STATISTICAL PROCESS CONTROL WITH SPECIAL REFERENCE TO MULTIVARIABLE PROCESSES AND SHORT RUNS

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DECLARATION

I hereby declare that :

(i) the following thesis contains only my original work which has not been submitted previously, in whole or in part, in respect of any other academic award, and

(ii) due acknowledgment has been made in the text of the thesis to all other material used.

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ABSTRACT

The quest for control and the subsequent pursuit of continuous quality improvement in the manufacturing sector, due to increasingly keen competition, has stimulated interest in statistical process control (SPC). Whilst traditional SPC techniques are well suited to the mass production industries, their usefulness in short run or low volume manufacturing environments is questionable. The major problem with short-run SPC is lack of data for estimation of the control parameters. In view of this limitation, many alternatives and adaptations of existing techniques have been devised. However, these efforts have largely been devoted to monitoring and controlling univariate processes.

In practice, the quality of manufactured products is often determined by reference to several quality characteristics which are correlated. Under these circumstances, it is necessary to use multivariate quality control procedures which take the correlational structure into consideration. Although this area has received considerable attention in the literature, most of the published work assumes that prior information about the process parameters is available. This assumption is rarely the case in the short run environment.

This thesis is primarily concerned with the development of multivariate quality control procedures that can be effectively used in situations where prior estimates of the process parameters are unavailable. For completeness, some better alternatives to previously proposed procedures are also provided for the case where the process parameters are assumed known in advance of production. These techniques are intended for detecting a shift in the mean vector, the variance-covariance matrix and other process disturbances. Using the proposed procedures, control can be initiated early in

production, whether or or not prior information about the process parameters is available.

The techniques presented for controlling the mean vector of multivariate processes utilize the probability integral transformation technique in order to produce sequences of independent or approximately independent standard normal variables. This offers greater flexibility than the 2-stage procedures recommended by some authors in the design of control charts for the unknown parameter case. Apart from the conventional rule that signals when a plotted value exceeds either of the 3-sigma limits, run tests as well as the methods of Cumulative Sum (CUSUM) and Exponentially Weighted Moving Average (EWMA) can be used. A simulation study indicates that the techniques, with the usual decision rule imposed, are particularly useful for 'picking up' a persistent change in the process mean vector when subgroup data are used, even if prior information about the process parameters is not available. For detecting step shifts and linear trends based on individual observations, two specifically designed EWMA charts based on similarly transformed variables but which use a different estimator of the process variance-covariance matrix are found to be much more effective than other competing procedures.

For dispersion control, use is made of the independent statistics that result from the decomposition of variance-covariance matrices and the modified likelihood ratio statistic for testing the equality of several covariance matrices, for the cases with known and unknown dispersion parameters respectively. The proposed techniques are based on some aggregate-type indices computed from such independent variables. It is found that these techniques outperform previously proposed procedures for many sustained shifts in the process variance-covariance matrix. In addition, it is demonstrated that the dispersion

control chart, for the known parameters case, is more sensitive to certain shifts than that which involves separate charting of the standardized variances of the principal components or the individual variables resulting from the partitioning of the variance-covariance matrices. The proposed techniques also possess some practical advantages over existing procedures. In particular, better control over the false signal rate, ease of locating control limits and identification of the nature of process changes.

To satisfactorily describe the capability of multivariate processes, a multivariate capability index is required. This thesis describes three approaches to designing capability indices for multivariate normal processes. Three process capability indices are presented and some simple rules provided for interpreting the ranges of values they take. The development of one index involves the projection of a process ellipse, containing at least a specified proportion of products, on to its component axes. The other two are based on Bonferroni and Sidak's multivariate normal probability inequalities in their constructions. A comparison indicates that the latter two are superior to the former and that the Sidak-type capability index is marginally better than that based on the Bonferroni Inequality. An approximate test is developed for the Sidak-type capability index. A possible method of forming robust multivariate capability indices based on multivariate Chebyshev-type inequalities is also considered.

In addition, whilst not multivariate, a statistical comparison is made between the adjusted $\overline{X}$ and R charting technique and the method of 'pre-control'. These techniques are suitable for application in the short run environment since they do not require accumulation of process data for calculation of the control limits but instead determine their *pseudo* limits based on given specifications. The results of comparison show that the former are superior in many circumstances.

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CHAPTER 1
INTRODUCTION

1.1 Introduction to SPC

Dr. Walter Shewhart (1931) introduced the notion of statistical process control (SPC), and in particular control charts, as a means of monitoring industrial processes and controlling the quality of manufactured products. These and other statistical tools have proven useful in many industries.

Regardless of the nature and the state of an industrial process, any measurable characteristics of the process or the manufactured product exhibit a certain amount of variability. In SPC, a distinction is often made between two types of variability : one due to *common* causes and the other that results from *special* or *assignable* causes. Examples of common causes are machining operations, setting-up methods, measurement systems and atmospheric conditions. Variability due to these factors is either non-controllable or cannot be reduced or eliminated economically. On the other hand, special causes include, amongst others, machine failure, tool wear, defective material and operator error, which are preventable or at least correctable or controllable. The primary objective of SPC is to help detect the presence of these extraneous sources of variability so that timely corrective actions can be taken. In this manner, it is hoped that the production process will be capable of meeting given product specifications consistently and economically.

Once the common-cause or inherent variability has been quantified, control charts can be used to determine when and whether or not special causes affecting the process under consideration are present. A process is said to be in a state of *statistical control* or

simply *in control* if it is free from the influence of such factors. Otherwise, it is said to be *out of control*. In statistical terms, this means an in-control *process variable* has a constant distribution. In applications, however, it is generally considered sufficient for the *process parameters* such as the mean, μ and the standard deviation, σ to remain constant.

A control chart portrays the values of some chosen statistic in chronological order and provides graphical evidence of when and whether or not process troubles occur. As shown in Figure 1.1, a typical control chart is a plot of a *control statistic* against the sample number and consists of a center line (CL), a lower control limit (LCL) and an upper control limit (UCL).

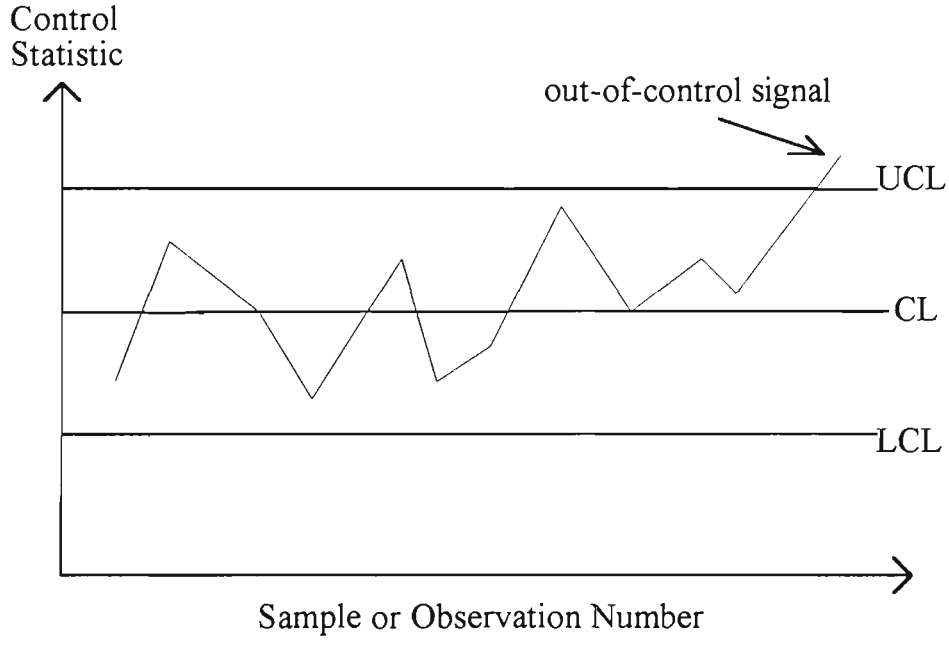


Figure 1.1. A typical control chart

Common examples of a control statistic are the individual observations of a quality characteristic (X), sample mean ($\bar{X}$), sample range (R), sample standard deviation (S), sample proportion of defective items (p) and sample fraction of nonconformities (C). The

control limits, which depend on the process or control parameters, act as the thresholds for the plotted values beyond which out of control conditions are indicated. These limits may or may not be symmetric and it is also possible that only a single control limit is used under certain circumstances. In any event, the control limits are usually determined such that the resulting *false signal* rate can be tolerated. A false signal occurs when the value of the control statistic plots outside the control limits whilst in fact the process under consideration is in control.

In practice, the control parameters are estimated based on data collected from a process assumed to have been in control. Certain rules of thumb or practical experience are often employed to determine the necessary amount of calibration data. The commonly recommended approach is then to 'plug in' these sample estimates to the formulae for the control limits. Thus, unless the set of calibration data is reasonably large, there is often no assurance that the resulting estimated control limits will behave essentially like the known limits (see Quesenberry (1993)).

The effectiveness of a control procedure is often determined based on its associated *run length* (*RL*) distribution. RL refers to the number of samples taken (since the last signal) before a signal is triggered. As a summary measure of sensitivity, the average number of samples required for the detection of a change in the control parameters, termed the *average run length* (*ARL*), has commonly been used. Using a simplified ecomonic model, Ghost, Reynolds and Hui (1981) showed that ARL is the most important measure associated with the RL distribution that determines the effectiveness of a control procedure. They also noted that other measures may be of interest for some applications. Quesenberry (1995d) demonstrated that for comparison of various competing procedures, if RL distributions are known to be geometric before and

after a shift in the process parameters, ARL is an appropriate performance criterion because the run length distribution function, $\Pr\{RL \leq k\}$ at any fixed value k is a strictly monotonic function of ARL irrespective of shift size. On the other hand, if some of the RL distributions are not geometric or unknown, even combining information on the standard deviation of run length (*SDRL*) with ARL, as considered by some authors, may lead to misleading conclusions about their relative performance. This issue has either been dismissed or overlooked by numerous authors.

The idea of SPC can be extended to a more practical situation in which several correlated quality characteristics are monitored simultaneously. Under these circumstances, procedures which take into account the correlational structure of the individual variables are required in order to correctly reflect the process status.

1.2 Problems of Traditional SPC for Short Production Runs

Shewhart control charts have enjoyed considerable popularity as control and monitoring tools. These and more recent variations of them, enable production operators to detect process troubles or out-of-control situations before they become critical. Appropriate corrective action can then be initiated to prevent further deterioration in process operation and so avoid a negative impact on product quality. While these techniques are well suited to the mass production industries, their usefulness in low volume manufacturing environments is subject to debate.

Besides the continuance of traditional small job shops, there has been, even in the mass production industries, an increased demand for more frequent production changes and a consequential proliferation of short runs. Attempts made to apply traditional control charting techniques in such environments are plagued with difficulties. The

essential problems facing those seeking to provide useful statistical tools for application in the short production run environment are those of machine 'warm up', control parameter estimation and parameter changes between the manufacture of different product types.

In short-run environments, the control limits for traditional control charts, such as $\overline{X}$ and R charts, often cannot be located in the usual manner due to insufficient data. Thus, one might consider estimating the control limits based on a much smaller number of samples or subgroups than usually recommended. However, it has been adequately demonstrated in the literature that this practice is not reliable. Amongst others, Quesenberry (1993) provided a most reasonable evaluation of the effect of estimated control limits on the overall run length performance for conventional $\overline{X}$ and X charts. He showed by means of simulation, that the rate of false alarms after short runs, increases, and much larger sets of calibration data are required so that the resulting estimates of the control limits for these charts are practically the same as their *true* values. However, these requirements can rarely be met for small batch manufacturing.

The problems of lack of process performance data are further aggravated by process 'warm up', which is perhaps the most important and yet least considered obstacle to meaningful and successful application of traditional control charts in small lot production. This phenomenon is a common and dominant feature of short-run processes, as instability after set-up or reset often constitutes a large proportion of production run time. Neglecting this fact and using sample data from such a period to obtain control limits will often lead to erroneous conclusions regarding past, current and future states of the process. Murray and Oakland (1988) demonstrated this using simulation, specifically, if process variability increases during the calibration period, an out-of-control process

will often appear to be in control, as reflected by either a standard deviation or range chart with the usual decision rules imposed.

Another practical reality that characterizes short run environments, is the diversity of products made. If separate control charts are maintained for each type of product, the system becomes unwieldy.

1.3 Multivariate Quality Control

Due to the rapid development of data-acquisition and computer technology, it is not uncommon in many industrial situations to monitor on-going performance of a manufacturing process with respect to more than one process or product characteristics. If the product characteristics are correlated and no consideration is made of their joint distribution, the use of separate control charts for each of them can be misleading. Specifically, the related variables, when studied separately, may appear to be in statistical control but appear out of control when considered in a multivariate context. Montgomery and Wadsworth (1972) and Alt and Smith (1990) illustrated this for the case of $p = 2$ variables. Under these circumstances, it seems necessary to consider the use of multivariate quality control procedures which take into account the covariance structure of the quality characteristics.

Over the last 2 decades, the problem of multivariate quality control has received considerable attention in the literature. A review of this work can be found in Alt et al.(1990) and Jackson (1985) and this has been updated recently by Wierda (1994) and Lowry and Montgomery (1995). Besides the problem of lack of data for parameter estimation in low volume and short-run environments, multivariate SPC procedures suffer from practical drawbacks such as computational complexities and difficulties in

interpreting the out-of-control signals. Other issues include use and understanding, complexity of the distribution theory involved and statistical efficiency. Some previously proposed procedures such as the dispersion control technique based on generalized variance of Alt et al.(1990) are fundamentally flawed because they are unlikely to 'pick up' certain shifts in the dispersion parameters. The computational aspect of multivariate control procedures can be handled by computers but other problems remain and need further investigation.

1.4 Thesis Objectives

The main thrust of this thesis is to present some multivariate quality control procedures that can be effectively used in situations where prior estimates of the process parameters are unavailable. For completeness, some better alternatives to existing multivariate SPC procedures are also provided for known parameter situations. The types of process change considered include step shift in both the process mean vector and the variance-covariance matrix and linear trend. Some simulations, as well as previously published data, are used to assess the practicality of the models and the methods. Where appropriate, the relative control performance of proposed and currently existing procedures are also evaluated.

Statistical process control is often considered with no reference to product specifications. It merely ensures that the process under focus is in a state of statistical control so that its behaviour is predictable under normal operating conditions. In practice, however, the quality and the 'acceptability' of the products is determined by conformance to given specifications. Having confirmed that the process is in control, the next step is thus to measure the process capability. Process capability can often be

conveniently summarized by an index. Whilst substantial efforts have been devoted to the development of capability indices for univariate situations in the literature, the work on multivariate process capability study is at a relatively rudimentary stage. In order to satisfactorily describe the capability of multivariate processes, some indices are therefore presented.

In addition to the above, the relative merits of the *adjusted* $\overline{X}$ *and* R charting technique and *Pre-control* are provided for the univariate situation. These tools are attractive for application in short-run environments since they do not require accumulation of process data for computation of the control limits but instead determines *pseudo* limits by reference to given specifications.

CHAPTER 2
LITERATURE REVIEW

2.1 Introduction

In view of the limitations of traditional SPC techniques, many alternatives and adaptations of them have been devised for the express purpose of removing the barriers between SPC and short production runs. Recently, a review of the literature on the use of SPC in batch production has been presented by Al-Salti and Statham (1994). Much of their article is devoted to a particular aspect of short-run SPC, namely, the use of data transformation techniques. This chapter, however, gives a more comprehensive review of techniques proposed for short runs along with some possible methods and provides its own contribution to the debate. Whilst the focus is on univariate processes, wherever appropriate consideration is also given to the multivariate environment for which little has appeared in the literature.

It should be pointed out that process monitoring in the chemical industries (eg. Nomikos and MacGregor (1995), Doganaksoy, Schmee and Vandeven (1996)) are, in most cases, not applicable to short-run SPC which assumes small production volume of the monitored product. In chemical or continuous process industries, various products or grades of the same product are manufactured in lots or batches and production of the same product continues, although intermittently, followed by batches of other products or grades and so forth. Thus, adequate historical data from past succesful batches are usually available for calibrating the in-control behaviour of the quality/product variables which are usually measured once at the end of each batch run and the *process* variables which are monitored during each batch run.

For ease of presentation, the techniques to be reviewed are grouped into several general approaches and discussed under appropriate headings although there is, of course, inevitable overlap between the groups.

2.2 Adjusting Control Limits Based On The Number of Subgroups or Observations

In the event of only a small number of subgroups being available yet where early control of the process is still desirable, Hillier (1964,1967,1969) and Yang and Hillier (1970) proposed adjusting the control limits for Shewhart-type variable control charts, both for retrospective (stage I) testing and for future (stage II) control in such a way that the predetermined probability of a type I error is preserved. A similar approach, based on individual observations, has also been presented by Roes, Does and Schurink (1993). It should be pointed out, however, that the limits so adjusted do *not* always ensure that the resulting probability of a type I error for each future subgroup is as desired. In fact, this probability, as well as the adjusted limits, vary stochastically with the mean and the dispersion of the calibration sample. As such, the resulting control procedure may either be too conservative, causing delay in reacting to process troubles, or too stringent, giving rise to too many false alarms. Note also that, as opposed to setting the limits by the conventional method, even if the estimates of the process parameters are accurate, the resulting control limits are wider than necessary ! Furthermore, since the estimated control limits may not be close to the nominal values, additional run rules such as those discussed by Nelson (1984) cannot be used indiscriminately. Other drawbacks are the number of calculations required and the likelihood of misinterpreting the information contained in the control charts, arising from the use of different control limits for stage I

and stage II, as well as changing control limits after every couple of subgroups (Ermer and Born (1989)).

A generalization of this approach to the control of the mean vector of a multivariate normal process, based on the well known Hotelling T^2 statistic, was presented by Alt, Goode and Wadsworth (1976) and Tracy, Young and Mason (1992) for the cases when subgroup data and individual measurements are used respectively. Some issues of importance regarding the use of such a composite measure to monitor the stability of multivariate processes were raised, for example, in Hawkins (1991,1993).

More recently, Scholz and Tosch (1994) proposed updating the parameter estimates and the stage II control limits for the individual values charts based on the *student-t* statistic and T^2 charts respectively after *every* future observation. In order to reduce the effect of any systematic process behaviour on the estimates of the dispersion parameters, they suggested using a moving variance of successive observations for the individual values chart and an analogous procedure for estimating the in-control process variance-covariance matrix, Σ that is required for computing the value of the T^2 statistic. The latter procedure uses vector differences of successive observations to form the estimator of Σ as follows :-

$$\tilde{\mathbf{S}}_n = \frac{1}{2(n-1)} \sum_{i=1}^{n-1} (\mathbf{X}_{i+1} - \mathbf{X}_i)(\mathbf{X}_{i+1} - \mathbf{X}_i)^{\mathrm{T}}$$

where $\mathbf{X}_i$ and n denote respectively the ith observation vector and the number of previous observation vectors. This estimator was considered by Holmes and Mergen (1993) and compared to other estimators including the usual sample covariance matrix by Sullivan and Woodall (1995) for application in the retrospective stage. It was

demonstrated that, step shifts in the mean vector and linear trends are more likely to be detected with the use of this estimator.

King (1954) has also presented a similar approach for analysis of past data or retrospective testing using $\overline{X}$ charts with control limits based on average subgroup range except that the control chart factors are derived in such a way that the joint probability of a false alarm, γ is as required instead of the individual probability of a false signal for each initial subgroup. He gave a nomogram for a selected range of common subgroup sizes and numbers of subgroups from which an appropriate value of the control chart factor can be obtained for the case where $\gamma = 0.05$. Except when limits are constructed based on 3 or 4 subgroups, the given factors were obtained through simulation by ignoring the random fluctuations of the average subgroup range $\overline{R}$. As such, the validity of the given factors is questionable. Furthermore, this method can only be useful if nomograms or tables for other values of γ, which might be preferable in practice, are widely available.

2.3 Control Charts Based On Individual Measurements

If the problem is one of short production runs and lack of data, a possible solution might be use of individual readings in place of averages. The use of individual readings is natural anyway if there is no natural subgrouping of the observed data. Two basic types of chart employing individual measurements are:

(i) Individual values and Moving Range (*I-MR*) charts or *X-MR* charts.

(ii) Target Individual-Moving Range (*Target I-MR*) charts or the ΔX-*MR* charts.

These charts attempt to maximise the information obtained from the limited amount of available data. Apart from being suitable for processes with limited output within a single set-up, they can be used in situations where :

- processing time per unit item is long or the data accumulation rate is slow,
- testing or measurement is expensive or time consuming,
- testing is destructive.

The first of these two charting methods has been around for many years and is well documented (see for eg., Grant and Leavenworth (1980)). Burr (1954) suggested this as one of the possible methods which can cater for short production runs. In his proposal, he advocates use of 2-σ control limits rather than the conventional 3-σ limits for the individual values chart to compensate for its lack of sensitivity to mean shifts. The method has also been considered by Nugent (1990) for use in short-run manufacturing environments.

The second, as its name implies, differs slightly from the first in that the target or the nominal specification is subtracted from the measurements before they are plotted. Ermer et al.(1989) highlighted some practical merits of this charting method in comparison to other existing 'short-run' techniques. However, this method will not work in circumstances where no target is available as often occurs with products having a one-sided specification.

The fundamental problem of lack of data has not been adequately addressed by the above authors, except for the indication that estimation of the process parameters and subsequent computation of the control limits should be based on data from either the present run or on previous runs of identical or similar products. In fact, no clear

guidelines are provided as to how many observations are sufficient for the purpose of estimating the control limits.

For both types of chart, the control limits may be determined based on successive moving ranges of size 2. Several other estimators of process spread such as that given by Roes et al.(1993) can also be used for this purpose. Although it is well known that the sample standard deviation provides the most efficient estimate of the inherent process variability for a stable normal process, the average moving range is used because it is not only computationally simpler but it can also safeguard against the likely events of trends, cycles or other irregular patterns in the calibration data (i.e it minimizes the inflationary effects from these conditions and hence the inherent process variability will not be overestimated). However, besides re-emphasizing the fact that displaying a moving range chart will only cause confusion due to correlation between consecutive moving ranges, Roes et al.(1993) substantiated Nelson's (1982) view that doing so has no real added value because the chart of individual values contains almost all the information available. If the process measurements are known to be independently normally distributed and a state of statistical control has been achieved, as demonstrated by the retrospective use of *I-MR* or *Target I-MR* charts, Cryer and Ryan (1990) suggested that, for future process monitoring, control limits for individual values charts should be estimated based on the sample standard deviation instead of the average moving range, due to its relatively superior efficiency.

As for $\overline{X}$ charts, additional run rules can be effectively applied to the individual values charts to identify non-random variations or systematic process changes, hence providing better protection against potential process problems, provided the estimated control limits do not differ considerably from the 'true' limits. Of course, there is an

increased false alarm rate associated with these additional control rules but if power of the charts is of paramount importance and outweighs the costs of searching needlessly for non-existing assignable causes, the use of these rules can be useful.

The major practical benefits that can be gained by using individual readings instead of averages are :

- Measurements can be seen, compared to specification limits and easily understood.
- Substantial savings in time and cost may be accrued as a result of less sampling, testing or measurement.
- Improved employee involvement in decision making and problem solving which could be catalytic in bringing about quality and productivity improvements, as a result of operators having better appreciation of the techniques in use.

Individual values charts with conventional control limits should, however, be considered with reference to statistical efficiency. First, and most importantly, their sensitivity to substantial shifts in process average is less than that of the usual $\overline{X}$ charts. Although greater sensitivity may be gained by the use of narrower limits or additional warning lines, such sensitivity is gained at the expense of increasing the chance of false lack of control indications.

The second problem with individual values charts centres around the normality assumption of the process distribution. If the underlying distribution is not normal, this will tend to distort the interpretation of the control limits. On the other hand, $\overline{X}$ values will tend to normality fairly rapidly by virtue of the Central Limit Theorem, provided the underlying distribution isn't too skewed.

2.4 Mixing Production Lots and Normalizing Process Output Data

Recent developments in the use of statistical process control in multi-component and low-volume manufacturing environments have focussed mainly on studying and monitoring the *process* irrespective of the type of parts or products being manufactured. The basic idea with this approach is that values for different products or components being assessed on the basis of the same quality characteristic but with different design specifications can be plotted together on the same chart, provided that they are the output of a *homogeneous* process. Homogeneity means that the components should, technically, be machined under similar conditions, for example, in terms of cutting tools, tool holders, component holding methods and setting-up methods etc (Al-Salti and Aspinwall (1991)). The same principle applies to chemical manufacturing processes where similar chemicals are produced in small batches on an, 'as needed' basis. Emphasis on process homogeneity is important in order to reduce or eliminate the effect of variability due to extraneous sources, thus ensuring only inherent variation exists. This avoids erroneous appraisal of the process and any process irregularities can be more readily detected. In addition, the measurement process should be adequate enough, in terms of accuracy, and carried out in a consistent manner for every measured component.

Several papers (Al-Salti et al.(1991), Armitage and Wilharm (1988), Bothe (1989), Burr (1989), Crichton (1988), Nugent (1990), Thompson (1989)) have given accounts of this approach. It is accomplished by means of data transformations which effectively eliminate the differences between the types of products or components. A list of possible transformation techniques that cover a claimed majority of manufacturing situations, along with conditions of use, is presented by Al-Salti, Aspinwall and Statham

(1992). A comparison of control charts based on the given techniques for a set of industrial data is also provided in the same paper. Koons and Luner (1991) outlined a related approach and illustrated the technique with a case study. Al-Salti et al.(1991) investigated the appropriateness of the moving average, moving range and cusum control charting techniques using transformed individual observations.

This approach of mixing production lots and normalizing process output data has been highlighted by many authors as an integral part of quality assurance and improvement strategies for two main reasons :

(a) Reduction of the number of control charts and charting effort required which results in time and cost savings as well as higher productivity.

(b) Time-related process changes such as runs, trends and cycles can be more readily detected since pertinent process data is not scattered over separate charts.

However, there are a number of critically important problems with this approach. As an example of some important issues that the previous authors either overlooked or dismissed, consider the so-called *Short Run* $\overline{X}$ & R charts as proposed by Bothe (1989,1990b), that have been extensively discussed in the literature. To 'control' the process mean and the process dispersion, Bothe suggested use of the statistics

$$\overline{\text{X}}\text{ PLOT POINT} = \overline{X}_{\text{pp}} = \frac{\overline{X} - \text{TARGET}\,\overline{\overline{X}}}{\text{TARGET}\,\overline{R}} \tag{2.1}$$

and

$$\text{R PLOT POINT} = R_{\text{pp}} = \frac{R}{\text{TARGET}\,\overline{R}} \tag{2.2}$$

respectively. The notation TARGET $\overline{\overline{X}}$ and TARGET $\overline{R}$ here denote respectively the target values of the mean and the range of the process distribution which can be determined in a number of possible ways. A related control charting method based on the assumption of *constant* standard deviation across different parts or products has also

been presented by Bothe (1990a) in another paper. This involves charting the statistic $(\overline{X} - \text{TARGET}\,\overline{\overline{X}})$ and the usual subgroup range resulting in so-called *Nominal* charts. When data are available from previous runs of identical or similar products, the author recommends estimating TARGET $\overline{\overline{X}}$ and TARGET $\overline{R}$ from these data (which will usually be small data sets). Without relevant historical data, he suggests taking TARGET $\overline{\overline{X}}$ equal to nominal specification and

$$\text{TARGET}\,\overline{R} = \frac{d_2(U-L)}{6C_{p(\text{goal})}} \tag{2.3}$$

where U, L, $C_{p(\text{goal})}$ and d_2 denote respectively the upper specification limit, lower specification limit, the target value of the process capability index, $C_p = \frac{U-L}{6\sigma}$ and the control chart factor (which is a function of the sample size, n).

Note that when $\overline{X}_{pp}$ and R_{pp} are computed from (2.1) and (2.2), the sequences of statistics plotted on each chart will be highly correlated. If TARGET $\overline{R}$ from (2.3) is used, one cannot possibly have any realistic idea of how this value relates to the process standard deviation. Even for this case, the plotted points are correlated. As a result, one does not know what kinds of point pattern to expect for a stable process, and thus one cannot possibly determine when a process is not stable.

As stated by Ermer et al.(1989), this normalizing approach also loses its appeal as a quality control procedure due to practical problems :

- Although the calculations for standardized charts are not difficult, they are more involved than traditional Shewhart charts.
- The coded data do not appear to have any physical meaning to production operators who usually have little background in statistics.

- Since the control limits for this chart never change, process improvements that are being implemented from time to time will not be reflected by the chart. In other words, no visual impression is available as to how much improvement has been made to the process.

Additionally, considerable effort and time have to be spent to obtain, review and revise the scaling factors if necessary. Furthermore, care must be taken to ensure that proper scaling factors are being used in each calculation to avoid misinterpretation regarding the stability of the controlled process.

2.5 Setup Variation and Measurement Errors Considerations

In any kind of machine set-up, whether manual or automatic, there exists a certain amount of natural variability in the setting regardless of how well it is performed. This issue must not be overlooked and set-up acceptance should be based on sound statistical principles. On a long term basis, for the same machine or production process, the set-up error, which is defined as the deviation of average output from the desired value, tends to fluctuate in a random manner around its expected value which is usually assumed to be zero. In every set-up, therefore, as long as the set-up error is within its natural spread, no adjustment is necessary and the production should be allowed to run to avoid over control. The set-up method can of course be improved to reduce this source of variation provided it is economically feasible to do so.

As demonstrated by Robinson (1991) and Bothe (1990b), in the presence of significant set-up variation, the use of Nominal and Short Run charts as reviewed in the preceding section is not appropriate. In particular, out of control conditions are indicated by the charts although the process is well in statistical control. These warning signals are

actually caused by the set-up variation rather than out of control conditions. To cope with this situation, Bothe suggested that two sets of scaling factors, i.e 'set-up' and 'run' factors should be used for each part number, he further discussed how they can be derived. Similarly, Robinson proposed that separate charts should be drawn for monitoring between-setup and within-setup variation in isolation. He and Robinson, Touw and Veewers (1993) illustrated this idea and exemplified how it can be achieved. For monitoring the first source of variation, he suggested charting the average value of the first five pieces (or the so called 'first offs') relative to nominal specification and the control limits for the resulting chart are derived by treating this as an individual values chart. As for control of within-setup variation, he suggested use of the traditional Range chart or a chart for the difference in average departure from nominal between the first five pieces and the last five pieces (or the 'last offs') for each short production run. He also mentioned use of the more practical 'Modified' control limits on the 'Deviation' chart to ensure that most individual items produced will conform to specifications, with the implicit assumption that $C_p > 1$. However, as he pointed out, this approach is not suitable if the specification band varies between runs.

When piece-to-piece variation is confounded with set-up variation, the Nominal and Short run charts do not provide an adequate means of reflecting the actual status of the process. By contrast, separate monitoring of these components of variation does not only give a real picture of the process but also provides some guidance as to what might need to be fixed when an out-of-control signal is present (Robinson (1991)).

In statistical process control, measurement error should also be given due consideration as this constitutes part of the inherent variation of a stable cause system. More simply stated, the natural variability observed in measured values of the quality

characteristic of any industrial product is due in part to the variability of the product and in part to the variability in the method of measurement. This latter is sometimes negligible, but at other times cannot be ignored without risk. If the standard deviation of measurement error varies in some systematic manner, both nominal and short run charts will give misleading signals, especially when the measurement error is relatively significant.

In his paper, Farnum (1992) incorporated this component of variation into some process models having made certain reasonable assumptions. He then developed a charting procedure for one particular model. This assumes nonconstant process and measurement error, more specifically the short run process has constant coefficient of variation coupled with a measurement system whose error variability is proportional to the true reading. The resulting procedure seeks to remove the differences in average and dispersion between various components to enable them to be monitored with the use of a single chart. As long as subgroup size does not change, this charting method yields a common set of control limits for every component, irrespective of their design specifications. Provided the appropriate model has been identified, these limits can be established early by utilizing data from different production lots in a predetermined manner. As with any charting method which plots different components on the same chart, caution should be exercised when sequence rules are applied.

2.6 'Self-Starting' Procedure Based on 'Running' Estimates of the Process Parameters

A series of articles by Quesenberry (1991a,b,c) presented an innovative approach to control, particularly pertinent to short-run processes and processes during the start-up

phase. The first paper considered independently and identically distributed (i.i.d) normal processes whereas the following two are devoted to monitoring processes with attribute data, namely, Binomial and Poisson processes, under various assumptions about the process parameters. Unlike the preceding approach, a non-linear transformation technique, specifically the *'Probability Integral Transformation'* technique, in conjunction with the usual linear transformation, was used to develop new charting procedures. These so-called 'Q' charting procedures enable production operators to begin monitoring the process essentially with the first units or samples of production whether or not prior knowledge of the process parameters is available. Consequently, the task of identifying and removing assignable causes, and thereby bringing the process into control, can begin at an earlier stage. For the case where no relevant data is available in advance of a production run, the control parameters are 'estimated' and 'updated' sequentially from the current data stream. These 'running' estimates, together with the immediately succeeding observations are in turn used to test whether the process remains stable.

Since the 'Q' statistics are either standard normal variables with independent observations or approximately so, the resulting charts can all be constructed using the same scale and with the same control limits irrespective of the type of product being monitored, thus simplifying charting administration. Additional run rules can also be used to detect any non-random patterns on such standardized charts which suggest various process instabilities. When subgroups or sampling inspection units vary in size, it is well understood that this situation is difficult to handle by classical methods. By contrast, the control limits and interpretation of point patterns for 'Q' charts are not affected by a varying sample size.

The behaviour of these charts for particular situations was studied using simulated data. The results show that 'Q' charts based on known and unknown parameters, are in close agreement with each other after the first few points, for in-control processes. As for processes with sustained shift in a parameter, the points on 'Q' charts which update the parameter estimates progressively from the data sequence will eventually settle into a pattern indicative of an in-control process. However, Quesenberry (1995a) quoted standard results as saying, that under common assumptions, the traditional 1-of-1 test (i.e one point outside the $3-\sigma$ control limits) on each of the 'Q' charts has the maximum possible detection capability on the next observation after a shift, among all control schemes with equal probability of a false alarm. As such, these classical 'Q' charts are recommended by some authors including Quesenberry himself for the problem of outlier detection.

Castillo and Montgomery (1994) showed that the strength of the signal from 'Q' charts for variables (with unknown mean but with known standard deviation) when a persistent step change in mean occurs depends on both the number of samples *before* and *after* the shift. It was also demonstrated that, as a consequence of this, the ARL performance of the 'Q' charts is poor in some cases. However, the accuracy of their simulation results is doubted. For instance, the run length distribution of the 'Q' charts considered in Table 1 for $\delta = 0$ has a known geometric distribution with mean of 370.4 and standard deviation of 369.9 but the simulated values given are 410.8 and 383.6 respectively. Another problem with this paper is the use of ARL as a performance criterion. It was mentioned in chapter 1 that the only case where ARL is a useful criterion is when the run length distribution is geometric. As most of the run length distributions involved in Quesenberry's (1991a) paper are *not* geometric after a shift in

the control parameters, their results are of limited usefulness in assessing the overall performance of 'Q' charts.

Due to the discrete nature of Binomial and Poisson processes, some comparisons were made between the 'Q' charts, standard normalizing charts and charts using other transformation techniques in the goodness of their normal approximations for the known parameter case. Generally, it is found that charts based on 'Q' transformations are superior. As for the unknown parameter case, Quesenberry simply stated that it is unlikely for other charts to perform as well as the proposed techniques due to their excellent theoretical properties.

Since no simple recursive formula is available and highly sophisticated computations are involved, implementation of the 'Q' control scheme requires computing facilities and complex algorithms. Fortunately, these algorithms are widely available and have been built into most of the commercial statistical software packages such as *S-plus*.

Like any other methods involving transformation, the resulting plotted points on 'Q' charts do not appear to have any physical meaning to production operators. Besides, 'Q' charts with unknown parameters fail to reflect both the process-tolerance incompatibilities and severe off-target conditions which occur right from the beginning of production runs. Therefore, unless close examination of the raw data is carried out, timely corrective actions will likely not be initiated until a considerable number of defects have been produced. This problem arises because such charts are designed to ensure that the process under surveilance is in a state of statistical control and hence they are unlikely to indicate any process trouble if no change in the process parameters takes

place, even though the process is incapable or substantially off-target immediately after set-up.

It is perhaps worth noting that there is an error in Quesenberry's (1991a) paper that has not been pointed out correctly in the literature. The sequence of control statistics given by formula (12) of this paper, i.e

$$Q_i\left(\overline{X}_i\right) = \Phi^{-1}\left[\mathbf{G}_{n_1+n_2+\ldots+n_i}\left(\frac{\sqrt{n_i}\left(\overline{X}_i - \mu_0\right)}{S_{0,i}}\right)\right]$$

with $$S_{0,i}^2 = \frac{\sum_{\alpha=1}^{i}\sum_{j=1}^{n_\alpha}\left(X_{\alpha j} - \mu_0\right)^2}{n_1 + n_2 + \ldots + n_i} \qquad i = 2,3,\ldots$$

where the notation used here is as defined in the original paper, is in fact *not* a sequence of i.i.d standard normal variables as claimed by Quesenberry (1995a,d) because the ith subgroup mean $\overline{X}_i$ is *not* independent of the standard deviation estimate $S_{0,i}$ and successive arguments of the normalizing transformation are correlated. Thus, using this formula indiscriminately can be misleading. To form a sequence of approximately i.i.d standard normal variables, simply replace n_i and $S_{0,i}$ by n_{i-1} and $S_{0,i-1}$ respectively in the above formula.

Recently, the RL properties of the 1-of-1 and three tests for point patterns made on the 'Q' charts as well as specially designed CUSUM and EWMA charts based on the values of 'Q' statistics, for detecting a one step shift in a control parameter, were studied by Quesenberry (1995a,b,c). The major conclusions drawn from this work are

- the classical 1-of-1 test has poor sensitivity
- the test that signals when 4 out of 5 consecutive points are beyond one standard deviation in the same direction is a reasonable choice

- the EWMAQ and CUSUMQ tests are the most sensitive and are about comparable in overall performance

In accordance with the empirical findings, he also made some recommendations for practical application. For more insight into 'Q' charting procedures and an interesting debate of their potential and limitations, see the discussions of these techniques by Castillo (1995), Hawkins (1995), Farnum (1995), Woodall, Crowder and Wade (1995) and the response from Quesenberry (1995d).

A logical extension of the 'Q' charting techniques to the control of the mean level of a multivariate normal process when prior estimates of the process parameters are not available has also been presented by Tang (1995). The author demonstrated that this method is particularly useful for 'picking up' a sustained shift in the mean vector when subgroup data are used.

In fact, this dynamic approach of charting was first perceived by Hawkins (1987). He proposed two related CUSUM procedures based on transformed individual readings for checking the constancy of process average and variability, along with some implementation details and illustrative examples. These were proposed as substitutes for the standard CUSUM procedures which generally assume known parameters. The proposed method provides another useful alternative for controlling, particularly, short run processes as it does not require knowledge of process parameters in advance of production runs and eliminates the need for a separate preliminary study.

In order to effectively apply the CUSUM procedure, it is well understood that successive values for which the sum is accumulated should be independent and identically distributed. For this reason, the following transformation formula (attributable

to Wallace (1959)) was suggested to obtain a sequence of independent and approximately standard normal variables, Z_j's, :-

$$Z_j = \left(\frac{8j-15}{8j-13}\right)\left[(j-2)\ln\left(1+\frac{T_j^2}{j-2}\right)\right]^{\frac{1}{2}}$$

where $T_j = \sqrt{\frac{(j-1)}{j}}\left(\frac{X_j - \overline{X}_{j-1}}{S_{j-1}}\right)$

X_j : jth individual reading

$\overline{X}_j$: mean of the first j readings

S_j : standard deviation of the first j readings

By maintaining a cumulative sum of successive Z_j's, starting from the 3rd observation, and using the established control rule, process mean stability can thus be monitored progressively without having to wait until adequate process performance data has built up. However, this method should not be used indiscriminately. Careful examination of the above transformation formula reveals that the resulting Z_j's are always positive and can be regarded as 'folded' standard normal variables which can assume positive values only. In fact, this normal approximation formula was originally considered by Wallace (1959) for converting *upper* tail values of the student-t distribution to corresponding standard normal deviates. Hence, the need for a modification to the formula is indicated. The minor change necessary is simply the addition of a negative sign to the transformed value Z_j if T_j is less than zero. For purposes of controlling process dispersion, Hawkins suggested using the scale CUSUM given in his previous paper (Hawkins (1981)) which involves cumulative summing of the following quantities :-

$$V_j = \frac{\left(\sqrt{|Z_j|} - 0.822\right)}{0.349}$$

He made some efforts to justify his recommendation for a 'self-starting' CUSUM over the adoption of a CUSUM procedure based on some start-up calibration data. These included consideration of the average run length properties of the two methods. His simulation results indicate that 'self-starting' CUSUM procedures are superior to those obtained with some 25 special start-up values, not to mention the short run situations where usually much less than this is available for initiating conventional cusum charts.

A final concern about this approach is its likely lack of robustness to both non-normality and to the presence of outliers in the underlying distribution of process measurements. Without previous data, there is often no assurance that the process output will conform reasonably to a normal distribution. The question arises, therefore, as to what effect departures from the normality assumption will have upon performance. Hawkins (1987) argued, by quoting others' results, that his method works for non-normal heavy-tailed data with little loss in ARL performance.

The latter issue is particularly pertinent as a sequence of measurements (which might include occasional outliers) is used simultaneously both for process control and to refine parameter estimates. Apparently, incorporating unknowingly occasional valid extreme observations into the estimates of the process parameters will cause inflation or deflation of them. This can have a substantial negative impact on the performance of the control method. To cope with this as well as to protect the CUSUM method (which is intended primarily for 'picking up' sustained mean shifts of small magnitude) from signals generated solely by isolated outliers, Hawkins (1987) suggested a robustification

approach using 'winsorisation'. 'Winsorizing' the measurements means that any measurement beyond a preset threshold will be set equal to it and used in subsequent calculations. In this manner, 'winsorisation' reduces or limits the effect of outliers on the parameter estimates and the properties of the control charts. This idea was further discussed by Hawkins (1993b) who also examined the relationship between the 'winsorizing' constants for 'self-starting' CUSUM and CUSUM based on a large process performance study. He stated that this method provides good protection against outliers with little additional cost in computational effort. In the same paper, he also showed that, 'winsorizing' causes little loss to CUSUM procedures in responsiveness to actual mean shifts for various sets of clean, contaminated and clean-contaminated data.

Note that, for monitoring process dispersion based on individual observations, the Exponentially Weighted Mean Squares (EWMS) and Exponentially Weighted Moving variance (EWMV) by MacGregor and Harris (1993) may be used following some adaptations. The latter is particularly useful for processes with drifting means when it is used in conjunction with the conventional EWMA. In the absence of historical data, these monitoring procedures may be initiated at the 3rd observation where the first two observations are used to provide some initial estimates of the mean and the standard deviation, resulting in some essentially 'self-starting' procedures. Approximate control limits for the resulting techniques may either be obtained algebraically or by means of simulation.

2.7 Control Based on Exponentially Weighted Moving Averages

Castillo et al.(1994) proposed two alternative methods as improvements to the 'Q' charting technique, for monitoring the deviations from target based on individual

measurements where the process standard deviation is assumed unknown. These methods are based on some *exponentially weighted moving average* (EWMA) type control statistics. The first method results from a straightforward adaptation of the standard EWMA control algorithm with the smoothing factor, λ chosen to be 0.1, the initial value of the EWMA statistic Z_0 equated to the specified or known target, μ_0 and the unknown standard deviation, σ estimated sequentially in some suggested manner. The resulting EWMA statistic

$$Z_t = (1-\lambda)Z_{t-1} + \lambda X_t \qquad ,t = 1,2,\ldots$$

is plotted on a control chart with limits

$$\mu_0 \pm L\sqrt{\frac{\lambda}{2-\lambda}}\hat{\sigma}_t$$

where $$\hat{\sigma}_t = \frac{S_t}{c_4}$$

with $$S_1 = X_1 - \mu_0, \qquad S_t = \sqrt{\tfrac{1}{t-1}\sum_{i=1}^{t}(X_i - \bar{X}_t)^2}, \quad t = 2,3,\ldots$$

c_4 is a control chart factor depending on t that can be found from most of the standard text books on SPC and L denotes a constant that is chosen to achieve specified run length performance. The authors also noted the use of the more exact transient control limits (for the tth observation) given by

$$\mu_0 \pm L\sqrt{\frac{\lambda}{2-\lambda}\left[1-(1-\lambda)^{2t}\right]}\hat{\sigma}_t$$

The other control algorithm was derived from the well known *Kalman* model (see, for example, Crowder (1989)) upon noting that the assumed i.i.d in-control model for the process measurements can be represented by a special case of the Kalman model. The resulting control statistic is given by the following recursive expression :-

$$Z_t = (1-\lambda_t)Z_{t-1} + \lambda_t X_t; \quad Z_0 = \mu_0$$

This latter was referred to as the *adaptive Kalman filtering* control method since the smoothing factors or Kalman weights, λ_t's change adaptively according to

$$\lambda_t = \frac{\hat{q}_{t-1}}{\hat{q}_{t-1} + \hat{\sigma}_t^2} \tag{2.4}$$

The variance components of the model, including the process variance, σ^2 and the *posterior* variance of the process mean after the tth observation, $q_t = \lambda_t \sigma^2$, are estimated and updated from the data sequence as follows :-

$$\hat{\sigma}_1^2 = S_1^2, \qquad \hat{\sigma}_t^2 = S_t^2 \qquad t = 2,3,\ldots$$

$$\hat{q}_t = \hat{q}_{t-1}(1-\lambda_t).$$

Following Crowder (1989), the control limits for this method were given as

$$\mu_0 \pm L\sqrt{\hat{q}_t} \tag{2.5}$$

The initial value $\hat{q}_0$ was found to have practically no effect on the ARL performance of the method provided it is greater than zero. A variant of this method has also been presented by the same authors (Castillo and Montgomery (1995)) where the $\hat{\sigma}_t^2$ in (2.4) and L in (2.5) are respectively replaced by 1 and $K\hat{\sigma}_t$, and K is a constant chosen to yield specified ARL performance. In the same paper, they provided some guidelines for the economic design of the control scheme with respect to K and the number of observations to take in a batch. The objective of the economic model is to minimize the expected total cost per batch (which comprises sampling/inspection cost, cost associated with false alarms and cost of running an out-of-control process) subject to certain in-control run length performance requirements.

