529845075810	$7.2170995291109818E-06$

Table C.53: Interpolation Method - Down and Out call option for Adaptive node points and 8 time steps with $\epsilon = 10^{-9}$, $\sigma = 0.20$, $r = 0.08$, $T = 0.25$, strike of $100 and barrier of $75. The values are calculated in this table are performed in double precision.

Asset Price ($)	η	Partitions Used	IPM	Error
80	10^{-3}	386	0.0683063772075618	$7.1135609362429821E-04$
	10^{-4}	409	0.0687782529016035	$2.3948039958263676E-04$
	10^{-5}	430	0.0688465414940495	$1.7119180713660517E-04$
	10^{-6}	451	0.0688557767585354	$1.6195654265074941E-04$
	10^{-7}	472	0.0688569633657301	$1.6076993545608483E-04$
	10^{-8}	493	0.0688571136846980	$1.6061961648816724E-04$
	10^{-9}	511	0.0688571300196840	$1.6060328150216668E-04$
90	10^{-3}	447	1.0224393313096838	$3.0144028242598542E-03$
	10^{-4}	461	1.0250851667585097	$3.6856737543403129E-04$
	10^{-5}	452	1.1459434441326730	$1.2048970999872927E-01$
	10^{-6}	500	1.0254496931976311	$4.0409363126492348E-06$
	10^{-7}	523	1.0254527260822395	$1.0080517042626580E-06$
	10^{-8}	502	1.4629638180944728	$4.3751008396052915E-01$
	10^{-9}	562	1.0254529641116374	$7.7002230633449464E-07$
100	10^{-3}	474	5.0097140215873459	$7.2665846750655239E-03$
	10^{-4}	494	5.0161803320614933	$8.0027420091763335E-04$
	10^{-5}	510	5.0169316216436100	$4.8984618801484192E-05$
	10^{-6}	531	5.0169719219845765	$8.6842778343476645E-06$
	10^{-7}	553	5.0169802502349148	$3.5602749637320130E-07$
	10^{-8}	570	5.0169811870119183	$5.8074950681774595E-07$
	10^{-9}	596	5.0169812240959644	$6.1783355306821441E-07$
110	10^{-3}	492	12.6080067369330528	$1.2441765049986397E-02$
	10^{-4}	515	12.6191594245831684	$1.2890773998713856E-03$
	10^{-5}	535	12.6203148781615173	$1.3362382152160013E-04$
	10^{-6}	551	12.6204437312712390	$4.7707118006057314E-06$
	10^{-7}	576	12.6204465123836265	$1.9895994129992545E-06$
	10^{-8}	603	12.6204478464727288	$6.5551030992150316E-07$
	10^{-9}	618	12.6204479207741027	$5.8120893653423877E-07$
120	10^{-3}	487	22.0482525571904660	$1.8307644416643010E-02$
	10^{-4}	516	22.0647160840671859	$1.8441175399230847E-03$
	10^{-5}	544	22.0663738108224941	$1.8639078461779590E-04$
	10^{-6}	566	22.0665404520127701	$1.9749594340479071E-05$
	10^{-7}	592	22.0665585954243753	$1.6061827340374535E-06$
	10^{-8}	614	22.0665588752758559	$1.3263312531597293E-06$
	10^{-9}	644	22.0665590692659244	$1.1323411847063980E-06$

Table C.54: Interpolation Method - Down and Out call option for Adaptive node points and 8 time steps with $\epsilon = 10^{-10}$, $\sigma = 0.20$, $r = 0.08$, $T = 0.25$, strike of \$100 and barrier of \$75. The values are calculated in this table are performed in double precision.