It was demonstrated using simulation, that for a particular choice of the design parameters; λ, $\hat{q}_0$ and L, the first two methods have better ARL performance than the corresponding 'Q' chart for 'picking up' off-target conditions from start-up, especially when the size of the deviation (in multiple of standard deviations), δ, is small. It was also found that, these methods (with the particular choice of the design parameters) are superior to the classical Shewhart chart (classical in the sense that the process parameters are assumed known) in terms of ARL for small δ. Furthermore, it was observed that, using the same criterion, the first control method has a comparable performance to the classical EWMA chart. Despite these, the choice of the design parameters are quite arbitrary and no consideration is made of the optimality of the design for these control techniques. In addition, it is necessary to consider more complete profile of the RL distributions instead of only the use of ARL (and SDRL) for the reasons given in the first chapter.

Another contribution along this line comes from Wasserman (1994) who proposed the use of the so-called *Dynamic* EWMA control chart. This technique differs from the former methods in the charting convention and the manner in which the variance components are estimated. In particular, the Dynamic EWMA chart compares the specified target μ_0 with the varying control limits (constructed from the posterior distribution of the process mean) given by

$$\mathrm{LCL}_t = Z_t - L\hat{\sigma}_t\sqrt{\lambda_t}$$

$$\mathrm{UCL}_t = Z_t + L\hat{\sigma}_t\sqrt{\lambda_t}$$

where $\lambda_t = \dfrac{\hat{q}_{t-1}}{\hat{q}_{t-1} + \hat{\sigma}^2_{t-1}}$, $\hat{q}_t = \dfrac{\lambda_t \hat{\sigma}^2_t}{1-\lambda}$,

$$\hat{\sigma}_t^2 = \frac{\beta_t}{\alpha_t}, \quad \alpha_t = \alpha_{t-1} + 1, \quad \beta_t = \beta_{t-1} + (1-\lambda)(X_t - Z_{t-1})^2.$$

An out-of-control condition is signalled by the chart if the target μ_0 is outside these limits. Note that this is equivalent to plotting and comparing Z_t's with the control limits

$$\mu_0 \pm L\hat{\sigma}_t\sqrt{\lambda_t}$$

Note also that the equations for estimating σ^2 and q_t are derived from Bayesian considerations. α_0 and β_0 are the parameters for the chosen Gamma distributional prior of the precision parameter, $\frac{1}{\sigma^2}$. λ represents the prespecified steady state value of the EWMA smoothing parameter. Wasserman and Sudjianto (1993) developed a similar technique for detecting the presence of linear trend in process measurements. This is based on the second order dynamic linear model, 2-DLM (a special case of the Kalman model) defined by the following observation and system equations :

Observation : $X_t = \mu_t + v_t, \qquad t = 1,2,\ldots$

System : $\mu_t = \mu_{t-1} + b_{t-1} + w_{1t}, \qquad t = 1,2,\ldots$

$b_t = b_{t-1} + w_{2t}, \qquad t = 1,2,\ldots$

where X_t and μ_t denote respectively the observed value and the process mean of the quality variable at time t or for the tth sampling inspection unit and b_t represents the level change or trend parameter in the interval $(t-1, t]$. The observation error, v_t and the random errors in the system equations, w_{1t} and w_{2t}, are assumed to be distributed as

$$v_t \sim N(0, \mathrm{V}), \qquad w_{1t} \sim N(0, \mathrm{W}_1), \qquad w_{2t} \sim N(0, \mathrm{W}_2)$$

where V, W_1 and W_2 are some constants. As with Dynamic EWMA control chart, the resulting technique involves charting the varying control limits,

$$\mathrm{LCL}_t = \hat{\mu}_t - L\sqrt{C_{1,t}}$$

$$\mathrm{UCL}_t = \hat{\mu}_t + L\sqrt{C_{1,t}}$$

and signal the departure of process mean from its in-control value or speficied target μ_0 if these limits do not enclose μ_0. $\hat{\mu}_t$ and $C_{1,t}$ in the above expressions are updated recursively using the following equations :

$$\hat{\mu}_t = \lambda_{1,t} X_t + (1-\lambda_{1,t})(\hat{\mu}_{t-1} + \hat{b}_{t-1})$$

$$\hat{b}_t = \lambda_{2,t}(X_t - \hat{\mu}_{t-1}) + (1-\lambda_{2,t})\hat{b}_{t-1}$$

$$\lambda_{1,t} = \frac{R_{1,t}}{R_{1,t} + \mathrm{V}}, \quad \lambda_{2,t} = \frac{R_{3,t}}{R_{1,t} + \mathrm{V}}$$

$$C_{1,t} = \lambda_{1,t}\mathrm{V}, \quad C_{2,t} = R_{2,t} - \lambda_{2,t} R_{3,t}, \quad C_{3,t} = \lambda_{2,t}\mathrm{V}$$

$$R_{1,t} = C_{1,t-1} + 2C_{3,t-1} + C_{2,t-1} + W_{1,t}, \quad R_{2,t} = C_{2,t-1} + W_{2,t}, \quad R_{3,t} = C_{2,t-1} + C_{3,t-1} + W_{3,t}$$

$$\mathbf{W}_t = \begin{pmatrix} W_{1,t} & W_{3,t} \\ W_{3,t} & W_{2,t} \end{pmatrix} = \frac{1-\varphi}{\varphi}\mathbf{G}\mathbf{C}_{t-1}\mathbf{G}^{\mathrm{T}} = \frac{1-\varphi}{\varphi}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} C_{1,t-1} & C_{3,t-1} \\ C_{3,t-1} & C_{2,t-1} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

where $\hat{\mu}_0 = \mu_0$, $\hat{b}_0 = 0$, and φ, the so-called *discount factor*, is chosen to yield specified ARL performance. $\mathbf{C}_0$ is the variance-covariance matrix for the prior distribution of the vector of process mean and trend parameter and is chosen to reflect the uncertainty of the prior estimates, $\hat{\mu}_0$ and $\hat{b}_0$. If V is unknown, it is suggested estimating it sequentially using a Bayesian procedure. Using a set of simulated data, the resulting technique was shown to be more effective than the classical individual values chart and the 'Q' chart with known mean and unknown variance for detecting small process trends. Table 1 of the same paper also indicates that the *transient* 2-DLM control charts (with *observational* variance assumed known) tend to react more quickly

to small process trends than the steady state 2-DLM and the conventional EWMA charts.

A control method based on adaptive Kalman filtering, coupled with a *tracking signal* feature, has also been proposed by Castillo et al.(1994) as an alternative to the 'Q' chart for the case where both μ and σ are unknown. It was noted that besides being capable of 'picking up' sudden mean shifts, this control procedure can also be used to detect other out-of-control conditions such as linear process trends. In order to use this method, some prior estimates of μ (denoted by $\hat{\mu}_0$) and σ (denoted by $\hat{\sigma}_0$) are required. In the short-run environment, reasonably accurate estimates may not be available. However, it was shown using simulation, that this control technique can provide better ARL performance than the 'Q' chart especially for early shifts of large sizes (more than 2 standard deviations) if the estimate of μ differs from the true value by less than 1.5 standard deviation and σ is under or over-estimated by less than 50%. Again, since the RL distributions for both procedures after the shift are unknown, the results may not reflect their actual relative performance correctly.

The method involves the computation of the so-called *smoothed error statistic* and the *smoothed mean absolute deviation*, defined respectively as (Trigg and Leach (1967)) :-

$$Q(t) = \alpha e_1(t) + (1-\alpha)Q(t-1); \quad Q(0) = 0$$

$$\Delta(t) = \alpha|e_1(t)| + (1-\alpha)\Delta(t-1); \quad \Delta(0) = 0.8\hat{\sigma}_0\sqrt{2/(2-\lambda_t)}$$

where the one-step-ahead forecast error is

$$e_1(t) = X_t - \hat{\mu}_{t-1}$$

and the variance of the process is updated sequentially by

$$\hat{\sigma}_t^2 = \alpha\hat{\sigma}_{t-1}^2 + (1-\alpha)\left(X_t - \overline{X}_{t-1}\right)^2$$

For this control technique, a signal is triggered when

$$\left|\frac{Q(t)}{\Delta(t)}\right| > L,$$

the control limit, L being between 0 and 1.

As the computational effort involved is substantial, implementation of the above control algorithms requires computerisation. No guidelines about the choice of the design parameters to achieve desired operating performance are available.

All the methods discussed thus far either ignore or give inadequate consideration to the problem of process 'warm up', when the process is invariably unstable. On the other hand, the following approach, which determines the 'control' limits based on given specifications, appears to be capable of handling this problem effectively.

2.8 Deriving 'Control' Limits From Specifications

Without the data necessary to set up conventional control charts, compounded by the problem of process 'warm up', it makes some sense to use product specifications to provide control information. A control technique that pre-determines its 'control' limits by reference only to the specifications rather than requiring an accumulation of data for computation of control limits, is known as 'pre-control' (P.C).

P.C was first proposed by Shainin (1954) as an alternative to various traditional on-line quality control methods and, in particular, as an improvement to $\overline{X}$ and R charts. It provides a simple and flexible tool for process monitoring as well as set-up approval, particularly in the low volume manufacturing environment. P.C is conceptually different from traditional charting techniques in that it focusses directly on preventing non-

conforming units from occurring rather than on maintaining a process in a state of statistical control. The details of operation and the practical merits of this method are given in Chapter 6.

A new type of control chart which originates from the idea of 'pre-control', named the 'Balance' chart (B.C) has been introduced by Thomas (1990). B.C can be used in several different modes. When B.C is used in 'pre-control' mode, it eliminates the need for estimating the process parameters but instead, derives 'pseudolimits' (±pL) from the specified tolerance. In B.C, successive measurements from the process on a certain quality characteristic are classified as -1, 0 or 1 according to their values relative to ±pL and specification boundaries. Cumulative recording and plotting of these data about a target line give information both on the process 'accuracy' and 'precision'. A mathematical derivation of the control limits which define the maximum deviation of the plot from the target line, and the maximum number of positive and negative changes from the start of the run is provided in the same paper. In addition, several rules governing the maximum number of changes in a given run length were developed to indicate the possible presence of process troubles. With manual charting, however, too many supplementary rules will complicate the interpretation of the Balance chart.

Besides data scoring, B.C has the unique feature that the operating rules and control limits are common to every application of the chart. Thus, it has great potential for computerisation.

Like Pre-Control, this technique does not require exact measurements, but only needs to know into which 'band' the measurements fall. In order to justify his recommendation, Thomas also provides comparison of the Balance chart and the $\overline{X}$

chart operating characteristics for a mean shift of 1 standard deviation, along with some illustrative examples which clearly show that B.C possesses higher sensitivity.

The last two charting methods falling in this category were presented by Bayer (1957) (see also Sealy (1954)) and Maxwell (1953). Both methods are essentially the same, as they are adaptations of the Nominal $\overline{X}$ & R charts with limits derived on the assumption that the process is just capable of meeting the specification. The only difference between them is that the latter expresses the coded measurements and 'control' limits in terms of 'cells'.

Representing the specification band by 10 cells, the resulting 'cell' chart has constant control limits regardless of the specification or the actual process capability, provided the sample size remains unchanged. Thus, it is possible to have just one chart per machine on which all parts having possibly different specifications processed can be controlled. However, these methods of control charting cannot handle one-sided specification situations.

A comparison between these *adjusted* $\overline{X}$ & R charts and 'pre-control', based on certain statistical grounds is presented in Chapter 6. The results of the comparison indicate that the former are superior in many circumstances.

2.9 Adjusting Set-up Continuously Based On Process Output

As a substitute for conventional SPC for low volume production, an entirely different approach was proposed in Lill, Chu and Chung (1991), 'Statistical Setup Adjustment' (SSA). This represents a form of 'feedback control' where the deviation from the desired dimension or error of the measured output characteristic, is used to calculate the best possible adjustment to be made in a machine set-up, starting with the first piece

produced. As such, it is not a set-up approval method but one which provides an algorithm as to how much adjustment should be made as each of the successive observations arises. Methods are also presented to minimize the number of adjustments, to avoid early false signals and to anticipate the effects of a known trend such as tool wear.

As discussed earlier, in the presence of significant set-up variation, Robinson (1991) and Bothe (1990b) proposed separate monitoring of the set-up processes and their subsequent runs. If the set-up varies from the desired setting but is within predictable limits, no machine adjustment is necessary. This is due to the fact that such corrective action is not only uneconomical, but would probably result in a greater percentage of defects. SSA differs from this method in that it does not accept the risk of inaccurate set-up as a consequence of natural set-up variation which inevitably exists, but is constantly 'forcing' the set-up value to the desired dimension. This approach is, therefore, in line with Taguchi's idea of quality loss, i.e emphasis is placed on the uniformity of product quality characteristic about its target value rather than on mere conformance to specifications.

In SSA, both the machine variations and set-up errors are modelled with conceptual normal populations. From available information and experience, a 'maximum likelihood' estimator of the set-up error can be obtained and hence the correct adjustment derived. However, determination of the standard deviation of set-up variability based on subjective judgement, as suggested, leads to doubts about its reliability. In fact, it is possible to obtain such an estimate directly from the available data.

In this work, the implicit assumption is made that set-up is the critical or dominant 'system' that largely determines quality of the output. In other words, defects

are the direct result of the accuracy of tools or precision of adjustment of the set-up. Therefore, theoretically, SSA does not provide protection against mean shifts or increase in process spread due to some special causes during the production run.

If set-up is the dominant cause system, this method works provided the effects of adjustments are manifested instantaneously and in full. The realization of this, however, requires dynamic machine control with automatic inspection feedback and measurable means of adjustment.

2.10 Monitoring Process Input Parameters

By monitoring the process output, traditional SPC and the approaches discussed above, at best, indicate only when production is not free of troubles. In many instances, when an out-of-control condition is indicated, numerous corrective measures are possible and the correct course of action is not always obvious. As such, delay in preventing waste is inevitable. For small lot production, this can be regarded as the same shortcoming as 'post mortem' inspection !

In view of this limitation, recent research into the area of applying SPC in low volume manufacturing environments has given up trying to monitor the process output but instead has concentrated on the process inputs (Foster (1988) and Thompson (1989)). Foster presented this idea for controlling highly technical or time consuming processes where corrective measures for unacceptable work are often uncertain or even unknown. The implementation strategy for the suggested approach involves the creation of a 'true' process by compiling a 'Master Process Requirements List' from all specifications used for a particular process, selection of the vital few critical input parameters to be monitored and process capability evaluation.

2.11 Economically Optimal Control Procedures

While, traditionally, the development of SPC techniques has been mainly concerned with statistical efficiency, the ultimate objective of any process control strategy is cost reduction as a result of reduced scrap, rework and rejects, improved product quality and increased productivity. This objective may be accomplished by having an economically optimum policy governing the process monitoring, adjustment and maintenance activities. In the light of this, over the last four decades, a considerable amount of study has been devoted to the design of process control methods with respect to economic criteria. Various process models and cost structures have been proposed and the corresponding optimal control strategies derived. However, much of the theoretical work on incorporating cost considerations into the design of process control procedures has been undertaken implicitly in the context of long production runs.

The economic decision models currently available for on-line quality control can be broadly classified into two types. These are economic-process-control models and economic models for traditional SPC. In their paper, Adams and Woodall (1989) distinguished between these two types and highlighted some similarities and differences between them. A thorough review of the literature on the latter was provided by Montgomery (1980). Ho and Case (1994) supplemented this work by presenting more detailed and complete discussions of different models and aspects of economic design for traditional SPC, and by summarizing the published work on economic designs of control charts covering the period from 1981 to 1991. For typical examples of the former, see Box and Jenkins (1963), Box, Jenkins and MacGregor (1974), Bather (1963) and Taguchi (1981).

Crowder (1992) considered a short run economic-process-control model in which observations on a certain measured quality characteristic of the product are assumed to be generated by an integrated moving average (IMA(1,1)) process and the costs involved consist of the usual quadratic loss of process mean being off-target and the fixed cost for each adjustment. He also made the assumptions that any adjustment made to the process has a known effect (i.e no adjustment error) and that an adjustment changes the process mean instantaneously or before the next sample measurement is taken (i.e no process dynamics or inertia). Sampling cost and sampling interval were not formally considered. Furthermore, deterministic drift and step or cyclical changes were not taken into consideration.

The proposed model seeks to find the sequence of adjustments, $a_t's$, which minimizes the total expected loss, $L(n)$, incurred throughout the production run as given by the following expression :-

$$L(n) = E\left\{\sum_{t=1}^{n}\left(c_1\mu_t^2 + c_2\delta(a_{t-1})\right)\right\}$$

where c_1 is the cost parameter associated with any squared deviation of process mean, μ_t from target (assumed, without loss of generality, to be 0), c_2 represents the cost of adjustment irrespective of magnitude, n is the terminating sample number and

$$\delta(a) = 1 \quad if\ a \neq 0$$
$$= 0 \quad if\ a = 0$$

Using dynamic programming or the backwards induction technique, the author derived an algorithm which enables the optimal control or adjustment strategy (i.e the optimal sequence of adjustments, $a_t's$) to be obtained numerically. An approximation formula was also given for the case where the total number of inspections, $n \leq 10$ and

the cost ratio, $c = \frac{c_2}{c_1} > 200$. In general, his results can be stated as follows. The resulting decision procedure as to when and how much adjustment should be made is based on the Bayes (or Posterior) estimate of the current process mean. It was also found that the 'control' or adjustment limits are changing with time and becoming wider towards the end of a production run, in contrast to the fixed limits proposed by some for the asymptotic case. This solution, he stressed, is consistent with the philosophy of traditional SPC in that it calls for adjustments only when the process mean is substantially off-target. In addition, it is found to be intuitively reasonable as the 'widening' action limits will decrease the likelihood of performing economically unjustifiable adjustments or maintenance near the end of a production run. In the same paper, Crowder demonstrated, by an example, that using the infinite-run (fixed) limits for the short-run problem with relatively large adjustment costs can significantly increase the total expected cost.

Woodward and Naylor (1993) also presented an approach to short run SPC which takes economic factors into consideration. In this work, a normal linear model is assumed in which three components of variation are involved. These are the set-up, adjustment (or resetting) and inherent process variabilities. The model also implies that there is no delay for any adjustment to take effect, no occurrence of parameter changes within a machine set-up or production run and that the process standard deviation is constant irrespective of product types. In comparison to that of Crowder, these authors proposed a more realistic cost structure which includes the following components :

- inspection cost
- rework cost

- scrapping cost
- cost associated with adjustment
- quadratic loss of being off-target

They considered a sequential scheme with three possible control actions at each decision point and attempted to derive a control rule using Bayesian methods such that the decision made at any stage of the sequential procedure minimizes the expected loss over all possible future decisions based on a given cost function. However, the solution of this optimal control problem is not straightforward and requires the use of techniques such as backwards induction. As they stated, in practice, it is impossible to find an optimal rule for this control plan because of the complexity introduced by the three-way decision structure. In view of this, a simplification in which a control decision is only made at two stages was considered. Even with this simplified scheme, determination of the decision boundaries remains complicated and requires a great deal of numerical computation.

For both the proposed economic process control models, some prior knowledge of the process parameters such as the variance terms or availability of some relevant historical data for their estimation is assumed. Woodward et al. described a method to quantify the historical information pertinent to their model. In practice, this could be a problem because historical data for this purpose is rarely sufficient in the short-run environment.

Another practical problem with these SPC approaches is the difficulty in specifying the cost parameters. This is due to the fact that some of the cost factors are intangible. For example, it is difficult to figure out the value of the cost parameters associated with the quadratic loss of being off-target and the loss due to process

downtime (as a consequence of process adjustment), even by someone who has substantial knowledge of production and of the cost involved. As a first step to the implementation of these economic decision models, it is advisable to carry out a sensitivity analysis of the models to identify the critical parameters and subsequently exercise greater caution in their determination. However, this is a time-consuming exercise.

CHAPTER 3

MEAN CONTROL FOR MULTIVARIATE NORMAL PROCESSES [1]

3.1 Introduction

The majority of the statistical process control (SPC) techniques proposed to date for controlling the mean of a multivariate process are based on Hotelling's (1947) χ^2 or T^2-type statistic. Other multivariate control procedures, including the use of principal components and multivariate cusum (MCUSUM) techniques, were reviewed by Jackson (1991). A multivariate version of the exponentially weighted moving average chart, referred to as the MEWMA chart, has also been presented by Lowry, Woodall, Champ and Rigdon (1992). Apart from these, some techniques that are designed to provide protection against changes in the process covariance matrix have been presented, for example, by Montgomery et al. (1972) and Alt and Bedewi (1986). In recent years, some attempts have also been made to develop control techniques which can both detect process irregularities and identify the actual set of out-of-control variables, taking into account the correlational structure of the quality characteristics (see for eg., Doganaksoy, Faltin and Tucker (1991), Hawkins (1993a) and Hayter and Tsui (1994)). All of this work, however, assumes that the process mean, μ and the process covariance matrix, Σ are known or that they can be reliably estimated prior to full scale production. Thus, these techniques do not readily lend themselves to applications in the short-run or

[1] Part of the material from this chapter is based on the paper entitled 'Mean control for multivariate processes with specific reference to short runs', *Proceedings of the International Conference on Statistical Methods and Statistical Computing for Quality and Productivity Improvement*, August 1995, Seoul, Korea.

low-volume manufacturing environment where data for estimating the process parameters are invariably unavailable.

The main thrust of this chapter is to present some 'self-starting' and unified procedures, for monitoring the stability, and in particular, the mean level of a multivariate normal process where prior estimates of the process parameters are not available, such as frequently occurs in short-run, low-volume and multi-product manufacturing environments. The proposed techniques also facilitate the control of long-run processes at an earlier stage. For completeness, control procedures are also given for the cases where μ, Σ or both are assumed known in advance of the production runs. In addition, two EWMA procedures specifically designed for detecting sustained mean shifts and linear trends are considered. These are shown to be superior to some competing procedures. The total discourse of this chapter is in the context of discrete items manufacture.

3.2 Methodological Basis

One of the desirable properties for any SPC procedure is that the in-control behaviour is predictable. This is the case if successive values of the control statistic are independent realizations of a known constant distribution or at least approximately so under in-control conditions. Another desirable property is the capacity to use additional run rules with the associated control chart to help identify any non-random patterns otherwise not apparent. The control statistics presented in the next section possess these properties. As the arguments involved in establishing the distributional properties of these control statistics are similar, this section considers only a particular case, namely,

that based on individual observations where the process mean vector and variance-covariance matrix are unknown prior to production.

Let $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_r$ be independent p-variate random observation vectors which have the same covariance structure and are distributed as,

$$\mathbf{X}_j \sim N_p(\mu_j, \Sigma), \qquad j = 1,2,\ldots,k$$

where Σ denotes the unknown non-singular variance-covariance matrix. Next, define

$$\mathbf{V}_j = \mathbf{X}_j - \overline{\mathbf{X}}_k \qquad j = 1,2,\ldots,k$$

where $$\overline{\mathbf{X}}_k = \frac{1}{k}\sum_{j=1}^{k} \mathbf{X}_j .$$

We wish to test the hypothesis,

$$H_0 : \mu_1 = \mu_2 = \ldots\ldots = \mu_k = \mu \text{ (say)} \quad vs. \quad H_A : \text{not all } \mu_j\text{'s are equal}$$

Under H_0 and the assumption of a *constant* process variance-covariance matrix,

$$\mathbf{V} = \left(\mathbf{V}_1^T \quad \mathbf{V}_2^T \quad \ldots \quad \mathbf{V}_k^T\right)^T \sim N_{kp}(\mu_V, \Sigma_V)$$

where $$\mu_V = \mathbf{0}_{kp\times 1}, \qquad \Sigma_V = \Sigma \otimes \mathbf{C},$$

$\otimes$ denotes the (*left*) *Kronecker Product* (Graybill (1983), p.216) and

$$\mathbf{C} = \begin{pmatrix} \frac{k-1}{k} & -\frac{1}{k} & \cdots & \cdots & \cdots & -\frac{1}{k} \\ -\frac{1}{k} & \frac{k-1}{k} & \cdots & \cdots & \cdots & -\frac{1}{k} \\ \vdots & & \ddots & & & \vdots \\ \vdots & & & \ddots & & \vdots \\ -\frac{1}{k} & \cdots & \cdots & \cdots & \cdots & \frac{k-1}{k} \end{pmatrix}_{k\times k}$$

Some linear combinations of the $p \times 1$ component vectors $\mathbf{V}_1, \mathbf{V}_2, \ldots\ldots, \mathbf{V}_k$ that are uncorrelated are now obtained using the standard principal components approach. Consider the following linear combinations :

$$
\begin{aligned}
\mathbf{Y}_1 &= a_{11}\mathbf{V}_1 + a_{12}\mathbf{V}_2 + \ldots\ldots + a_{1k}\mathbf{V}_k \\
\mathbf{Y}_2 &= a_{21}\mathbf{V}_1 + a_{22}\mathbf{V}_2 + \ldots\ldots + a_{2k}\mathbf{V}_k \\
&\vdots \qquad\qquad \vdots \qquad\qquad \vdots \\
&\vdots \qquad\qquad \vdots \qquad\qquad \vdots \\
\mathbf{Y}_k &= a_{k1}\mathbf{V}_1 + a_{k2}\mathbf{V}_2 + \ldots\ldots + a_{kk}\mathbf{V}_k
\end{aligned}
$$

In matrix notation, these are represented by the following equation :

$$
\mathbf{Y} = \Gamma\mathbf{V} = (\mathbf{I} \otimes \mathbf{A})\mathbf{V} \tag{3.1}
$$

where $\mathbf{I}$ is a $p \times p$ identity matrix and

$$
\mathbf{A} = \begin{pmatrix}
a_{11} & a_{12} & \cdots & \cdots & a_{1k} \\
a_{21} & a_{22} & \cdots & \cdots & \vdots \\
\vdots & & & & \vdots \\
\vdots & & & & \vdots \\
a_{k1} & \cdots & \cdots & \cdots & a_{kk}
\end{pmatrix}_{k \times k}
$$

Thus, the variance-covariance matrix of $\mathbf{Y} = \begin{pmatrix} \mathbf{Y}_1^{\mathrm{T}} & \mathbf{Y}_2^{\mathrm{T}} & \cdots & \mathbf{Y}_k^{\mathrm{T}} \end{pmatrix}^{\mathrm{T}}$ is

$$
\begin{aligned}
\Sigma_{\mathbf{Y}} &= \Gamma \Sigma_{\mathbf{V}} \Gamma^{\mathrm{T}} \\
&= \Sigma \otimes \mathbf{A}\mathbf{C}\mathbf{A}^{\mathrm{T}}.
\end{aligned} \tag{3.2}
$$

To produce vectors $\mathbf{Y}_1, \mathbf{Y}_2, \ldots, \mathbf{Y}_k$ which are uncorrelated, $\mathbf{A}$ in (3.2) is chosen to diagonalize the symmetric matrix $\mathbf{C}$. One choice for $\mathbf{A}$ is

$$
\mathbf{A} = \begin{pmatrix}
\frac{1}{\sqrt{k}} & \frac{1}{\sqrt{k}} & \cdots & \cdots & \cdots & \cdots & \cdots & \frac{1}{\sqrt{k}} \\
\frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} & 0 & \cdots & \cdots & \cdots & 0 \\
\frac{1}{\sqrt{4(4\text{-}1)}} & \frac{1}{\sqrt{4(4\text{-}1)}} & \frac{1}{\sqrt{4(4\text{-}1)}} & \frac{-3}{\sqrt{4(4\text{-}1)}} & 0 & \cdots & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\frac{1}{\sqrt{k(k-1)}} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \frac{-(k-1)}{\sqrt{k(k-1)}}
\end{pmatrix}
$$

where the rows of the matrix are the normalized eigenvectors of $\mathbf{C}$. Substituting $\mathbf{A}$ into (3.1) results in the following linear combinations :

$$
\begin{aligned}
&\mathbf{Y}_1 = \mathbf{0},\\
&\mathbf{Y}_2 = \tfrac{1}{\sqrt{2}}(\mathbf{V}_1 - \mathbf{V}_2) = \tfrac{1}{\sqrt{2}}(\mathbf{X}_1 - \mathbf{X}_2)\\
&\mathbf{Y}_3 = \tfrac{1}{\sqrt{6}}(\mathbf{V}_1 + \mathbf{V}_2 - 2\mathbf{V}_3) = \tfrac{1}{\sqrt{6}}(\mathbf{X}_1 + \mathbf{X}_2 - 2\mathbf{X}_3)\\
&\vdots \qquad\quad \vdots \qquad \vdots \qquad\quad \vdots \qquad \vdots\\
&\vdots \qquad\quad \vdots \qquad \vdots \qquad\quad \vdots \qquad \vdots\\
&\mathbf{Y}_k = \frac{1}{\sqrt{k(k-1)}}\left[\sum_{j=1}^{k-1}\mathbf{V}_j - (k-1)\mathbf{V}_k\right] = \frac{1}{\sqrt{k(k-1)}}\left[\sum_{j=1}^{k-1}\mathbf{X}_j - (k-1)\mathbf{X}_k\right]
\end{aligned}
$$

Note that the transformation results in a new set of uncorrelated random vectors $\mathbf{Y}_2, \mathbf{Y}_3, \ldots\ldots, \mathbf{Y}_k$, one less than the set of original random vectors. This is due to the fact that the transformation is subject to a constraint, namely, the sum of the component vectors, $\mathbf{V}_1, \mathbf{V}_2, \ldots\ldots, \mathbf{V}_k$ is equal to a zero vector leading to $rank(\Sigma_{\mathrm{V}}) = kp - p = k(p-1)$. It is also clear from (3.2) that Σ_{Y} is a 'quasi-diagonal' matrix with diagonal submatrices Σ except for the first one which is a zero matrix.

Since the resulting transformed vectors $\mathbf{Y}_2, \mathbf{Y}_3, \ldots\ldots, \mathbf{Y}_k$ are linear combinations of multivariate normal vectors and Σ_{Y} is a quasi-diagonal matrix as mentioned above, they are mutually independent with common variance-covariance matrix Σ. As the kth observation vector $\mathbf{X}_k$, the sample mean vector $\overline{\mathbf{X}}_{k-1}$ and variance-covariance matrix $\mathbf{S}_{k-1} = \dfrac{1}{k-2}\sum_{i=1}^{k-1}(\mathbf{X}_i - \overline{\mathbf{X}}_{k-1})(\mathbf{X}_i - \overline{\mathbf{X}}_{k-1})^{\mathrm{T}}$ of the first k-1 observations are independent,

$$
\begin{aligned}
A_k &= \mathbf{Y}_k^{\mathrm{T}}(\mathbf{S}_{k-1})^{-1}\mathbf{Y}_k\\
&= \frac{1}{k(k-1)}\left[\sum_{j=1}^{k-1}\mathbf{X}_j - (k-1)\mathbf{X}_k\right]^{\mathrm{T}}\mathbf{S}_{k-1}^{-1}\left[\sum_{j=1}^{k-1}\mathbf{X}_j - (k-1)\mathbf{X}_k\right]\\
&= \left(\tfrac{k-1}{k}\right)(\mathbf{X}_k - \overline{\mathbf{X}}_{k-1})^{\mathrm{T}}\mathbf{S}_{k-1}^{-1}(\mathbf{X}_k - \overline{\mathbf{X}}_{k-1}) \qquad\qquad k = p+2, \ldots\ldots
\end{aligned}
$$

are easily seen to be distributed as (Anderson (1984), p.163)

$$A_k \sim \frac{(k-2)p}{(k-1-p)}F_{p;k-1-p}$$

where $F_{v1,v2}$ denotes the F-distribution with $v1$ and $v2$ degrees of freedom. Before proceeding to show that successive $A_k's$ are statistically independent, a number of remarks will be made. Note that A_k is the familiar T^2-type statistic that can be used to check the consistency of $\mathbf{X}_k$ as coming from the same normal distribution as the sample of k-1 preceding observations. This statistic possesses certain optimal properties, as reported in Anderson (1984, pp.181). Of all tests whose power depend on $\frac{k-1}{k}(\mu_k-\mu)^T\Sigma^{-1}(\mu_k-\mu)$, the test based on A_k can be shown to be uniformly most powerful for testing the equality of mean vectors of the two normal populations from which $\mathbf{X}_k$ and the sample of k-1 preceding observations are drawn, when the unknown variance-covariance matrix Σ is assumed to be constant (see Lehmann (1959)). It can also be shown, using a theorem of Stein (1956), that this test is admissible i.e, there exists no other test which is better or uniformly more powerful. Besides, note that the denominator degree of freedom of the F-distribution associated with A_k varies according to k. If the values of the $A_k's$ are plotted in different scales, they are likely to give a misleading visual impression about the status of the process. Thus, it is advantageous to transform the $A_k's$ into a sequence of independent and identically distributed variables, preferably having standard normal distribution, because any anomalous process behaviour will then show up more clearly in the resulting control chart. This can also be assisted by means of additional run rules.

To establish mutual independence of successive $A_k's$, it is first proved that they are *pairwise* independent. Clearly,

$$\mathbf{Y}_k \sim N_p(\mathbf{0},\Sigma),\ \mathbf{Y}_{k+1} \sim N_p(\mathbf{0},\Sigma),\ \ (k-2)\mathbf{S}_{k-1} \sim W_p(k-2,\Sigma)$$

are mutually independent. The notation $W_p(v,\Sigma)$ here denotes the p-dimensional *Wishart* distribution with v degrees of freedom and parameter Σ. Let $\mathbf{D}$ be a non-singular matrix such that $\mathbf{D}\Sigma\mathbf{D}^{\mathrm{T}} = \mathbf{I}$ and define

$$\mathbf{Y}_k^* = \mathbf{D}\mathbf{Y}_k \ , \quad \mathbf{Y}_{k+1}^* = \mathbf{D}\mathbf{Y}_{k+1} \ , \quad \mathbf{S}_{k-1}^* = \mathbf{D}\mathbf{S}_{k-1}\mathbf{D}^{\mathrm{T}}.$$

Thus,

$$\mathbf{Y}_k^* \sim N_p(\mathbf{0},\mathbf{I}), \quad \mathbf{Y}_{k+1}^* \sim N_p(\mathbf{0},\mathbf{I}), \quad (k-2)\mathbf{S}_{k-1}^* \sim W_p(k-2,\mathbf{I})$$

and their independence is preserved. Due to the invariance property of the transformation,

$$A_k = \mathbf{Y}_k^{\mathrm{T}}\mathbf{S}_{k-1}^{-1}\mathbf{Y}_k = \mathbf{Y}_k^{*\mathrm{T}}\left(\mathbf{S}_{k-1}^*\right)^{-1}\mathbf{Y}_k^*,$$

$$A_{k+1} = \mathbf{Y}_{k+1}^{\mathrm{T}}\mathbf{S}_k^{-1}\mathbf{Y}_{k+1} = \mathbf{Y}_{k+1}^{*\mathrm{T}}\left(\mathbf{S}_k^*\right)^{-1}\mathbf{Y}_{k+1}^*.$$

Noting that $(k-2)\mathbf{S}_{k-1}^* = (k-1)\mathbf{S}_k^* - \mathbf{Y}_k^*\mathbf{Y}_k^{*\mathrm{T}}$ and using an identity for matrix inverses (Press(1982), Binomial Inverse Theorem, p.23), A_k may be expressed as follows :

$$A_k = \frac{(k-2)}{(k-1)}\left[\mathbf{Y}_k^{*\mathrm{T}}\left(\mathbf{S}_k^*\right)^{-1}\mathbf{Y}_k^* + \frac{\left(\mathbf{Y}_k^{*\mathrm{T}}\left(\mathbf{S}_k^*\right)^{-1}\mathbf{Y}_k^*\right)^2}{(k-1)-\mathbf{Y}_k^{*\mathrm{T}}\left(\mathbf{S}_k^*\right)^{-1}\mathbf{Y}_k^*}\right].$$

As $\mathbf{Y}_k^{*\mathrm{T}}\left(\mathbf{S}_k^*\right)^{-1}\mathbf{Y}_k^*$ is independent of $\mathbf{S}_k^*$ (Srivastava and Khatri (1979), Theorem 3.6.6, p.94) and $\mathbf{Y}_{k+1}^*$, it is also independent of any function of $\mathbf{S}_k^*$ and $\mathbf{Y}_{k+1}^*$. Thus, $\mathbf{Y}_k^{*\mathrm{T}}\left(\mathbf{S}_k^*\right)^{-1}\mathbf{Y}_k^*$ is independent of $A_{k+1} = \mathbf{Y}_{k+1}^{*\mathrm{T}}\left(\mathbf{S}_k^*\right)^{-1}\mathbf{Y}_{k+1}^*$ and it follows immediately that A_k (a function of $\mathbf{Y}_k^{*\mathrm{T}}\left(\mathbf{S}_k^*\right)^{-1}\mathbf{Y}_k^*$) is independent of A_{k+1}. Similarly, it can be shown that any pair of A_k's are pairwise independent. Note that,

$$A_{k+2} = k\mathbf{Y}_{k+2}^{*\mathrm{T}}\left[(k-1)\mathbf{S}_k^* + \mathbf{Y}_{k+1}^*\mathbf{Y}_{k+1}^{*\mathrm{T}}\right]^{-1}\mathbf{Y}_{k+2}^*$$

and since $\mathbf{Y}_k^{*\mathrm{T}}\left(\mathbf{S}_k^*\right)^{-1}\mathbf{Y}_k^*$ is independent of $\mathbf{S}_k^*$, $\mathbf{Y}_{k+1}^*$ and $\mathbf{Y}_{k+2}^*$, it is clear that A_k is also independent of A_{k+2} . By induction, A_{k+m} and A_{k+n} $(m \neq n)$ are independent.

Using the result of pairwise independence, it is now possible to proceed to show that they are mutually independent. As $\mathbf{Y}_k^{*\mathrm{T}}\left(\mathbf{S}_k^*\right)^{-1}\mathbf{Y}_k^*$, $\mathbf{S}_k^*$, $\mathbf{Y}_{k+1}^*$ and $\mathbf{Y}_{k+2}^*$ are mutually independent, their joint probability density function (p.d.f) is

$$f\left(\mathbf{Y}_k^*\mathbf{S}_k^{*-1}\mathbf{Y}_k^*, \mathbf{S}_k^*, \mathbf{Y}_{k+1}^*, \mathbf{Y}_{k+2}^*\right) = g\left(\mathbf{Y}_k^{*\mathrm{T}}\mathbf{S}_k^{*-1}\mathbf{Y}_k^*\right)h\left(\mathbf{S}_k^*\right)q\left(\mathbf{Y}_{k+1}^*\right)z\left(\mathbf{Y}_{k+2}^*\right)$$

where g, h, q and z represent their marginal probability density functions respectively. The joint moment generating function of

$$A_k = \mathbf{Y}_k^{*\mathrm{T}}\mathbf{S}_{k-1}^{*-1}\mathbf{Y}_k^* = f_1\left(\mathbf{Y}_k^{*\mathrm{T}}\mathbf{S}_k^{*-1}\mathbf{Y}_k^*\right)$$

$$A_{k+1} = \mathbf{Y}_{k+1}^{*\mathrm{T}}\mathbf{S}_k^{*-1}\mathbf{Y}_{k+1}^* = f_2\left(\mathbf{S}_k^*, \mathbf{Y}_{k+1}^*\right)$$

$$A_{k+2} = \mathbf{Y}_{k+2}^{*\mathrm{T}}\mathbf{S}_{k+1}^{*-1}\mathbf{Y}_{k+2}^* = f_3\left(\mathbf{S}_k^*, \mathbf{Y}_{k+1}^*, \mathbf{Y}_{k+2}^*\right)$$

where f_1, f_2 and f_3 are some fuctions of the indicated arguments, is given by

$$M_{A_k, A_{k+1}, A_{k+2}}(t_1, t_2, t_3) = \int_{\substack{all\ possible \\ (A_k, A_{k+1}, A_{k+2})}} e^{t_1A_k + t_2A_{k+1} + t_3A_{k+2}} Q(A_k, A_{k+1}, A_{k+2})\, dA_k\, dA_{k+1}\, dA_{k+2}$$

where $Q\left(A_k, A_{k+1}, A_{k+2}\right)$ denotes the joint p.d.f of A_k, A_{k+1} and A_{k+2}. Integrating over the *original* space gives,

$$M_{A_k,A_{k+1},A_{k+2}}(t_1,t_2,t_3)$$

$$= \int_{\substack{all\ possible \\ (Y_k^*,Y_{k+1}^*,Y_{k+2}^*,S_k^*)}} e^{t_1 f_1\left(\mathbf{Y}_k^{*\mathrm{T}}\mathbf{S}_k^{*-1}\mathbf{Y}_k^*\right)+t_2 f_2\left(S_k^*,Y_{k+1}^*\right)+t_3 f_3\left(S_k^*,Y_{k+1}^*,Y_{k+2}^*\right)} f\left(\mathbf{Y}_k^{*\mathrm{T}}\mathbf{S}_k^{*-1}\mathbf{Y}_k^*,\mathbf{S}_k^*,\mathbf{Y}_{k+1}^*,\mathbf{Y}_{k+2}^*\right) d\mathbf{Y}_k^* d\mathbf{Y}_{k+1}^* d\mathbf{Y}_{k+2}^* d\mathbf{S}_k^*$$

$$= \int_{\substack{all\ possible \\ (Y_k^*,Y_{k+1}^*,Y_{k+2}^*,S_k^*)}} e^{t_1 f_1\left(\mathbf{Y}_k^{*\mathrm{T}}\mathbf{S}_k^{*-1}\mathbf{Y}_k^*\right)+t_2 f_2\left(S_k^*,Y_{k+1}^*\right)+t_3 f_3\left(S_k^*,Y_{k+1}^*,Y_{k+2}^*\right)} g\left(\mathbf{Y}_k^{*\mathrm{T}}\mathbf{S}_k^{*-1}\mathbf{Y}_k^*\right) h\left(\mathbf{S}_k^*\right) q\left(\mathbf{Y}_{k+1}^*\right) z\left(\mathbf{Y}_{k+2}^*\right) d\mathbf{Y}_k^* d\mathbf{Y}_{k+1}^* d\mathbf{Y}_{k+2}^* d\mathbf{S}_k^*$$

As $\mathbf{Y}_k^{*\mathrm{T}}\mathbf{S}_k^{*-1}\mathbf{Y}_k^*$ and $\mathbf{S}_k^*$ are independent and the region or space for two independent variables over which the joint density is not zero must factor (see Lindgren (1973), p.96 and Hogg and Craig (1971), p.78), from the above,

$$M_{A_k,A_{k+1},A_{k+2}}(t_1,t_2,t_3)$$

$$= \int_{\substack{all\ possible \\ \mathbf{Y}_k^{*\mathrm{T}}\mathbf{S}_k^{*-1}\mathbf{Y}_k^*}} e^{t_1 f_1\left(\mathbf{Y}_k^{*\mathrm{T}}\mathbf{S}_k^{*-1}\mathbf{Y}_k^*\right)} g\left(\mathbf{Y}_k^{*\mathrm{T}}\mathbf{S}_k^{*-1}\mathbf{Y}_k^*\right) d\left(\mathbf{Y}_k^{*\mathrm{T}}\mathbf{S}_k^{*-1}\mathbf{Y}_k^*\right)$$

$$\bullet \int_{\substack{all\ possible \\ (S_k^*,Y_{k+1}^*,Y_{k+2}^*)}} e^{t_2 f_2\left(S_k^*,Y_{k+1}^*\right)+t_3 f_3\left(S_k^*,Y_{k+1}^*,Y_{k+2}^*\right)} h\left(\mathbf{S}_k^*\right) q\left(\mathbf{Y}_{k+1}^*\right) z\left(\mathbf{Y}_{k+2}^*\right) d\mathbf{S}_k^* d\mathbf{Y}_{k+1}^* d\mathbf{Y}_{k+2}^*$$

$$= M_{A_k}(t_1).M_{A_{k+1},A_{k+2}}(t_2,t_3)$$

$$= M_{A_k}(t_1).M_{A_{k+1}}(t_2).M_{A_{k+2}}(t_3) \qquad \because A_{k+1} \text{ and } A_{k+2} \text{ are independent}$$

where M_{A_k} , $M_{A_{k+1}}$ and $M_{A_{k+2}}$ are the moment generating functions of A_k, A_{k+1} and A_{k+2} respectively. Hence, A_k, A_{k+1} and A_{k+2} are *mutually independent*. The proof can be extended to any set of A_k's in a similar manner.

3.3 Monitoring the Mean of Multivariate Normal Processes

Like the 'Q' charting approach previously proposed by Quesenberry (1991) for controlling univariate normal processes, the techniques presented in this paper involve the use of the *probability integral transformation* of some (scaled) quadratic forms in order to produce sequences of independent or approximately independent standard

normal variables. These procedures essentially enable charting to commence with the first units or samples of production whether or not prior knowledge of the process parameters is available. For the case where no relevant data is available prior to a production run, the process parameters μ and Σ are 'estimated' and 'updated' sequentially from the current data stream. These dynamic estimates, together with the next observation or subgroup are in turn used to determine whether the process is stable.

In the following sub-sections, control statistics for use with individual observations and subgroup data are given for the cases when both, either or neither of the process parameters μ and Σ are assumed known in advance of production.

3.3.1 Control Charts Based on Individual Measurements

Let $\mathbf{X}_1, \mathbf{X}_2, \ldots$ be the vectors of measurements on p quality characteristics for products produced in time sequence. Assume that these observation vectors are independently and identically distributed having been collected from a p-variate normal $N_p(\mu, \Sigma)$ process. Further, let $\overline{\mathbf{X}}_k$, $\mathbf{S}_k$ and $\mathbf{S}_{\mu,k}$ denote respectively the mean vector, the usual and the 'mean-dependent' covariance matrices of the first k observations as defined below :

$$\overline{\mathbf{X}}_k = \frac{1}{k}\sum_{i=1}^{k} \mathbf{X}_i$$

$$\mathbf{S}_k = \frac{1}{k-1}\sum_{i=1}^{k}(\mathbf{X}_i - \overline{\mathbf{X}}_k)(\mathbf{X}_i - \overline{\mathbf{X}}_k)^{\mathrm{T}}$$

$$\mathbf{S}_{\mu,k} = \frac{1}{k}\sum_{i=1}^{k}(\mathbf{X}_i - \mu)(\mathbf{X}_i - \mu)^{\mathrm{T}}$$

These values can be updated sequentially using the following recursive formulae :

$$\overline{\mathbf{X}}_k = \frac{1}{k}\left[(k-1)\overline{\mathbf{X}}_{k-1} + \mathbf{X}_k\right] \qquad k = 2,3,.....$$

$$\mathbf{S}_k = \frac{(k-2)}{(k-1)}\mathbf{S}_{k-1} + \frac{1}{k}\left(\mathbf{X}_k - \overline{\mathbf{X}}_{k-1}\right)\left(\mathbf{X}_k - \overline{\mathbf{X}}_{k-1}\right)^{\mathrm{T}} \qquad k = 3,4,.....$$

$$\mathbf{S}_{\mu,k} = \frac{k-1}{k}\mathbf{S}_{\mu,k-1} + \frac{1}{k}\left(\mathbf{X}_k - \mu\right)\left(\mathbf{X}_k - \mu\right)^{\mathrm{T}} \qquad k = 2,3,.....$$

Additionally, the following notation will be used :-

$\Phi(\bullet)$: distribution function of a standard normal variable

$\Phi^{-1}(\bullet)$: inverse of the standard normal distribution function

$\chi^2_\nu(\bullet)$: distribution function of a *chi-square* variable with ν degrees of freedom

$F_{\nu 1,\nu 2}(\bullet)$: distribution function of an F variable with $\nu 1$ numerator degrees of freedom and $\nu 2$ denominator degrees of freedom.

The appropriate control statistics are now presented as follows :

Case (I) : Both μ and Σ known

$$Z_k = \Phi^{-1}\left[\chi^2_p(T_k)\right] \qquad k = 1,2,.....$$

$$\text{where} \quad T_k = \left(\mathbf{X}_k - \mu\right)^{\mathrm{T}} \Sigma^{-1}\left(\mathbf{X}_k - \mu\right) \tag{3.3}$$

Case (II) : μ unknown, Σ known

$$Z_k = \Phi^{-1}\left[\chi^2_p(T_k)\right] \qquad k = 2,3,.....$$

$$\text{where} \quad T_k = \left(\tfrac{k-1}{k}\right)\left(\mathbf{X}_k - \overline{\mathbf{X}}_{k-1}\right)^{\mathrm{T}} \Sigma^{-1}\left(\mathbf{X}_k - \overline{\mathbf{X}}_{k-1}\right) \tag{3.4}$$

Case (III) : μ known, Σ unknown

$$Z_k = \Phi^{-1}\left[F_{p,k-p}(T_k)\right] \qquad k = p+1,.....$$

$$\text{where} \quad T_k = \frac{(k-p)}{p(k-1)}(\mathbf{X}_k - \mu)^{\mathrm{T}}\mathbf{S}_{\mu,k-1}^{-1}(\mathbf{X}_k - \mu) \tag{3.5}$$

Case (IV) : Both μ and Σ unknown

$$Z_k = \Phi^{-1}\left[F_{p,k-1-p}(T_k)\right] \qquad k = p+2,.....$$

$$\text{where} \quad T_k = \left(\frac{(k-1)(k-1-p)}{kp(k-2)}\right)(\mathbf{X}_k - \overline{\mathbf{X}}_{k-1})^{\mathrm{T}}\mathbf{S}_{k-1}^{-1}(\mathbf{X}_k - \overline{\mathbf{X}}_{k-1}) \tag{3.6}$$

As shown above, for case (I), a value of Z_k corresponds to $\mathbf{X}_k$ for all values of $k = 1,2,....$ However, no value of Z_k corresponding to the first observation $\mathbf{X}_1$ is calculated for case (II). This is due to the fact that the unknown process mean vector μ has to be estimated from $\mathbf{X}_1$ before its constancy can be subsequently monitored. For case (III), the monitoring procedure begins after $p+1$ observations. When the process parameters are unknown, control is initiated at the $(p+2)$th observation. In this case, no value is plotted prior to the $(p+2)$th observation because the sample covariance matrix $\mathbf{S}_{k-1}$, used in formula (3.6), is not positive definite and hence is not invertible for k less than $p+2$. For the latter two cases, if p is small relative to the volume of production, the effect of not charting the first few observations on the performance of the procedures is negligible. If p is relatively large, it is recommended that the quality characteristics be partitioned into smaller groups and similar procedures applied to them so that monitoring can begin sooner. The overall false alarm rate from these multiple charts can be adjusted using Bonferroni inequalities. Note that, there will be situations where historical data for

estimating the process parameters are available but insufficient for us to assume that the in-control values of the process parameters, particularly the covariance matrix, are essentially known. Under these circumstances, such data may be used to provide more stable estimators and to initiate process monitoring sooner, as considered by Quesenberry(1991a) for the univariate setting.

It is clear from the preceding section, that $\{Z_k\}$ for each case is a sequence of independently and identically distributed (i.i.d) normal variables with mean 0 and standard deviation 1 under the stable in-control normality assumption. The distributional property obtained for case (IV) is the most pertinent result for short run SPC.

As the plot statistics for all cases are sequences of i.i.d. $N(0, 1)$ variables, the resulting control charts can be constructed using the same scale and with the common Shewhart control limits ±3. In addition, supplementary run rules can be employed to reveal any assignable causes hidden in the point patterns of the charts. Although the argument statistics $T_k's$ can be plotted instead of the transformed variables $Z_k's$, this practice is not recommended because for those cases other than case (I), the use of $T_k's$ involves the added complexity of varying control limits. The use of these procedures is illustrated later.

3.3.2 Control Charts Based on Subgroup Data

In practice, the use of subgroup data is often preferable to individual measurements even in situations where only a limited amount of data are available. This is due to the fact that the resulting control charts are more sensitive to substantial shifts in the process average and that the subgroup mean (vector) is less affected by departure from the underlying normality assumption, by virtue of the central limit theorem.

Adaptation of the above formulae yields appropriate control statistics for monitoring the stability of the process mean vector based on subgroup data. Discussion will be restricted to the case of *constant* subgroup size. Let $\mathbf{X}_{ij}$, n and k denote respectively the jth observation vector of the ith subgroup, the common subgroup size and the subgroup number, and define the following quantities :

$$\overline{\mathbf{X}}_{(i)} = \frac{1}{n}\sum_{j=1}^{n}\mathbf{X}_{ij}$$

$$\overline{\overline{\mathbf{X}}}_{(k)} = \frac{1}{k}\sum_{i=1}^{k}\overline{\mathbf{X}}_{(i)} = \frac{1}{k}\left[(k-1)\overline{\overline{\mathbf{X}}}_{(k-1)} + \overline{\mathbf{X}}_{(k)}\right]$$

$$\mathbf{S}_{(i)} = \frac{1}{n-1}\sum_{j=1}^{n}\left(\mathbf{X}_{ij} - \overline{\mathbf{X}}_{(i)}\right)\left(\mathbf{X}_{ij} - \overline{\mathbf{X}}_{(i)}\right)^{\mathrm{T}}$$

$$\mathbf{S}_{\mu}^{(k)} = \frac{1}{kn}\sum_{i=1}^{k}\sum_{j=1}^{n}\left(\mathbf{X}_{ij} - \mu\right)\left(\mathbf{X}_{ij} - \mu\right)^{\mathrm{T}}$$

$$= \frac{1}{k}\left[(k-1)\mathbf{S}_{\mu}^{(k-1)} + \frac{1}{n}\sum_{j=1}^{n}\left(\mathbf{X}_{kj} - \mu\right)\left(\mathbf{X}_{kj} - \mu\right)^{\mathrm{T}}\right]$$

$$\mathbf{S}_{pooled}^{(k)} = \frac{1}{k}\sum_{i=1}^{k}\mathbf{S}_{(i)} = \frac{1}{k}\left[(k-1)\mathbf{S}_{pooled}^{(k-1)} + \mathbf{S}_{(k)}\right]$$

where $\mathbf{S}_{\mu}^{(0)} = \mathbf{S}_{pooled}^{(0)} = \mathbf{0}$. The same technique is now applied to transform the sample mean vectors $\overline{\mathbf{X}}_{(i)}$'$s$ to standard normal variables for each of the cases considered above for individual measurements.

Case (I) : Both μ and Σ known

$$Z_k = \Phi^{-1}\left[\chi_p^2(T_k)\right] \qquad k = 1,2,\ldots\ldots$$

$$\text{where } T_k = n\left(\overline{\mathbf{X}}_{(k)} - \mu\right)^{\mathrm{T}} \Sigma^{-1}\left(\overline{\mathbf{X}}_{(k)} - \mu\right) \tag{3.7}$$

Case (II) : μ unknown, Σ known

$$Z_k = \Phi^{-1}\left[\chi_p^2(T_k)\right] \qquad k = 2,3,\ldots\ldots$$

$$\text{where } T_k = \left(\frac{n(k-1)}{k}\right)\left(\overline{\mathbf{X}}_{(k)} - \overline{\overline{\mathbf{X}}}_{(k-1)}\right)^{\mathrm{T}} \Sigma^{-1}\left(\overline{\mathbf{X}}_{(k)} - \overline{\overline{\mathbf{X}}}_{(k-1)}\right) \tag{3.8}$$

Case (III) : μ known, Σ unknown

Two alternative techniques which employ different process covariance matrix estimates will be considered for this case. The first one incorporates μ into its 'running' estimate of Σ whereas the other ignores knowledge of μ and is based on the pooled estimate $\mathbf{S}_{pooled}^{(k)}$. It will be seen in section 3.5 that the latter gives better run length performance.

(a) Uses μ in estimation of Σ

$$Z_k = \Phi^{-1}\left[F_{p,n(k-1)-p+1}(T_k)\right] \qquad k = 2,3,\ldots\ldots \quad , \; n \geq p$$

$$\text{where } T_k = \left(\frac{n(k-1)-p+1}{p(k-1)}\right)\left(\overline{\mathbf{X}}_{(k)} - \mu\right)^{\mathrm{T}} \left(\mathbf{S}_{\mu}^{(k-1)}\right)^{-1}\left(\overline{\mathbf{X}}_{(k)} - \mu\right) \tag{3.9}$$

(b) Uses Pooled Sample Covariance Matrix

$$Z_k = \Phi^{-1}\left[F_{p,k(n-1)-p+1}(T_k)\right] \qquad k = 1,2,\ldots\ldots \quad , \; n \geq p+1$$

$$\text{where } T_k = \left[\frac{n[k(n-1)-p+1]}{pk(n-1)}\right]\left(\overline{\mathbf{X}}_{(k)} - \mu\right)^{\mathrm{T}} \left(\mathbf{S}_{pooled}^{(k)}\right)^{-1}\left(\overline{\mathbf{X}}_{(k)} - \mu\right) \tag{3.10}$$

Case (IV) : Both μ and Σ unknown

$$Z_k = \Phi^{-1}\left[F_{p,k(n-1)-p+1}(T_k)\right] \qquad k = 2,3,..... \qquad , n \geq \frac{p}{2}+1$$

$$\text{where } T_k = \left(\frac{n(k-1)(k(n-1)-p+1)}{k^2 p(n-1)}\right)\left(\overline{\mathbf{X}}_{(k)} - \overline{\overline{\mathbf{X}}}_{(k-1)}\right)^{\mathrm{T}}\left(\mathbf{S}^{(k)}_{pooled}\right)^{-1}\left(\overline{\mathbf{X}}_{(k)} - \overline{\overline{\mathbf{X}}}_{(k-1)}\right) \qquad (3.11)$$

Under the assumption that the $\mathbf{X}_{ij}$'s are independent observation vectors obtained from a common process with a $N_p(\mu,\Sigma)$ distribution, the control statistics given by (3.7), (3.8) and (3.9) are sequences of i.i.d. $N(0, 1)$ variables (see section 3.2). For those given by (3.10) and (3.11), successive plotted values arise from a standard normal distribution but they are correlated due to the use of the pooled sample covariance matrix, $\mathbf{S}^{(k)}_{pooled}$. Sequences of independently distributed variables can be obtained for these latter cases by replacing the $\mathbf{S}^{(k)}_{pooled}$ by the sample covariance matrix of the current subgroup, $\mathbf{S}_{(k)}$. This is not considered, however, because it is found that the performances of the resulting charts are poor, even when some additional run rules are used. Furthermore, ignoring the issue of dependence among the Z_k's of (3.10) and (3.11) has no significant effect on the false signal rate. In particular, the probability of getting a false signal from *any* one of the first 50 subgroups using both techniques was simulated for various combinations of p and n based on 5,000 runs when either 3-sigma limits or only the upper 99.73th probability limit is used. The results, which are tabulated in Table 3.1, appear to agree well with the nominal value of $1-0.9973^{50} = 0.1264$. The reason the above criterion is chosen as a measure of in-control performance instead of the usual average run length is that the control statistics given by (3.10) and (3.11) are primarily concerned with short production run situations in which the total number of

samples rarely exceeds 50. Furthermore, note that for case (IV), it is possible for each individual sample covariance matrix to be of less than full rank provided the (common) sample size is not less than $\frac{p}{2}+1$. This is because for the pooled covariance matrix, $\mathbf{S}_{pooled}^{(k)}$ to be distributed as a positive definite scaled Wishart matrix such that T_k is well defined, the total degrees of freedom for the k possibly rank deficient sample covariance matrices that form $\mathbf{S}_{pooled}^{(k)}$, $k(n-1)$ must be at least p, or equivalently, $n \geq \frac{p}{k}+1$ (see Sullivan et al.(1995)). For this condition to hold for all k, starting from $k=2$, the minimum sample size is thus $n=\left[\frac{p}{2}+1\right]$ where $[\bullet]$ denotes the greatest integer function.

A remark should be made about the computation of the argument statistics for (3.3) to (3.11) above which involve evaluation of the inverse of a matrix. In fact, each of these arguments can be expressed as a quotient of two determinants, thus eliminating the need for inverting either the known process covariance matix or some estimates of it (see for example, Morrison (1976), p.134). For instance, the argument statistic of (3.6) has the following alternative expression

$$T_k = \frac{\left|\mathbf{S}_{k-1} + \frac{(k-1)(k-1-p)}{kp(k-2)}\left(\mathbf{X}_k - \overline{\mathbf{X}}_{k-1}\right)\left(\mathbf{X}_k - \overline{\mathbf{X}}_{k-1}\right)^{\mathrm{T}}\right|}{\left|\mathbf{S}_{k-1}\right|} - 1$$

Evaluating an expression of this form is much more convenient than that of its original form especially when the number of quality characteristics, p is large.

For some general guidelines on using the above control charting approach, readers are referred to the article by Quesenberry (1991a) from which the ideas of this present work originate.

TABLE 3.1. Probability of a False Signal from Any One of the First 50 Subgroups.

		Control Statistic	
p	n	(3.10)	(3.11)
2	3	0.1340 *	0.1196
		(0.1242) †	(0.1266)
	4	0.1240	0.1270
		(0.1172)	(0.1284)
	5	0.1208	0.1220
		(0.1176)	(0.1152)
3	4	0.1200	0.1160
		(0.1186)	(0.1146)
	5	0.1224	0.1194
		(0.1288)	(0.1168)
	6	0.1284	0.1202
		(0.1186)	(0.1248)
5	6	0.1190	0.1210
		(0.1256)	(0.1250)
	7	0.1222	0.1230
		(0.1260)	(0.1210)
	8	0.1192	0.1218
		(0.1220)	(0.1114)

* Unbracketed values correspond to the use of *upper* control limit only with $\alpha = 0.0027$.

† Bracketed values correspond to the use of two-sided 3-sigma control limits.

3.4 Examples

In this section, use of the proposed techniques are illustrated by some numerical examples based on simulated data as well as using data from a previously published article. The examples presented here are not intended to cover all possible situations, however, they provide some insight into the behaviour of the proposed techniques under various circumstances.

Example 3.4.1

The first illustration uses the formulae for *individual* measurements. 30 observations have been generated from a bivariate normal distribution with the following parameters

$$\mu = \begin{pmatrix} 10 \\ 15 \end{pmatrix}, \qquad \Sigma = \begin{pmatrix} 1 & 1.275 \\ 1.275 & 2.25 \end{pmatrix}.$$

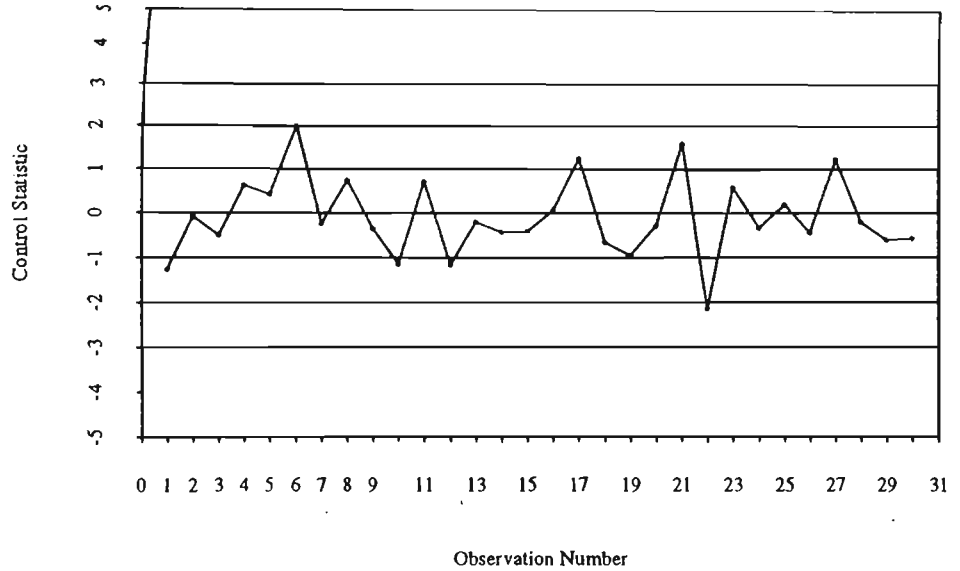
These data are shown in Table 3.2, along with the computed values of the control statistics (3.3) to (3.6). The corresponding control charts have also been constructed as shown in Figures 3.1(a) to 3.1(d). It should be noted that the use of 3-σ control limits in this and subsequent examples is merely for the purpose of illustration. In practice, it may be preferable to use narrower control limits or only the upper control limit in line with the traditional Hotelling T^2 charting approach.

Note that for the cases with some unknown parameters, the corresponding statistics were computed using the values of μ and Σ as given above. Note also that since $p = 2$, calculations of the plotted values of the control statistics (3.4), (3.5) and (3.6) have been started with the 2nd, 3rd and 4th observations respectively. As shown in the figures, none of the plotted points exceed the control limits for each of the control

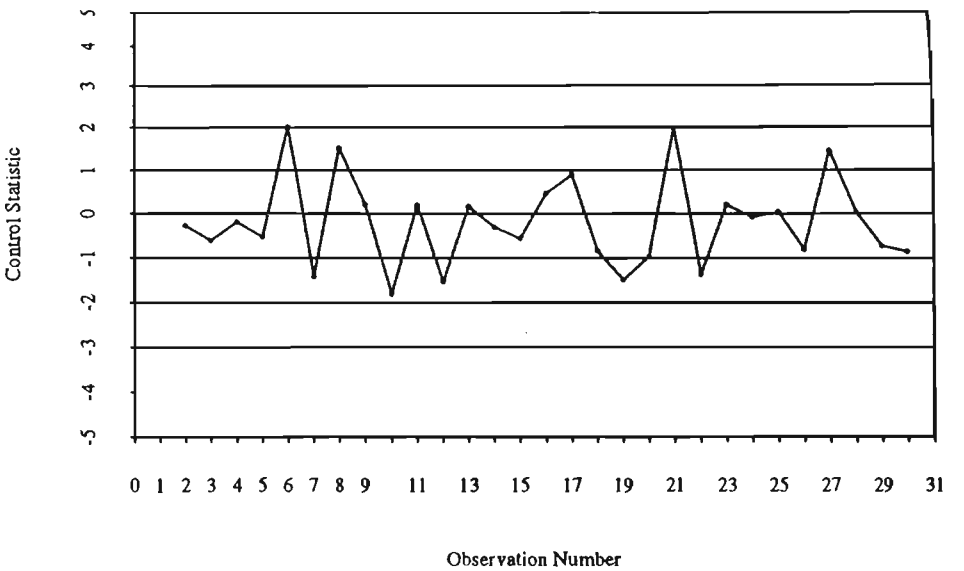
charts. This is as expected because the data for this example can be regarded as having been collected from an in-control process. It is also interesting to note that after the first few observations, the movement of the charted points are very similar for all cases. This phenomenon is typical for in-control multivariate normal processes.

TABLE 3.2. Simulated Data and Values of The Control Statistics based on Individual Measurements for Example 3.4.1

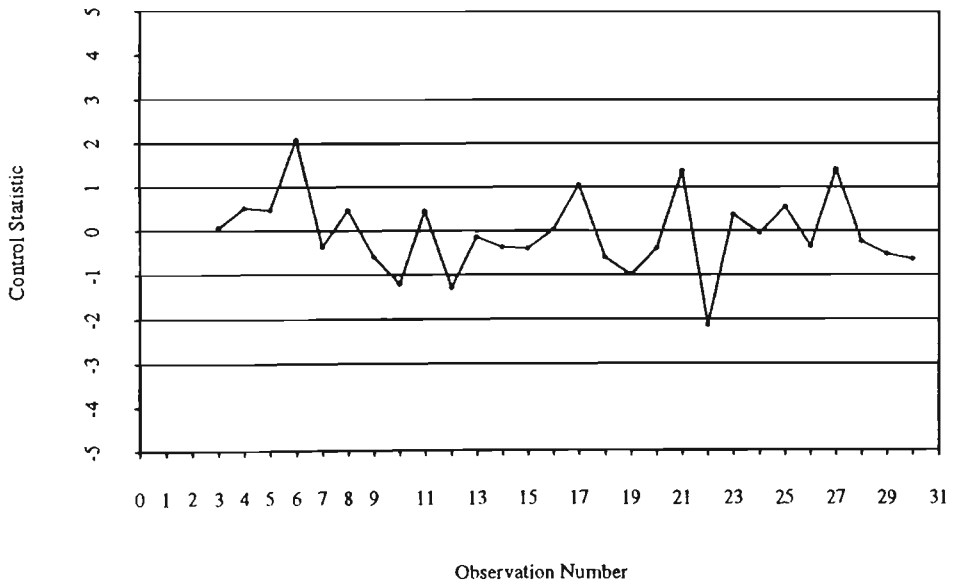
Obs. No.	Variable 1	Variable 2	Control Statistic (3.3)	(3.4)	(3.5)	(3.6)
1	10.39	15.70	-1.27	NA	NA	NA
2	9.02	14.19	-0.08	-0.28	NA	NA
3	9.28	13.71	-0.50	-0.62	0.07	NA
4	8.67	14.04	0.61	-0.19	0.52	-0.32
5	9.22	14.99	0.40	-0.55	0.46	-0.21
6	8.82	11.54	1.98	1.99	2.09	1.56
7	9.07	14.14	-0.24	-1.39	-0.37	-1.52
8	11.70	17.31	0.73	1.50	0.47	1.83
9	10.92	16.35	-0.35	0.22	-0.60	-0.07
10	9.64	14.84	-1.15	-1.80	-1.21	-1.91
11	8.31	12.85	0.70	0.18	0.45	-0.01
12	9.56	14.23	-1.16	-1.55	-1.29	-1.56
13	10.39	14.74	-0.22	0.15	-0.14	0.19
14	10.11	15.84	-0.43	-0.30	-0.36	-0.33
15	9.53	13.79	-0.42	-0.57	-0.39	-0.55
16	11.15	16.77	0.05	0.46	0.01	0.45
17	8.03	11.89	1.24	0.88	1.05	0.72
18	9.68	14.04	-0.67	-0.86	-0.62	-0.72
19	9.57	14.11	-0.95	-1.48	-1.00	-1.39
20	9.05	13.58	-0.29	-0.98	-0.40	-1.01
21	12.40	18.15	1.59	1.98	1.37	1.80
22	10.17	15.17	-2.14	-1.37	-2.12	-1.49
23	8.41	13.00	0.58	0.22	0.38	0.07
24	10.31	16.11	-0.33	-0.07	-0.03	0.08
25	9.66	13.57	0.19	0.05	0.55	0.44
26	9.50	13.78	-0.44	-0.80	-0.35	-0.62
27	10.91	17.65	1.22	1.44	1.40	1.52
28	10.59	16.43	-0.20	0.03	-0.23	-0.11
29	9.71	14.05	-0.62	-0.73	-0.54	-0.55
30	9.48	13.85	-0.60	-0.87	-0.66	-0.86



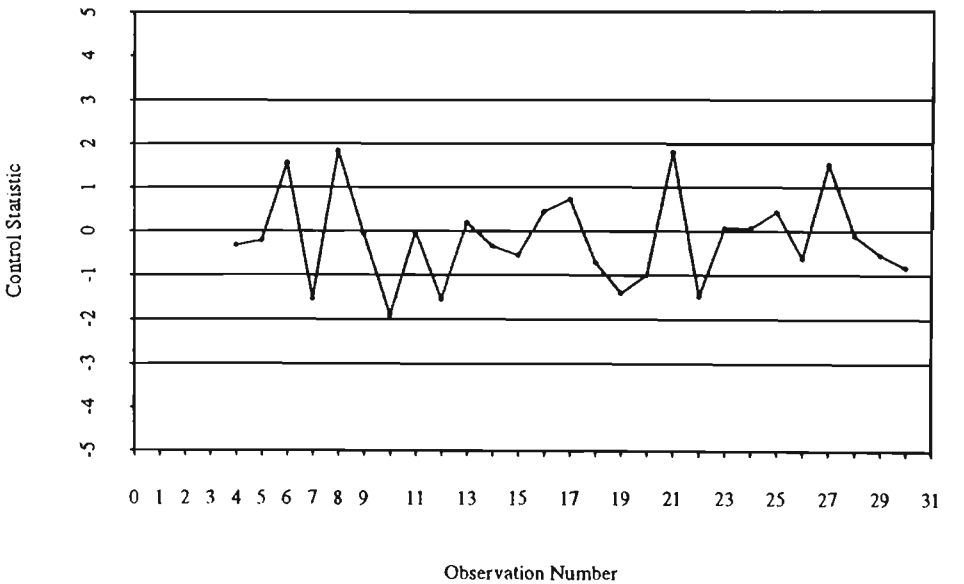
(a) Known Parameters



(b) Unkown Mu, Known Sigma



(c) Known Mu, Unknown Sigma



(d) Unknown Parameters

Figure 3.1. Multivariate Control Charts for Example 3.4.1.

Example 3.4.2

Next, in order to demonstrate the behaviour of the control charts based on *subgroup* data for a *stable* process, 120 observations have been generated from a trivariate normal distribution with

$$\mu = \begin{pmatrix} 5 \\ 12 \\ 15 \end{pmatrix} \quad \text{and} \quad \Sigma = \begin{pmatrix} 0.0625 & 0.10625 & 0.05 \\ 0.10625 & 0.25 & 0.136 \\ 0.05 & 0.136 & 0.16 \end{pmatrix}.$$

These observations are grouped into samples of size $n = 4$ and the corresponding values of the control statistics (3.7), (3.8), (3.9), (3.10) and (3.11) are calculated and plotted in Figures 3.2(a) to 3.2(e). Note that except for the case with known parameters and when the statistic (3.10) is used, charting begins with the 2nd subgroup.

Again, as shown in the figures, the point patterns of the resulting control charts are similar after the first few points. This similarity in the point patterns will generally be the case for other in-control processes and the appearance of the control charts will be more similar as the subgroup size increases. Following Quesenberry's (1991) suggestion, since μ and Σ are not likely to be known precisely, the safer approach to charting is to use (3.6) and (3.11) which do not assume known values for the process parameters.

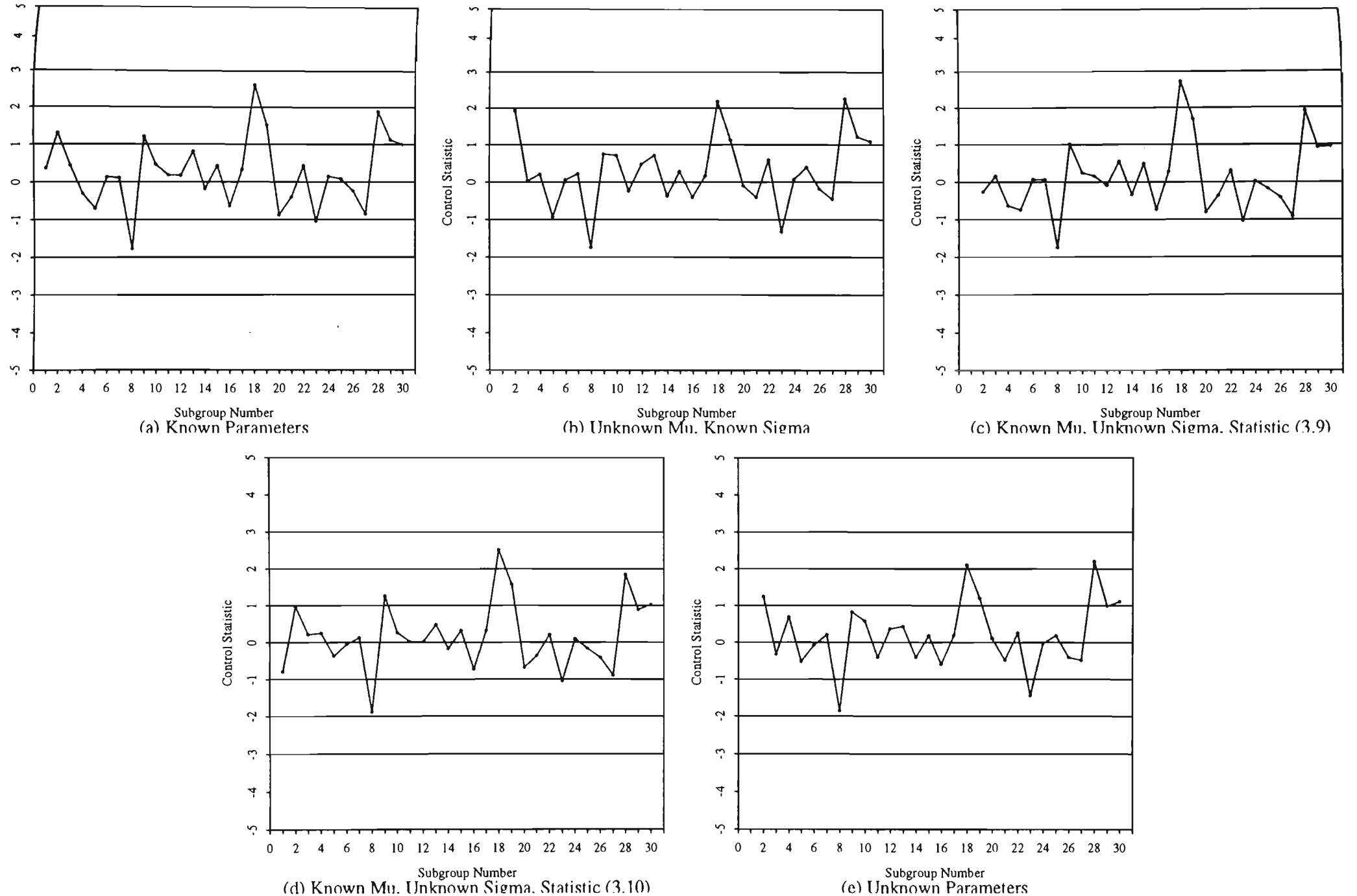


Figure 3.2. Multivariate Control Charts for Example 3.4.2.

Example 3.4.3

In this example, two charts are shown in order to compare the control techniques based on *subgroup* data for the cases when the parameters μ and Σ are assumed known and when they are both unknown. Figures 3.3(a) and 3.3(b) show these control charts for 30 samples of 3 observations each from a bivariate process where the mean of the 1st variable increases by 1.5 standard deviations after the 8th sample. The first 8 samples were generated from a $N_2(\mu, \Sigma)$ process distribution with

$$\mu = \begin{pmatrix} 10 \\ 23 \end{pmatrix} \qquad \text{and} \qquad \Sigma = \begin{pmatrix} 0.25 & 0.375 \\ 0.375 & 1 \end{pmatrix}$$

whereas samples 9 to 30 were simulated from $N_2(\mu_{new}, \Sigma)$ where

$$\mu_{new} = \begin{pmatrix} 10.75 \\ 23 \end{pmatrix}.$$

Note that the correct values of μ and Σ to use for computing statistic (3.7) are those before the shift occurs. The figures show that whilst the shift is large enough to trigger out-of-control signals from both charts as soon as subgroup 9 is observed, that based on unknown parameters gradually settles into a pattern indicative of in-control conditions. This is due to the fact that the corresponding control statistic utilizes the current data stream to estimate the unknown values of the process parameters sequentially, causing the effect of parameter changes to 'dilute' as more out-of-control data are incorporated into the parameter estimates. It should be noted, that if an outlier or out-of-control observation (or subgroup) is present, that observation should be removed from subsequent computations. If this is not done, the parameter estimates will be distorted, causing an out-of-control process to appear in-control or vice versa.

Figure 3.3(a). Multivariate Control Chart for Example 3.4.3, Known Parameters.

Figure 3.3(b). Multivariate Control Chart for Example 3.4.3, Unknown Parameters.

Example 3.4.4

As a preliminary investigation of the effect of changes in process standard deviations on the *individual* values control charts, 30 observations are considered that have been generated from a trivariate normal process where the standard deviations of two of the variables double after the 13th observation. Observations 1 to 13 were generated from $N_3(\mu, \Sigma)$ with

$$\mu = \begin{pmatrix} 8 \\ 25 \\ 27 \end{pmatrix} \quad \text{and} \quad \Sigma = \begin{pmatrix} 0.01 & 0.042 & 0.0825 \\ 0.042 & 1.44 & -0.18 \\ 0.0825 & -0.18 & 2.25 \end{pmatrix}$$

while the remaining subsequent observations were generated from $N_3(\mu, \Sigma_{new})$ where

$$\Sigma_{new} = \begin{pmatrix} 0.01 & 0.084 & 0.165 \\ 0.084 & 5.76 & -0.72 \\ 0.165 & -0.72 & 9 \end{pmatrix}.$$

The individual values control charts for the case when both or neither of the process parameters are assumed known were constructed based on these simulated data and these are displayed in Figures 3.4(a) and 3.4(b) respectively.

As shown, the change in the process standard deviations causes a spike on both of the control charts at the 15th and the 17th observation respectively. However, the signal from the latter is less pronounced than that corresponding to the known parameter case. This example demonstrates that the individual values control chart, which does not assume known values of the process parameters, can also detect changes in the process covariance matrix Σ.

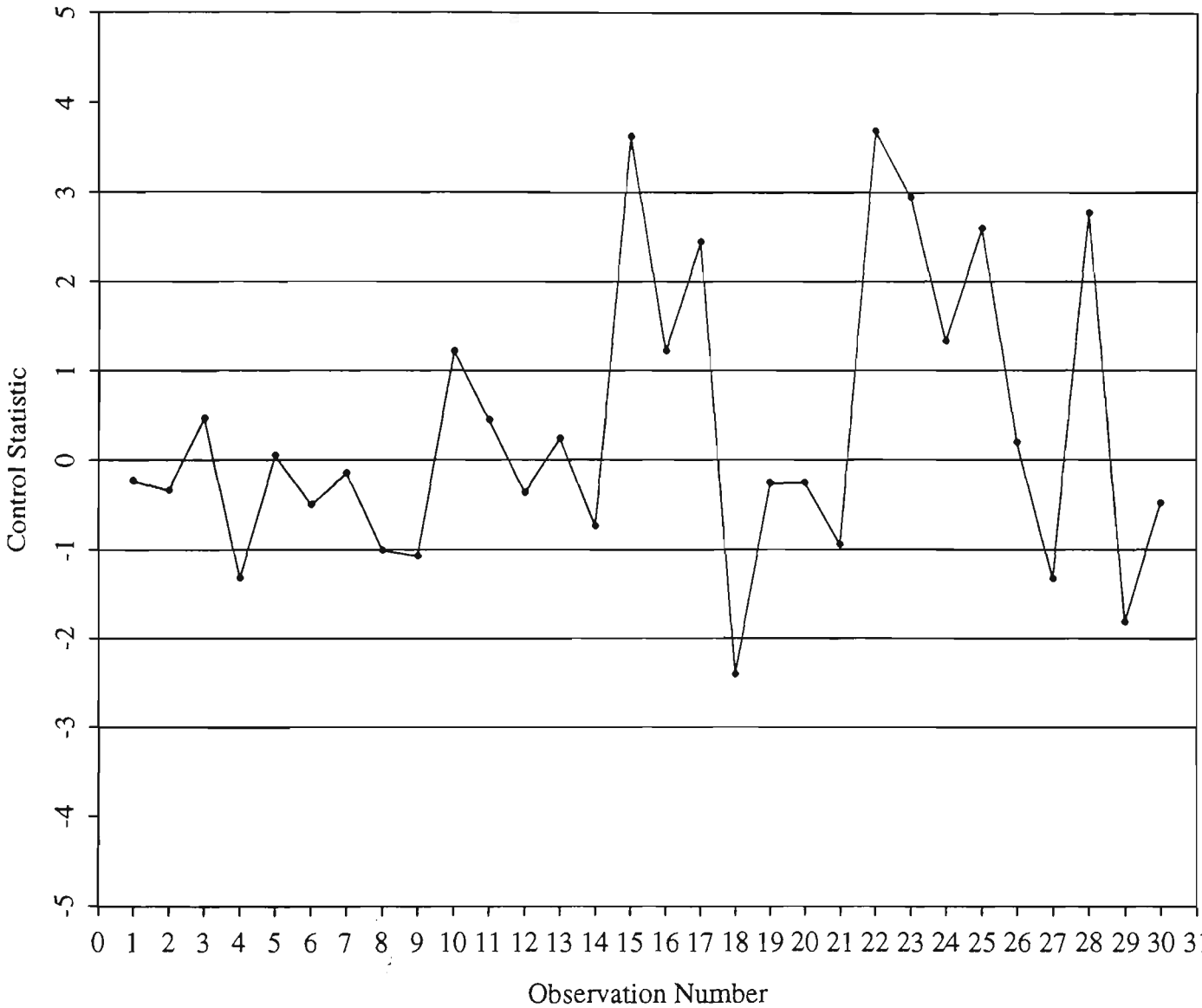


Figure 3.4(a). Multivariate Control Chart for Example 3.4.4, Known Parameters.

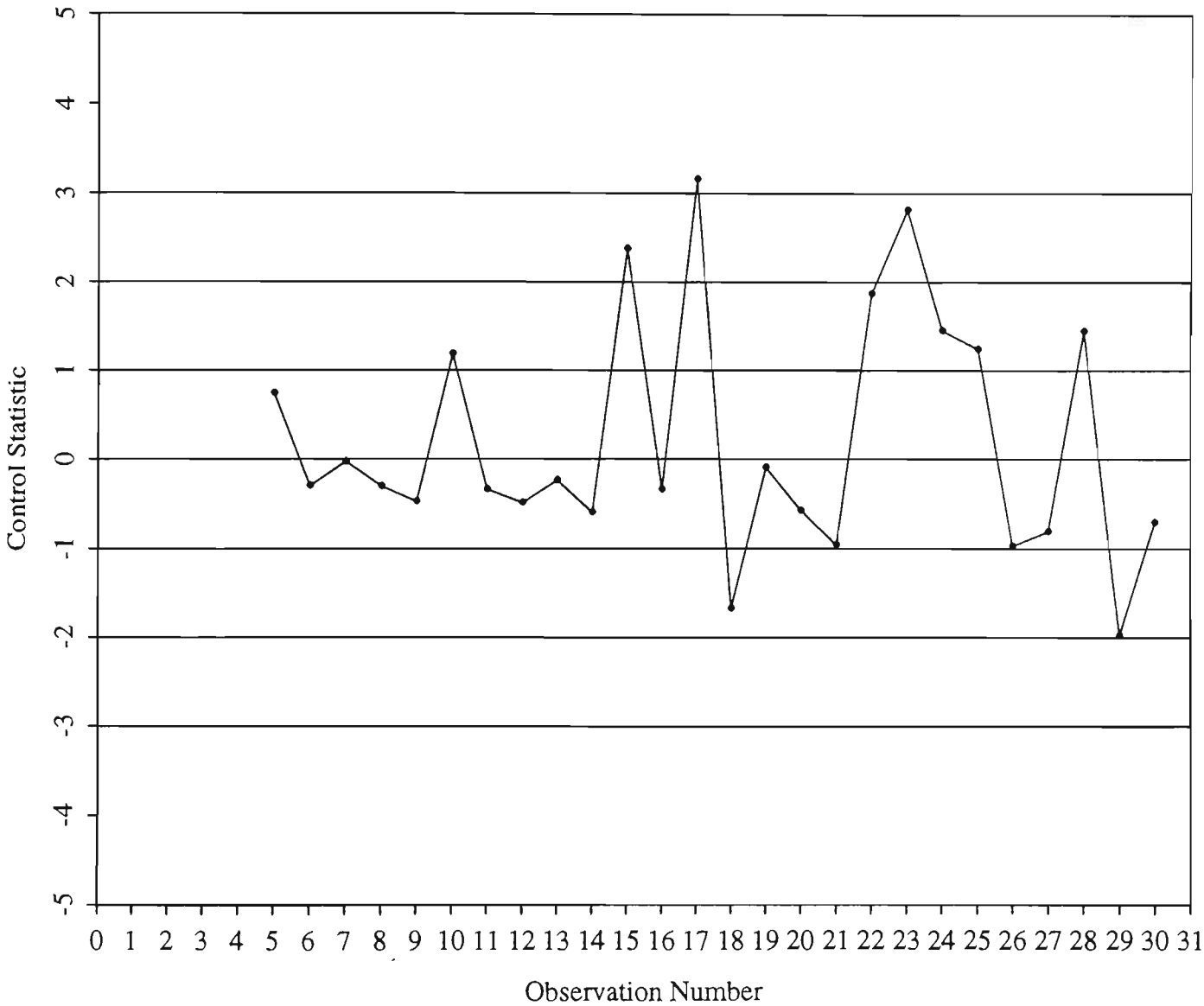


Figure 3.4(b). Multivariate Control Chart for Example 3.4.4, Unknown Parameters.

Example 3.4.5

To illustrate the usefulness of individual values control procedure for the unknown parameters case, consider the data analysed by Holmes et al.(1993) which gives the composition of 'grit' manufactured by a plant in Europe. These data are reproduced in Table 3.3 where the columns headed L, M, and S show the percentage classified as large, medium and small respectively, for each of the 56 observations. Since the percentages sum to 100 and the sample variance-covariance matrix will not be invertible under this condition, we analyse the data in the first two columns only, as Holmes et al. did, using a technique based on statistic (3.6). The computed values of this statistic are provided in the same table and the associated control chart is shown in Figure 3.5. Note that, as opposed to Holmes et al.'s retrospective testing for process stability based on all these 56 observations, the proposed charting is done essentially in real time i.e it begins as soon as the 4th observation is available. Thus, this approach has the potential to react to process instability sooner than the former method.

As shown in the figure, a signal is triggered by the 26th observation which, according to Holmes et al., actually occurred in the presence of process troubles. This observation is thus deleted from subsequent computations. To stress this point, the charted points corresponding to this and the next observation are disconnected. Although the plotted point for the 45th observation which was also affected by some identified special causes does not exceed the 3-σ control limits, using the common run rule that signals when 2 of 3 consecutive points exceed 2-σ limits (in this case ± 2) on the same side of the center line, the process trouble is detected as soon as the next observation is available. This additional rule provides enhanced detection capability with little loss in the in-control RL performance, especially for short run situations (refer to Palm (1990)

for the in-control RL probabilities of the combined test above). It is, in fact, the ability to recognise many anomalous point patterns through, for instance, the use of such additional run rules that the strength of the proposed charting technique lies.

Table 3.3. Holmes and Mergen (1993)'s Data and Values of Statistic (3.6).

i	L	M	S	Z_i
1	5.4	93.6	1	NA
2	3.2	92.6	4.2	NA
3	5.2	91.7	3.1	NA
4	3.5	86.9	9.6	0.6399
5	2.9	90.4	6.7	-0.4774
6	4.6	92.1	3.3	-1.4148
7	4.4	91.5	4.1	-2.0361
8	5	90.3	4.7	-0.1776
9	8.4	85.1	6.5	2.7482
10	4.2	89.7	6.1	-1.1743
11	3.8	92.5	3.7	-0.7038
12	4.3	91.8	3.9	-1.3520
13	3.7	91.7	4.6	-1.0359
14	3.8	90.3	5.9	-0.8824
15	2.6	94.5	2.9	0.5530
16	2.7	94.5	2.8	0.2870
17	7.9	88.7	3.4	1.4587
18	6.6	84.6	8.8	1.4113
19	4	90.7	5.3	-1.3677
20	2.5	90.2	7.3	0.6618
21	3.8	92.7	3.5	-0.7556
22	2.8	91.5	5.7	-0.2284
23	2.9	91.8	5.3	-0.4814
24	3.3	90.6	6.1	-0.5848
25	7.2	87.3	5.5	0.8209
26	7.3	79	13.7	3.2867*
27	7	82.6	10.4	2.0908
28	6	83.5	10.5	1.4377
29	7.4	83.6	9	1.0241
30	6.8	84.8	8.4	0.3840
31	6.3	87.1	6.6	-0.4525
32	6.1	87.2	6.7	-0.6524
33	6.6	87.3	6.1	-0.2495
34	6.2	84.8	9	0.3005
35	6.5	87.4	6.1	-0.3970
36	6	86.8	7.2	-0.7454
37	4.8	88.8	6.4	-1.6929
38	4.9	89.8	5.3	-1.9147
39	5.8	86.9	7.3	-0.7932
40	7.2	83.8	9	0.5805
41	5.6	89.2	5.2	-0.9938
42	6.9	84.5	8.6	0.2369
43	7.4	84.4	8.2	0.3382
44	8.9	84.3	6.8	1.3784
45	10.9	82.2	6.9	2.4500
46	8.2	89.8	2	2.0966
47	6.7	90.4	2.9	0.7397
48	5.9	90.1	4	-0.3457
49	8.7	83.6	7.7	0.6670
50	6.4	88	5.6	-1.1449
51	8.4	84.7	6.9	0.3555
52	9.6	80.6	9.8	1.4025
53	5.1	93	1.9	0.8303
54	5	91.4	3.6	-0.2968
55	5	86.2	8.8	0.4030
56	5.9	87.2	6.9	-1.4174

* This observation is removed from computation of subsequent Z_i's.

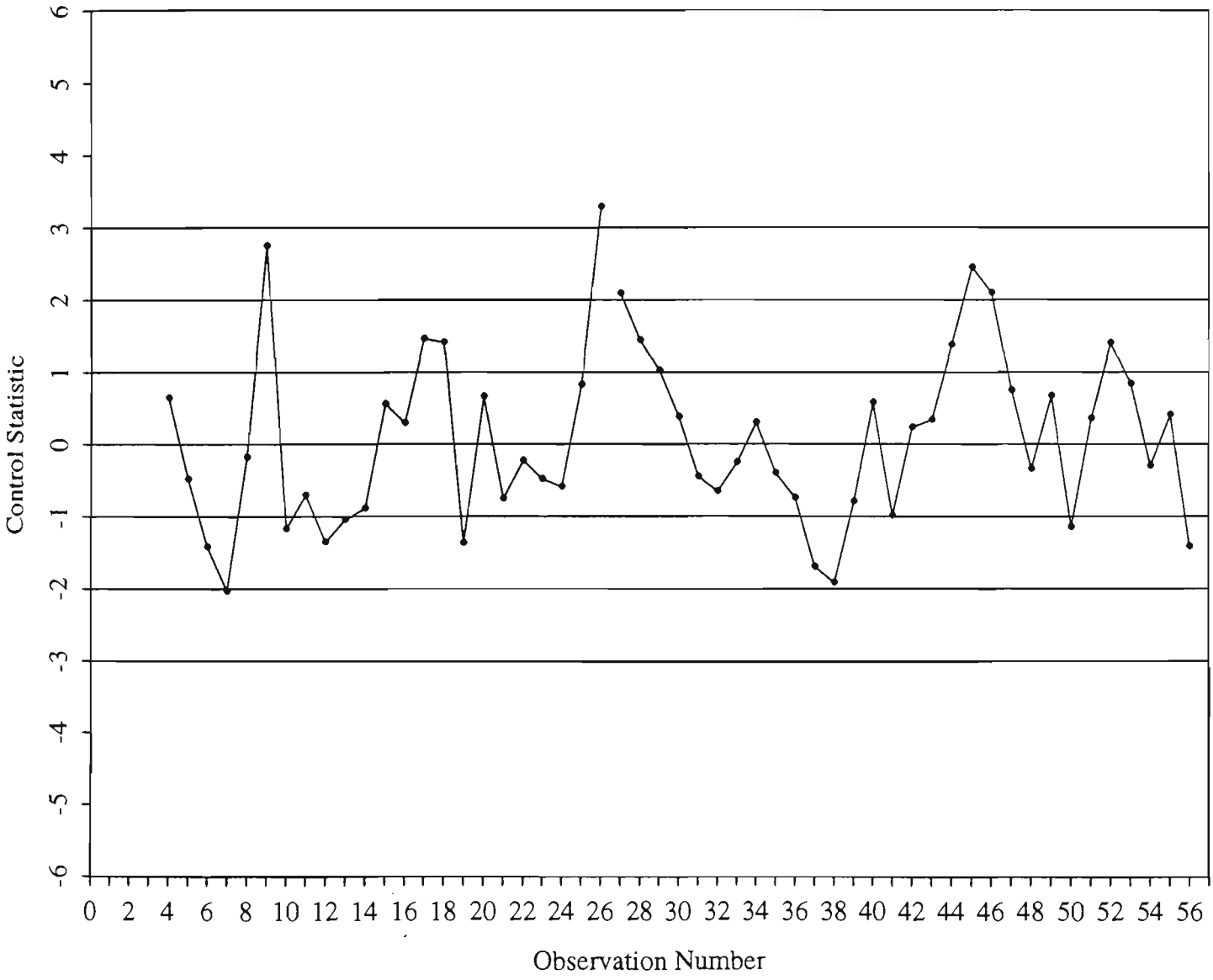


Figure 3.5. Multivariate Control Chart for Holmes and Mergen's Data.

Example 3.4.6

To see how the control technique with unknown parameters that is based on *subgroup* data performs in comparison to an existing method, consider the data given by Alt et al.(1976). These authors presented formulae to compute the control limits for the T^2-type control chart based on a small number of subgroups, both for retrospective and future testing of the mean level of a multivariate normal process and used the data to illustrate the use of these so-called *small sample probability limits*. The data consists of measurements on $p = 2$ quality characteristics which are grouped into subgroups of size $n = 10$. Due to limited space, the summary data are not reproduced here. However, for ease of comparison, the values of the T^2-type statistic, the stage 1 (retrospective) and stage 2 (future) control limits are given in Table 3.4, together with the results obtained using the technique (3.11) proposed in this paper. Note that the stage 1 and stage 2 control limits are set at $\alpha = 0.001$ and $\alpha = 0.005$ respectively.

As shown in the table, the use of the proposed technique also results in an abnormally large value of the control statistic for subgroup 4 as when Alt et al.'s method is used, indicating that the process was out-of-control when this sample was taken. Similarly, subgroup 8 is also detected as being out-of-control using both control procedures.

TABLE 3.4. Values of Alt et al.'s Test Statistic, Small Sample Probability Limits and Control Statistic (3.11) for Example 3.4.6.

Subgroup No.	Alt et al.'s Stage 1 Statistic	Stage 1 UCL	Revised Value of Alt et al's Statistic	Revised Stage 1 UCL	Control Statistic (3.11)
1	0.009	1.3268	0.327	0.9546	NA
2	1.147	1.3268	0.264	0.9546	1.0758
3	0.136	1.3268	0.034	0.9546	-1.4909
4	4.901*	1.3268	NA		5.0192*
5	0.632	1.3268	0.057	0.9546	-0.5821†
Subgroup No.	Alt et al.'s Stage 2 ‡ Statistic	Stage 2 UCL ‡			
6	0.392	1.5906			0.7766
7	0.197	1.5906			0.5906
8	4.594*	1.5906			4.8528*
9	0.190	1.5906			0.4045†
10	0.226	1.5906			0.3248
11	0.410	1.5906			1.0596
12	0.460	1.5906			0.9490

* These numbers exceed their respective UCLs indicating the presence of assignable causes.

† These and subsequent values are calculated after removing the out-of control subgroups immediately preceding them.

‡ These are based on subgroups 1, 2, 3 and 5.

3.5 Control Performance

It is shown in Appendix (A.1), (A.2) and (A.3), that the *statistical* performance of the techniques presented above depend on the following parameter(s) (scalar, vector or matrix) for each of the given types of process changes (besides the *change point r*, i.e the observation or sample after which the change takes place) :

(a) A sustained shift in the mean vector from μ to μ_{new} whilst Σ remains unchanged

$$\lambda = \sqrt{(\mu_{new} - \mu)^T \Sigma^{-1} (\mu_{new} - \mu)}$$

(b) A sustained shift in covariance matrix from Σ to Σ_{new} whilst μ remains unchanged

Eigenvalues, $\lambda_1,\ldots,\lambda_p$, of $\Sigma_{new}\Sigma^{-1}$ or $\Sigma^{-1}\Sigma_{new}$

(c) A simultaneous sustained shift in mean vector from μ to μ_{new} and covariance matrix from Σ to Σ_{new}

$$\eta = \Sigma^{-\frac{1}{2}}(\mu_{new} - \mu) \quad \text{and} \quad \Omega = \Sigma^{-\frac{1}{2}}\Sigma_{new}\Sigma^{-\frac{1}{2}}$$

Note that $\Sigma^{\frac{1}{2}}$ here denotes the symmetric *square root matrix* of Σ such that $\Sigma = \Sigma^{\frac{1}{2}}\Sigma^{\frac{1}{2}}$ (Johnson and Wichern (1988), p.51) and $\Sigma^{-\frac{1}{2}} = \left(\Sigma^{\frac{1}{2}}\right)^{-1}$. The importance of these results is clear when one realizes that the effort for determining the control performance of the proposed techniques is greatly reduced. For instance, in order to determine the performance under the first type of process change, it may be assumed, without loss of generality, that $\mu = (0,0,\ldots,0)^T$, $\Sigma = \mathbf{I}$ and μ_{new} subsequently

considered in the form of $\mu_{new} = (\lambda,0,\ldots,0)^T$ for various values of λ. Note that λ is sometimes referred to as the *noncentrality parameter* (eg., Lowry et al.(1992)). Some issues of importance regarding the use of λ are discussed in the same paper (see also Pignatiello and Runger (1990)).

In this section, we will consider only the simplest type of process change, namely, a persistent change in the process mean vector. The performance of the proposed techniques are evaluated on the basis of *probability of detection within* $m = 5$ *successive observations or subgroups* by means of simulation. This is chosen as the performance criterion instead of the common measure of ARL because, as demonstrated in the examples using simulated data, the first few observations or subgroups after the change are the critical ones. If the mean shift is not detected within the first few observations or subgroups after its occurrence, it is even more unlikely that this will be 'picked up' by subsequent observations or subgroups because of the 'diluting' effect. In addition, the run length distributions for the techniques with some unknown parameters are not geometric, so ARL is not a suitable performance criterion (see Quesenberry (1993,1995d)). Furthermore, this paper is particularly concerned with short production runs or low volume manufacturing and as such early response of the techniques to any process anomalies or irregularities is a crucial factor.

It should be pointed out that only an upper control limit is used in the simulation. This approach is used because the control techniques are intended primarily for 'picking up' changes in the mean vector and it appears that any such change is likely to result in unusually large values for the control statistics. In practice, however, it might be preferable to use both lower and upper control limits because the former can provide protection against occasional changes in the variance-covariance matrix and other

process disturbances which may cause abnormally small values for the control statistics. The limit is set at the 99.73th percentage point so that the false alarm rate for the proposed techniques equates to that of the traditional Shewhart charts with 3-sigma limits. As a partial check of the simulation, we have included results for those cases with known parameters.

The results for the individual values control techniques obtained through 10,000 simulation runs are tabulated in Table 3.5 for various combinations of p, λ and r. In Table 3.6, the results for those techniques based on subgroup data are given. Note that the exact probabilities for techniques (3.3) and (3.7) are obtainable from the noncentral chi-square distribution tables or standard statistical software packages. The simulation results for these techniques are found to agree well with the theoretical values. For instance, the theoretical probabilities for technique (3.3) are 0.0569, 0.3452, 0.8571 and 0.9972 respectively for $\lambda = 1, 2, 3$ and 4 when $p = 3$. These are very close to the corresponding figures in Table 3.5.

As shown in Table 3.6, the control techniques based on subgroup data for the cases with at least some unknown parameters can be expected to perform as well as the technique with known parameters under this type of process change especially when the noncentrality parameter, λ is larger than or equal to 2. For instance, using control statistic (3.10) and (3.11) with a subgroup size of $n = 6$ when $p = 5$, the probabilities of 'picking up' a mean shift of $\lambda = 2$ which occurs after the 10th subgroup, within 5 consecutive subgroups, are respectively 0.9966 and 0.9402. For smaller values of λ, these control techniques can also be expected to perform reasonably well relative to the technique corresponding to the known parameter case. For example, these probabilities are 0.3812 and 0.2834 respectively when statistics (3.10) and (3.11) are used, as

compared to 0.4513 for the known parameter case. As for control based on individual observations, those techniques with some unknown parameters have poor performance relative to that based on known parameters when λ and r are small and p is large. However, the performance of these individual values control charts improves with increasing value of λ and r. As shown in Table 3.5, the probability of detecting a shift of $\lambda = 5$ for a bivariate process within 5 successive observations using statistic (3.6), which does not assume known values for the process parameters, is 0.8514 if $r = 20$. Apart from these, a number of points can be noted from the tables. It is found that except for control statistic (3.10), the proposed techniques decline in performance according to the number of unknown parameters on which they are based. Specifically, those based on known values of the process parameters have the best performance as expected, followed by those with unknown mean vector μ, those with unknown covariance matrix Σ and those with both process parameters unknown. Note also that, for the same λ and r, the performance of the individual values control techniques become worse as p increases. Finally, it can be seen from Table 3.6 that using statistic (3.10) is always better or as good as statistic (3.9) although the former ignores knowledge of μ in the estimation of Σ.

TABLE 3.5. Probability of Detection Within $m = 5$ Subsequent Observations.

			Control Statistic			
λ	r	p	(3.3)	(3.4)	(3.5)	(3.6)
1	10	2	0.0720	0.0492	0.0328	0.0245
		3	0.0537	0.0399	0.0260	0.0223
		5	0.0417	0.0306	0.0197	0.0172
	20	2	0.0736	0.0588	0.0437	0.0367
		3	0.0545	0.0432	0.0363	0.0304
		5	0.0403	0.0345	0.0267	0.0247
2	10	2	0.4430	0.2560	0.0807	0.0541
		3	0.3437	0.2033	0.0553	0.0364
		5	0.2500	0.1428	0.0342	0.0228
	20	2	0.4357	0.3230	0.1660	0.1353
		3	0.3456	0.2525	0.1172	0.0925
		5	0.2508	0.1830	0.0755	0.0614
3	10	2	0.9153	0.7050	0.1887	0.1325
		3	0.8550	0.6014	0.1191	0.0827
		5	0.7477	0.4752	0.0570	0.0408
	20	2	0.9124	0.8074	0.4011	0.3314
		3	0.8533	0.7220	0.2863	0.2333
		5	0.7530	0.5920	0.1747	0.1385
4	10	2	0.9993	0.9687	0.3735	0.2728
		3	0.9966	0.9354	0.2483	0.1967
		5	0.9880	0.8762	0.1026	0.0722
	20	2	0.9994	0.9921	0.7040	0.6213
		3	0.9960	0.9796	0.5624	0.4837
		5	0.9900	0.9456	0.3576	0.2962
5	10	2	1	0.9996	0.6114	0.4880
		3	1	0.9982	0.4292	0.3076
		5	1	0.9930	0.1560	0.0984
	20	2	1	1	0.9096	0.8514
		3	1	1	0.8122	0.7366
		5	0.9999	0.9988	0.5866	0.5074
6	10	2	1	1	0.8180	0.6980
		3	1	1	0.6304	0.4832
		5	1	0.9998	0.2624	0.1568
	20	2	1	1	0.9862	0.9702
		3	1	1	0.9500	0.9162
		5	1	1	0.8094	0.7468

TABLE 3.6. Probability of Detection Within $m = 5$ Subsequent Subgroups.

				Control Statistic				
λ	r	p	n	(3.7)	(3.8)	(3.9)	(3.10)	(3.11)
1	10	2	3	0.2970	0.1794	0.1670	0.2102	0.1276
		3	4	0.3411	0.1981	0.1918	0.2381	0.1356
		5	6	0.4549	0.2692	0.2938	0.3638	0.2063
	20	2	3	0.2925	0.2128	0.2088	0.2192	0.1712
		3	4	0.3410	0.2521	0.2469	0.2787	0.2025
		5	6	0.4513	0.3256	0.3367	0.3812	0.2834
2	10	2	3	0.9864	0.8772	0.7356	0.8680	0.6512
		3	4	0.9966	0.9399	0.8637	0.9582	0.7996
		5	6	1	0.9908	0.9716	0.9966	0.9402
	20	2	3	0.9880	0.9460	0.8992	0.9440	0.8566
		3	4	0.9966	0.9774	0.9582	0.9816	0.9402
		5	6	1	0.9984	0.9952	0.9997	0.9926
3	10	2	3	1	1	0.9924	0.9990	0.9796
		3	4	1	1	0.9991	1	0.9979
		5	6	1	1	1	1	1
	20	2	3	1	1	1	1	0.9994
		3	4	1	1	1	1	0.9979
		5	6	1	1	1	1	1

3.6 Detecting Step Shifts and Linear Trends Using A Robust Estimator of Σ and EWMA

As demonstrated by examples 3.4.4 and 3.4.5, when a sustained shift in a process parameter occurs, the technique presented for the unknown parameter case that is based on subgroup data will either 'pick up' the process change within the first few samples or it will not signal at all due to the effect of incorporating out-of-control observations in the estimation of the process variance-covariance matrix, Σ. The latter event is likely to occur when the shift takes place early in the production and the shift size (as measured by λ) is small. This problem is particularly acute when the corresponding individual values control technique is used.

Apart from one-step shifts, there are situations where the process mean vector, μ changes linearly with time or with the chronological order of the observations or samples. A *linear trend* in the multivariate observations is defined by Chan and Li (1994) as the event which occurs when there exists a constant p-dimensional vector β such that $\beta\mu$ is a linear function of the sample or observation number. As a special case of this, a linear trend that occurs after the rth observation is represented by

$$\mathbf{X}_i = \mu_i + \varepsilon_i$$

$$\text{where } \mu_i = \begin{cases} \mu & i = 1,\ldots,r \\ \mu + i\theta & i = r+1,\ldots \end{cases} \tag{3.12}$$

θ and ε_i's denote respectively a constant vector characterising the linear trend and the i.i.d $N_p(\mathbf{0},\Sigma)$ vectors of random errors. This is the model that is considered herein. If the linear trend occurs soon after the commencement of production, the control procedures for the unknown Σ case, particularly those based on individual observations

are likely to be ineffective since the associated sample estimates of Σ appear to be adversely affected by such systematic process changes.

In order to minimize the effect of step shifts, trends and other types of process irregularities on the Σ estimates, it is suggested that the estimation procedure recommended by Holmes et al.(1993) and Scholz et al.(1994) is used. This alternative is analogous to the use of successive squared differences for the univariate situation. When k observations are available, the suggested estimator of the process variance-covariance matrix is given by

$$\widetilde{\mathbf{S}}_k = \frac{1}{2(k-1)} \sum_{i=1}^{k-1} (\mathbf{X}_{i+1} - \mathbf{X}_i)(\mathbf{X}_{i+1} - \mathbf{X}_i)^{\mathrm{T}}.$$

This was shown to be unbiased by Sullivan et al.(1995) for situations where the observation vectors are i.i.d.

In this section, we examine the appropriateness of using EWMA computed from the sequence of *normalized* T^2 statistics based on $\widetilde{\mathbf{S}}_k$ estimator and individual observations as a control procedure, for the case where μ is either specified or unknown. To do this, first note that Scholz et al. approximate the distribution of $\widetilde{\mathbf{S}}_k$ with

$$f_k \widetilde{\mathbf{S}}_k \dot{\sim} W_p(f_k, \Sigma)$$

where $f_k = \frac{2(k-1)^2}{3k-4}$. In addition, they stated that $\overline{\mathbf{X}}_k$ is independent of $\widetilde{\mathbf{S}}_k$. Thus, it follows that

$$T^*_{k+1,\mu} = \frac{f_k - p + 1}{f_k p} (\mathbf{X}_{k+1} - \mu)^{\mathrm{T}} \widetilde{\mathbf{S}}_k^{-1} (\mathbf{X}_{k+1} - \mu) \dot{\sim} F_{p, f_k - p + 1}$$

and

$$T^*_{k+1} = \frac{k(f_k - p + 1)}{f_k p(k+1)} (\mathbf{X}_{k+1} - \overline{\mathbf{X}}_k)^{\mathrm{T}} \widetilde{\mathbf{S}}_k^{-1} (\mathbf{X}_{k+1} - \overline{\mathbf{X}}_k) \dot{\sim} F_{p, f_k - p + 1}.$$

Note also that the $T^*_{k,\mu}$'s (and T^*_k's) are approximately independent when k becomes large. Thus, as in section 3.3, these statistics can be transformed to sequences of *approximately* i.i.d standard normal variables $\{Z^*_{k,\mu}\}$ and $\{Z^*_k\}$ as follows :

$$Z^*_{k,\mu} = \Phi^{-1}\left[F_{p,f_{k-1}-p+1}\left(T^*_{k,\mu}\right)\right]$$

and $$Z^*_k = \Phi^{-1}\left[F_{p,f_{k-1}-p+1}\left(T^*_k\right)\right].$$

It is now possible to form EWMA statistics with $Z^*_{k,\mu}$ and Z^*_k taken as the inputs. The resulting techniques for the known and unknown μ case are based on the EWMA statistics, EWMAZ1_k and EWMAZ1U_k, given respectively by

$$\text{EWMAZ1}_k = \gamma Z^*_{k,\mu} + (1-\gamma)\text{EWMAZ1}_{k-1}$$

and $$\text{EWMAZ1U}_k = \gamma Z^*_k + (1-\gamma)\text{EWMAZ1U}_{k-1}.$$

For ease of subsequent discussion, these techniques are referred to as EWMAZ1 and EWMAZ1U respectively. Note that using these procedures, process monitoring begins with the kth observation where EWMAZ1_{k-1} and EWMAZ1U_{k-1} are set to 0 and k is the smallest integer greater than $\dfrac{(3p+5)+\sqrt{(p-1)(9p-17)}}{4}$. The values of EWMAZ1_k and EWMAZ1U_k are plotted on a chart with control limits at $\pm h\sqrt{\gamma/(2-\gamma)}$ where the smoothing constant γ and the control limits factor h are chosen to achieve specified in-control run length performance. Using the results obtained by Quesenberry (1995a), these design parameters are set to 0.25 and 2.9 respectively to give the control limits at ± 1.096. These same EWMA parameters are used in all the work reported subsequently in this chapter. This combination of γ and h gives an in-control ARL of 372.6 and an ARL of 5.18 for a shift of 1.5 standard deviation in the mean of a normal variable. Since

the proposed procedures are *not* based on sequences of *exactly* i.i.d standard normal variables, their in-control RL distributions will be generally different from those expected. However, as shown later, the effect of approximating $\{Z^*_{k,\mu}\}$ and $\{Z^*_k\}$ with sequences of i.i.d $N(0,1)$ variables on the in-control RL performance of the associated EWMA procedures appears to be insignificant, at least for the first 50 observations, the selected combination of γ and h, and the dimensions considered.

In order to illustrate the use of the proposed EWMA procedures, consider again the data of Holmes et al.(1993) which are shown in Table 3.3. As before, only the first two columns of the data, namely, those giving the percentages of large (L) and small (S) 'grits', are analysed. As the process mean vector is not specified, the appropriate statistic to be used for this problem is EWMAZ1U_k. Since

$$\frac{(3p+5)+\sqrt{(p-1)(9p-17)}}{4} = \frac{(3\times 2+5)\sqrt{(2-1)(9\times 2-17)}}{4} = 3$$

for this situation, the control procedure is initiated at the 4th observation by letting $\text{EWMAZ1U}_3 = 0$. The resulting EWMAZ1U chart is shown in Figure 3.6. As shown in the figure, this chart issues out-of-control signals at observations 27, 29, 45, 46 and 52. Note that these observations are removed before calculating the control statistic for their subsequent observations. To emphasize this, the points corresponding to observations 28, 30, 46, 47 and 53 are disconnected from the immediately preceding ones. It can also be seen that observations 28 and 30 almost trigger a signal. They have $\text{EWMAZ1U}_{28} = 1.083$ and $\text{EWMAZ1U}_{30} = 1.081$ which are very close to the upper

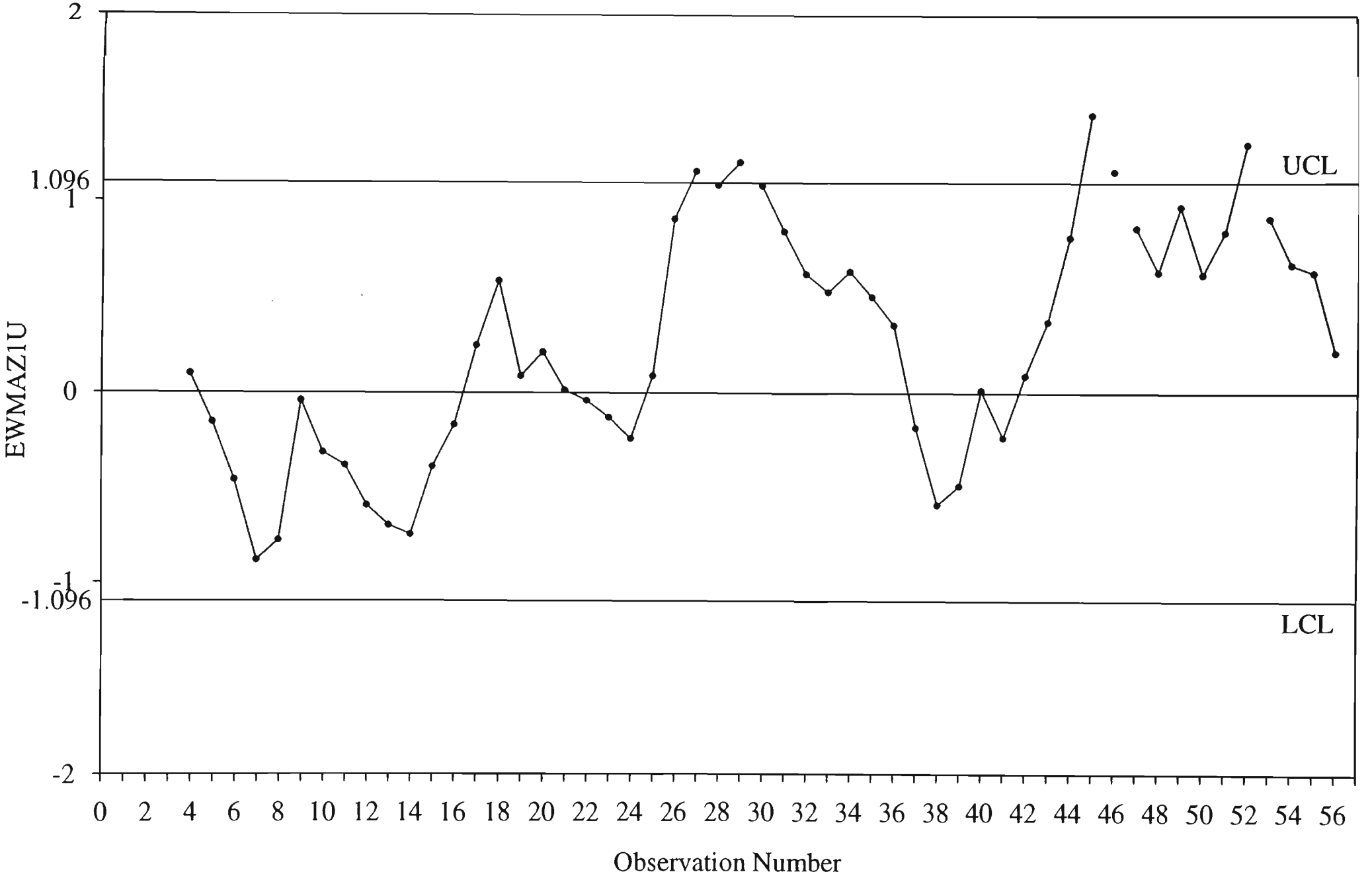


Figure 3.6. EWMAZ1U Chart for Holmes and Mergen's (1993) Data.

control limit of 1.096. In comparison to the use of statistic (3.6) (see Example 3.4.5) and the retrospective tests of Holmes et al. and Sullivan et al.(1995) using the $\widetilde{\mathbf{S}}_k$ estimator, there are more signals from the EWMAZ1U chart for this set of data. These findings are not surprising since it was found by Sullivan et al. that the difference between the mean vectors of the first 24 and the last 32 observations are statistically significant (an evidence of a shift in the process mean vector following the 24th observation), while the 'within' variance-covarince matrices are not statistically different.

Next, to see how the proposed procedures respond to linear trends, 50 bivariate observations have been generated from the process model (3.12) with k = 1, $\mu = \mathbf{0}$, $\Sigma = \mathbf{I}$ and $\theta = (0.3, 0)^{\mathrm{T}}$. Besides EWMAZ1 and EWMAZ1U charts, M charts specifically designed by Chan et al.(1994) for detecting such process changes have also been constructed based on the same data. These are all shown in Figure 3.7. The latter procedures are developed based on *projection pursuit* and linear regression techniques. These procedures involve charting the values of certain statistics based on *moving* samples of d observations. In this example, the size of the moving samples used is $d = 7$. If μ is specified, the sequence of control statistics involved is given by

$$M_k = \frac{\mathbf{W}^{\mathrm{T}}\mathbf{G}_k\mathbf{U}_k^{-1}\mathbf{G}_k\mathbf{W}}{\mathbf{W}^{\mathrm{T}}\mathbf{W} - \mathbf{W}^{\mathrm{T}}\mathbf{G}_k\mathbf{U}_k^{-1}\mathbf{G}_k\mathbf{W}}, \qquad k = d,\ d+1,\ldots$$

where $\mathbf{W}$ is a d-dimensional vector with the ith element being $\frac{2i-d-1}{2}$, $\mathbf{G}_k = (\mathbf{X}_{k-d+1} - \mu,\ \cdots\ \mathbf{X}_k - \mu)^{\mathrm{T}}$, $\mathbf{U}_k = \mathbf{G}_k^{\mathrm{T}}\mathbf{G}_k$ and $d > p$. The values of M_k are plotted on a chart with upper control limit at $F_{p,d-p,\alpha}$, the upper 100αth percentile of an F distribution with p and $d-p$ degrees of freedom. If the process mean vector is

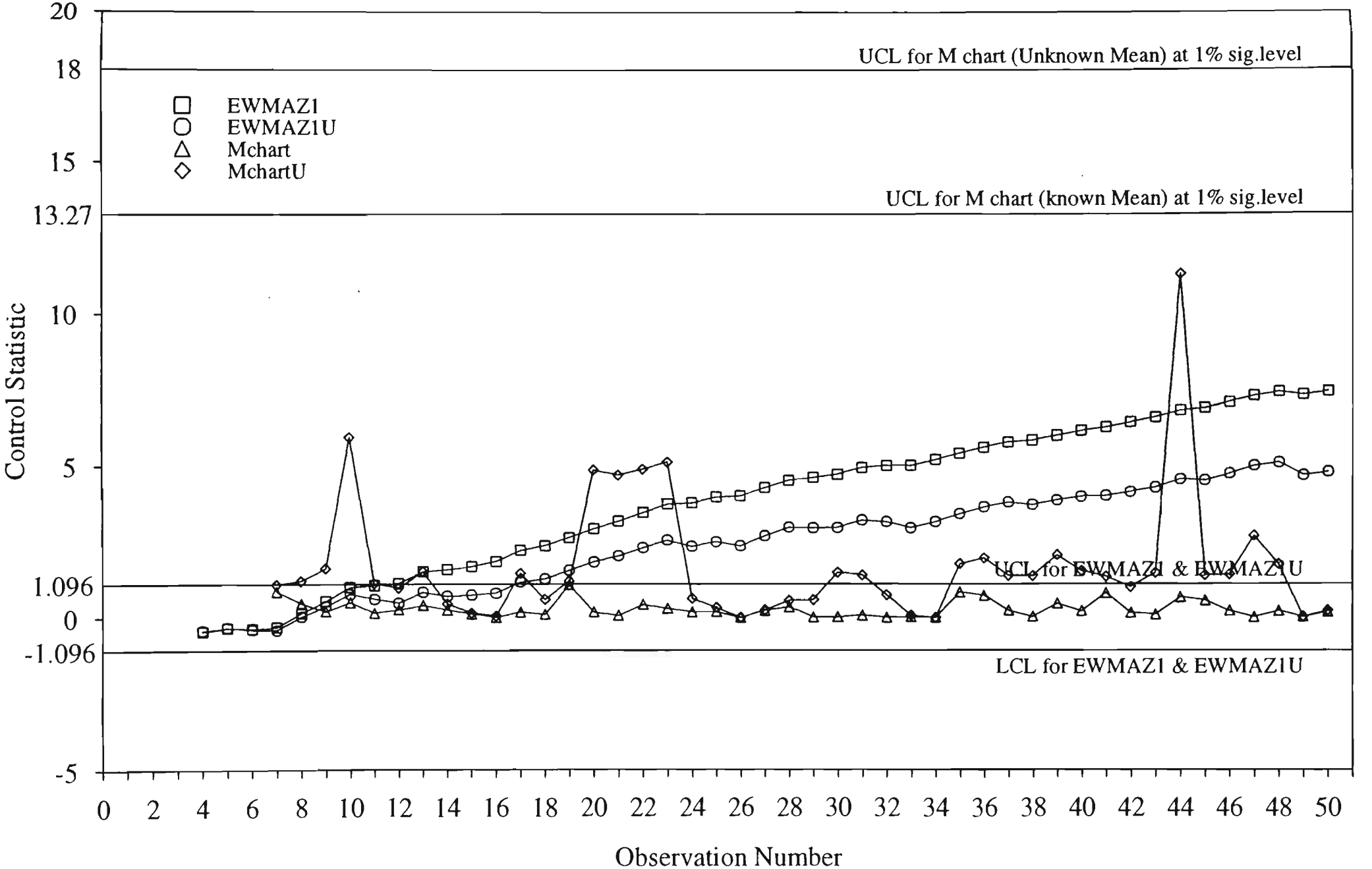


Figure 3.7. EWMAZ1, EWMAZ1U and M Charts for Bivariate Data with Linear Trend.

unknown, the procedure is similar except that μ and $F_{p,d-p,\alpha}$ should be replaced by the mean vector of the current moving sample,

$$\overline{\mathbf{X}}_{d,k} = \frac{1}{d}\sum_{i=k-d+1}^{k}\mathbf{X}_i$$

and $F_{p,d-p-1,\alpha}$ respectively, with the condition that $d > p+1$. As shown in the figure, although the process trend begins at the 2nd observation, no signal is generated by both the M charts with known and unknown μ when a 1% significance level is used. In contrast, EWMAZ1 and EWMAZ1U control procedures 'pick up' the trend at observations 11 and 17 respectively. In addition, the upward trends of the charted points for these EWMA charts provide very strong visual evidence of process troubles. This point pattern is typical for processes affected by linear trends. Note that since $d = 7$, both the M charts have been started from the 7th observation. Note also that the values of the M chart with specified μ are generally lower than the corresponding values for the μ unknown case. As will be seen later, this technique is very ineffective. This is due, in part, to the effect of estimating the in-control value of Σ based on

$$\begin{aligned}\mathbf{U}_k &= \mathbf{G}_k^{\mathrm{T}}\mathbf{G}_k \\ &= \sum_{i=k-d+1}^{k}(\mathbf{X}_i - \mu)(\mathbf{X}_i - \mu)^{\mathrm{T}},\end{aligned}$$

which apparently incorporates both the 'local' and the 'long term' variabilities in the presence of systematic process changes.

In order to provide more insight into the relative performance of the proposed and other procedures including the M charts, some simulation results for the RL probabilities, $\Pr(RL \le k)$, are presented. These are all based on 2000 replications, giving

a maximum standard error of $\sqrt{\frac{0.5(1-0.5)}{2000}} = 0.0112$ which occurs when the probability being estimated is actually 0.5. For the case where μ is assumed known, two other procedures similar to EWMAZ1, abbreviated hereafter as EWMAZ2 and EWMAZ3 respectively are considered. The first uses the EWMA computed from the $Z_k's$ of (3.5) whereas the other is based on the EWMA computed from the following sequence of statistics :

$$Z_k = \Phi^{-1}\left[F_{p,k-1-p}\left(\frac{(k-1-p)}{p(k-2)} (\mathbf{X}_k - \mu)^{\mathrm{T}} \mathbf{S}_{k-1}^{-1} (\mathbf{X}_k - \mu) \right) \right], \qquad k = p+2,.... \quad (3.13)$$

As for the case with unknown μ, a similar EWMA procedure with the $Z_k's$ of (3.6) taken as inputs is also considered. For convenience, this technique is referred to as EWMAZ2U.

For a linear trend that occurs immediately (i.e. r = 1), the simulated run length probabilities are shown graphically in Figures 3.8 to 3.13 for $k = 1(1)50$ and various combinations of p and the trend parameter $\lambda_{trend} = \sqrt{\theta^{\mathrm{T}} \Sigma^{-1} \theta}$. In fact, using the similar arguments as that for a step shift, it can be shown that the statistical performance of all the control procedures considered above depend on λ_{trend} under linear trend conditions as specified in (3.12). Note that the run lengths here are measured from the 1st observation although several observations are necessary for initiating the stated procedures. Note also that statistical comparison of various control techniques can be misleading if their in-control RL distributions differ considerably. However, as shown in Figures 3.8(a) to 3.10(a), the difference between the in-control RL distributions for EWMAZ1, EWMAZ2 and EWMAZ3 are practically insignificant for $k = 1(1)50$ and p = 2, 3 and 5. The same is true for EWMAZ1U and EWMAZ2U (see Figures 3.11(a) to

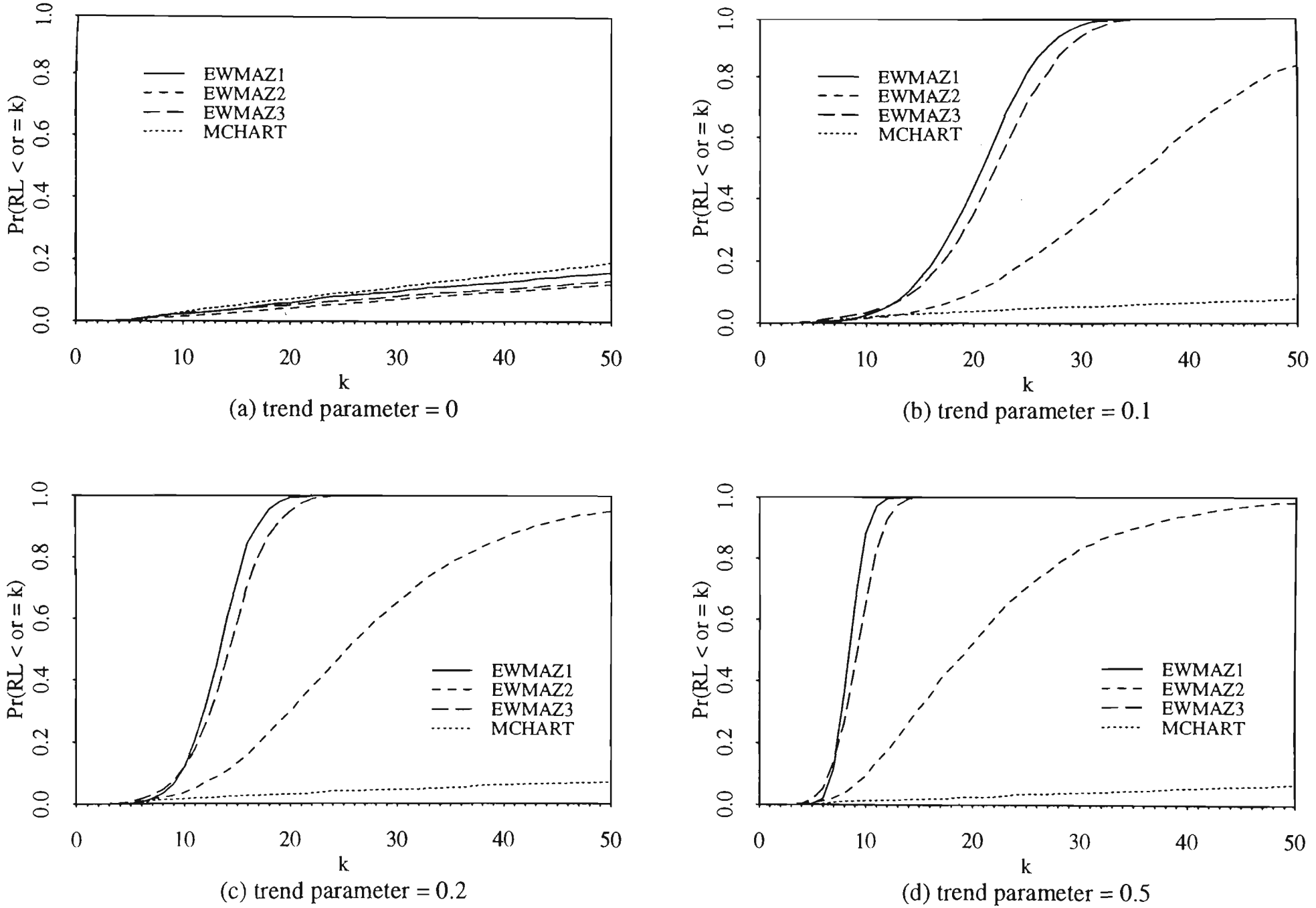


Figure 3.8. Run Length Probabilities, Known Mean Vector, p = 2, d = 7, a = 0.03.

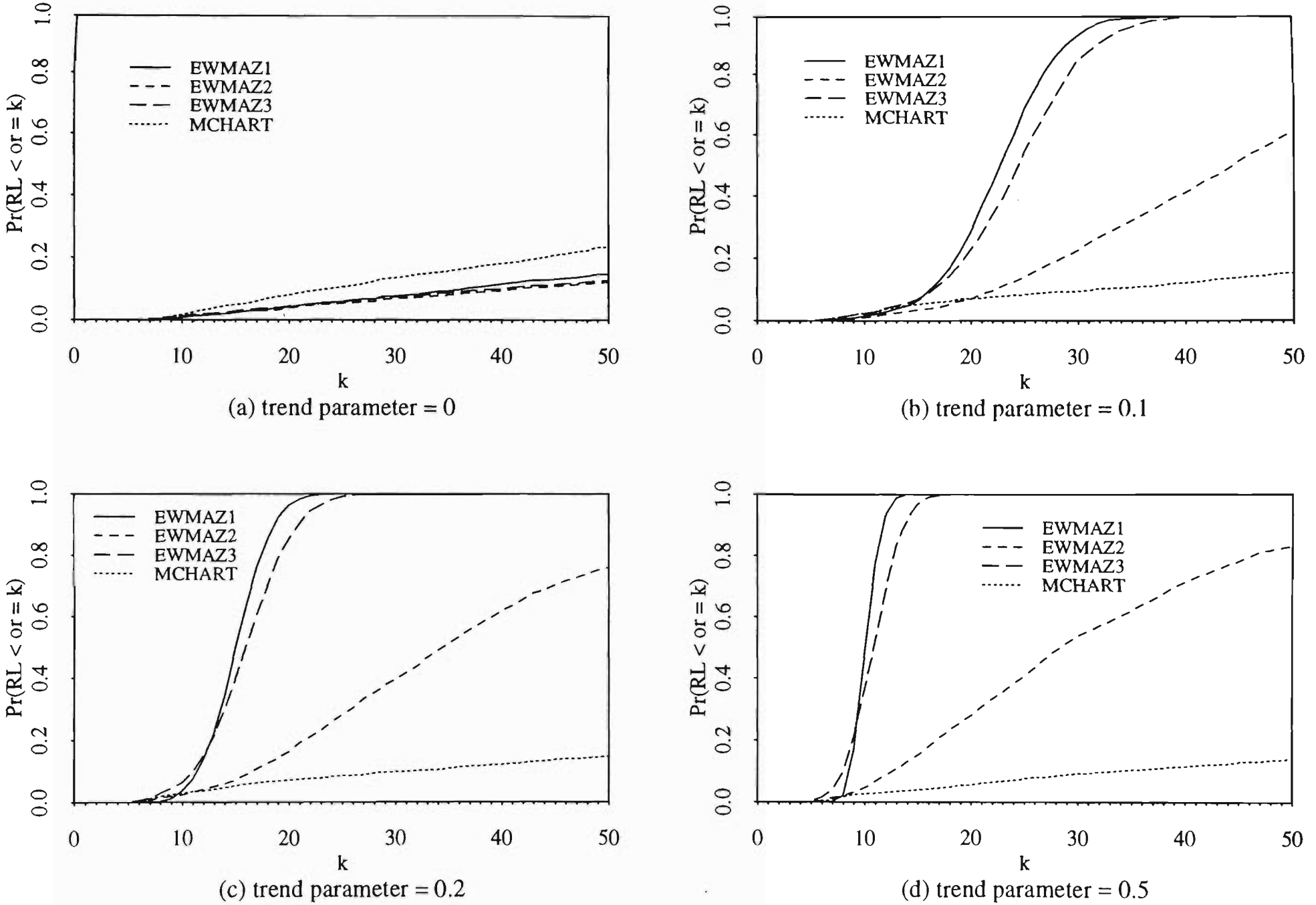
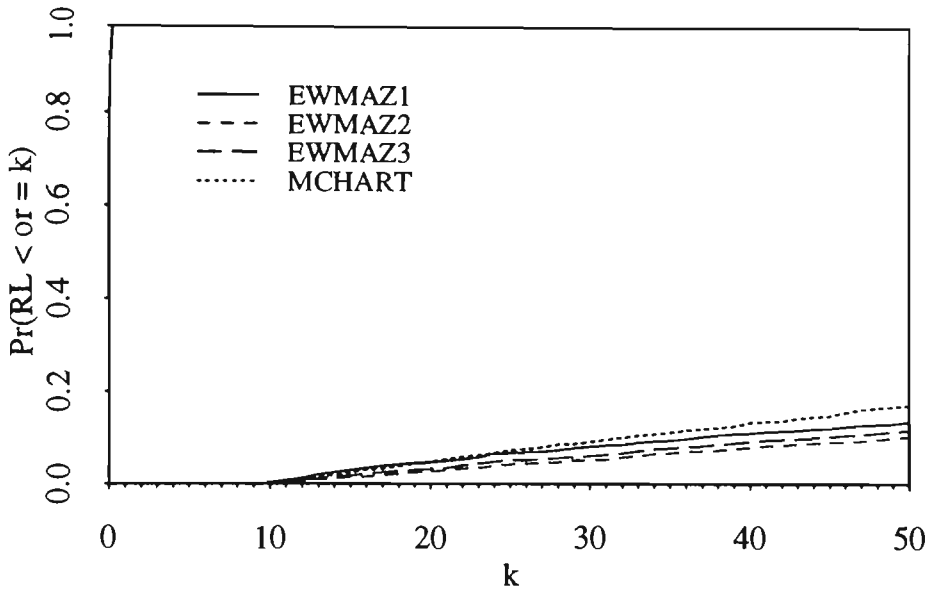
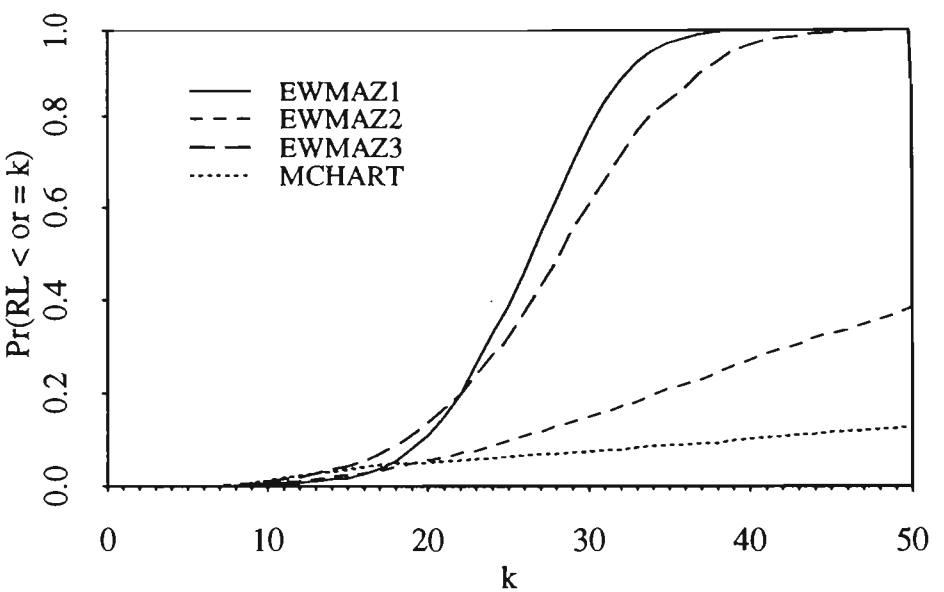


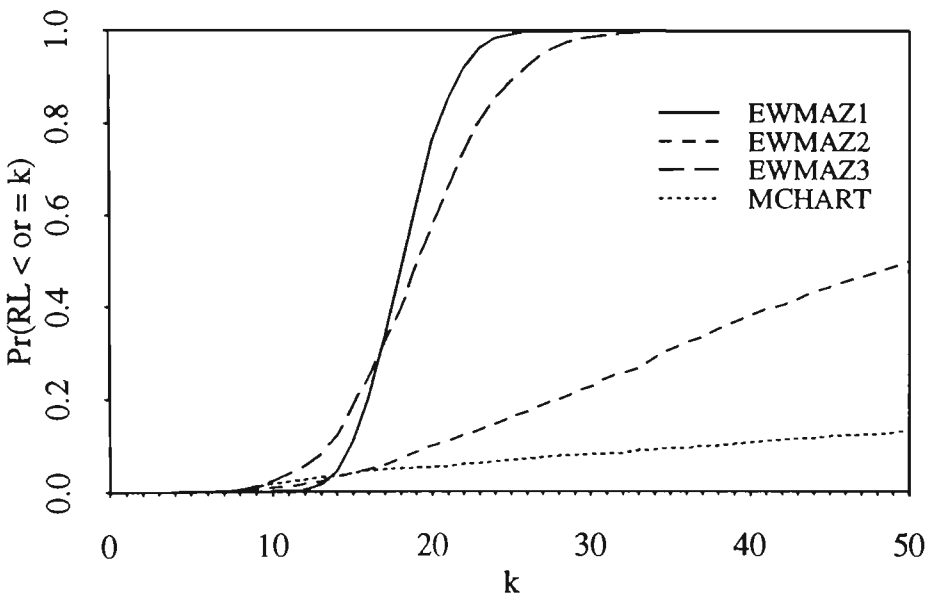
Figure 3.9. Run Length Probabilities, Known Mean Vector, p = 3, d = 8, a = 0.02.



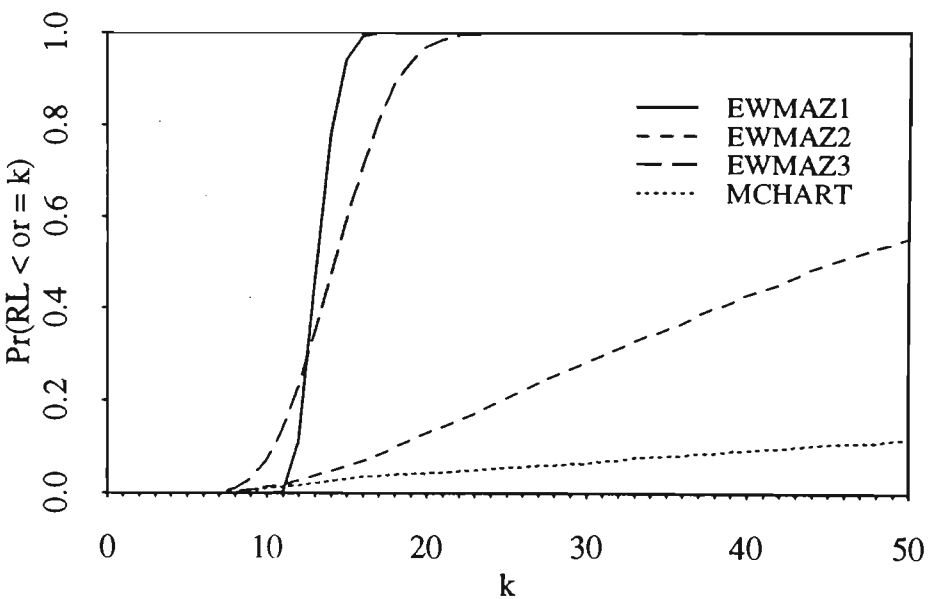
(a) trend parameter = 0



(b) trend parameter = 0.1



(c) trend parameter = 0.2



(d) trend parameter = 0.5

Figure 3.10. Run Length Probabilities, Known Mean Vector, p = 5, d = 10, a = 0.005.

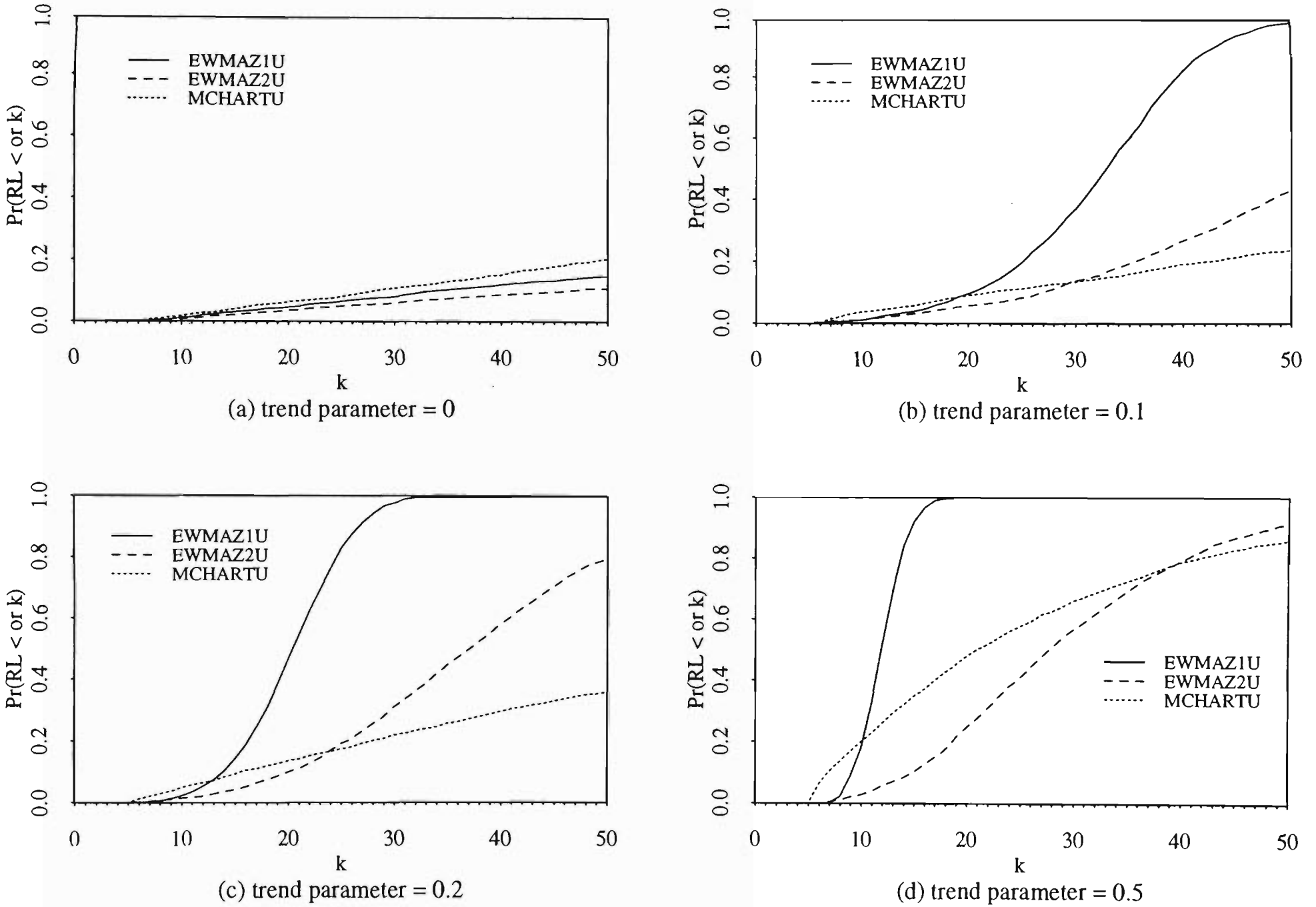
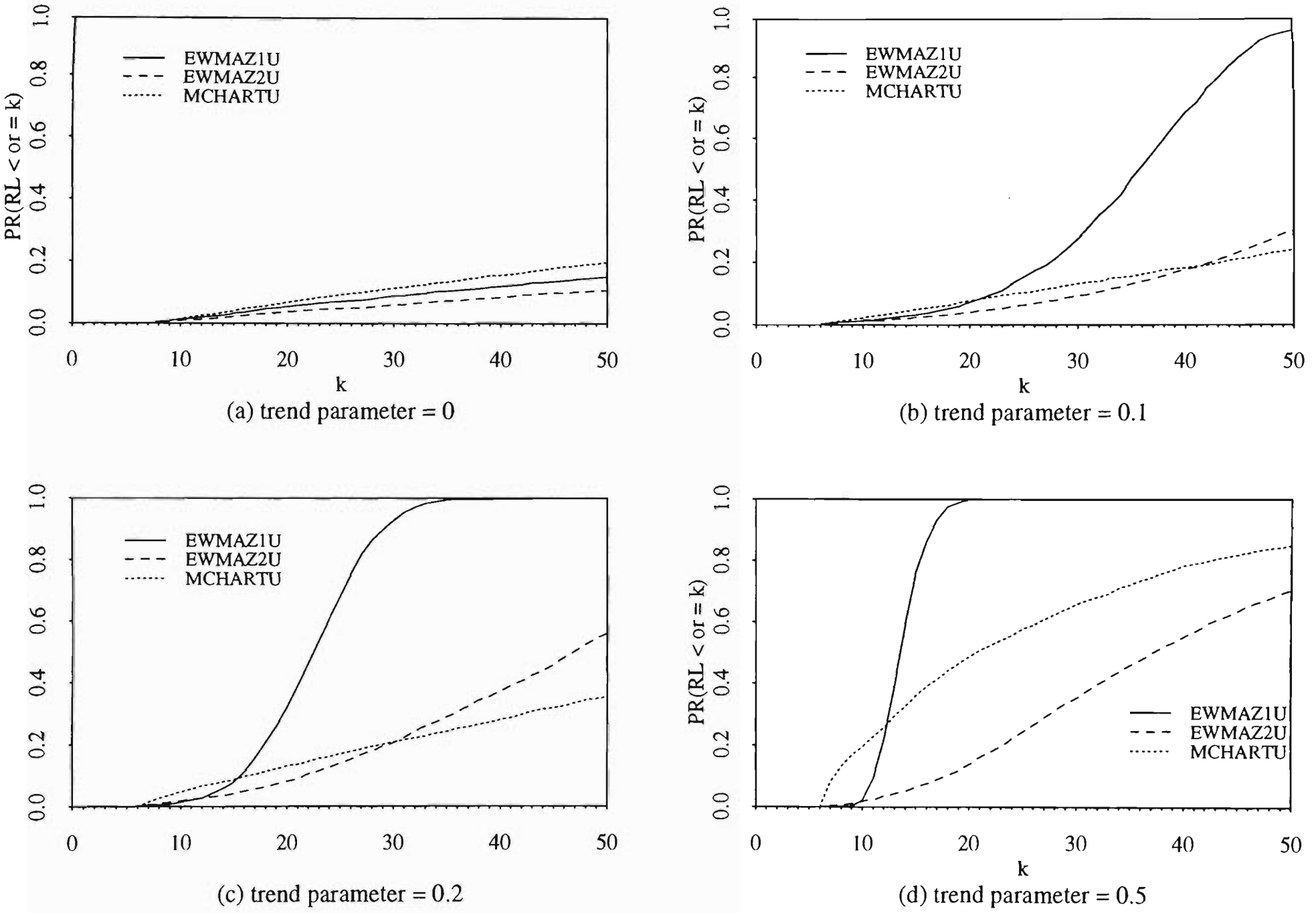


Figure 3.11. Run Length Probabilities, Unknown Mean Vector, p = 2, d = 7, a = 0.02.



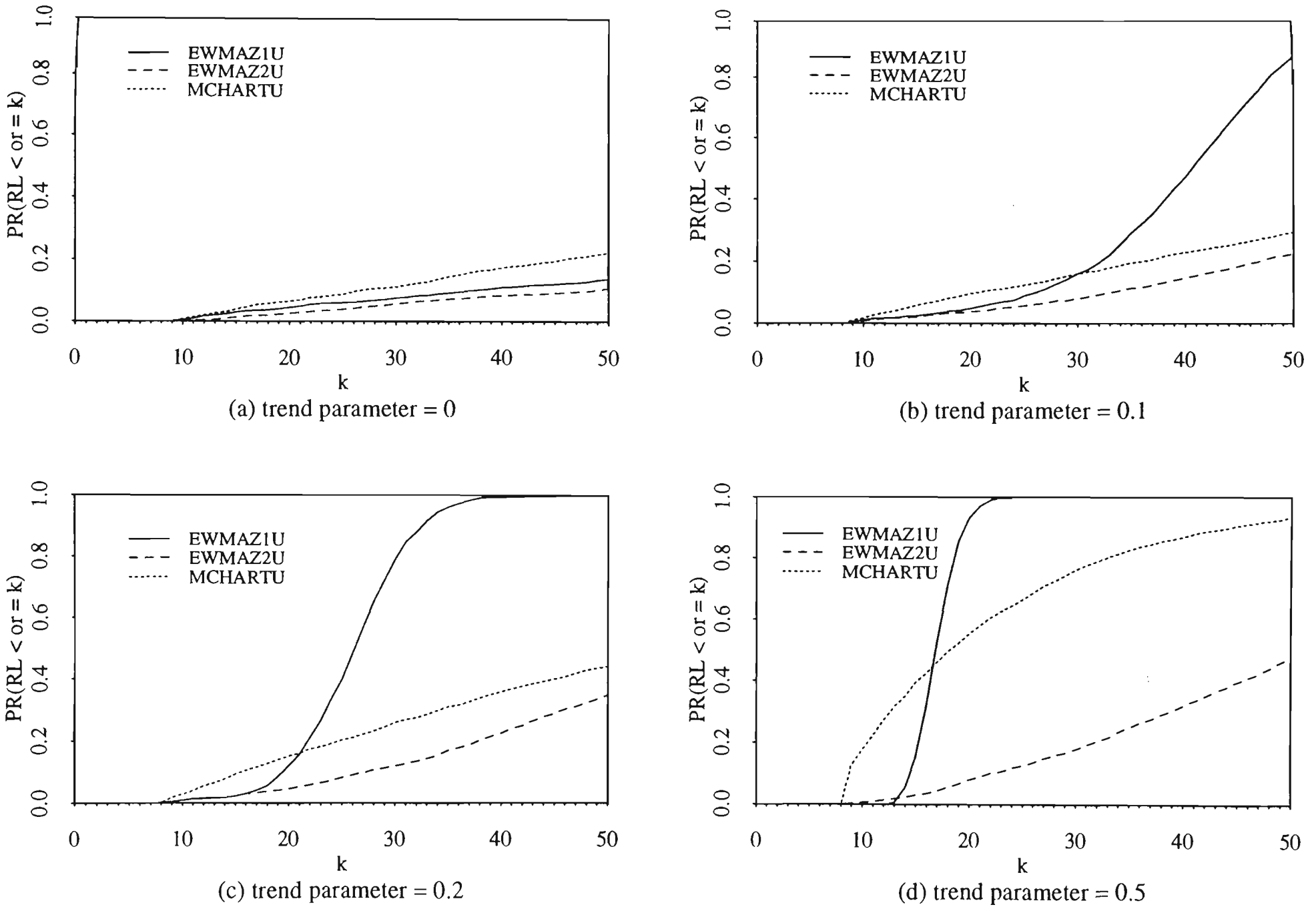
(a) trend parameter = 0

(b) trend parameter = 0.1

(c) trend parameter = 0.2

(d) trend parameter = 0.5

Figure 3.12. Run Length Probabilities, Unknown Mean Vector, p = 3, d = 8, a = 0.01.



(a) trend parameter = 0

(b) trend parameter = 0.1

(c) trend parameter = 0.2

(d) trend parameter = 0.5

Figure 3.13 . Run Length Probabilities, Unknown Mean Vector, p = 5, d = 10, a = 0.005.

3.13(a)). This is found to be generally the case for any dimension, p. For the M charts, the significance level, α, used in each case was 'tuned' to give approximately the same or slightly worse in-control RL performance than that for the proposed procedure. For the case with specified or known μ, the values of α used in the simulation are 0.03, 0.02 and 0.005 respectively for p = 2, 3 and 5. For the unknown μ case, these are 0.02, 0.01 and 0.005.

As shown in Figures 3.8 to 3.10, EWMAZ1 is far superior to EWMAZ2 and the M charting technique for all the cases considered. It is also seen to be better than EWMAZ3 except when k is small in which case the difference in their RL probabilities is marginal. In general, these procedures rank in performance in the order of EWMAZ1, EWMAZ3, EWMAZ2 and the M charting technique. EWMAZ2 performs much better than the M chart especially when p is small and λ_{trend} is large. Apart from these, it can be seen from the figures that the RL probability, $\Pr(RL \le k)$, of the M charting technique decreases as the trend parameter, λ_{trend} increases ! This counter-intuitive phenomenon suggests that the technique is of little value. For the unknown μ case, Figures 3.11 to 3.13 reveal that EWMAZ1U has much better RL performance than EWMAZ2U irrespective of p and λ_{trend}. It is also observed that the proposed procedure is considerably better than the corresponding M charting technique except when both p and λ_{trend} are large in which case no definitive conclusion can be made. Limited simulation using other values of d, α and λ_{trend} yields similar conclusion. For instance, when $p = 5$ and $\lambda_{trend} = 0.5$, the run length probability, $\Pr(RL \le k)$, of the former method is relatively higher for $k \ge 16$ but is smaller for k from 9 to 15. However, the observed inferiority of the proposed technique for small values of k is compensated for

by its lower likelihood of false alarms. Another interesting point that can be noticed from the figures is that the RL probabilities for the M chart are higher for the unknown μ case than for the known μ case. Thus, if process changes are anticipated to be in the form of linear trends and an M chart is to be used, it is advisable to use that which does not assume known value of μ even though μ is specified or known.

In the comparison above, it is assumed that the trends occur immediately after the process set-up. This is, however, not always the case in practice. For deferred trends, the EWMA procedures : EWMAZ1, EWMAZ2, EWMAZ3, EWMAZ1U and EWMAZ2U can be expected to improve in performance since the sample covariance estimate used with each is based on some in-control observations and has greater degrees of freedom. However, it is readily seen that this deferral of trends has no effect on the RL properties of the M charting techniques. Thus, it appears that the proposed procedures outperform the M charting techniques for most circumstances.

The results for step shifts are shown in Figures 3.14 to 3.19 for various combinations of p, r and λ. As shown in Figures 3.14 to 3.16, EWMAZ1 is far superior to EWMAZ2 and is as good as or significantly better than EWMAZ3. For $r = 10$ and $\lambda = 2$ or 3, the RL probabilities of EWMAZ1 within short runs are slightly lower than those for the other two procedures. For the case with unknown parameters, it is evident from Figures 3.17 to 3.19 that EWMAZ1U is more likely to 'pick up' the shift irrespective of p, r and λ.

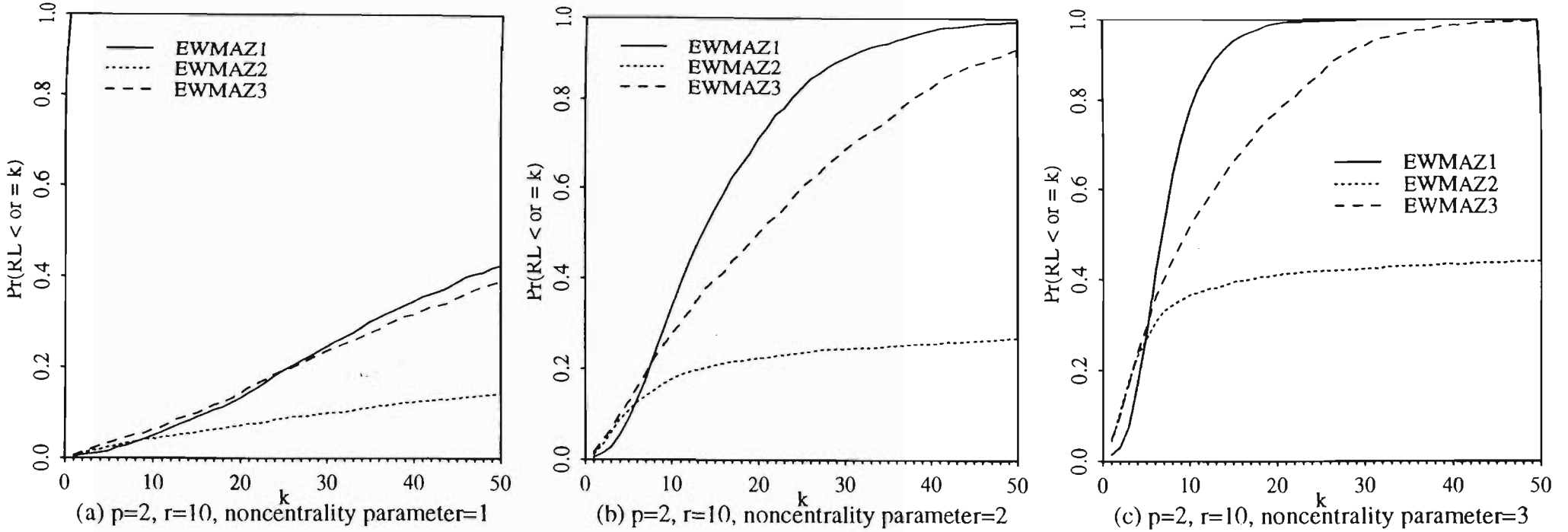
Although $Z^*_{k,\mu}$, Z^*_k and similarly transformed variables as given by (3.5), (3.6) and (3.13) can be used in their own rights, some limited simulations not reported here indicate that using the associated EWMA procedures results in significantly better RL performance for step shift and linear trend conditions. However, control charts based on

statistics (3.5) and (3.6) are good as the filters to isolate outliers and they will exhibit various discernible point patterns for other erroneous processes. Thus, it is recommended that in practice these charts be used and supplemented by EWMAZ1 and EWMAZ1U respectively.

Note that, EWMAZ1 and EWMAZ1U are also sensitive to other process disturbances including changes in the variance-covariance structure of the measured variables. To distinguish between location and scale changes, a reasonable approach is to construct maximum likelihood ratio or Schwarz information criterion (SIC) statistics for the respective out-of-control models (see Chen and Gupta (1994, 1995)) and use the *p*-value as an indicator of the most probable type of change. This is an interesting problem which needs further investigation and it will not be considered further here.

3.7 Computational Requirements

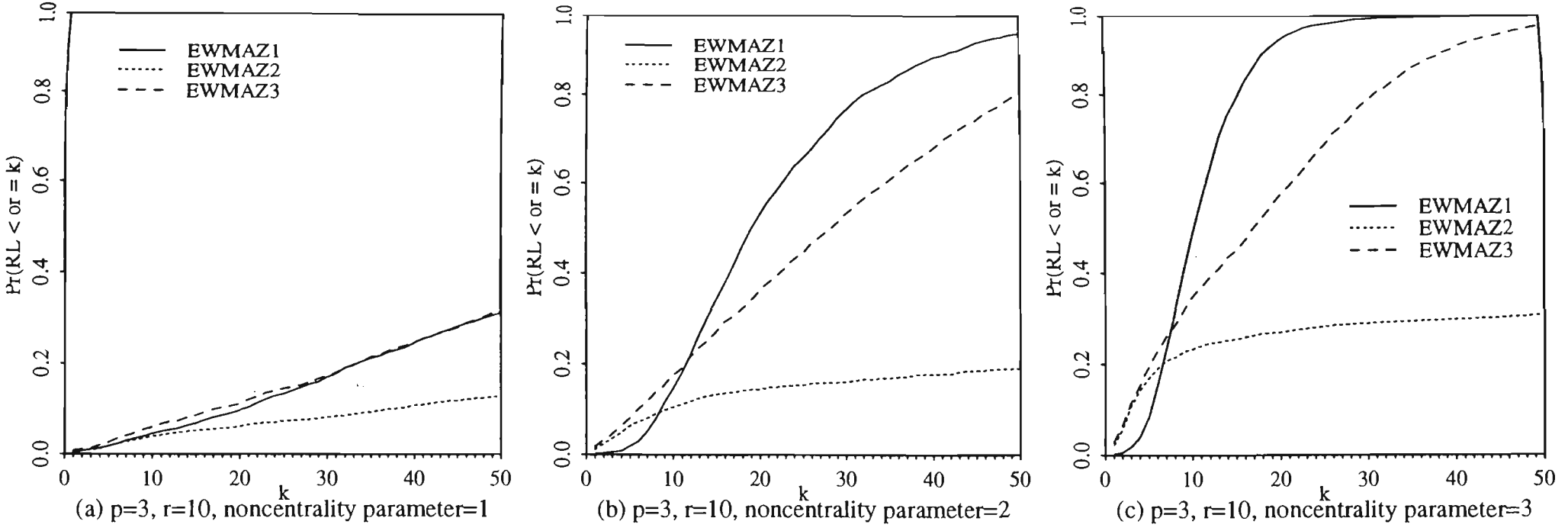
In this work, evaluation of the standard normal distribution, its inverse, chi-square and F distribution functions are required in order to compute the trasformed Z_k statistics. In addition, computation of the argument statistics T_k's involve matrix multiplication and inversion. To implement the proposed control scheme, therefore, requires some fairly complex algorithms. Fortunately, these are widely available and have been built into most of the commercial statistical software packages. The simulated data and the charts in this and the next chapter were generated and made by the authour using programs written in *S-plus*.



(a) p=2, r=10, noncentrality parameter=1 (b) p=2, r=10, noncentrality parameter=2 (c) p=2, r=10, noncentrality parameter=3

(d) p=2, r=20, noncentrality parameter=1 (e) p=2, r=20, noncentrality parameter=2 (f) p=2, r=20, noncentrality parameter=3

Figure 3.14. Run Length Probabilities for Step Shifts in Mean Vector



(a) p=3, r=10, noncentrality parameter=1

(b) p=3, r=10, noncentrality parameter=2

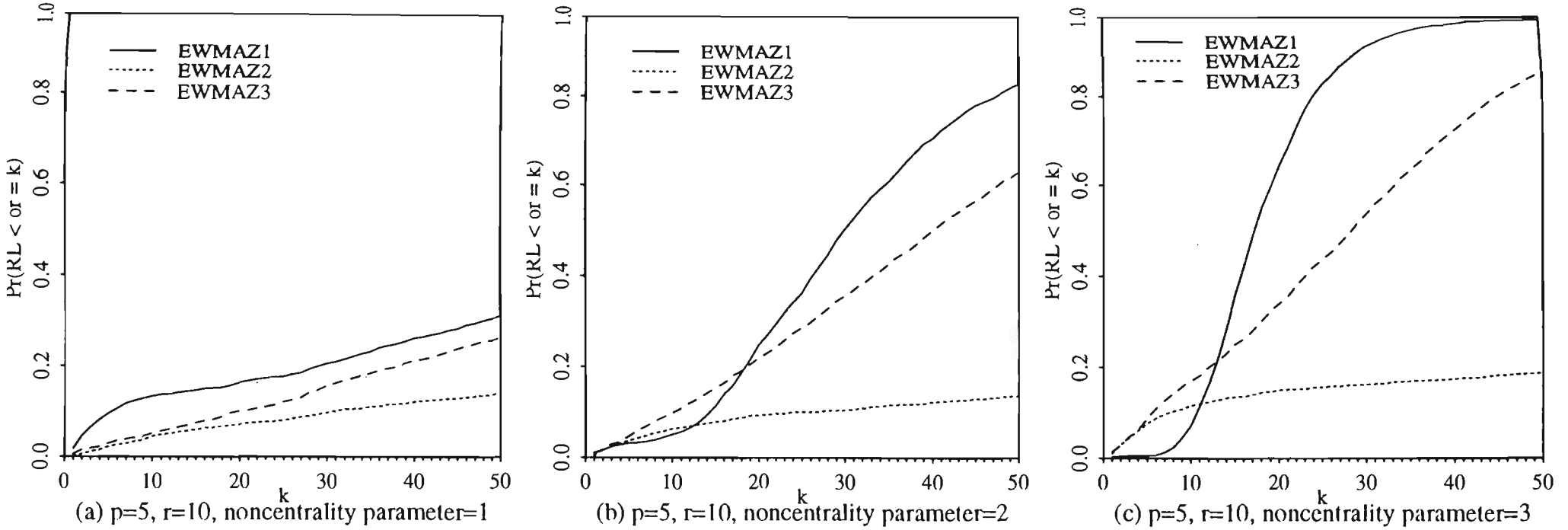
(c) p=3, r=10, noncentrality parameter=3

(d) p=3, r=20, noncentrality parameter=1

(e) p=3, r=20, noncentrality parameter=2

(f) p=3, r=20, noncentrality parameter=3

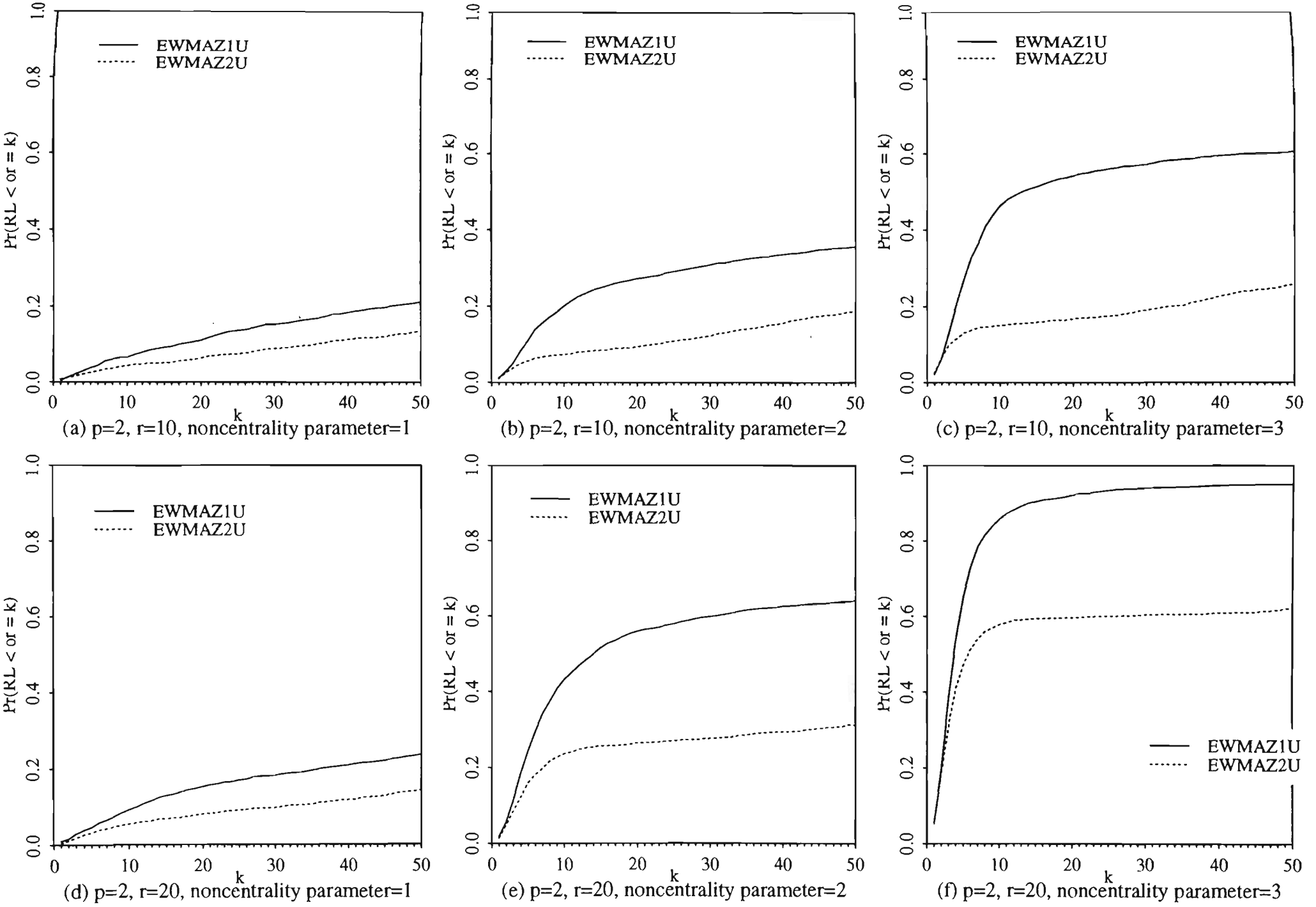
Figure 3.15. Run Length Probabilities for Step Shifts in Mean Vector



(a) p=5, r=10, noncentrality parameter=1 (b) p=5, r=10, noncentrality parameter=2 (c) p=5, r=10, noncentrality parameter=3

(d) p=5, r=20, noncentrality parameter=1 (e) p=5, r=20, noncentrality parameter=2 (f) p=5, r=20, noncentrality parameter=3

Figure 3.16. Run Length Probabilities for Step Shifts in Mean Vector



(a) p=2, r=10, noncentrality parameter=1

(b) p=2, r=10, noncentrality parameter=2

(c) p=2, r=10, noncentrality parameter=3

(d) p=2, r=20, noncentrality parameter=1

(e) p=2, r=20, noncentrality parameter=2

(f) p=2, r=20, noncentrality parameter=3

Figure 3.17. Run Length Probabilities for Step Shifts in Mean Vector

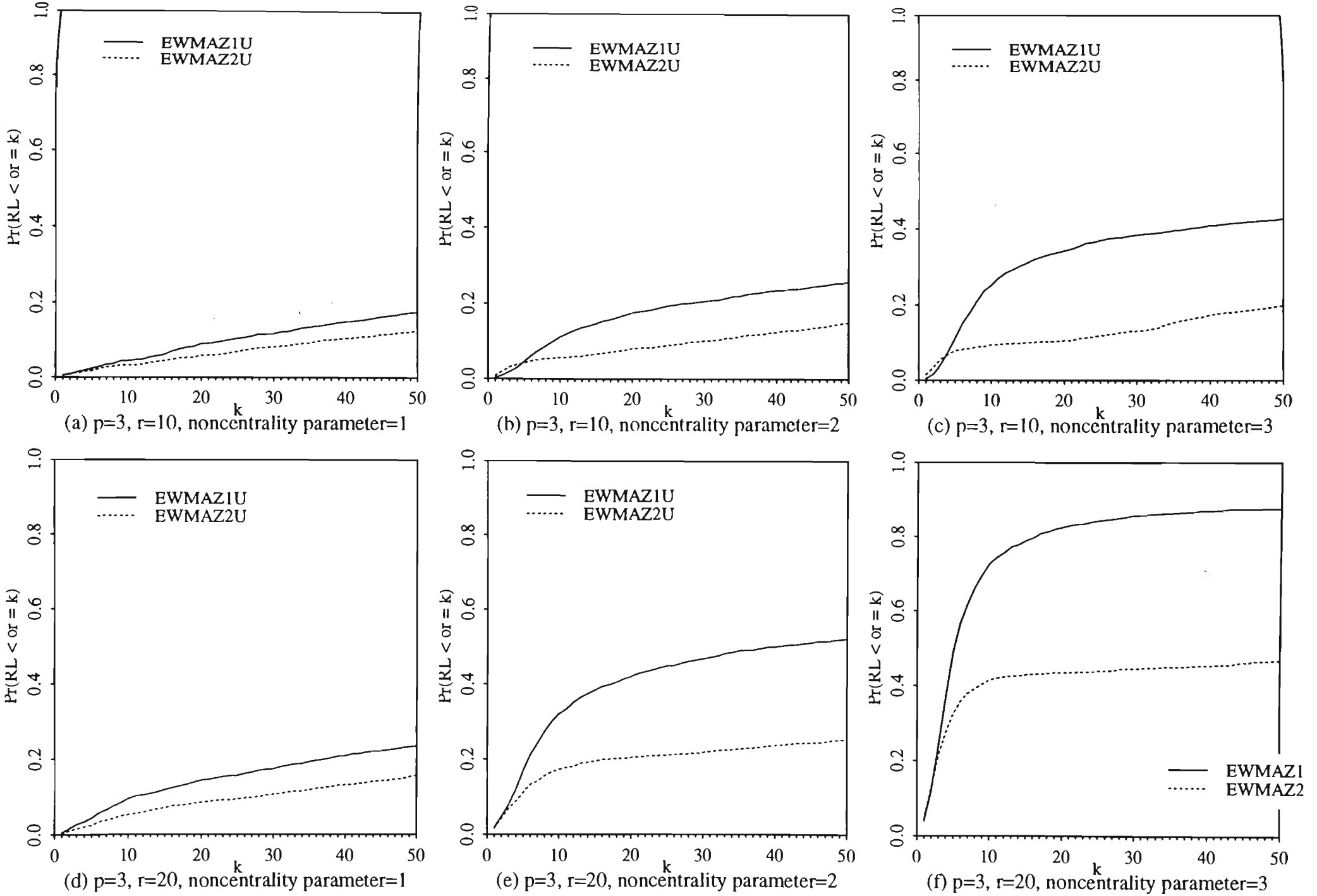


Figure 3.18. Run Length Probabilities for Step Shifts in Mean Vector

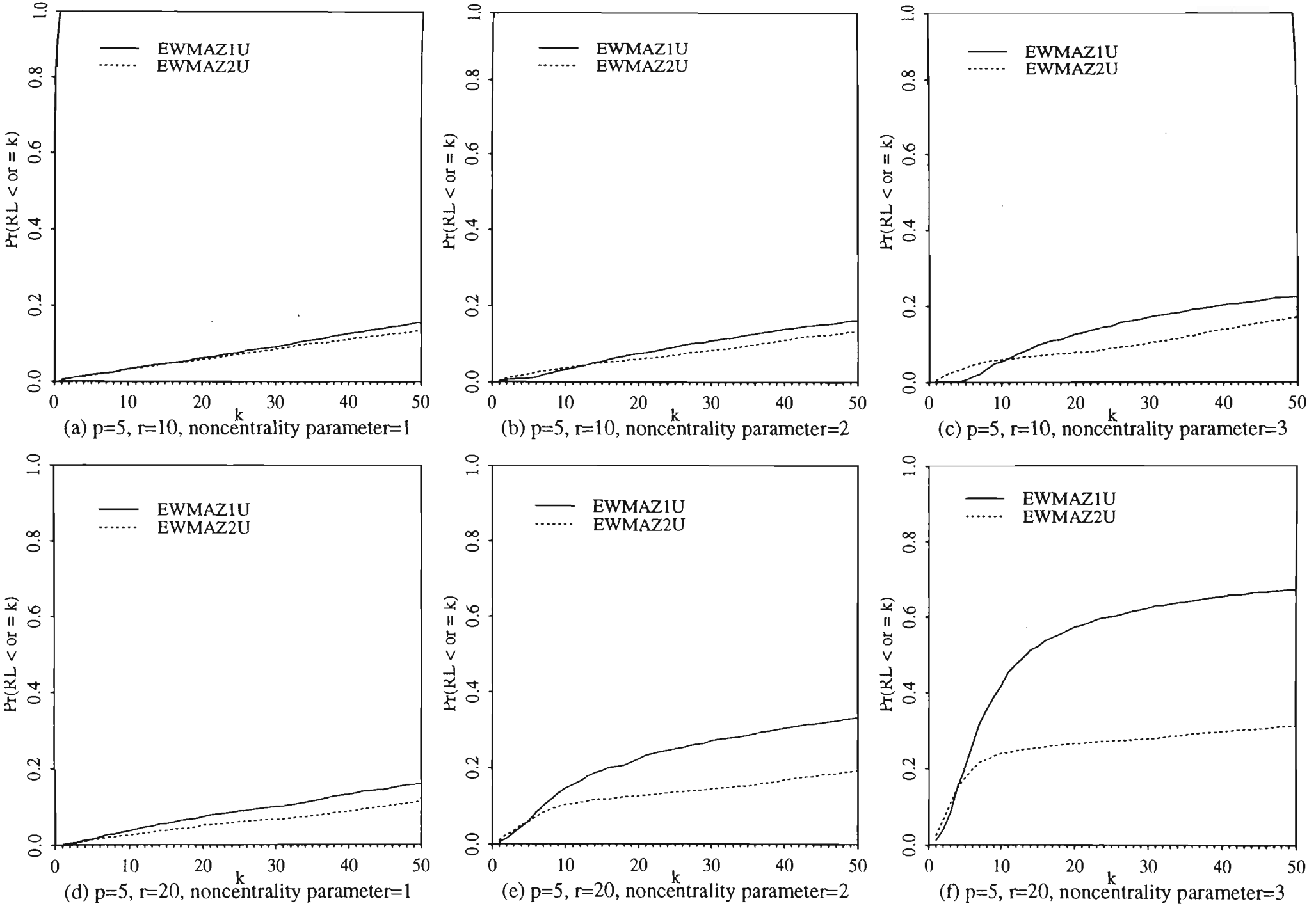


Figure 3.19. Run Length Probabilities for Step Shifts in Mean Vector

CHAPTER 4

DISPERSION CONTROL FOR MULTIVARIATE NORMAL PROCESSES [1]

4.1 Introduction

Over the last decade, the problem of multivariate quality control has received considerable attention in the literature (see for eg., Woodall and Ncube (1985), Murphy (1987), Healy (1987), Crosier (1988), Pignatiello et al.(1990), Doganaksoy et al.(1991), Sparks (1992), Tracy et al.(1992), Lowry et al.(1992), Hawkins (1991,1993), Hayter et al.(1994), Chan et al.(1994) and Mason, Tracy and Young (1995)). This work has focussed on the detection of parameter changes, departures from distributional assumptions, and the identification of out-of-control variables. Most of the work is based on the assumption that the observation vectors, $\mathbf{X}_i's$ are independently and identically distributed (i.i.d) multivariate normal variables and that the true values of the process parameters, in particular the process variance-covariance matrix, Σ, are known. In chapter 3, a control procedure for monitoring the mean level of multivariate normal processes for situations where prior information about the in-control process parameters is unavailable has been presented. It was demonstrated that the procedure is particularly useful when subgroup data are used.

Whilst substantial work has been devoted to the control of the process mean vector, μ, very little emphasis has been placed on the importance of monitoring and controlling Σ. In fact, the issue is a formidable one due to the complexity of the

[1] This chapter is based on the papers entitled 'Dispersion Control for Multivariate Processes', *Australian Journal of Statistics 38 (3)*, pp.235-251, 1996 and 'Dispersion Control for Multivariate Processes - Some Comparisons', *Australian Journal of Statistics 38 (3)*, pp.253-273, 1996.

distribution theory involved. One exception is the paper by Alt et al.(1986) who proposed two control techniques for Σ; one based on the likelihood ratio principle and the other that makes use of the sample generalized variance, which is sometimes taken as a measure of dispersion or spread of multivariate processes. Although traditional multivariate control charts such as the Hotelling χ^2 or T^2 charts may signal certain shifts in Σ (see Hawkins (1991)), other particular changes in Σ will remain undetected. This is also true for the technique based on generalized variance. For instance, if Σ shifts in such a way that the resulting process region (i.e the ellipsoidal region in which almost all observations fall) is contained completely within the undisturbed one, this 'shrunken' process is unlikely to be detected by a χ^2 chart, especially when the sample size is small. In addition, Hawkins (1991) stated that 'measures based on quadratic forms (like T^2) also confound mean shifts with variance shifts and require quite extensive analysis following a signal to determine the nature of the shift'. Note that the 'T^2' term that he used actually refers to the more commonly called χ^2 statistic which uses (presumably) the true value of the process covariance matrix. When 'special' or 'assignable' causes affecting both process parameters are present, it is also possible that the effect of the mean (vector) shift is masked or 'diluted' by the accompanying change in the variance-covariance matrix.

The purpose of this chapter is to present some control procedures for the dispersion of multivariate normal processes based on subgroup data. Special attention is drawn to the situations where prior information about Σ is not available as is often the case in situations of short production runs, which have become increasingly prevalent. When Σ is specified or assumed known, the proposed procedure involves the

decomposition of the sample covariance matrix and uses the resulting independent components, which have meaningful interpretations, as the bases for checking the constancy of the process covariance matrix. Another possible approach is also outlined for this case. As for the case where Σ is unknown in advance of production, the proposed procedure is adapted from the step-down test of Anderson (1984, p.417) which is based on the decomposition of the likelihood ratio statistic for testing the equality of several covariance matrices. When these procedures are used together with Hotelling χ^2 or T^2-type charts, they supplement the latter by providing independent information about the stability of the process covariance matrix. Furthermore, these techniques effectively replace existing procedures to provide enhanced detection of general shifts in Σ.

This chapter is organized into five subsequent sections. In 4.2, the underlying methodology is presented. In 4.3, appropriate control statistics are given for both the cases regarding prior knowledge or lack of prior knowledge of the process covariance matrix. Methods to cope with rank deficient problem which arises from the use of sample sizes not exceeding the number of quality variables are briefly considered in section 4.4. Comparisons are made between the proposed techniques and various competing procedures in section 4.5. In 4.6, an illustrative example is presented. In the last section, the effect of incorporating independent components from the decomposition to form an aggregate-type statistic, on the control performance, is examined for the known Σ case. The total discourse is given in the context of the manufacture of discrete items.

4.2 Methodology

Suppose that the vectors of observations on p correlated product characteristics, $\mathbf{X}_i's$ follow a multivariate normal $N_p(\mu,\Sigma)$ distribution with mean vector μ and

covariance matrix Σ when the process is operating under stable conditions. In practice, the validity of this assumption should be checked using, for example, a multivariate normal goodness-of-fit test (Gnanadesikan (1977)). The aim here is to develop control procedures for monitoring and controlling the *dispersion* of such a multivariate process based on *rational* subgroups where the sample size, n may vary. It is assumed in this and the next section that $n > p$ so that the suggestion works. When this is not the case, little adaptation of the proposed procedures as discussed in section 4.4 is required.

In order to provide more flexibility, ease of implementation and better control of the false alarm rate than existing procedures, as well as to facilitate the interpretation of out-of-control signals, it is suggested that the sample variance-covariance matrix be partitioned into various statistically independent components having physical interpretation and known distributions. These components are then used to indicate the stability of the process covariance matrix.

It is well known, that under the stable or in-control normality assumption, the sample covariance matrix, $\mathbf{S}$, multiplied by the factor $(n-1)$ follows the *Wishart* distribution with parameters $(n-1)$ and Σ, denoted by

$$(n-1)\mathbf{S} \sim W_p(n-1, \Sigma)$$

Let the sample and population covariance matrices be similarly expressed in partitioned form as follows :-

$$\mathbf{S} = \left(\begin{array}{c|c} \mathbf{S}_{11} & \mathbf{S}_{12} \\ \hline \mathbf{S}_{21} & \mathbf{S}_{22} \end{array}\right) = \left(\begin{array}{c|c} S_1^2 & \mathbf{S}_{12}^{T} \\ \hline \underset{\sim}{\mathbf{S}_{12}} & \mathbf{S}_{22} \end{array}\right), \qquad \Sigma = \left(\begin{array}{c|c} \Sigma_{11} & \Sigma_{12} \\ \hline \Sigma_{21} & \Sigma_{22} \end{array}\right) = \left(\begin{array}{c|c} \sigma_1^2 & \underset{\sim}{\Sigma_{12}^{T}} \\ \hline \underset{\sim}{\Sigma_{12}} & \Sigma_{22} \end{array}\right)$$

where S_1^2 (σ_1^2), $\underset{\sim}{\mathbf{S}_{12}}$ $(\underset{\sim}{\Sigma_{12}})$ and $\mathbf{S}_{22}$ (Σ_{22}) denote respectively the sample (population) variance of the 1st variable, the vector of sample (population) covariances between the

1st and each of the remaining variables, and the sample (population) covariance matrix excluding the 1st variable. Next, define

$$\begin{aligned}\mathbf{S}_{22\bullet 1} &= \mathbf{S}_{22} - \mathbf{S}_{21}\mathbf{S}_{11}^{-1}\mathbf{S}_{12} \\ &= \mathbf{S}_{22} - \frac{\underset{\sim}{\mathbf{S}_{12}}\,\underset{\sim}{\mathbf{S}_{12}^{\mathrm{T}}}}{S_1^2},\end{aligned}$$

then according to a well-known theorem (see for eg., theorem 6.4.1, p.120, Giri (1977), where $\Sigma_{(22)} - \Sigma_{(21)}\Sigma_{(11)}^{-1}\Sigma_{(12)}$ in (c) should be replaced by $\Sigma_{(11)} - \Sigma_{(12)}\Sigma_{(22)}^{-1}\Sigma_{(21)}$ using his notation),

(i) $\mathbf{S}_{22\bullet 1}$ is *independent* of $\left(S_1^2, \underset{\sim}{\mathbf{S}_{12}^{\mathrm{T}}}\right)$,

(ii) $(n-1)S_1^2 \sim \sigma_1^2 \chi_{n-1}^2$,

(iii) $(n-1)\mathbf{S}_{22\bullet 1} \sim W_{p-1}\left(n-2, \Sigma_{22\bullet 1}\right)$ where $\Sigma_{22\bullet 1} = \Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12}$

$$= \Sigma_{22} - \frac{\underset{\sim}{\Sigma_{12}}\,\underset{\sim}{\Sigma_{12}^{\mathrm{T}}}}{\sigma_1^2}.$$

(iv) The *conditional* distribution of $\dfrac{\underset{\sim}{\mathbf{S}_{12}}}{S_1^2}$, given $S_1^2 = s_1^2$, is $N\left(\dfrac{\underset{\sim}{\Sigma_{12}}}{\sigma_1^2}, \dfrac{\Sigma_{22\bullet 1}}{(n-1)s_1^2}\right)$.

Note that $\mathbf{S}_{22\bullet 1}$ and $\Sigma_{22\bullet 1}$ here denote respectively the conditional sample and population covariance matrices of the last (p-1) variables given the 1st. Note also that $\dfrac{\underset{\sim}{\mathbf{S}_{12}}}{S_1^2}$ and $\dfrac{\underset{\sim}{\Sigma_{12}}}{\sigma_1^2}$ represent respectively the vectors of sample and population regression coefficients when each of the last (p-1) variables is regressed on the 1st variable.

Furthermore, S_1^2, $\frac{\mathbf{S}_{12}}{\sim}$ and $\mathbf{S}_{22\bullet 1}$ may be regarded as *independent* components following the above decomposition of the sample covariance matrix $\mathbf{S}$. Further, decomposing $\mathbf{S}_{22\bullet 1}$ in the same manner yields $S^2_{2\bullet 1}$ (the conditional sample variance of the 2nd variable given the 1st one), $\frac{\mathbf{S}_{12\bullet 1}}{\sim}$ (the vector of regression coefficients when each of the last $(p-2)$ variables is regressed on the 2nd whilst the 1st variable is held fixed) and $\mathbf{S}_{3,4,\ldots,p\bullet 1,2}$ (the conditional sample covariance matrix of the last $(p-2)$ variables given the first two) which are independently distributed as

$$S^2_{2\bullet 1} = \left(1 - R^2_{12}\right) S^2_2 \sim \frac{(1-\rho^2_{12})\sigma^2_2}{(n-1)} \chi^2_{n-2} ,$$

$$\frac{\mathbf{S}_{12\bullet 1}}{\sim} \Big/ S^2_{2\bullet 1} = s^2_{2\bullet 1} \sim N\left(\frac{\Sigma_{12\bullet 1}}{\sim}, \frac{\Sigma_{3,\ldots,p\bullet 1,2}}{(n-1)s^2_{2\bullet 1}} \right) ,$$

and $$\mathbf{S}_{3,\ldots,p\bullet 1,2} \sim \frac{1}{(n-1)} W_{p-2}\left(n-3, \Sigma_{3,\ldots,p\bullet 1,2}\right) ,$$

where R^2_{12} and ρ^2_{12} are respectively the squares of the sample and population correlations between the 1st and the 2nd variables. Repeating the above procedures until further decomposition is impossible results in p *scaled* chi-square variables S_1^2, $S^2_{j\bullet 1,\ldots,j-1}$, $j = 2,\ldots,p$ and p-1 *conditional* (univariate or multivariate) normal variables which are independent and have meaningful interpretations. $S^2_{j\bullet 1,\ldots,j-1}$ here denotes the conditional sample variance of the jth variable given the first j-1 variables.

Note that the ordering of the variables is *not* unique. In fact, there are $p!$ possible permutations each of which results in $(2p-1)$ terms in the decomposition. If all of these

$p! \times (2p-1)$ variables are used as the control statistics, there will be a multitude of control charts even when p is quite small. For instance, when $p=3$, there will be $p!(2p-1)=30$ terms in total that can be obtained from the various decompositions. When $p=8$, this number increases to $p!(2p-1)=604800$ which clearly renders the approach impractical ! Furthermore, there are component variables in common to the various partitionings and terms that reflect essentially the same information. Therefore, one particular arrangement of the variables is deemed to be adequate for the purpose of decomposition. It is suggested that the choice of this should reflect the relative importance of the variables involved. In particular, the variables should be arranged in decreasing order of importance from 1 to p. For the case of a 'cascade' process as described by Hawkins (1993), the variables should be arranged from the most 'upstream' (being the 1st) to the most 'downstream' one (being the last) so that a shift in a variable will not be masked by the accompanying change in the downstream variables.

If $\sum$ is specified or assumed known in advance of production, the statistics obtained in the above manner can of course be used separately to monitor the dispersion of the multivariate process. However, due to independence, these statistics can be combined into a single aggregate-type control statistic as considered in the next section. In practice, if the latter approach is adopted, it is recommended that the values of the individual statistics be retained for post-signal analysis.

To illustrate the above idea, consider the case of p = 3 product characteristics. Using conventional notation, the sample covariance matrix of n observations on these variables is given by

$$\mathbf{S} = \begin{pmatrix} S_1^2 & R_{12}S_1S_2 & R_{13}S_1S_3 \\ R_{12}S_1S_2 & S_2^2 & R_{23}S_2S_3 \\ R_{13}S_1S_3 & R_{23}S_2S_3 & S_3^2 \end{pmatrix}.$$

Letting $\mathbf{S_{11}} = S_1^2$, $\mathbf{S_{12}} = (R_{12}S_1S_2,\ R_{13}S_1S_3)$, $\mathbf{S_{22}} = \begin{pmatrix} S_2^2 & R_{23}S_2S_3 \\ R_{23}S_2S_3 & S_3^2 \end{pmatrix}$ and

proceeding as previously, we have

$$\begin{aligned}(n-1)\mathbf{S_{22.1}} &= (n-1)\left[\mathbf{S_{22}} - \mathbf{S_{21}}\mathbf{S_{11}^{-1}}\mathbf{S_{12}}\right] \\ &= (n-1)\begin{pmatrix} S_2^2(1-R_{12}^2) & S_2S_3(R_{23}-R_{12}R_{13}) \\ S_2S_3(R_{23}-R_{12}R_{13}) & S_3^2(1-R_{13}^2) \end{pmatrix} \sim W_2\left(n-2, \Sigma_{22\cdot 1}\right)\end{aligned}$$

independently distributed of $(\mathbf{S_{11}}, \mathbf{S_{12}}) = \left(S_1^2,\ R_{12}S_1S_2,\ R_{13}S_1S_3\right)$ and

$$(n-1)S_1^2 \sim \sigma_1^2\chi^2(n-1)\ . \tag{4.1}$$

$$\begin{aligned}(\mathbf{S_{21}}\mathbf{S_{11}^{-1}} \mid \mathbf{S_{11}} = \mathbf{s_{11}}) &\equiv \left(\frac{R_{12}S_2}{S_1},\ \frac{R_{13}S_3}{S_1}\right)^{\mathrm{T}} \Big/ S_1^2 = s_1^2 \ \sim N\left(\frac{\Sigma_{21}}{\sigma_1^2},\ \frac{\Sigma_{22\cdot 1}}{(n-1)s_1^2}\right) \\ &\sim N\left(\left(\frac{\rho_{12}\sigma_2}{\sigma_1},\ \frac{\rho_{13}\sigma_3}{\sigma_1}\right)^{\mathrm{T}},\ \frac{\Sigma_{22\cdot 1}}{(n-1)s_1^2}\right)\end{aligned} \tag{4.2}$$

where

$$\begin{aligned}\Sigma_{22\cdot 1} &= \left[\Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12}\right] \\ &= \begin{pmatrix} \sigma_2^2(1-\rho_{12}^2) & \sigma_2\sigma_3(\rho_{23}-\rho_{12}\rho_{13}) \\ \sigma_2\sigma_3(\rho_{23}-\rho_{12}\rho_{13}) & \sigma_3^2(1-\rho_{13}^2) \end{pmatrix}.\end{aligned}$$

Note that $\mathbf{S_{21}}\mathbf{S_{11}^{-1}} = (R_{12}S_2/S_1,\ R_{13}S_3/S_1)^{\mathrm{T}}$ represents the vector of regression coefficients when each of the 2nd and 3rd variables is regressed on the 1st variable. Further, decomposing $\mathbf{S_{22\cdot 1}}$ in the same manner yields independent components,

$$\left(S_{2\cdot 1}^2\ ,\ R_{32\cdot 1}S_{2\cdot 1}S_{3\cdot 1}\right) = \left(S_2^2(1-R_{12}^2),\ S_2S_3(R_{23}-R_{12}R_{13})\right)$$

and $$S^2_{3\bullet 1,2} = S^2_3\left(1 - R^2_{3(1,2)}\right)$$

where

$$(n-1)S^2_2\left(1-R^2_{12}\right) \sim \sigma^2_2\left(1-\rho^2_{12}\right)\chi^2_{n-2} \tag{4.3}$$

$$\left(\frac{S_3(R_{23}-R_{12}R_{13})}{S_2\left(1-R^2_{12}\right)} \Big/ S^2_2\left(1-R^2_{12}\right) = s^2_2(1-r^2_{12})\right) \sim N\left(\frac{\sigma_3(\rho_{23}-\rho_{12}\rho_{13})}{\sigma_2(1-\rho^2_{12})}, \frac{\sigma^2_3(1-\rho^2_{3(1,2)})}{(n-1)s^2_2(1-r^2_{12})}\right) \tag{4.4}$$

$$(n-1)S^2_3\left(1-R^2_{3(1,2)}\right) \sim \sigma^2_3\left(1-\rho^2_{3(1,2)}\right)\chi^2_{n-3} \tag{4.5}$$

and $R^2_{3(1,2)}$ and $\rho^2_{3(1,2)}$ denote respectively the sample and population multiple R^2 when the 3rd variable is regressed on the first two variables. Note that $\frac{S_3(R_{23}-R_{12}R_{13})}{S_2\left(1-R^2_{12}\right)}$ is an unbiased estimate of the slope coefficient for the regression of the 3rd variable on the 2nd variable whilst the first variable is held fixed.

It is suggested that, if Σ is known, the statistics given in (4.1), (4.2), (4.3), (4.4) and (4.5) should all be used to provide protection against changes in the process covariance matrix Σ. It is advocated using all these components instead of only S^2_1, $S^2_{2\bullet 1}$ and $S^2_{3\bullet 1,2}$ because $\left(R_{12}S_2/S_1,\ \ R_{13}S_3/S_1\right)^{\mathrm{T}}$ and $\frac{S_3(R_{23}-R_{12}R_{13})}{S_2\left(1-R^2_{12}\right)}$ may reveal some changes in Σ that may not be reflected by the former statistics. For instance, if the 3rd quality characteristic is independent of others i.e $\rho^2_{3(12)} = \rho^2_{13} + \frac{(\rho_{12}\rho_{13}-\rho_{23})^2}{1-\rho^2_{12}} = 0$, and $(\sigma_2,\ \rho_{12})$ shifts to $(\sigma_{2new},\ \rho_{12new})$ such that $\frac{\sigma^2_{2new}(1-\rho^2_{12new})}{\sigma^2_2(1-\rho^2_{12})} = 1$, then this change is unlikely to be detected when only S^2_1, $S^2_{2\bullet 1}$ and $S^2_{3\bullet 1,2}$ are used because their

respective distributions are not distorted under these circumstances. However, this change induces a shift in the slope coefficient for the regression of the 2nd quality characteristic on the first. Therefore, it is possible to 'pick up' such a change if the vector of population regression coefficients $(\rho_{12}\sigma_2/\sigma_1,\ \rho_{13}\sigma_3/\sigma_1)^T$ is also monitored based on the corresponding vector of sample regression coefficients $(R_{12}S_2/S_1,\ \ R_{13}S_3/S_1)^T$ which is known to be bivariate normal for fixed S_1^2 under the in-control and normality assumption (see (4.2)). If the traditional Hotelling χ^2 chart based on these coefficient vectors is used, it is readily seen that its statistical performance depends on the noncentrality parameter

$$\lambda = \frac{(n-1)S_1^2\left(\pm\sqrt{\sigma_{2new}^2 - \sigma_2^2(1-\rho_{12}^2)} - \rho_{12}\sigma_2\right)^2}{\sigma_1^2\sigma_2^2(1-\rho_{12}^2)}$$

where the +ve sign is used when $\rho_{12new} > 0$ and the –ve sign otherwise. Thus, whilst the use of the control statistics S_1^2, $S_{2\bullet 1}^2$ and $S_{3\bullet 1,2}^2$ are unlikely to register the change, it is clear that the probability of detection by the Hotelling χ^2 chart may increase depending on the value of S_1 for the current subgroup, the sample size, n, as well as the dispersion parameters. The same is true if the aggregate-type control statistic as given in the next section is used.

As an alternative, the following method of decomposition may be employed. Let $\mathbf{S}_{<j>}$ and $\Sigma_{<j>}$ be the upper left-hand square submatrices of $\mathbf{S}$ and Σ respectively, of order j. Also, let $\mathbf{S}_{[j]}$ and $\Sigma_{[j]}$ denote respectively the sample and population

covariance matrices of the jth, 1st, 2nd, ... and $(j\text{-}1)$th variable in the order indicated. In addition, let $\underset{\sim}{\mathbf{S}}_j$ and $\underset{\sim}{\sigma}_j$ be such that

$$\mathbf{S}_{[j]} = \begin{pmatrix} S_{jj} & \mathbf{S}_j^{\mathrm{T}} \\ \underset{\sim}{\mathbf{S}}_j & \mathbf{S}_{<j-1>} \end{pmatrix} \quad \text{and} \quad \Sigma_{[j]} = \begin{pmatrix} \sigma_{jj} & \underset{\sim}{\sigma}_j^{\mathrm{T}} \\ \underset{\sim}{\sigma}_j & \Sigma_{<j-1>} \end{pmatrix}, \qquad j = 2,\ldots,p.$$

where $S_{jj} = S_j^2$ and $\sigma_{jj} = \sigma_j^2$. Repeatedly applying theorem 6.4.1 of Giri (1977) to $\mathbf{S}_{[j]}$, starting with $j = p$ and decreasing in steps of 1, results in the following $2p-1$ (conditionally) independent statistics :

$$S^2_{j\bullet 1,\ldots,j-1} \sim \frac{\sigma^2_{j\bullet 1,\ldots,j-1}}{(n-1)} \chi^2_{n-j}, \qquad j = 1,\ldots,p.$$

$$\mathbf{S}^{-1}_{<j-1>} \underset{\sim}{\mathbf{S}}_j \sim N_{j-1}\left(\Sigma^{-1}_{<j-1>} \underset{\sim}{\sigma}_j, \ \frac{\sigma^2_{j\bullet 1,\ldots,j-1}}{(n-1)} \mathbf{S}^{-1}_{<j-1>} \right), \qquad j = 2,\ldots,p.$$

where $S^2_{j\bullet 1,\ldots,j-1} = S_{jj} - \underset{\sim}{\mathbf{S}}_j^{\mathrm{T}} \mathbf{S}^{-1}_{<j-1>} \underset{\sim}{\mathbf{S}}_j$ and $\sigma^2_{j\bullet 1,\ldots,j-1} = \sigma_{jj} - \underset{\sim}{\sigma}_j^{\mathrm{T}} \Sigma^{-1}_{<j-1>} \underset{\sim}{\sigma}_j$ are respectively the conditional sample and population variances of the jth variable given the first $j-1$ variables. Note that $\mathbf{S}^{-1}_{<j-1>} \underset{\sim}{\mathbf{S}}_j$ is the $(j-1)$ dimensional vector estimating the regression coefficients of the jth variable regressed on the first $(j-1)$ variables (see Mason et al.(1995)). Note also that $S^2_{1\bullet 0}$ and $\sigma^2_{1\bullet 0}$ are taken to be S_1^2 and σ_1^2 respectively.

The hypothesis $H_0 : \Sigma = \Sigma_0$ may be tested based on these statistics for each subgroup in a sequential or step-down manner. At the jth step, the component hypothesis $\sigma^2_{j\bullet 1,\ldots,j-1} = (\sigma^2_{j\bullet 1,\ldots,j-1})_0$ is tested at the α_j significance level by means of a chi-square test based on

$$\frac{S^2_{j\bullet 1,\cdots,j-1}}{(\sigma^2_{j\bullet 1,\ldots,j-1})_0}. \qquad (4.6)$$

If there is failure to reject this sub-hypothesis, then $\underset{\sim}{\sigma}_j = \left(\underset{\sim}{\sigma}_j\right)_0$ (or $\Sigma^{-1}_{<j-1>}\underset{\sim}{\sigma}_j = \left(\Sigma^{-1}_{<j-1>}\right)_0\left(\underset{\sim}{\sigma}_j\right)_0$) is tested at significance level δ_j on the assumption that $\Sigma_{<j-1>} = \left(\Sigma_{<j-1>}\right)_0$. The test statistic,

$$\left(\mathbf{S}^{-1}_{<j-1>}\underset{\sim}{\mathbf{S}}_j - \left(\Sigma^{-1}_{<j-1>}\right)_0\left(\underset{\sim}{\sigma}_j\right)_0\right)^{\mathrm{T}} \frac{(n-1)\mathbf{S}_{<j-1>}}{\left(\sigma^2_{j\bullet 1,\ldots,j-1}\right)_0}\left(\mathbf{S}^{-1}_{<j-1>}\underset{\sim}{\mathbf{S}}_j - \left(\Sigma^{-1}_{<j-1>}\right)_0\left(\underset{\sim}{\sigma}_j\right)_0\right) \qquad (4.7)$$

is a χ^2_{j-1} variable if the component hypothesis is true. If there is failure to reject this component hypothesis, then the $(j+1)$th step is taken. The hypothesis $H_0{:}\Sigma = \Sigma_0$ is accepted provided there is failure to reject all the $2p-1$ component hypotheses. The overall significance level of this test for each subgroup is then given by

$$1-\prod_{j=1}^{p}(1-\alpha_j)\prod_{j=2}^{p}(1-\delta_j).$$

Anderson (1984, p.417-418) has presented such an approach for testing the equality of covariance matrices as an alternative to the standard maximum likelihood ratio procedure, with the unknown parameters replaced by appropriate estimates based on previous subgroups and other suitable adjustments made. The resulting statistics for all successive subgroups follow Snedecor-*F* distributions and were shown by this author to be stochastically independent (Anderson (1984), theorem 10.4.2, p.414). Although this method is not proposed in the context of SPC, it can be used for monitoring the stability of the process covariance matrix for which the true in-control value is *unknown* and cannot be reliably estimated. Following the conventional approach, however, a

single control chart based on all these statistics is considered instead of using them separately. This control technique, which is particularly useful for short production runs and low volume manufacturing, is discussed in detail in the next section.

4.3 Monitoring the Dispersion of Multivariate Processes

The techniques now presented involve use of the *probability integral transformation* in order to produce sequences of independent chi-square variables (see Quesenberry (1991a)). The suggested approach permits the monitoring of various components resulting from the decomposition of the covariance matrix on a single chart.

For uniformity of notation and ease of presentation, define $\mathbf{S}_{*k}$ (Σ_{*k}) and $\underset{\sim}{\mathbf{S}}_{v,u}$ ($\underset{\sim}{\sigma}_{v,u}$) respectively as the sample (population) covariance matrix of the *last k* variables and the vector of sample (population) covariances between the vth variable and each of the *first u* variables. Accordingly, the sample and population covariance matrices are expressible as

$$\mathbf{S} = \left(\begin{array}{c|c} \mathbf{S}_{<j-1>} & \underset{\sim}{\mathbf{S}}_{j,j-1}\ ,\cdots,\ \underset{\sim}{\mathbf{S}}_{p,j-1} \\ \hline \underset{\sim}{\mathbf{S}}^{\mathrm{T}}_{j,j-1} & \\ \vdots & \mathbf{S}_{*p-j+1} \\ \underset{\sim}{\mathbf{S}}^{\mathrm{T}}_{p,j-1} & \end{array}\right) \quad \text{and} \quad \Sigma = \left(\begin{array}{c|c} \Sigma_{<j-1>} & \underset{\sim}{\sigma}_{j,j-1}\ ,\cdots,\ \underset{\sim}{\sigma}_{p,j-1} \\ \hline \underset{\sim}{\sigma}^{\mathrm{T}}_{j,j-1} & \\ \vdots & \Sigma_{*p-j+1} \\ \underset{\sim}{\sigma}^{\mathrm{T}}_{p,j-1} & \end{array}\right)$$

where $\mathbf{S}_{<j>}$ ($\Sigma_{<j>}$) denotes the sample (population) covariance matrix of the *first j* variables and $\underset{\sim}{\mathbf{S}}_{j,j-1} = \underset{\sim}{\mathbf{S}}_{j}$ ($\underset{\sim}{\sigma}_{j,j-1} = \underset{\sim}{\sigma}_{j}$) as defined in the preceding section. The conditional sample variance of the jth variable given the first $j-1$ variables, is then given by

$$S^2_{j\bullet 1,\ldots,j-1} = S^2_j - \mathbf{S}^{\mathrm{T}}_{\underset{\sim}{j,j-1}} \mathbf{S}^{-1}_{<j-1>} \mathbf{S}_{\underset{\sim}{j,j-1}} \,. \tag{4.8}$$

Similarly, the corresponding population parameter is

$$\sigma^2_{j\bullet 1,\ldots,j-1} = \sigma^2_j - \sigma^{\mathrm{T}}_{\underset{\sim}{j,j-1}} \Sigma^{-1}_{<j-1>} \sigma_{\underset{\sim}{j,j-1}} \,. \tag{4.9}$$

In terms of variances and multiple correlation coefficients, these are expressible as $S^2_{j\bullet 1,\ldots,j-1} = S^2_j(1-R^2_{j(1,\ldots,j-1)})$ and $\sigma^2_{j\bullet 1,\ldots,j-1} = \sigma^2_j(1-\rho^2_{j(1,\ldots,j-1)})$. The conditional sample and population covariance matrices of the last $p-j+1$ variables given the remaining $j-1$ variables are respectively

$$\mathbf{S}_{j,\ldots,p\bullet 1,\ldots,j-1} = \mathbf{S}_{*p-j+1} - \left(\mathbf{S}_{\underset{\sim}{j,j-1}} \quad \cdots \quad \mathbf{S}_{\underset{\sim}{p,j-1}}\right)^{\mathrm{T}} \mathbf{S}^{-1}_{<j-1>} \left(\mathbf{S}_{\underset{\sim}{j,j-1}} \quad \cdots \quad \mathbf{S}_{\underset{\sim}{p,j-1}}\right) \tag{4.10}$$

and

$$\Sigma_{j,\ldots,p\bullet 1,\ldots,j-1} = \Sigma_{*p-j+1} - \left(\sigma_{\underset{\sim}{j,j-1}} \quad \cdots \quad \sigma_{\underset{\sim}{p,j-1}}\right)^{\mathrm{T}} \Sigma^{-1}_{<j-1>} \left(\sigma_{\underset{\sim}{j,j-1}} \quad \cdots \quad \sigma_{\underset{\sim}{p,j-1}}\right) \tag{4.11}$$

Apart from these, let $\underset{\sim}{\boldsymbol{d}}_j$ and $\underset{\sim}{\theta}_j$ $(j = 2,\ldots, p)$ denote respectively the vectors of sample and population regression coefficients when each of the last p-j+1 variables is regressed on the (j-1)th variable whilst the remaining j-2 variables are held fixed. Then, these are given by the following expressions :-

$$\underset{\sim}{\boldsymbol{d}}_j = \frac{\left\{\left(S_{j-1,j} \quad \cdots \quad S_{j-1,p}\right) - \mathbf{S}^{\mathrm{T}}_{\underset{\sim}{j-1,j-2}} \mathbf{S}^{-1}_{<j-2>} \left(\mathbf{S}_{\underset{\sim}{j,j-2}} \quad \cdots \quad \mathbf{S}_{\underset{\sim}{p,j-2}}\right)\right\}^{\mathrm{T}}}{S_{j-1,j-1} - \mathbf{S}^{\mathrm{T}}_{\underset{\sim}{j-1,j-2}} \mathbf{S}^{-1}_{<j-2>} \mathbf{S}_{\underset{\sim}{j-1,j-2}}} \tag{4.12}$$

and

$$\underset{\sim}{\theta}_j = \frac{\left\{\left(\sigma_{j-1,j} \quad \cdots \quad \sigma_{j-1,p}\right) - \underset{\sim}{\sigma}^{\mathrm{T}}_{j-1,j-2} \Sigma^{-1}_{<j-2>} \left(\underset{\sim}{\sigma}_{j,j-2} \quad \cdots \quad \underset{\sim}{\sigma}_{p,j-2}\right)\right\}^{\mathrm{T}}}{\sigma_{j-1,j-1} - \underset{\sim}{\sigma}^{\mathrm{T}}_{j-1,j-2} \Sigma^{-1}_{<j-2>} \underset{\sim}{\sigma}_{j-1,j-2}} \qquad (4.13)$$

Note that $\underset{\sim}{d}_2$ and $\underset{\sim}{\theta}_2$ should be interpreted as the vectors of unconditional sample and population regression coefficients when each of the last p-1 variables is regressed on the 1st variable and these are given by

$$\underset{\sim}{d}_2 = \frac{\left(S_{12} \quad \cdots \quad S_{1p}\right)^{\mathrm{T}}}{S_{11}} \quad \text{and} \quad \underset{\sim}{\theta}_2 = \frac{\left(\sigma_{12} \quad \cdots \quad \sigma_{1p}\right)^{\mathrm{T}}}{\sigma_{11}}$$

respectively.

In addition to the above, the following notation will be used :-

$\Phi^{-1}(\bullet)$: inverse of the standard normal distribution function

$\chi^2_\nu(\bullet)$: distribution function of a chi-square variable with ν degrees of freedom

$F_{\nu 1;\nu 2}(\bullet)$: distribution function of an F variable with $\nu 1$ numerator and $\nu 2$ denominator degrees of freedom

In the following subsections, control statistics for monitoring the stability of the process covariance matrix are presented for the case where either Σ is known or unknown. In order to specify the chronological order of the subgroups, the sample statistics are indexed with an additional subscript enclosed within a bracket.

Case (I) : Σ known

In practice, the process parameters, in particular the true value of the process covariance matrix Σ, is never known exactly. Instead, it is estimated based on a

presumably large enough set of relevant data that have been collected during the period in which the production process is assumed to be stable or in control. It will be assumed for current purposes that Σ is known precisely prior to production. In this case, the appropriate control statistic is

$$T_k = \sum_{j=1}^{2p-1} Z_{j(k)}^2 \qquad k = 1,2,\ldots \tag{4.14}$$

where

$$Z_{1(k)} = \Phi^{-1}\left\{\chi^2_{n_k-1}\left[\frac{(n_k-1)S^2_{1(k)}}{\sigma_1^2}\right]\right\},$$

$$Z_{j(k)} = \Phi^{-1}\left\{\chi^2_{n_k-j}\left[\frac{(n_k-1)S^2_{j\bullet 1,\ldots,j-1(k)}}{\sigma^2_{j\bullet 1,\ldots,j-1}}\right]\right\}, \qquad j = 2,\ldots,p.$$

$$Z_{p+1(k)} = \Phi^{-1}\left\{\chi^2_{p-1}\left[(n_k-1)S^2_{1(k)}\left(\underset{\sim}{d}_{2(k)} - \underset{\sim}{\theta}_2\right)^{\mathrm{T}}\Sigma^{-1}_{2,\ldots,p\bullet 1}\left(\underset{\sim}{d}_{2(k)} - \underset{\sim}{\theta}_2\right)\right]\right\},$$

$$Z_{p+j-1(k)} = \Phi^{-1}\left\{\chi^2_{p-j+1}\left[(n_k-1)S^2_{j-1\bullet 1,\ldots,j-2(k)}\left(\underset{\sim}{d}_{j(k)} - \underset{\sim}{\theta}_j\right)^{\mathrm{T}}\Sigma^{-1}_{j,\ldots,p\bullet 1,\ldots,j-1}\left(\underset{\sim}{d}_{j(k)} - \underset{\sim}{\theta}_j\right)\right]\right\}, \quad j = 3,\ldots,p.$$

It is readily seen from the foregoing discussion, that under the in-control and normality assumption, $\{Z_{j(k)}\}$, $j = 1,\ldots,2p-1$ are sequences of independently and identically distributed (*i.i.d*) standard normal variables, whence the T_k's are independent χ^2_{2p-1} variables. Although control charts may be constructed based on the arguments of $Z_{j(k)}$'s, this is not considered a viable option due to the proliferation of charts that results even when p is fairly small. Besides, it is found that combining the $Z_{j(k)}$'s in the proposed manner results in better control performance for certain shifts in Σ.

Note that since the arguments of the normalizing transformation are independent chi-square variables, a single aggregate-type control statistic may be obtained by summing them. The resulting test is commonly called the *sum test*. Similarly, the sum may be taken over the transformed statistics $Z_{j(k)}$'s giving a sequence of independent $N(0,\ 2p-1)$ variables. In either case, however, certain deviations of the process covariance matrix from the specified Σ are likely to be missed by the resulting techniques. In particular, if Σ shifts in such a way that the values of some $Z_{j(k)}$'s tend to be larger whilst others tend to be smaller than that attributable to common causes, then this type of change is unlikely to be detected by the resulting charts. To provide protection against such changes, it is suggested that the $Z_{j(k)}$'s be squared before summation as in formula (4.14), and only an upper control limit is then necessary. It should be noted that a similar technique can also be developed based on the alternative partitioning method outlined in the foregoing section. This is, however, not considered further because it is found that the proposed technique always performs better.

Case (II) : Σ unknown

In the absence of prior information about the process parameters, a natural solution is to estimate the various components resulting from the decomposition of Σ sequentially from the data stream of the current production. The resulting estimates, together with the corresponding observations from the next sample are then used to test whether or not Σ remains constant.

Before proceeding, define the quantities :-

$$N_{j,k} = \sum_{i=1}^{k} (n_i - j) \ ,$$

$$S^2_{j,k\,(pooled)} = \frac{1}{N_{j,k}} \sum_{i=1}^{k} (n_i - 1) S^2_{j\bullet 1,\ldots,j-1(i)} \;, \qquad j = 1,\ldots,p.$$

with $S^2_{1\bullet 0(i)} = S^2_{1(i)}$,

$$\underset{\sim}{\mathbf{U}}_{j,k} = \frac{1}{k^2} \sum_{i=1}^{k} (n_i - 1)^{-1} \mathbf{S}^{-1}_{<j>(i)} \;, \qquad j = 1,\ldots,p-1.$$

$$\underset{\sim}{\mathbf{V}}_{j,k} = \frac{1}{k} \sum_{i=1}^{k} \mathbf{S}^{-1}_{<j-1>(i)} \, \underset{\sim}{\mathbf{S}}_{j,j-1(i)} \;, \qquad j = 2,\ldots,p.$$

These values can be updated respectively through the following recursive equations :-

$$S^2_{j,k+1(pooled)} = \frac{1}{N_{j,k+1}} \left[N_{j,k} S^2_{j,k\,(pooled)} + (n_{k+1} - 1) S^2_{j\bullet 1,\ldots,j-1(k+1)} \right] ,$$

$$\underset{\sim}{\mathbf{U}}_{j,k+1} = \frac{1}{(k+1)^2} \left[k^2 \, \underset{\sim}{\mathbf{U}}_{j,k} + (n_{k+1} - 1)^{-1} \mathbf{S}^{-1}_{<j>(k+1)} \right] ,$$

and $$\underset{\sim}{\mathbf{V}}_{j,k+1} = \frac{1}{k+1} \left[k \, \underset{\sim}{\mathbf{V}}_{j,k} + \mathbf{S}^{-1}_{<j-1>(k+1)} \, \underset{\sim}{\mathbf{S}}_{j,j-1(k+1)} \right] .$$

The appropriate control statistic for this case is then given by

$$T_k = \sum_{j=1}^{2p-1} Z^2_{j(k)} \;, \qquad k = 2,\ldots \qquad (4.15)$$

where

$$Z_{j(k)} = \Phi^{-1} \left[F_{n_k - j; N_{j,k-1}} \left[\frac{(n_k - 1) S^2_{j\bullet 1,\ldots,j-1(k)}}{(n_k - j) S^2_{j,k-1\,(pooled)}} \right] \right] , \qquad j = 1,\ldots,p.$$

and

$$Z_{p+j-1(k)} = \Phi^{-1} \left[F_{j-1; N_{j,k}} \left[\frac{\left(\mathbf{S}^{-1}_{<j-1>(k)} \underset{\sim}{\mathbf{S}}_{j,j-1(k)} - \underset{\sim}{\mathbf{V}}_{j,k-1} \right)^{\mathrm{T}} \left((n_k - 1)^{-1} \mathbf{S}^{-1}_{<j->1(k)} + \underset{\sim}{\mathbf{U}}_{j-1,k-1} \right)^{-1} \left(\mathbf{S}^{-1}_{<j-1>(k)} \underset{\sim}{\mathbf{S}}_{j,j-1(k)} - \underset{\sim}{\mathbf{V}}_{j,k-1} \right)}{(j-1) S^2_{j,k\,(pooled)}} \right] \right]$$

$$, \quad j = 2,\ldots,p.$$

Note that when $k = 2$, the argument of $Z_{p+j(k)}$ is

$$\frac{\left(\mathbf{S}^{-1}_{<j-1>(2)}\mathbf{S}_{j,\underset{\sim}{j-1}(2)}-\mathbf{S}^{-1}_{<j-1>(1)}\mathbf{S}_{j,\underset{\sim}{j-1}(1)}\right)^{T}\left((n_1-1)^{-1}\mathbf{S}^{-1}_{<j-1>(1)}+(n_2-1)^{-1}\mathbf{S}^{-1}_{<j-1>(2)}\right)^{-1}\left(\mathbf{S}^{-1}_{<j-1>(2)}\mathbf{S}_{j,\underset{\sim}{j-1}(2)}-\mathbf{S}^{-1}_{<j-1>(1)}\mathbf{S}_{j,\underset{\sim}{j-1}(1)}\right)}{(j-1)S^2_{j,2\,(pooled)}}$$

where

$$S^2_{j,2(pooled)} = \frac{(n_1-1)S^2_{j\bullet 1,\ldots,j-1(1)}+(n_2-1)S^2_{j\bullet 1,\ldots,j-1(2)}}{(n_1+n_2-2j)}.$$

This is different from that given by Anderson (1984, p.418, expression (21)) which apparently contains a typographical error.

When the process covariance matrix is constant, $\left\{Z_{j(k)}\right\}$, $j = 1,\ldots,2p-1$ are independent sequences of i.i.d $N(0,1)$ variables (see Theorem 10.4.2, p.414 of Anderson(1984)). Thus, the T_k's here are again distributed as χ^2_{2p-1} variables. Note that although the arguments of the $Z_{j(k)}$'s have different degrees of freedom, the control limit for the resulting technique remains constant for successive subgroups. Note also that, using this technique, process monitoring can begin with the second subgroup without having to wait until considerable process performance data have been accumulated for computation of the unknown Σ.

It should be pointed out that the proposal in this chapter is not the only method for combining the information in independent tests. In fact, many other procedures have been presented in the literature. The most widely discussed method is attributed to Fisher (1950). Using this technique, the overall null hypothesis, $H_0: \bigcap_{j=1}^{q} H_{0j}$ is rejected at α level of significance if

$$-2\sum_{j=1}^{q} \ln P_j > \chi^2_{2q,1-\alpha}$$

where P_j's denote the p-values of the independent tests for the q sub-hypotheses H_{0_j}'s. Another common procedure is based on the minimum significance level of the independent tests as suggested by Tippett (1931). For this method, H_0 is rejected at α level of significance if

$$\min\{P_1,\ldots,P_q\} \le 1-(1-\alpha)^{1/q}.$$

We note that our procedure of forming a single control statistic is actually a special case of Lancaster's (1961) method which rejects H_0 at α level of significance if

$$\sum_{j=1}^{q} \Gamma_{\nu_j}^{-1}\left(1-P_j\right) \ge \Gamma_{\sum_{j=1}^{q}\nu_j,\,1-\alpha}$$

where $\Gamma_{\nu}^{-1}(\bullet)$ and $\Gamma_{\nu,\,a}$ denote respectively the inverse and the $100a$th percentile of a Gamma distribution with parameters (ν, 1/2). This is so when $\nu_j = 1/2$ for $j = 1,\ldots,q$. A review of these and other test combination procedures is given in Folks (1984). Previous studies indicate that Fisher and Tippett methods are good omnibus procedures if the test statistics for the sub-hypotheses are independent chi-square or F variables. However, the results of these studies do not apply in the present context since the problem here does not involve non-central chi-square or F variables under the alternative hypothesis.

In this chapter, no attempt has been made to determine the optimal choice of combination method except for the simulation study of the relative performance of Fisher, Tippett and the proposed procedures as reported in section 4.5. Other methods such as the weighted average of normal scores obtained from independent statistics and the weighted version of Fisher's method are not considered due to the arbitrary choice of

the weighting constants. If equal weights are used, the former is equivalent to the commoly called Inverse Normal procedure. This technique, which is based on the sum, $\sum_{j=1}^{2p-1} Z_{j(k)}$ for the problem under consideration, is expected to be ineffective for the reason given earlier. Similarly, the Bayes or likelihood ratio procedures derived by Koziol and Perlman (1978) and Marden (1982) for combining independent noncentral chi-square and F statistics respectively are inappropriate since the alternative hypotheses in question do not fit the specified models. Even if the models are appropriate, determination of critical values for the resulting techniques can be quite involved. The same is true for the other techniques not considered here including those based on the sum of *chi* statistics (see Koziol et al.(1978)).

4.4 Rank-Deficient Problem

As stated earlier, the proposed techniques are applicable only if the sample size exceeds the number of jointly monitored variables or the sample covariance matrix is of full rank. However, these techniques can be adapted to the rank-deficient situations as described below.

For mathematical convenience, suppose that samples of *equal* size n are used where $n \leq p$. In this case, simply transform the sample covariance matrices, $\mathbf{S}_{(i)}$'s to matrices of reduced dimension, $\mathbf{W}_{(i)}$'s as follows :

$$\mathbf{W}_{(i)} = \mathbf{A}\mathbf{S}_{(i)}\mathbf{A}^{\mathrm{T}}$$

where $\mathbf{A}$ is a full rank $(n\text{-}1)\times p$ matrix of constants. By theorem 3.3.7 of Srivastava et al.(1979),

$$(n-1)\mathbf{W}_{(i)} \sim W_{n-1}\left(n-1, \mathbf{A}\Sigma\mathbf{A}^{\mathrm{T}}\right).$$

Following this transformation, the problem becomes one that checks the constancy of the variance-covariance matrix, Σ through $\mathbf{A}\Sigma\mathbf{A}^{\mathrm{T}}$. The same methods of decomposition and combination of the resulting independent statistics can then be applied to the full rank matrices, $\mathbf{W}_{(i)}$'s and the associated likelihood ratio statistics for the known and unknown Σ cases respectively. Although dimensional reduction inevitably leads to some loss in information and the choice of $\mathbf{A}$ is somewhat arbitrary, this approach can be used irrespective of the sample size (provided $n \geq 2$) and valid control limits can be easily located.

Notice that as a special case of the transformation, $\mathbf{W}_{(i)}$ becomes the sample covariance matrix of the first n-1 quality characteristics if

$$\mathbf{A} = \left(\mathbf{I}_{(n-1)\times(n-1)}, \quad \mathbf{0}_{(n-1)\times p}\right).$$

This choice of $\mathbf{A}$ is undesirable since it sacrifices the information regarding the last $p-n+1$ variables. If Σ is known, a reasonable choice of $\mathbf{A}$ is that whose rows consist of the normalized eigenvectors corresponding to the largest n-1 eigenvalues, $\lambda_1,\ldots,\lambda_{n-1}$, of Σ. Under this transformation, $\mathbf{W}_{(i)}$ becomes the sample covariance matrix of the first n-1 principal components and $\mathbf{A}\Sigma\mathbf{A}^{\mathrm{T}} = \Lambda = diag\left(\lambda_1,\ldots,\lambda_{n-1}\right)$.

Other methods that can be used for the known Σ case include those which are based on *all* the principal components, any set of linearly independent combinations of the quality variables and the union-intersection principle (Srivastava et al. (1979)).

4.5 Comparisons

In this section, the relative performance of the proposed techniques are tested against some competing procedures. For the case where Σ is assumed known, comparison is made between the proposed technique, the associated Fisher's and Tippett's procedures, the modified likelihood ratio test (MLRT), the $|\mathbf{S}|^{1/2}$ charting technique as well as a possible method based on principal components. MLRT and $|\mathbf{S}|^{1/2}$ charting procedure appear to be the most widely discussed techniques for monitoring the dispersion of multivariate normal processes (see for eg., Alt et al.(1986, 1990)). The last method is included merely because it appears to be a reasonable technique in situations where principal components possess meaningful physical interpretations. In fact, as stated in Jackson (1989), this phenomenon is very common in industrial situations. In addition, Crosier (1988) has identified situations where '..... if the mean shifts, it does so along the major axis.....'. Under these circumstances, it is expected that changes in Σ may well occur along some of the principal axes, namely the variances of some principal components may shift. As for the unknown Σ case, the decomposition methods (Tippett, Fisher and the proposed techniques) are compared with the modified likelihood ratio test for the equality of covariance matrices (MLRTECM) as given, for eg., in Anderson (1984, p.405). For clarity of subsequent discussion, these techniques are briefly reviewed.

MLRT is an unbiased version of the likelihood ratio test of $H_0: \Sigma = \Sigma_0$ against $H_A: \Sigma \neq \Sigma_0$ with the sample size, n, replaced by the number of degrees of freedom, $n-1$. This test rejects the null hypothesis H_0 and suggests a departure of the process

covariance matrix from the standard or the known value Σ_0 when the test or control statistic, W^* exceeds $W^*_{p,n,\alpha}$ where

$$W^* = -p(n-1) - (n-1)\ln|\mathbf{S}| + (n-1)\ln|\Sigma_0| + (n-1)\mathrm{tr}\left(\Sigma_0^{-1}\mathbf{S}\right) . \qquad (4.16)$$

α and $W^*_{p,n,\alpha}$ denote respectively the false signal rate and the upper 100α th percentage point of W^* which depends on p and n. The distributional theory involved with this technique is prohibitively complicated thus limiting its practicality. Although W^* can be approximated by a chi-square distribution with $\frac{p(p+1)}{2}$ degrees of freedom when n is large, and the exact upper 1% and 5% percentage points of W^* have been tabulated, for example, in Anderson (1984), for p = 2(1)10 and various values of n, these may be of little value in the context of control charting. In practice, the control or monitoring procedures are likely to be based on samples not sufficiently large to justify the use of the chi-square approximation. Besides, it may be preferable to have a control technique with false signal rates considerably smaller than 1% or 5%. Although MLRT is admissible (Giri (1977), p.186), it will be seen later that this technique is inferior to the proposed technique for all of the cases considered.

The other competing procedure is based on the use of the square root of the generalized sample covariance matrix, $|\mathbf{S}|^{1/2}$. The resulting chart can be regarded as a multivariate analogue of the univariate S chart. When p = 2, $|\mathbf{S}|^{1/2}$ is distributed as a scaled chi-square variable under the stable or in-control multivariate normality assumption (Anderson (1984), p.264). Thus, control limits may be set at

$$\mathrm{LCL} = \frac{|\Sigma_0|^{1/2}\chi^2_{2(n-2),\alpha/2}}{2(n-1)}$$

$$\text{and} \quad \text{UCL} = \frac{|\Sigma_0|^{1/2}\chi^2_{2(n-2),1-\alpha/2}}{2(n-1)}, \tag{4.17}$$

where $\chi^2_{\nu,\delta}$ denotes the 100δ th percentage point of the chi-square distribution with ν degrees of freedom. For higher dimensions, Alt et al. (1986) suggested the use of the 3-sigma limits as given by the following formulae :

$$\text{LCL} = (b_3 - 3b_4^{1/2})|\Sigma_0|^{1/2}$$

$$\text{and} \quad \text{UCL} = (b_3 + 3b_4^{1/2})|\Sigma_0|^{1/2} \tag{4.18}$$

where

$$b_3 = \left(\frac{2}{n-1}\right)^{p/2}\prod_{i=1}^{p}\left[\Gamma\left(\frac{n-i+1}{2}\right)\Big/\Gamma\left(\frac{n-i}{2}\right)\right] \tag{4.19}$$

$$\text{and} \quad b_4 = \frac{1}{(n-1)^p}\left\{\prod_{i=1}^{p}(n-i) - 2^p\left[\prod_{i=1}^{p}\left[\Gamma\left(\frac{n-i+1}{2}\right)\Big/\Gamma\left(\frac{n-i}{2}\right)\right]\right]^2\right\}. \tag{4.20}$$

The use of this charting technique is ill-advised because :-

(i) it is incapable of detecting changes in Σ such that $|\Sigma|$ remains constant.

(ii) the formula (4.18) above yields negative values for the lower control limit for most practical values of p and n. If LCL is thus set to zero as suggested by Alt et al.(1990), this means no protection is provided by the resulting $|\mathbf{S}|^{1/2}$ chart against a decreasing value in $|\Sigma|$.

(iii) the associated false alarm rate is considerably larger than the nominal value of 0.0027 especially when n is relatively small. Refer to Table 4.1 for the false signal rates for various practical combinations of p and n. These figures are obtained based on 10,000 simulation runs except when p = 3 or 4 in which case the entries are

found by numerical integration. Note that, in some cases, the false signal rate is as large as 2%.

Table 4.1. False Signal Rate of $|S|^{1/2}$ Chart with '3-sigma' Limits

n \ p	3	4	5	6	7	8	9	10
4	0.0203							
5	0.0174	0.0206						
6	0.0154	0.0188	0.0203					
7	0.0139	0.0172	0.0187	0.0219				
8	0.0128	0.0159	0.0191	0.0197	0.0202			
9	0.0118	0.0149	0.0141	0.0191	0.0199	0.0203		
10	0.0111	0.0140	0.0178	0.0192	0.0207	0.0220	0.0205	
15	0.0087	0.0110	0.0130	0.0143	0.0157	0.0170	0.0204	0.0180
20	0.0085	0.0093	0.0108	0.0121	0.0148	0.0166	0.0164	0.0140
25	0.0071	0.0079	0.0108	0.0123	0.0125	0.0121	0.0124	0.0169
30	0.0056	0.0075	0.0106	0.0111	0.0098	0.0119	0.0137	0.0114

To illustrate the first two points, consider the case of a trivariate process. In this case, the population generalized variance is expressible as

$$\begin{aligned}|\Sigma| &= \sigma_1^2\sigma_{2\bullet 1}^2\sigma_{3\bullet 1,2}^2 \\ &= \sigma_1^2\sigma_2^2(1-\rho_{12}^2)\sigma_3^2(1-\rho_{3(1,2)}^2) \\ &= \sigma_1^2\sigma_2^2(1-\rho_{12}^2)\sigma_3^2\left\{1-\left[\rho_{13}^2+\frac{(\rho_{23}-\rho_{12}\rho_{13})^2}{1-\rho_{12}^2}\right]\right\},\end{aligned}$$

subject to the constraint

$$1-\rho_{12}^2-\rho_{13}^2-\rho_{23}^2+2\rho_{12}\rho_{13}\rho_{23}>0.$$

Thus, it is readily seen that, theoretically, it is possible for $|\Sigma|$ to remain constant or even decrease in the presence of process troubles. For example, suppose that $\sigma_1=\sigma_2=\sigma_3=1$, $\rho_{12}=0.1$, $\rho_{13}=0.7$ and $\rho_{23}=0.5$ are specified as the in-control values, but in fact σ_3 is 3 times as large as that specified and ρ_{23} is 0.7553, then $|\Sigma|$ remains at the value 0.32. If σ_3 doubles instead, then this results in a smaller value of

$|\Sigma|$. In either case, the departure is unlikely to be 'picked up' by the $|\mathbf{S}|^{1/2}$ charts presented by Alt et al.(1986).

It is perhaps worth noting that, for p = 3 or 4, control limits for the $|\mathbf{S}|^{1/2}$ chart can be obtained numerically at any desired α level. In this case, the control limits are given by

$$\text{LCL} = \frac{k_{\alpha/2}}{2^{p-2}(n-1)^{p/2}}|\Sigma_0|^{1/2} \qquad \text{and} \qquad \text{UCL} = \frac{k_{1-\alpha/2}}{2^{p-2}(n-1)^{p/2}}|\Sigma_0|^{1/2} \tag{4.21}$$

where k_δ is a numerical solution to the integral equation,

$$\int_0^\infty G_{2(n-p)}\left(k_\delta / x^{(p-2)/2}\right) g_{(p-2)(n-p+2)}(x)dx = \delta\,. \tag{4.22}$$

The notation $g_\nu(\bullet)$ and $G_\nu(\bullet)$ here denote respectively the probability density and the distribution function of a chi-square variable with ν degrees of freedom. k_δ is obtained in this manner using *Mathematica* version 2.2 (Wolfram (1991)) and is given to 4 significant digits in Table 4.2 for p = 3 and 4, and various combinations of δ and n.

The last technique considered for the known Σ case is aimed at changes of Σ along the principal axes. It involves charting the sum of the *standardized* variances for the principal components, abbreviated hereafter as *SSVPC*, which is known to be distributed as a *scaled* chi-square variable under the multivariate normal assumption with the hypothesized Σ. The value of SSVPC may be calculated from the familiar equivalent statistic $(n-1)tr\left(\mathbf{S}\Sigma_0^{-1}\right)$ (see Appendix A.6). Note that, unlike all other control procedures considered in this section, this technique is applicable even in situations where $n \le p$. It has also been shown by Kiefer and Schwartz (1965) that it is admissible. Although separate monitoring of the variances for the different principal components is

possible, this is not considered due to the large number of charts to be kept when p is large. It is also for meaningful comparison with other techniques involving a single chart that separate monitoring of the variances is not considered. However, in practice, it is the recommendation that these individual variances be retained in order to facilitate the interpretation of out-of-control signals from the SSVPC charts.

Table 4.2. $|\mathbf{S}|^{1/2}$ **Control Chart Factor,** k_δ

		δ					
p	n	0.00135	0.0025	0.005	0.995	0.9975	0.99865
3	4	.0034	.0063	.0126	20.98	24.48	27.73
	5	.1513	.2082	.2994	34.48	38.87	43.18
	6	.6842	.8573	1.112	48.15	53.81	58.90
	7	1.673	2.007	2.473	62.25	68.86	74.70
	8	3.117	3.652	4.367	76.85	84.35	91.01
	9	5.050	5.780	6.770	91.92	100.3	107.7
	10	7.417	8.375	9.659	107.5	116.7	124.8
	15	25.39	27.76	30.73	191.7	204.9	216.5
	20	52.80	56.65	61.51	285.6	302.6	317.4
4	5	.0108	.0200	.0401	90.46	108.0	124.5
	6	.5297	.7308	1.055	169.3	195.9	220.5
	7	2.620	3.303	4.318	260.4	296.2	329.1
	8	6.981	8.427	10.49	364.3	409.6	450.4
	9	14.10	16.60	20.03	480.6	535.7	585.0
	10	24.36	28.13	33.30	609.1	674.2	732.7
	15	132.1	146.0	164.1	1427	1546	1652
	20	350.0	379.1	416.6	2522	2703	2862

In the absence of prior information about Σ, Alt et al.(1990) suggested using the MLRT and $|\mathbf{S}|^{1/2}$ control charts with the unknown process dispersion parameters in (4.16) and (4.18) being replaced by some unbiased estimates based on current production data. Under these circumstances, the data used for estimating the unknown parameters are likely to be either small or moderate data sets. Accordingly, the false signal rates of the resulting techniques are not fixed but instead vary stochastically with the parameter estimates. This causes some difficulties when a comparison is to be made

between these and the proposed technique. Thus only MLRTECM, which was originally proposed in the context of hypotheses testing, is considered.

The criterion for MLRTECM is

$$W = \left(\sum_{i=1}^{q} n_i - q\right) \ln\left|\mathbf{S}_{pooled}\right| - \sum_{i=1}^{q} (n_i - 1) \ln\left|\mathbf{S}_{(i)}\right| \qquad (4.23)$$

where $\mathbf{S}_{(i)}$, n_i and $\mathbf{S}_{poooled}$ denote respectively the ith sample covariance matrix, its associated sample size and the pooled covariance matrix based on the q samples. It was shown by Anderson (1984) that it is an admissible test if $n_i > 2p-1$, $i = 1,\ldots,q$. For equal sample sizes $n_i = n$, this test rejects the null hypothesis that all the q samples are drawn from populations with the same covariance matrix if $W > W_{q,p,n,\alpha}$ where $W_{q,p,n,\alpha}$ denotes the critical value at the $100\alpha\%$ significance level. The upper 5% points of W have been tabulated, for example, in Anderson (1984) for various combinations of p, q and n. There are two problems with the use of MLRTECM in the context of SPC. The existing tables for the percentiles of W are incomplete in regard to other values of p, q, n and α which are required for SPC applications. Besides, successive values of W are correlated and thus the in-control behaviour of the resulting technique is somewhat unpredictable. However, it is found that ignoring this issue does not appear to have any remarkable effect on the overall false signal rate. Therefore, a reasonable comparison can still be made between this and the decomposition procedures by using the same α value for all the methods.

It is shown in Appendices A.4, A.6 and A.7 respectively that the statistical performance of MLRT, SSVPC and MLRTECM (for a step shift in Σ from Σ_0 to Σ_1) depend on the eigenvalues, $\lambda_1,\ldots,\lambda_p$ of $\Sigma_0^{-\frac{1}{2}}\Sigma_1\Sigma_0^{-\frac{1}{2}}$, or equivalently, of $\Sigma_0^{-1}\Sigma_1$ or

$\Sigma_1\Sigma_0^{-1}$, whereas that of the $|\mathbf{S}|^{1/2}$ charting technique depends on Σ_0 and Σ_1 only through the product of these eigenvalues, $\lambda_1\lambda_2...\lambda_p$ (see A.5). Note that $\lambda_1\lambda_2...\lambda_p$ represents the ratio of the 'equal-content' volumes of the process ellipsoids corresponding to Σ_1 and Σ_0 respectively. In addition, Nagao (1967) and Das Gupta (1969) independently showed that the power function of the MLRT is monotonically increasing with respect to $|\lambda_i - 1|$ $\forall i$. Furthermore, note that the MLRT, SSVPC, MLRTECM and $|\mathbf{S}|^{1/2}$ charting procedures are invariant w.r.t. the transformation $\mathbf{X}^* = \Gamma\mathbf{X} + \mu$ where Γ is any nonsingular matrix. This is not true with the decomposition techniques except when Γ is diagonal. By letting $\Gamma = diag\left(\frac{1}{\sigma_1},...,\frac{1}{\sigma_p}\right)$, it is readily seen that the values of Fisher, Tippett and the proposed statistics are the same whether they are computed from the covariance or correlation matrix (see A.8 and A.9), and this is also true for the other invariant procedures considered above. However, difficulties arise when an attempt is made to compare their operating characteristics, since the statistical properties of the decomposition techniques do not appear to be completely determined by the eigenvalues of $\Sigma_0^{-\frac{1}{2}}\Sigma_1\Sigma_0^{-\frac{1}{2}}$. To provide a 'sensible' comparison, it is therefore necessary to consider several possible combinations of Σ_0 and Σ_1 which yield the same eigenvalues $\lambda_1,...,\lambda_p$ (and thus the same *eigenproduct* $\lambda_1\lambda_2...\lambda_p$) and determine the 'average' performance of the decomposition techniques relative to that of the competing procedures. Note, however, that only Σ_0'*s* in the form of correlation matrix need to be considered.

A somewhat systematic way of studying the relative performance of the various competing techniques is to use several arbitrary orthonormal matrices Γ's that

diagonalize $\Sigma_0^{-1/2}\Sigma_1\Sigma_0^{-1/2}$ giving the same diagonal matrix of eigenvalues, $\Lambda = diag(\lambda_1,\ldots,\lambda_p)$. For a given Σ_0, Λ and Γ, Σ_1 is then determined from

$$\Sigma_1 = \Sigma_0^{1/2}\Gamma^T\Lambda\Gamma\Sigma_0^{1/2}.$$

Amongst others, the orthonormal matrix that diagonalizes Σ_0, $\Gamma = \Gamma_0$ is used. This corresponds to situations where the shift is along the principal axes, namely, the variances of some principal components either increase or decrease. Under these circumstances, the eigenvalues of $\Sigma_0^{-1/2}\Sigma_1\Sigma_0^{-1/2}$ are the ratios of variances for the principal components, after and before the shift, and hence the control performance of MLRT, MLRTECM, SSVPC and $|\mathbf{S}|^{1/2}$ depend on them (see A.10). Note that if the eigenvalues of $\Sigma_0^{-1/2}\Sigma_1\Sigma_0^{-1/2}$ are identical, then Σ_1 is the same irrespective of Γ. In this case, $\Sigma_1 = \lambda\Sigma_0$ where λ is the common eigenvalue. This situation may occur in practice as a consequence of all variances increasing proportionally and the correlational structure of the variables remaining the same (see Healy (1987)).

The techniques considered for the known Σ case above are compared on the basis of the probability of detecting a *persistent* shift in the process covariance matrix from Σ_0 to Σ_1. It is assumed that this change, as well as the shift in the process mean vector if any, occurs somewhere during the time interval between the sampling of two adjacent subgroups. Accordingly, the covariance matrices for samples or subgroups taken after the shift are characterized by a common distribution and the mean shift has no effect on this distribution. Of course, if desired, the average run lengths (ARLs) of the various techniques can be determined as the reciprocals of their corresponding probabilities of detection. As for the unknown Σ case, a more complete profile of the run length (RL) properties is required. This is because the out-of-control RL

distributions of all techniques under consideration are not geometric so that the ARL is not a suitable performance criterion (see Quesenberry (1993,1995d)). Furthermore, since these techniques estimate the in-control dispersion parameters sequentially from the current data stream and the efficiency of these estimates increases as more data are incorporated into the computations, it can be expected that their RL performance depends on when the shift takes place. As such, the relative performance of MLRTECM, Fisher, Tippett and the proposed techniques are evaluated based on $\Pr(RL \leq k)$ for a step change in Σ after the rth subgroup, for various combinations of r and k.

The results for the known Σ case are tabulated in Tables 4.3, 4.4 and 4.5 respectively for the cases where ($p = 3$, $n = 4$), ($p = 4$, $n = 5$) and ($p = 5$, $n = 8$). These are based on simulations consisting of 5000 iterations each. Thus, the estimated maximum standard error of the results is $\sqrt{\frac{\hat{\Pr}(1-\hat{\Pr})}{5000}} \approx 0.0071$ which occurs when the estimated probability, $\hat{\Pr} = 0.5$. For the unknown Σ case, the results are based on 2000 simulation runs each (with maximum standard error ≈ 0.0112) and these are shown in Tables 4.6, 4.7 and 4.8 respectively for ($p = 2$, $n = 3$), ($p = 3$, $n = 4$) and ($p = 4$, $n = 5$). Here, it is assumed that the control procedures are based on subgroups of equal size, n. The false signal rate associated with each of the control schemes is fixed at 0.0027, in line with the tradional control charting approach. Note that the control limits for MLRT and MLRTECM are obtained to sufficient accuracy using the work of Davis and Field (1971). The control limits for the $|\mathbf{S}|^{1/2}$ chart are determined from Table 4.2 for $p = 3$, 4 and by means of simulation consisting of 100,000 iterations for $p = 5$ and $n = 8$. Note also that 2-sided control limits with equal significance level on each side are used with SSVPC.

Table 4.3. Power Comparison of MLRT, $|S|^{1/2}$, SSVPC and Decomposition (Proposed, *Fisher* and Tippett) Techniques for $p = 3$, $n = 4$ and $\alpha = 0.0027$.

Eigenvalues of $\Sigma_0^{-1}\Sigma_1$	Power					
	MLRT	$\lvert S\rvert^{1/2}$	SSVPC	[a] Decomposition Methods		
				Max	Med	Min
4, 1, 1	0.0226	0.0206	0.1668	* **0.1820**; *0.1660*; 0.1494	**0.1286**; *0.1202*; 0.0968	**0.1254**; *0.1074*; 0.0832
				† **0.1622**; *0.1314*; 0.1138	**0.1474**; *0.1230*; 0.1100	**0.1346**; *0.1222*; 0.1034
				‡ **0.1390**; *0.1416*; 0.1216	**0.1334**; *0.1152*; 0.1044	**0.1262**; *0.1118*; 0.1036
6.25, 1, 1	0.0888	0.0340	0.3374	**0.3490**; *0.3336*; 0.3210	**0.3060**; *0.3168*; 0.2868	**0.2926**; *0.2842*; 0.2476
				0.3174; *0.3092*; 0.2786	**0.3122**; *0.2906*; 0.2540	**0.2874**; *0.2694*; 0.2508
				0.3010; *0.2988*; 0.2754	**0.2976**; *0.2832*; 0.2452	**0.2936**; *0.2768*; 0.2356
2.25, 2.25, 2.25	0.0102	0.0862	0.2140	----------	**0.1688**; *0.1546*; 0.1148	----------
4, 4, 4	0.1326	0.3182	0.6674	----------	**0.6014**; *0.6006*; 0.4622	----------
6.25, 6.25, 6.25	0.4276	0.5234	0.8944	----------	**0.8664**; *0.8524*; 0.7634	----------
[b] (1, 0.25, 4)	0.0342	0.0030	0.1298	----------	**0.1704**; *0.1510*; 0.1824	----------
				----------	**0.0886**; *0.1044*; 0.0998	----------
				----------	**0.1306**; *0.1182*; 0.1190	----------
(1, 0.16, 6.25)	0.1046	0.0028	0.2890	----------	**0.3324**; *0.3148*; 0.3366	----------
				----------	**0.2786**; *0.2522*; 0.2358	----------
				----------	**0.2732**; *0.2744*; 0.2706	----------
(1, 0.1, 10)	0.3026	0.0040	0.5036	----------	**0.5622**; *0.5580*; 0.5826	----------
				----------	**0.4610**; *0.4838*; 0.4522	----------
				----------	**0.4846**; *0.4542*; 0.4538	----------
(1, 0.05, 20)	0.6062	0.0042	0.7410	----------	**0.8066**; *0.7702*; 0.8008	----------
				----------	**0.7528**; *0.7526*; 0.7214	----------
				----------	**0.7430**; *0.7376*; 0.7290	----------

[a] $\Sigma_0 = \begin{pmatrix} 1 & 0.75 & 0.45 \\ 0.75 & 1 & 0.9 \\ 0.45 & 0.9 & 1 \end{pmatrix}$ is used.

[b] The entries for the decomposition methods are for the particular *permutations* of eigenvalues enclosed within the brackets.

* The entries in 1st block are for shifts along the principal axes.

† The entries in 2nd block are for shifts determined by $\Gamma = \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{6} & 1/\sqrt{6} & -2/\sqrt{6} \end{pmatrix}$.

‡ The entries in 3rd block are for shifts determined by $\Gamma = \begin{pmatrix} 0.44 & 0.85 & 0.27 \\ 0.78 & -0.52 & 0.35 \\ 0.44 & 0.06 & -0.90 \end{pmatrix}$.

Table 4.4. Power Comparison of MLRT, $|S|^{1/2}$, SSVPC and Decomposition (Proposed, *Fisher* and Tippett) Techniques for $p = 4$, $n = 5$ and $\alpha = 0.0027$.

Eigenvalues of $\Sigma_0^{-1}\Sigma_1$	Power					
	MLRT	$\|S\|^{1/2}$	SSVPC	[a] Decomposition Methods		
				Max	Med	Min
4, 1, 1, 1	0.0154	0.0214	0.1712	* **0.1782**; *0.1622*; 0.2084	**0.1664**; *0.1514*; 0.1665	**0.1452**; *0.1390*; 0.1538
				† **0.1850**; *0.1726*; 0.1876	**0.1530**; *0.1482*; 0.1511	**0.1326**; *0.1226*; 0.1258
				‡ **0.1656**; *0.1712*; 0.1416	**0.1480**; *0.1331*; 0.1292	**0.1350**; *0.1138*; 0.1004
6.25, 1, 1, 1	0.0688	0.0346	0.3812	**0.3996**; *0.3838*; 0.4270	**0.3790**; *0.3546*; 0.3653	**0.3502**; *0.3428*; 0.3570
				0.4010; *0.3602*; 0.4076	**0.3373**; *0.3370*; 0.3312	**0.3320**; *0.3228*; 0.2942
				0.3850; *0.3584*; 0.3380	**0.3448**; *0.3158*; 0.3276	**0.3280**; *0.3008*; 0.2694
2.25,2.25,2.25,2.25	0.0110	0.1358	0.3802	----------	**0.2970**; *0.2762*; 0.1686	----------
4, 4, 4, 4	0.1914	0.4480	0.8846	----------	**0.8300**; *0.8528*; 0.6470	----------
6.25,6.25,6.25,6.25	0.6144	0.6968	0.9842	----------	**0.9746**; *0.9680*; 0.9186	----------
[b](0.25, 0.25, 4, 4)	0.0810	0.0040	0.3218	----------	**0.4116**; *0.4088*; 0.3726	----------
				----------	**0.4150**; *0.3664*; 0.3478	----------
				----------	**0.4466**; *0.4352*; 0.3434	----------
(1, 1, 0.16, 6.25)	0.0932	0.0012	0.3114	----------	**0.3616**; *0.3570*; 0.3778	----------
				----------	**0.3742**; *0.3622*; 0.3454	----------
				----------	**0.3380**; *0.3074*; 0.3754	----------
(0.16,0.16,6.25,6.25)	0.3412	0.0038	0.6570	----------	**0.7208**; *0.7122*; 0.6816	----------
				----------	**0.7358**; *0.6858*; 0.6602	----------
				----------	**0.7587**; *0.7552*; 0.6340	----------
(1, 1, 0.1, 10)	0.3040	0.0036	0.5662	----------	**0.6116**; *0.5972*; 0.5984	----------
				----------	**0.6064**; *0.5990*; 0.5902	----------
				----------	**0.5868**; *0.5440*; 0.6006	----------

[a] $\Sigma_0 = \begin{pmatrix} 1 & 0.5 & 0.9 & 0.6 \\ 0.5 & 1 & 0.2 & 0.7 \\ 0.9 & 0.2 & 1 & 0.4 \\ 0.6 & 0.7 & 0.4 & 1 \end{pmatrix}$ is used.

[b] The entries for the decomposition methods are for the particular *permutations* of eigenvalues enclosed within the brackets.

* The entries in 1st block are for shifts along the principal axes.

† The entries in 2nd block are for shifts determined by $\Gamma = \begin{pmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 & 0 \\ 1/\sqrt{6} & 1/\sqrt{6} & -2/\sqrt{6} & 0 \\ 1/\sqrt{12} & 1/\sqrt{12} & 1/\sqrt{12} & -3/\sqrt{12} \end{pmatrix}$.

‡ The entries in 3rd block are for shifts determined by $\Gamma = \begin{pmatrix} 0.99 & 0.07 & 0.09 & 0.06 \\ -0.11 & 0.10 & 0.98 & 0.15 \\ -0.08 & 0.71 & -0.19 & 0.04 \\ 0.06 & 0.15 & 0.68 & -0.72 \end{pmatrix}$.

Table 4.5. Power Comparison of MLRT, $|S|^{1/2}$, SSVPC and Decomposition (Proposed, *Fisher* and Tippett) Techniques for $p = 5$, $n = 8$ and $\alpha = 0.0027$.

Eigenvalues of $\Sigma_0^{-1}\Sigma_1$	Power					
	MLRT	$\|S\|^{1/2}$	SSVPC	[a] Decomposition Methods		
				Max	Med	Min
4, 1, 1, 1, 1	0.0946	0.0320	0.2604	* **0.2356**; *0.2052*; 0.2174	**0.2206**; *0.2004*; 0.1702	**0.1926**; *0.1838*; 0.1550
				† **0.2788**; *0.2674*; 0.2474	**0.2450**; *0.2182*; 0.2302	**0.2206**; *0.2048*; 0.2040
				‡ **0.3174**; *0.3020*; 0.2472	**0.2362**; *0.2224*; 0.1834	**0.2176**; *0.1896*; 0.1768
9, 1, 1, 1, 1	0.5336	0.0844	0.7406	**0.7456**; *0.7306*; 0.7058	**0.7256**; *0.7124*; 0.6722	**0.7188**; *0.6872*; 0.6370
				0.7708; *0.7694*; 0.7476	**0.7626**; *0.7342*; 0.7208	**0.7262**; *0.7220*; 0.7042
				0.7838; *0.8050*; 0.7202	**0.7350**; *0.7478*; 0.6902	**0.7174**; *0.7006*; 0.6784
2.25,2.25,2.25,2.25,2.25	0.1356	0.4694	0.7560	----------	**0.5792**; *0.5738*; 0.3152	----------
4, 4, 4, 4, 4	0.8440	0.9232	0.9954	----------	**0.9926**; *0.9882*; 0.8998	----------
[b](1, 1, 1, 0.25, 4)	0.1332	0.0026	0.1766	----------	**0.1866**; *0.1740*; 0.1702	----------
				----------	**0.2990**; *0.2932*; 0.2414	----------
				----------	**0.2766**; *0.2850*; 0.2486	----------
(1, 4, 0.25, 0.25, 4)	0.4620	0.0008	0.4624	----------	**0.5422**; *0.5576*; 0.3652	----------
				----------	**0.6694**; *0.6672*; 0.4354	----------
				----------	**0.6244**; *0.6160*; 0.4072	----------
(1,6.25,0.16,0.16,6.25)	0.8692	0.0040	0.8156	----------	**0.9006**; *0.9052*; 0.7574	----------
				----------	**0.9344**; *0.9398*; 0.7618	----------
				----------	**0.9396**; *0.9410*; 0.7854	----------
(1, 1, 1, 0.1, 10)	0.7622	0.0042	0.7398	----------	**0.7780**; *0.7758*; 0.7098	----------
				----------	**0.8642**; *0.8502*; 0.7676	----------
				----------	**0.8532**; *0.8360*; 0.7972	----------

[a] $\Sigma_0 = \begin{pmatrix} 1 & 0.58 & 0.51 & 0.39 & 0.46 \\ 0.58 & 1 & 0.6 & 0.39 & 0.32 \\ 0.51 & 0.6 & 1 & 0.44 & 0.43 \\ 0.39 & 0.39 & 0.44 & 1 & 0.52 \\ 0.46 & 0.32 & 0.43 & 0.52 & 1 \end{pmatrix}$ is used.

[b] The entries for the decomposition methods are for the particular *permutations* of eigenvalues enclosed within brackets.

* The entries in 1st block are for shifts along the principal axes.

† The entries in 2nd block are for shifts determined by $\Gamma = \begin{pmatrix} 1/\sqrt{5} & 1/\sqrt{5} & 1/\sqrt{5} & 1/\sqrt{5} & 1/\sqrt{5} \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 & 0 & 0 \\ 1/\sqrt{6} & 1/\sqrt{6} & -2/\sqrt{6} & 0 & 0 \\ 1/\sqrt{12} & 1/\sqrt{12} & 1/\sqrt{12} & -3/\sqrt{12} & 0 \\ 1/\sqrt{20} & 1/\sqrt{20} & 1/\sqrt{20} & 1/\sqrt{20} & -4/\sqrt{20} \end{pmatrix}$.

‡ The entries in 3rd block are for shifts determined by $\Gamma = \begin{pmatrix} 0.88 & -0.48 & 0.00 & 0.00 & 0.00 \\ 0.47 & 0.87 & 0.11 & 0.08 & 0.07 \\ -0.07 & -0.13 & 0.52 & 0.59 & 0.59 \\ -0.02 & -0.03 & 0.85 & -0.34 & -0.41 \\ 0.00 & 0.00 & 0.04 & 0.72 & 0.69 \end{pmatrix}$.

Table 4.6. $\Pr(RL \le k)$ for a Change in Σ after the *r*th Subgroup, by MLRTECM and Decomposition (Proposed, *Fisher* and Tippett) Techniques for $p = 2$, $n = 3$ and $\alpha = 0.0027$.

Eigenvalues of $\Sigma_0^{-1}\Sigma_1$	r	k	MLRTECM	[a] Decomposition Methods	
				Max	Min
4, 1	10	5	0.0110	* *0.1215* (**0.1230**) 0.1315	*0.0955* (**0.0945**) 0.0705
				† *0.1675* (**0.1745**) 0.1925	*0.1040* (**0.1185**) 0.0990
				‡ *0.1610* (**0.1725**) 0.1730	*0.0895* (**0.0930**) 0.0800
		10	0.0205	*0.1830* (**0.1840**) 0.1835	*0.1350* (**0.1370**) 0.0965
				0.2025 (**0.2185**) 0.2490	*0.1505* (**0.1605**) 0.1210
				0.1955 (**0.2060**) 0.1985	*0.1390* (**0.1385**) 0.1150
	20	5	0.0125	*0.2265* (**0.2480**) 0.2415	*0.1965* (**0.1950**) 0.1475
				0.2515 (**0.2775**) 0.3050	*0.1935* (**0.2045**) 0.1870
				0.2180 (**0.2370**) 0.2320	*0.1975* (**0.1915**) 0.1480
		10	0.0165	*0.3315* (**0.3345**) 0.3285	*0.2810* (**0.2845**) 0.2400
				0.3290 (**0.3490**) 0.3675	*0.2950* (**0.2955**) 0.2715
				0.3270 (**0.3545**) 0.3720	*0.3035* (**0.3050**) 0.2530
9, 1	10	5	0.0360	*0.4730* (**0.4915**) 0.4960	*0.3445* (**0.3470**) 0.2445
				0.4635 (**0.5055**) 0.5450	*0.2990* (**0.3035**) 0.2490
				0.4445 (**0.4670**) 0.5225	*0.3120* (**0.3140**) 0.2270
		10	0.0525	*0.5360* (**0.5675**) 0.5780	*0.3920* (**0.3855**) 0.2855
				0.5360 (**0.5740**) 0.6150	*0.4065* (**0.4065**) 0.3270
				0.5355 (**0.5620**) 0.5940	*0.4090* (**0.4045**) 0.3240
	20	5	0.0485	*0.6510* (**0.6675**) 0.6760	*0.5475* (**0.5490**) 0.4175
				0.7210 (**0.7425**) 0.7680	*0.5800* (**0.5845**) 0.5130
				0.6565 (**0.6805**) 0.7105	*0.5685* (**0.5640**) 0.4530
		10	0.0970	*0.7760* (**0.7980**) 0.8085	*0.7030* (**0.6955**) 0.5535
				0.8055 (**0.8245**) 0.8490	*0.7190* (**0.7205**) 0.6295
				0.7765 (**0.8045**) 0.8090	*0.6955* (**0.6920**) 0.5730
4, 4	10	5	0.0265	---------- *0.3540* (**0.3575**) 0.3200 ----------	
		10	0.0425	---------- *0.4055* (**0.4160**) 0.3815 ----------	
	20	5	0.0330	---------- *0.5035* (**0.5135**) 0.4730 ----------	
		10	0.0645	---------- *0.6510* (**0.6705**) 0.6340 ----------	
6.25, 6.25	10	5	0.0675	---------- *0.5950* (**0.6110**) 0.5445 ----------	
		10	0.1090	---------- *0.6635* (**0.6820**) 0.6265 ----------	
	20	5	0.0940	---------- *0.8275* (**0.8405**) 0.7810 ----------	
		10	0.1905	---------- *0.9240* (**0.9320**) 0.8865 ----------	
9, 9	10	5	0.1465	---------- *0.8025* (**0.8100**) 0.7370 ----------	
		10	0.2270	---------- *0.8370* (**0.8535**) 0.7930 ----------	
	20	5	0.2365	---------- *0.9660* (**0.9725**) 0.9450 ----------	
		10	0.4120	---------- *0.9745* (**0.9785**) 0.9780 ----------	

[a] $\Sigma_0 = \begin{pmatrix} 1 & 0.5 \\ 0.5 & 1 \end{pmatrix}$ is used.

* The entries in 1st row are for shifts along the principal axes.

† The entries in 2nd row are for shifts determined by $\Gamma = \begin{pmatrix} -0.996 & 0.094 \\ -0.094 & -0.996 \end{pmatrix}$.

‡ The entries in 3rd row are for shifts determined by $\Gamma = \begin{pmatrix} 0.189 & 0.982 \\ 0.982 & -0.189 \end{pmatrix}$.

Table 4.7. $\Pr(RL \le k)$ for a Change in Σ after the rth Subgroup, by MLRTECM and Decomposition (Proposed, *Fisher* and Tippett) Techniques for $p = 3$, $n = 4$ and $\alpha = 0.0027$.

Eigenvalues of $\Sigma_0^{-1}\Sigma_1$	r	k	MLRTECM	[a] Decomposition Methods		
				Max	Med	Min
4, 1, 1	10	5	0.0100	*0.1560 **(0.1685)** 0.1605	0.1470 **(0.1545)** 0.1160	0.0935 **(0.0940)** 0.1040
				†0.1465 **(0.1740)** 0.2160	0.1235 **(0.1335)** 0.1380	0.0605 **(0.0575)** 0.0495
				‡0.1575 **(0.1750)** 0.1670	0.1305 **(0.1460)** 0.1500	0.0755 **(0.0745)** 0.0580
		10	0.0170	0.2010 **(0.2175)** 0.1915	0.1755 **(0.1815)** 0.1325	0.1130 **(0.1175)** 0.1220
				0.1880 **(0.2145)** 0.2780	0.1635 **(0.1770)** 0.2085	0.1160 **(0.1240)** 0.0895
				0.2375 **(0.2510)** 0.2740	0.1725 **(0.2035)** 0.2530	0.1180 **(0.1205)** 0.0845
	20	5	0.0135	0.2975 **(0.3130)** 0.2555	0.2540 **(0.2590)** 0.1965	0.1810 **(0.1935)** 0.1930
				0.2595 **(0.2935)** 0.3515	0.1065 **(0.2210)** 0.1990	0.1435 **(0.1445)** 0.1060
				0.2440 **(0.2665)** 0.2900	0.1970 **(0.2235)** 0.2575	0.1450 **(0.1440)** 0.1205
		10	0.0145	0.3665 **(0.3760)** 0.3405	0.3560 **(0.3705)** 0.2905	0.2565 **(0.2765)** 0.2805
				0.3270 **(0.3575)** 0.4685	0.3075 **(0.3410)** 0.3230	0.2280 **(0.2335)** 0.1705
				0.3705 **(0.3990)** 0.3970	0.3010 **(0.3305)** 0.3640	0.2055 **(0.2185)** 0.1640
9, 1, 1	10	5	0.0300	0.5560 **(0.5895)** 0.5840	0.5270 **(0.5405)** 0.3990	0.2575 **(0.2670)** 0.2350
				0.5455 **(0.6040)** 0.7005	0.4225 **(0.4535)** 0.4360	0.2710 **(0.2695)** 0.1550
				0.4990 **(0.5385)** 0.6120	0.4760 **(0.5360)** 0.5500	0.3110 **(0.3065)** 0.2325
		10	0.0495	0.6065 **(0.6480)** 0.6410	0.5740 **(0.5935)** 0.4795	0.3260 **(0.3505)** 0.3290
				0.6325 **(0.6855)** 0.7940	0.4920 **(0.5220)** 0.5320	0.3615 **(0.3640)** 0.2305
				0.5340 **(0.5955)** 0.6825	0.5310 **(0.5795)** 0.5970	0.3630 **(0.3760)** 0.2785
	20	5	0.0335	0.7570 **(0.7865)** 0.7705	0.7450 **(0.7565)** 0.6230	0.5455 **(0.5770)** 0.5240
				0.7255 **(0.7780)** 0.8590	0.6825 **(0.7055)** 0.6680	0.5350 **(0.5275)** 0.3250
				0.7630 **(0.8035)** 0.7915	0.6885 **(0.7440)** 0.7875	0.5155 **(0.5235)** 0.3615
		10	0.0785	0.8560 **(0.8785)** 0.8440	0.8465 **(0.8545)** 0.7500	0.6270 **(0.6590)** 0.6485
				0.8460 **(0.8915)** 0.9480	0.7860 **(0.8175)** 0.7855	0.6810 **(0.6785)** 0.4700
				0.8415 **(0.8650)** 0.8910	0.7870 **(0.8255)** 0.8775	0.6595 **(0.6560)** 0.5130
4, 4, 4	10	5	0.0420	----------	0.5945 **(0.6125)** 0.4750	----------
		10	0.0690	----------	0.6740 **(0.6840)** 0.5365	----------
	20	5	0.0450	----------	0.8220 **(0.8245)** 0.6780	----------
		10	0.0810	----------	0.9055 **(0.9230)** 0.8310	----------
9, 9, 9	10	5	0.3090	----------	0.9810 **(0.9845)** 0.9500	----------
		10	0.4460	----------	0.9890 **(0.9895)** 0.9540	----------
	20	5	0.4768	----------	1.0000 **(1.0000)** 0.9955	----------
		10	0.7655	----------	1.0000 **(1.0000)** 1.0000	----------

[a] $\Sigma_0 = \begin{pmatrix} 1 & 0.75 & 0.45 \\ 0.75 & 1 & 0.9 \\ 0.45 & 0.9 & 1 \end{pmatrix}$ is used.

* The entries in 1st row are for shifts along the principal axes.

† The entries in 2nd row are for shifts determined by $\Gamma = \begin{pmatrix} 0.87 & 0.23 & 0.44 \\ -0.36 & -0.31 & 0.88 \\ 0.34 & -0.92 & -0.19 \end{pmatrix}$.

‡ The entries in 3rd row are for shifts determined by $\Gamma = \begin{pmatrix} 0.85 & 0.04 & 0.53 \\ 0.31 & -0.85 & -0.43 \\ -0.43 & -0.53 & 0.73 \end{pmatrix}$.

Table 4.8. $\Pr(RL \le k)$ for a Change in Σ after the *r*th Subgroup, by MLRTECM and Decomposition (Proposed, *Fisher* and Tippett) Techniques for $p = 4$, $n = 5$ and $\alpha = 0.0027$.

Eigenvalues of $\Sigma_0^{-1}\Sigma_1$	r	k	MLRTECM	[a] Decomposition Methods		
				Max	Med	Min
4, 1, 1, 1	10	5	0.0055	*0.1800* **(0.2005)** 0.2730	*0.1565* **(0.1710)** 0.1695	*0.0580* **(0.0575)** 0.0570
				†*0.1705* **(0.1940)** 0.2530	*0.1315* **(0.1415)** 0.1475	*0.0690* **(0.0630)** 0.0650
				‡*0.1675* **(0.1885)** 0.2340	*0.1015* **(0.1125)** 0.1080	*0.0650* **(0.0635)** 0.0265
		10	0.0205	*0.2305* **(0.2585)** 0.3340	*0.2020* **(0.2200)** 0.2295	*0.0900* **(0.0925)** 0.0950
				0.2080 **(0.2420)** 0.3095	*0.1895* **(0.1985)** 0.1850	*0.1030* **(0.1060)** 0.0895
				0.2040 **(0.2435)** 0.3140	*0.1430* **(0.1465)** 0.1745	*0.0950* **(0.1005)** 0.0690
	20	5	0.0130	*0.3130* **(0.3365)** 0.4245	*0.2700* **(0.3055)** 0.3130	*0.1325* **(0.1350)** 0.1350
				0.2635 **(0.3130)** 0.3960	*0.2295* **(0.2515)** 0.2550	*0.1320* **(0.1405)** 0.0955
				0.2745 **(0.3175)** 0.4150	*0.2210* **(0.2415)** 0.2555	*0.1315* **(0.1435)** 0.1080
		10	0.0225	*0.4150* **(0.4650)** 0.5305	*0.3670* **(0.4015)** 0.4220	*0.1675* **(0.1835)** 0.2100
				0.3515 **(0.3925)** 0.5030	*0.3210* **(0.3460)** 0.3585	*0.2060* **(0.2160)** 0.1755
				0.3545 **(0.4015)** 0.5145	*0.3145* **(0.3380)** 0.3415	*0.2090* **(0.2145)** 0.1690
9, 1, 1, 1	10	5	0.0300	*0.6220* **(0.6895)** 0.7910	*0.5180* **(0.5640)** 0.5335	*0.1915* **(0.1990)** 0.1665
				0.6085 **(0.6740)** 0.8425	*0.4955* **(0.5355)** 0.5080	*0.2560* **(0.2765)** 0.2360
				0.5825 **(0.6540)** 0.7995	*0.4710* **(0.5020)** 0.4895	*0.2590* **(0.2655)** 0.1770
		10	0.0465	*0.6800* **(0.7465)** 0.8670	*0.5975* **(0.6395)** 0.6275	*0.2755* **(0.2830)** 0.2480
				0.6415 **(0.7105)** 0.8240	*0.5370* **(0.5840)** 0.5515	*0.3355* **(0.3450)** 0.2575
				0.6625 **(0.7390)** 0.8450	*0.5330* **(0.5765)** 0.5530	*0.3190* **(0.3295)** 0.2460
	20	5	0.0370	*0.8205* **(0.8820)** 0.9175	*0.7895* **(0.8210)** 0.8110	*0.4965* **(0.5185)** 0.4660
				0.7980 **(0.8565)** 0.9095	*0.7545* **(0.7870)** 0.7510	*0.5290* **(0.5400)** 0.4025
				0.7940 **(0.8735)** 0.9185	*0.7325* **(0.7685)** 0.7425	*0.5215* **(0.5335)** 0.3760
		10	0.0745	*0.9040* **(0.9285)** 0.9555	*0.8535* **(0.9015)** 0.8930	*0.5845* **(0.6100)** 0.5675
				0.8625 **(0.9090)** 0.9610	*0.8365* **(0.8625)** 0.8355	*0.6445* **(0.6715)** 0.5200
				0.8740 **(0.8980)** 0.9475	*0.8115* **(0.8505)** 0.8345	*0.6245* **(0.6320)** 0.4960
4, 4, 4, 4	10	5	0.0585	----------	*0.8190* **(0.8305)** 0.6405	----------
		10	0.1125	----------	*0.8595* **(0.9030)** 0.6985	----------
	20	5	0.0750	----------	*0.9690* **(0.9690)** 0.8380	----------
		10	0.1675	----------	*0.9960* **(0.9965)** 0.9200	----------
9, 9, 9, 9	10	5	0.5755	----------	*1.0000* **(1.0000)** 0.9865	----------
		10	0.7580	----------	*1.0000* **(1.0000)** 0.9980	----------
	20	5	0.7500	----------	*1.0000* **(1.0000)** 1.0000	----------
		10	0.9535	----------	*1.0000* **(1.0000)** 1.0000	----------

[a] $\Sigma_0 = \begin{pmatrix} 1 & 0.5 & 0.9 & 0.6 \\ 0.5 & 1 & 0.2 & 0.7 \\ 0.9 & 0.2 & 1 & 0.4 \\ 0.6 & 0.7 & 0.4 & 1 \end{pmatrix}$ is used.

* The entries in 1st row are for shifts along the principal axes.

† The entries in 2nd row are for shifts determined by $\Gamma = \begin{pmatrix} 0.96 & 0.11 & 0.26 & 0.03 \\ -0.27 & 0.11 & 0.94 & 0.17 \\ 0.04 & -0.66 & 0.22 & -0.72 \\ -0.06 & 0.74 & 0.02 & -0.67 \end{pmatrix}$.

‡ The entries in 3rd row are for shifts determined by $\Gamma = \begin{pmatrix} 0.96 & 0.12 & 0.26 & 0.04 \\ 0.28 & -0.23 & -0.87 & -0.33 \\ 0.01 & 0.62 & -0.42 & 0.66 \\ 0.07 & -0.74 & -0.04 & -0.67 \end{pmatrix}$.

For every set of eigenvalues $\lambda_1,\ldots,\lambda_p$ not enclosed within a bracket, the probabilities for the decomposition techniques are simulated for all possible permutations and the maximum, median and minimum values are tabulated, except for $p = 2$ in which case the median is not applicable. The results for the other control procedures are unaffected by these permutations. Note that for the cases where the eigenvalues are identical, the results for each of the decomposition techniques are theoretically the same irrespective of Σ_0 (see section 4.7). As shown in Tables 4.3, 4.4 and 4.5, the proposed technique is far more sensitive to all the shifts considered than the MLRT and $|\mathbf{S}|^{1/2}$ charting procedures. Note that even its worst performance in each case is significantly better than these procedures. For instance, when $p = 5$, $n = 8$ and the standard deviation of a principal component trebles, the results for the MLRT and $|\mathbf{S}|^{1/2}$ techniques are respectively 0.5336 and 0.0844 whereas the smallest probability of detection for the proposed procedure is 0.7188. It is also observed that the proposed technique is generally better than Fisher's and Tippett's procedures. As compared to SSVPC, the proposed technique is seen to be marginally worst in most cases where one of the eigenvalues are greater than 1 whilst others are 1. However, in cases where some eigenvalues are greater and others are smaller than 1 (a situation which typically occurs as a result of some but not all of the variances increasing), the proposed technique is almost always superior. Particularly notable is the situation when p is large. For instance, when $p = 5$, $n = 8$ and the eigenvalues are (1, 1, 1, 0.1, 10), the result for SSVPC is 0.7398 whereas those for the proposed technique are 0.7780, 0.8642 and 0.8532 respectively for the different Γ considered. Note that the exact probabilities for SSVPC and $|\mathbf{S}|^{1/2}$ charting technique (for $p = 3, 4$) are obtainable using a published program and

standard statistical software. The program of Davies (1980) can be used for finding the cumulative probability of SSVPC since the latter is distributed as a linear combination of independent chi-square variables with coefficients $\lambda_1,\ldots,\lambda_p$ (see A.11). If $\lambda_1,\ldots,\lambda_p$ are identical, resulting in a scaled chi-square distribution, then many of the statistical software packages currently available can be used. If p = 3 or 4, the cumulative probability of $|\mathbf{S}|^{1/2}$ can be determined by means of numerical integration. However, as a partial check of the simulation, the simulation results for these techniques are included. They are found to agree well with the theoretical values. For instance, the theoretical probabilities for $|\mathbf{S}|^{1/2}$ and SSVPC are (0.1333, 0.3828), (0.4575, 0.8873) and (0.7020, 0.9866) for p = 4, n = 5, $\alpha = 0.0027$ and $\lambda_1,\ldots,\lambda_p$ all equal to 2.25, 4 and 6.25 respectively. These are very close to the corresponding figures in Table 4.4.

As for the comparison for the unknown Σ case, Tables 4.6, 4.7 and 4.8 clearly reveal that the proposed technique and the associated Fisher's and Tippett's procedures are far superior to MLRTECM irrespective of the dimension p, the change point r, the eigenvalues $\lambda_1,\ldots,\lambda_p$ and the direction of the shift as specified by Γ. It can also be seen that the proposed technique is as good as or significantly better than Fisher's procedure. As compared to Tippett's procedure, the proposed technique is generally worse in terms of 'maximum performance'. However, the opposite is true when comparison is based on the 'medium' and the 'minimum performance'. Furthermore, the proposed technique is consistenly better than Tippett's procedure when the eigenvalues $\lambda_1,\ldots,\lambda_p$ are all equal. Like the known Σ case, limited comparisons using other values of p, n, r, Σ_0 and some arbitrarily chosen Γ's yielded similar conclusions.

It is perhaps worth noting that Calvin (1994) has developed a one-sided test of covariance matrix with a known null value. However, no attempt has been made to compare the operating characteristics of this and the proposed technique. A reasonable comparison cannot be made since the former is specifically designed for situations where the deviations take the form of $\Sigma = \Sigma_0 + \mathbf{B}$ ($\mathbf{B}$ is a symmetric positive definite matrix) whereas the proposed technique is not meant for any specific shifts. If the shift is anticipated to be of the form $\Sigma = \Sigma_0 + \mathbf{B}$, then the recommendation is to use the former technique.

4.6 An Example

In order to illustrate the behaviour of the proposed techniques and to compare them with previously proposed procedures, consider the hypothetical bomb manufacturing data analysed by Alt et al. (1986). The data consist of m = 15 samples of n = 10 observations on the overall length of the bomb base (X_1) and the depth to shoulder of bomb heads (X_2). The specified values of the dispersion parameters are $\sigma_1 = 0.00216$, $\sigma_2 = 0.00384$ and $\rho = -0.6$. Note that the values for samples 12, 13, 14 and 15 have been generated from an out-of-control process where both the process standard deviations, σ_1 and σ_2 were increased by 25%, 50%, 75% and 100% respectively. The sample values of the dispersion parameters are reproduced in Table 4.9. The authors found that samples 14 and 15 provide out-of-control indications when MLRT is used with $\alpha = 0.01$. For the $|\mathbf{S}|^{1/2}$ technique using either exact control limits given by (4.17) with $\alpha = 0.02$ or 3-sigma limits given by (4.18), it was found that only the 15th sample produces a signal.

Since the dispersion parameters are specified in this example, control statistic (4.14) can be used. The computed values of this statistic for all of the 15 samples are given in the same table and the associated control chart is shown in Figure 4.1(a). At $\alpha = 0.0027$, UCL = 14.16. Thus, this technique 'picks up' the increased process variability at both the 14th and the 15th sample, as the MLRT does. Note particularly the impact of increased variability on the 14th sample. For this sample, the value of the control statistic is infinity giving a very strong indication of process troubles !

Next, suppose that the in-control process variance-covariance matrix is unknown as in the retrospective stage of the two-stage control procedure considered by Alt et al. (1990). In this case, the authors suggested replacing the unknown parameters $|\Sigma_0|$ and Σ_0^{-1} in expression (4.16) for the MLRT statistic, W^*, by the unbiased estimates

$$|\mathbf{V}| = \frac{\sum_{i=1}^{m} |\mathbf{S}_{(i)}|}{m b_1} \quad \text{and} \quad \Lambda = \frac{(n-p-2)\sum_{i=1}^{m} \mathbf{S}_{(i)}^{-1}}{m(n-1)}$$

respectively where $b_1 = (n-1)^{-p} \prod_{i=1}^{p} (n-i)$ and treating the resulting values of W^* as the true values. If W^* exceeds appropriate percentage point or the control limit for any of the m samples, that sample is discarded and W^* are recalculated for the remaining samples using the revised value of m, $|\mathbf{V}|$ and Λ. This procedure is repeated until there are no other samples with W^* beyond the control limit. Similarly, for the 3-sigma $|\mathbf{S}|^{1/2}$ charts, they suggested substituting the unknown parameter $|\Sigma_0|^{1/2}$ with the unbiased estimate

$$\frac{\sum_{i=1}^{m}|\mathbf{S}_{(i)}|^{1/2}}{mb_3}$$

in expression (4.18) for the control limits where b_3 is given by (4.19). If any of the m samples give out-of-control signal, that sample is deleted and control limits are recomputed using the remaining samples. This procedure is continued until none of the values of $|\mathbf{S}|^{1/2}$ exceed the control limits. Unlike these methods, the technique presented for the unknown Σ case may be used as soon as the 2nd sample is available. Besides, the proposed technique does not involve recalculation of the control statistic and the revision of the control limits. The results for the respective techniques are provided in Table 4.9. In addition, the control chart associated with the proposed technique is shown in Figure 4.1(b). Note that for the same value of α, this control chart has the same control limit as that for the known Σ case. As shown in the figure, the out-of-control condition causes a spike on the control chart at the 14th sample. This out-of-control sample is removed before control statistic (4.15) is computed for the next sample. For the 3-sigma $|\mathbf{S}|^{1/2}$ chart, the estimated control limits are respectively $\mathrm{LCL} = 0$ and $\mathrm{UCL} = 3.41 \times 10^{-5}$. Note that LCL is set to 0 because its calculated value is negative. As shown in Table 4.9, this technique also reveals that sample 14 is affected by some special causes. After leaving out this sample and recalculating, the control limits become $\mathrm{LCL} = 0$ and $\mathrm{UCL} = 1.49 \times 10^{-5}$ where again LCL here is set to zero for the same reason. No further sample triggers out-of-control signal based on these revised limits. As for MLRT, the values of W^* with the 'plug in' estimates are larger than the upper 1% point of 12.38 (see Anderson (1984), p.641) for all of the 15 samples ! This is an unusual phenomenon that is likely to cause misinterpretation about the process status. However, it is seen from

Table 4.9 that the problem might be due in part to sample 14 since its W^* value of 2581.67 is unusually large. If this sample is dropped and W^* is recalculated for the remaining samples, then there is an evidence that the last sample is affected by some special causes since its W^* value of 12.65 is slightly larger than 12.38. No further out-of-control samples are found when this procedure is repeated. Note that the revised values of W^* are negative for a few samples. This is due to the use of the sample estimates in place of the unknown parameters in the expression for W^*.

Table 4.9. Alt and Bedewi (1986)'s Data and Values of Control Statistic (4.14), (4.15), MLRT statistic (unknown Σ) and $|\mathbf{S}|^{1/2}$.

Sample No.	S_1^2 ($\times 10^{-6}$)	S_2^2 ($\times 10^{-5}$)	S_{12} ($\times 10^{-6}$)	Statistic (4.14)	Statistic (4.15)	MLRT	Revised MLRT	$\lvert\mathbf{S}\rvert^{1/2}$ ($\times 10^{-6}$)
1	3.71	1.94	-4.40	2.65	NA	28.25	-0.40	7.24
2	3.59	1.46	-2.82	1.20	0.25	28.61	-0.11	6.67
3	5.56	1.31	-2.56	2.11	0.02	29.11	0.53	8.16
4	3.26	1.11	-2.96	0.93	1.98	29.76	0.84	5.24
5	6.70	1.94	-10.44	3.89	4.61	35.38	6.39	4.57
6	8.85	3.26	-14.13	4.40	0.89	30.77	2.20	9.41
7	7.93	2.04	-11.04	2.60	1.05	32.27	3.37	6.33
8	4.50	1.79	-6.29	0.13	0.10	28.81	-0.03	6.42
9	4.71	1.68	-3.26	1.95	1.13	28.18	-0.39	8.28
10	1.25	0.99	-1.60	5.32	1.30	35.35	6.32	3.13
11	4.89	2.54	-8.28	1.59	0.52	29.01	0.35	7.48
12	4.71	1.38	0.33	6.19	4.53	31.70	3.37	8.06
13	4.71	1.38	0.33	6.19	3.61	31.70	3.37	8.06
14	7.82	390.30	94.44	∞	∞	2581.70	NA	147.00
15	10.93	1.38	1.28	19.85	7.71	40.31	12.65	12.22

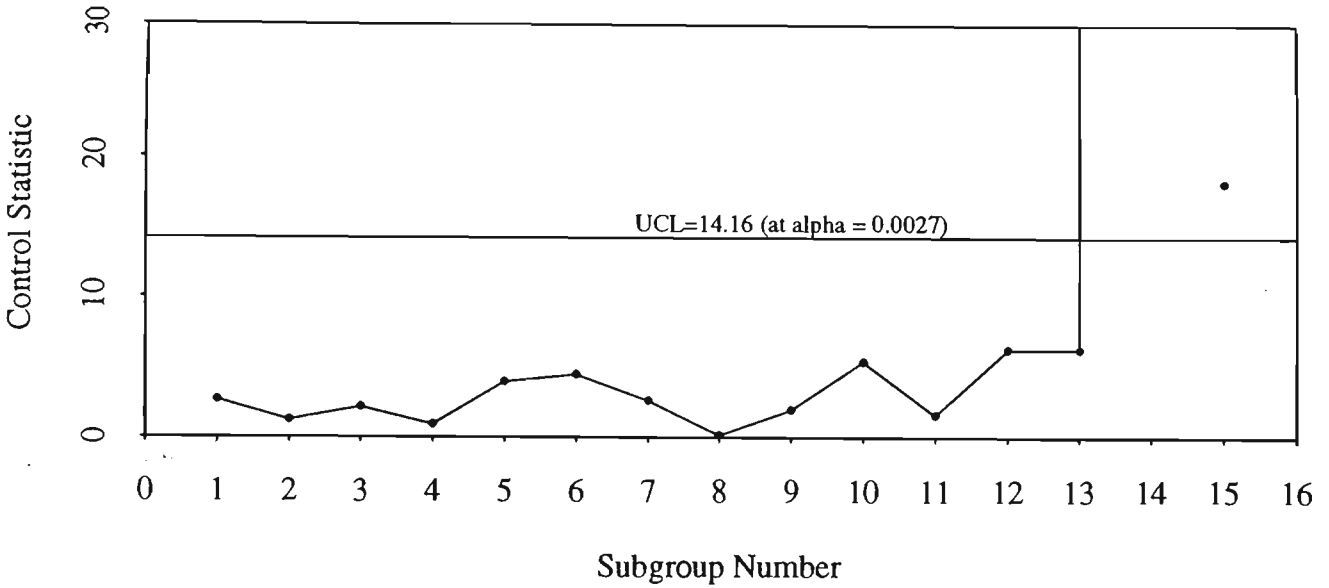


Figure 4.1(a). Dispersion Control Chart for Alt and Bedewi (1986)'s Data, Known Sigma.

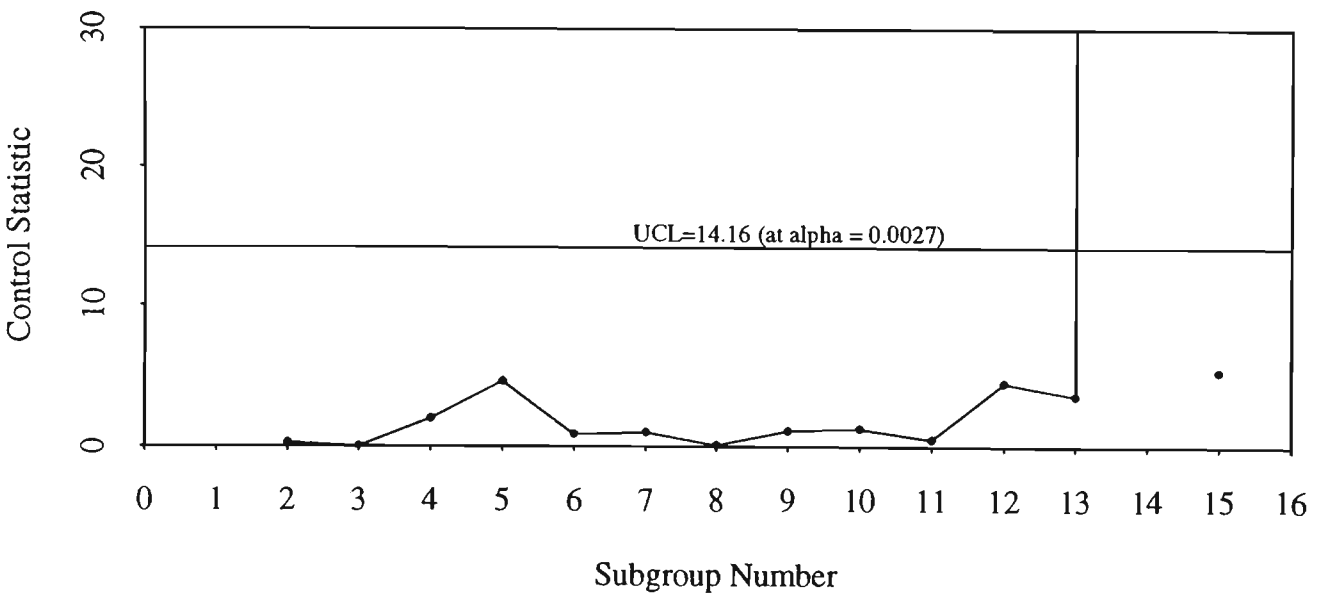


Figure 4.1(b). Dispersion Control Chart for Alt and Bedewi (1986)'s Data, Unknown Sigma.

4.7 Effect of Aggregation on Control Performance

As presented earlier, the proposed techniques involve the charting of some aggregate-type statistics formed using independent components resulting from the partitioning of the covariance matrices. Certainly, if the use of such a statistic incurs some loss in control performance, then it is preferable to chart the individual components separately though the improved performance is gained at the expense of increasing the charting effort. In this section, the effect of such 'aggregation' on the control performance, is briefly considered for the known Σ case. This effect is examined by comparing the probabilities of detection by the proposed technique with that based on the use of the individual components for certain shifts in Σ, having equated first their false signal rates. Note that the latter technique, abbreviated hereafter as the *IC* technique, involves the plotting of the following statistics :-

$$\frac{(n_k-1)S^2_{1(k)}}{\sigma^2_1} \tag{4.24}$$

$$\frac{(n_k-1)S^2_{j\bullet 1,\ldots,j-1(k)}}{\sigma^2_{j\bullet 1,\ldots,j-1}} \qquad j=2,\ldots,p. \tag{4.25}$$

$$(n_k-1)S^2_{1(k)}\left(\underset{\sim}{d}_{2(k)}-\underset{\sim}{\theta}_2\right)^{\mathrm{T}}\Sigma^{-1}_{2,\ldots,p\bullet 1}\left(\underset{\sim}{d}_{2(k)}-\underset{\sim}{\theta}_2\right) \tag{4.26}$$

and

$$(n_k-1)S^2_{j-1\bullet 1,\ldots,j-2(k)}\left(\underset{\sim}{d}_{j(k)}-\underset{\sim}{\theta}_j\right)^{\mathrm{T}}\Sigma^{-1}_{j,\ldots,p\bullet j-1}\left(\underset{\sim}{d}_{j(k)}-\underset{\sim}{\theta}_j\right) \qquad j=3,\ldots,p. \tag{4.27}$$

It may also be of some value to study the control performance of the technique based on the use of principal components. In the presence of Σ, it is reasonable to chart

the standardized variances of the principal components (multiplied by (n-1) each) separately. These are given by the diagonal elements of $(n-1)\Lambda_0^{-\frac{1}{2}}\Gamma_0 S\Gamma_0\Lambda_0^{-\frac{1}{2}}$ where Λ_0 and Γ_0^T denote respectively the diagonal matrix containing the eigenvalues of Σ_0 and the matrix of the corresponding eigenvectors. For ease of subsequent discussion, this technique is referred to as *ISVPC*.

Following a shift in the process covariance matrix Σ, the statistics (4.24) and (4.25) are readily seen to follow some scaled chi-square distribution whereas the Hotelling χ^2-type statistics in (4.26) and (4.27) can be shown to be generally distributed as linear combinations of independent noncentral chi-square variables (see A.11). Furthermore, note that the (conditional) independence of these statistics is preserved. Thus, given the program of Davies (1980), it is possible to determine the overall probability of 'picking up' any given shift in Σ by the use of these statistics. However, for mathematical convenience, only a special case is considered, namely, when the shifts take the form :-

$$\Sigma_1 = \lambda\Sigma_0,$$

a situation which has been noted earlier. Note that under these circumstances, all the above statistics are chi-square except for the scalar multiple λ. The same is true for the ISVPC technique. Thus, the statistical performance of these and the proposed technique depends on λ (besides p and n) irrespective of Σ_0.

For reasonable comparison, suppose that the significance levels associated with the control charts for the IC and ISVPC techniques are set to be *equal* to α^* so that the *overall* false signal rate, α, for each control scheme is the same as that of the proposed technique. Accordingly, the α^* for both techniques are respectively given by

$$\alpha^* = 1-(1-\alpha)^{\frac{1}{2p-1}} \quad \text{and} \quad \alpha^* = 1-(1-\alpha)^{\frac{1}{p}}.$$

Furthermore, both the lower and upper control limits are used with each chart and these are set at $\frac{\alpha^*}{2}$ and $1-\frac{\alpha^*}{2}$ probability levels respectively. Note that under this condition, the former technique is equivalent to Tippett's combination procedure as considered earlier. The power of these and the proposed technique are given to 4 decimal places in Table 4.10 for various combinations of p, n, λ and α. Note that the results for the proposed technique are obtained by means of 5000 simulation runs. Note also the proximity of the exact probabilities and the corresponding simulation results for the IC technique as given in Tables 4.3, 4.4 and 4.5. In all these cases, it is observed that the proposed technique is significantly better than the IC and ISVPC techniques. It is also seen that ISVPC ranks between the proposed and the IC technique in performance for all the given shifts. Although no attempt has been made to study their relative performance thoroughly, the results provide an indication that incorporating the individual components into a composite statistic in the suggested manner may well result in improved control performance.

Table 4.10. Power Comparison of Proposed, IC and ISVPC Charting Technique for Shifts in the Form of $\Sigma_1 = \lambda \Sigma_0$.

p	n	α	λ	Power		
				Proposed	IC	ISVPC
3	4	0.0027	2.25	0.1688	0.1149	0.1334
			4.00	0.6014	0.4666	0.5134
			6.25	0.8664	0.7596	0.7965
		0.01	2.25	0.2604	0.1995	0.2238
			4.00	0.6818	0.5928	0.6342
			6.25	0.8912	0.8404	0.8662
4	5	0.0027	2.25	0.2970	0.1669	0.2017
			4.00	0.8300	0.6468	0.7097
			6.25	0.9746	0.9127	0.9390
		0.01	2.25	0.4050	0.2781	0.3216
			4.00	0.8792	0.7672	0.8145
			6.25	0.9822	0.9542	0.9686
5	8	0.0027	2.25	0.5792	0.3078	0.3877
			4.00	0.9926	0.9036	0.9459
			6.25	1.0000	0.9962	0.9986
		0.01	2.25	0.7132	0.4648	0.5495
			4.00	0.9938	0.9566	0.9772
			6.25	1.0000	0.9989	0.9996

CHAPTER 5

CAPABILITY INDICES FOR MULTIVARIATE PROCESSES [1]

5.1 Introduction

Since the pioneering work of Kane (1986), there have been many articles published dealing with process capability indices. Some developments in process capability analysis are outlined by Rodriguez (1992) in a special issue of the Journal of Quality Technology, entirely devoted to the topic. In Marcucci and Beazley (1988), it was noted that 'an index for multidimensional situations.....is another outstanding problem.....'. Most of the relevant work to date has focussed on the developments of process capability indices for single product characteristics. In many manufacturing situations, the quality of a manufactured product is more often than not determined by reference to more than one product characteristic. Invariably manufacturing conditions are such that there is an inter-dependency in the development of these product characteristics. To discuss process capability under these circumstances then, requires a method that acknowledges this inter-dependency and constructs an index that incorporates knowledge of the covariance structure of the quality characteristics.

The most commonly used univariate capability indices are the C_p, C_{pk} and C_{pm} indices which are defined as :-

$$C_p = \frac{U - L}{6\sigma},$$

[1] This chapter is based on the paper entitled 'Capabilitity Indices for Multivariate Processes', *Technical Report 49 EQRM 14*, Department of Computer and Mathematical Sciences, Victoria University of Technology, December 1994.

$$C_{pk} = \min\left\{\frac{U-\mu}{3\sigma}, \frac{\mu-L}{3\sigma}\right\}$$

and $$C_{pm} = \frac{U-L}{6\sqrt{\sigma^2+(\mu-T)^2}}$$

where μ, σ, U, L and $T = \frac{U+L}{2}$ denote the process mean, standard deviation, upper and lower specification limits, and target respectively. The first is strictly concerned with process potential in that it makes no reference to the process mean, μ. However, they all essentially reflect process potential in that they implicitly assume a perfectly controlled process. For meaningful use of these indices to describe actual process behaviour consideration of their sampling distributions is necessary. Statistical issues of estimation and hypothesis testing and practical matters such as the use and interpretation of these indices have been extensively discussed in the literature (see for eg., Kushler and Hurley (1992), Franklin and Wasserman (1992), Pearn, Kotz and Johnson (1992), Barnett (1990) and Boyles (1991)). These indices are applicable for situations involving two-sided specifications but some adaptations for one-sided specifications can also be found in the literature.

After reviewing existing work on multivariate process capability indices, this chapter explores further the possibility of assessing multivariate process performance by using a single composite measure and describes three approaches for doing so. In particular, three bivariate process capability indices are proposed and some simple rules provided for interpreting the values they take. The relative effectiveness of the proposed indices as a comprehensive summary of process performance, with respect to all of the measured characteristics, is also provided. An approximate test for one of the proposed indices is developed. Possible methods of developing robust capability indices are also considered. The paper focuses on the commonly encountered situations in which the

measured characteristics of a process or a product have two-sided specifications forming a rectagular specification region. The extension of this work to situations involving unilateral or a mixture of unilateral and bilateral tolerances is a straightforward matter. The total discourse is given in the context of discrete item manufacturing.

5.2 A Review of Multivariate Capability Indices

Chan, Cheng and Spiring (1991) introduced a so-called multivariate version of the C_{pm} index which is defined as :

$$\underset{\sim}{C_{pm}} = \sqrt{\frac{np}{\sum_{i=1}^{n}(\mathbf{X}_i - \mathbf{T})^{\mathrm{T}}\mathbf{A}^{-1}(\mathbf{X}_i - \mathbf{T})}}$$

To do this, they made the assumption that the specification requirements for a v-variate process or product are prescribed in the form of an ellipsoidal region given by

$$(\mathbf{X}-\mathbf{T})^{\mathrm{T}}\mathbf{A}^{-1}(\mathbf{X}-\mathbf{T}) \le c^2$$

where $\mathbf{X}$, $\mathbf{T}$, $\mathbf{A}$ and c are respectively the v-dimensional random observation vector, some specified $v \times 1$ vector, a $v \times v$ positive definite matrix and a constant. These may either be the actual engineering requirements or that *created* from various forms of specifications in the suggested manner. For the latter case, it generally imposes more stringent requirements than actually needed.

As the definition of $\underset{\sim}{C_{pm}}$ involves the sample observations rather than being based on the process parameters (i.e the mean vector μ and the covariance matrix Σ), Pearn et al.(1992) stated, quite correctly, that it should be taken as an estimator (denoted $\underset{\sim}{\hat{C}_{pm}}$) of the following revised index :

$$\underset{\sim}{C_{pm}} = \sqrt{\frac{\nu}{E\left[(\mathbf{X}-\mathbf{T})^{\mathrm{T}}\mathbf{A}^{-1}(\mathbf{X}-\mathbf{T})\right]}}$$

Much of the discussion of Chan et al.(1990) was devoted to the test of $\underset{\sim}{C_{pm}} = 1$ based on the univariate statistic,

$$D = \sum_{i=1}^{n}(\mathbf{X}_i - \mathbf{T})^{\mathrm{T}}\mathbf{A}^{-1}(\mathbf{X}_i - \mathbf{T}) \ ,$$

which is distributed as a chi-square variable with $n\nu$ degrees of freedom under the multinormal assumption, with $\boldsymbol{\mu} = \mathbf{T}$ and $\Sigma = \mathbf{A}$. Note that the quadratic form $(\mathbf{X}-\mathbf{T})^{\mathrm{T}}\mathbf{A}^{-1}(\mathbf{X}-\mathbf{T})$ and D are distributed as linear combinations of independent noncentral chi-square variables (see Appendix A.11) under the alternative hypothesis. Thus, using the program of Davies (1980), it is possible to determine the power of the test and relate it to the expected proportion of items satisfying the ellipsoidal specification requirements. It is also worth noting that this work is more concerned with 'process capability analysis' rather than with the design of a unitless capability measure.

As in Chan et al.(1990), Pearn et al.(1992) considered a ν-variate process with specification requirements formulated as an ellipsoidal region and proposed the following capability indices,

$${}_{\nu}C_p^2 = \frac{c^2}{c_{\nu}^2}$$

and $${}_{\nu}C_{pm}^2 = \frac{{}_{\nu}C_p^2}{\left[1 + \frac{(\boldsymbol{\mu}-\mathbf{T})^{\mathrm{T}}\mathbf{A}^{-1}(\boldsymbol{\mu}-\mathbf{T})}{\nu}\right]} \ ,$$

as generalizations of the univariate C_p and C_{pm} indices. If $\boldsymbol{\mu} = \mathbf{T}$ and $\Sigma = \mathbf{A}$, then c_v^2 in the above definitions is equated to $\chi^2_{v,0.9973}$, the 99.73th percentile of the chi-square distribution with v degrees of freedom, otherwise, it is computed such that

$$\Pr\left\{(\mathbf{X}-\mathbf{T})^{\mathrm{T}}\mathbf{A}^{-1}(\mathbf{X}-\mathbf{T}) \le c_v^2\right\} = 0.9973 \ .$$

Note that both indices have the same value if $\boldsymbol{\mu} = \mathbf{T}$. These indices correctly reflect process capability in the sense that their values decrease with declining process performance. However, as noted in their paper, the essential problem with these indices lies in the estimation of them.

In view of the fact that it is unlikely to have specifications given as ellipsoids, Rodriguez (1992) suggested the direct estimation of the proportion of nonconforming items by integration of the multivariate normal density function over the rectangular specification region. Boyles (1994) also considered this alternative of estimating process capability and discussed its statistical and practical merits over a competing procedure which is based on simple binomial estimates. The total discussion is in the context of repeated *lattice-structured* measurements.

Unlike others, Hubele, Shahriari and Cheng (1991) proposed a capability *vector* for a bivariate normal process which consists of three components. The first is the ratio of the area of the *specification rectangle* to that of the *projected process rectangle*, giving an analogue of the univariate C_p index. The second component, is defined as the significance level computed from a T^2-type statistic which measures the relative location of the process centre and the target. The last component is designed to capture situations where one or more of the *process limits* fall beyond the corresponding specification limits. Although some efforts were made to demonstrate the usefulness of this capability

vector as a summary measure of the process performance, interpretation is sometimes difficult.

Other contributions come from Taam, Subbaiah and Liddy (1993) who proposed a multivariate capability index defined as

$$MC_{pm} = \frac{\text{Volume of } R_1}{\text{Volume of } R_2},$$

where R_1 and R_2 represent respectively the *modified tolerance region* (modified according to the process distribution) and the scaled 99.73% process region (scaled by the mean squared error, $\Sigma_{\mathbf{T}} = E\left[(\mathbf{X}-\mathbf{T})(\mathbf{X}-\mathbf{T})^{\mathrm{T}}\right]$). If the process follows a multivariate normal distribution, then the modified tolerance region here is the largest ellipsoid inscribing the original specification region and the scaled process region, R_2, is an ellipsoidal region represented by $(\mathbf{X}-\mu)^{\mathrm{T}}\Sigma_{\mathbf{T}}^{-1}(\mathbf{X}-\mu) \le \chi^2_{\nu,0.9973}$. Thus, under normality assumptions, this index becomes

$$\begin{aligned} MC_{pm} &= \frac{\text{Vol.}(R_1)}{\text{Vol.}(R_3)} \times \frac{1}{\left[1+(\mu-\mathbf{T})^{\mathrm{T}}\Sigma^{-1}(\mu-\mathbf{T})\right]} \\ &= \frac{MC_p}{D_{\mathbf{T}}}, \end{aligned}$$

where R_3 is the natural process ellipsoid containing 99.73% of items, $MC_p = \frac{\text{Vol.}(R_1)}{\text{Vol.}(R_3)}$ is an analogue of the univariate C_p (squared) index which measures the process potential and $D_{\mathbf{T}} = 1+(\mu-\mathbf{T})^{\mathrm{T}}\Sigma^{-1}(\mu-\mathbf{T})$ is a measure of process mean deviation from target. As stated by Taam et al.(1993), this is an analogue of the univariate C_{pm} (squared) index. Note also that this index is similar to ${}_{\nu}C^2_{pm}$, except in the manner in which the process potential and the deviation of mean from target are quantified. In terms of its ease of computation and general applicability, it is superior to the latter. Besides the fact

that it can be used for different types of specification region (see the example on *geometric dimensioning and tolerancing* (GDT) in the same paper), this index can be extended to non-normal processes provided the specifications are two-sided. This, however, entails the determination of the proper process and modified tolerance region and the resulting computations are likely to be complex. In the same paper, Taam et al.(1993) considered the estimation of this capability index. However, they simply replace the unknown mean vector μ and the covariance matrix Σ in the expression for the proposed index with the usual unbiased estimates and use $\chi^2_{\nu,0.9973}$ as the boundary of the process ellipsoid without taking into consideration issues such as unbiasedness, efficiency and uncertainty of the resulting capability index estimate. They also highlighted some similarities and differences between the proposed index (MC_{pm}), $\underset{\sim}{C}_{pm}$ and the bivariate capability vector proposed by Hubele et al.(1991). A major problem with this index is its potential to provide misleading conclusions. For instance, if the measured characteristics are not independent and the index value is 1 (as a result of the process being on-target and the volume of the process ellipsoid being the same as that of the modified tolerance region), there is no assurance that the process under consideration is capable of meeting the specifications consistently or can be expected to produce 99.73% of conforming items. This is in conflict with the statement made by Taam et al.(1993) that, '.....when the process is centered at the target and the capability index is 1, it indicates 99.73% of the process values lie inside the tolerance region.' The deficiency in this comment is illustrated in Figure 5.1 for a bivariate normal process.

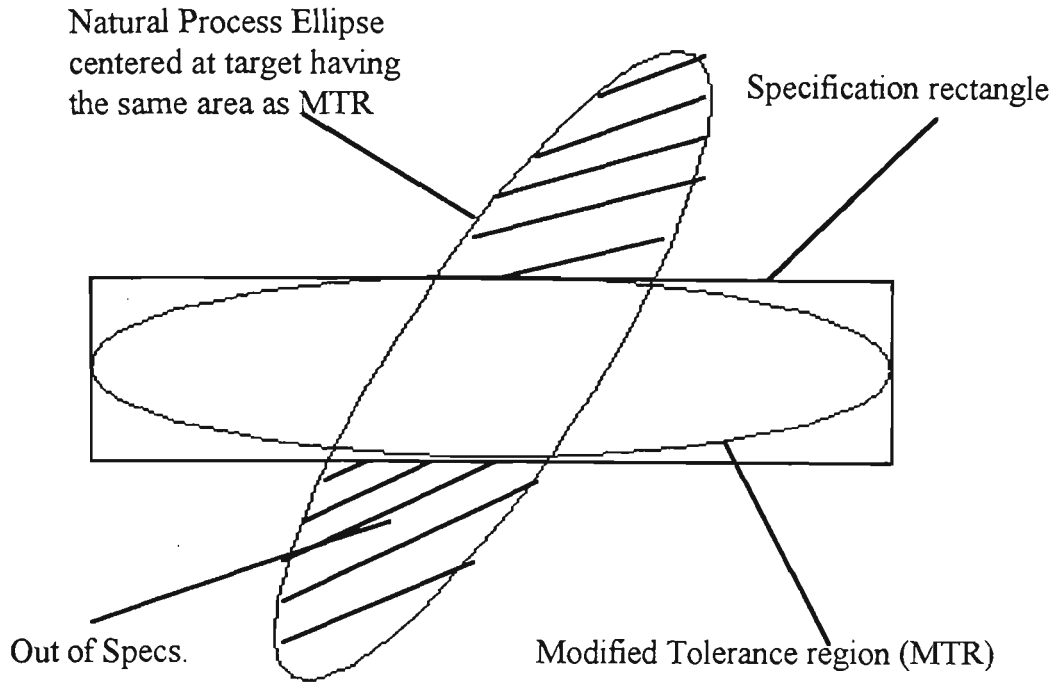


Figure 5.1. Graphical Illustration of An Incapable Bivariate Normal Process with $MC_{pm} = 1$

Boyles (1996) introduced the concept of *exploratory* capability analysis (ECA) which is aimed at capability improvement rather than assessment. This should be distinguished from the so-called *confirmatory* capability analysis (CCA) which involves formally assessing whether the process under consideration is capable of meeting the given specifications or not. ECA, essentially utilizes exploratory graphical data analysis techniques, such as boxplots, to reveal or to assist in identifying new opportunities for process improvement. Three real examples involving repeated measurements with lattice structure were used to illustrate the usefulness of the concept.

In another paper, Boyles (1994) proposed an *expository* technique of analyzing multivariate data using repeated measurements with a lattice structure where the number of measurements for the same characteristic on each part or product, v, may possibly exceed the number of inspected parts or products, n. He developed a class of *Direct Covariance* (DC) models corresponding to a general class of lattices and obtained some

positive definite estimates of the covariance matrix denoted by $\hat{\Sigma}_{DC}$ even when $n \leq v$. This property of positive definiteness for the estimated covariance matrix permits the computations of multivariate capability indices and estimated process yields which depend on Σ^{-1} when $n \leq v$ or when n is not much greater than v, in which case the usual sample covariance is ill-conditioned with respect to matrix inversion. He made some efforts to justify the use of the proposed model for process capability analysis. In particular, he demonstrated the superiority of employing $\hat{\Sigma}_{DC}$ to provide an estimate of the proportion of nonconforming units over the use of sample covariance and the empirical approach of simple binomial estimates. To do this he used sets of data from Boyles (1996) along with some simulation results.

5.3 Constructing A Multivariate Capability Index

With the assumption that the process under focus follows a multivariate normal distribution, consider the following approaches to the design of a multivariate process capability index. Before proceeding, it should be pointed out that, although these approaches have been widely discussed in simultaneous interval estimation problems (see, for example, Johnson et al. (1988) and Nickerson (1994)), they are used here in a different context.

The first approach entails the construction of a conservative p-dimensional process rectangle from the projection of an exact ellipsoid (ellipse if bivariate) containing a specified proportion of items on to its component axes. The edges of the resulting process rectangle (the process limits) are then compared with their corresponding specification limits. The associated index is defined in such a way that it is 1 if the process rectangle is contained within the p-dimensional specification rectangle with at

least one edge coinciding with its corresponding upper or lower specification limits, greater than 1 if the process rectangle is completely contained within the specification rectangle and less than 1 otherwise. A bivariate capability index developed using this approach is presented in the next section.

The second approach is based on the well known Bonferroni inequality. Unlike the first one, this approach actually requires only the weaker assumption of normality for each individual product characteristic. The capability index using this approach is defined in the same manner as above. The resulting process rectangular region having at least a specified proportion of conforming items is compared with the specification rectangle. The value of the proposed capability index reflects *conservatively* the process capability of meeting the specifications consistently. In fact, the assessment of process performance based on the Bonferroni inequality has been perceived by Boyles (1994) but it is used in a different way and context. It should also be pointed out, despite his statement to the contrary, that the given inequality

$$\pi_l + \pi_u \geq \pi$$

where

$$1-\pi = \Pr\left(-D_l \leq X_j \leq D_u,\ 1 \leq j \leq p \mid \mu, \Sigma\right),$$

$$1-\pi_l = \Pr\left(X_j \geq -D_l,\ 1 \leq j \leq p \mid \mu, \Sigma\right),$$

$$1-\pi_u = \Pr\left(X_j \leq D_u,\ 1 \leq j \leq p \mid \mu, \Sigma\right),$$

is not generally true.

Another approach utilizes the multivariate normal probability inequality given by Sidak (1967). It will be seen later, that a capability index constructed based on this

inequality and using arguments similar to the above, provides the best measure among all those proposed in this paper.

5.4 Three Bivariate Capability Indices

Suppose that the vector of the p product characteristics, $\mathbf{X} = \left(X_1, X_2, \ldots, X_p\right)^{\mathrm{T}}$ follows a multivariate normal distribution with mean vector $\boldsymbol{\mu} = \left(\mu_1, \mu_2, \ldots, \mu_p\right)^{\mathrm{T}}$ and covariance matrix Σ. Further, suppose that a manufactured product is considered *usable* if *all* its measured product characteristics are within their corresponding specification limits i.e $\mathrm{L}_j \le X_j \le \mathrm{U}_j$ for $j = 1, \ldots, p$. Let δ denote the proportion of unusable items produced that can be tolerated. Our aim is to obtain the relationship between the component means, the elements of the covariance matrix, δ and the specification limits of all the measured characteristics by solving the following integral equation :

$$\int_{\mathrm{L}_p}^{\mathrm{U}_p} \cdots \int_{\mathrm{L}_2}^{\mathrm{U}_2} \int_{\mathrm{L}_1}^{\mathrm{U}_1} f\left(x_1, x_2, \ldots, x_p\right) dx_1 dx_2 \ldots dx_p = 1 - \delta ,$$

so that an index can be defined that reliably reflects the actual process capability. Directly attempting to solve this equation is generally inadvisable due to computational difficulties, so some approximations are presented.

5.4.1 Projection of Exact Ellipsoid Containing a Specified Proportion of Products

It is known, for eg., Johnson et al.(1988) that, if $\mathbf{X} \sim N_p(\mu, \Sigma)$, the quadratic form $(\mathbf{X}-\mu)^T \Sigma^{-1}(\mathbf{X}-\mu) \sim \chi^2_p$. Thus, a region containing $100(1-\delta)\%$ of the products is the solid ellipsoid given by

$$(\mathbf{X}-\mu)^T \Sigma^{-1}(\mathbf{X}-\mu) \leq \chi^2_{p,1-\delta}.$$

As given by Nickerson (1994), the projection of the above ellipsoid on to each of its component axes is given by :

$$|x_j - \mu_j| \leq \sqrt{\chi^2_{p,1-\delta}}\,[j\text{th } \textit{diagonal element of } \Sigma]^{\frac{1}{2}}$$

or

$$\mu_j - \sqrt{\chi^2_{p,1-\delta}}\,\sigma_j \leq x_j \leq \mu_j + \sqrt{\chi^2_{p,1-\delta}}\,\sigma_j \qquad j = 1,2\ldots,p. \qquad (5.1)$$

Note that rewriting (5.1) yields the well known 100(1-δ)% simultaneous confidence interval for $\mu = (\mu_1, \mu_2, \ldots, \mu_p)^T$ based on a sample of size $n = 1$ when $\sigma_1, \sigma_2, \ldots, \sigma_p$ are known (Johnson et al.(1988)). As a special case, consider developing a capability index for bivariate processes, though it can easily be extended to the more general case. Note that, for $p = 2$, $\chi^2_{p,1-\delta} = -2\ln\delta$. Thus, we have from (5.1) that, the 'bivariate process limits' (i.e the limits beyond which at most $100\delta\%$ of items are expected to be produced) are,

$$x_j = \mu_j \pm \sigma_j \sqrt{-2\ln\delta} \qquad j = 1,2.$$

It follows, that for

$$\int_{L_2}^{U_2}\int_{L_1}^{U_1} f(x_1, x_2)\,dx_1 dx_2 \geq 1-\delta,$$

the following conditions need to be simultaneously satisfied :-

$$U_1 \geq \mu_1 + \sigma_1\sqrt{-2\ln\delta},$$
$$L_1 \leq \mu_1 - \sigma_1\sqrt{-2\ln\delta},$$
$$U_2 \geq \mu_2 + \sigma_2\sqrt{-2\ln\delta},$$
$$L_2 \leq \mu_2 - \sigma_2\sqrt{-2\ln\delta},$$

or equivalently,

$$\frac{U_1 - L_1}{2\left\{\sigma_1\sqrt{-2\ln\delta} + \left|\mu_1 - \frac{U_1+L_1}{2}\right|\right\}} \geq 1$$

and

$$\frac{U_2 - L_2}{2\left\{\sigma_2\sqrt{-2\ln\delta} + \left|\mu_2 - \frac{U_2+L_2}{2}\right|\right\}} \geq 1 .$$

Accordingly, the bivariate process capability index, $C_{pk}^{(2)}$, is defined as :-

$$C_{pk}^{(2)} = Min\left\{\frac{C_{p1}}{\frac{1}{3}\sqrt{-2\ln\delta} + \frac{|\mu_1 - T_1|}{3\sigma_1}}, \frac{C_{p2}}{\frac{1}{3}\sqrt{-2\ln\delta} + \frac{|\mu_2 - T_2|}{3\sigma_2}}\right\},$$

where C_{pj} and T_j $(j = 1,2)$ are respectively the univariate process capability indices (C_p) and the target values for the two product characteristics. Note that if the process is on-target i.e $\mu_1 = T_1$ and $\mu_2 = T_2$,

$$C_{pk}^{(2)} = \frac{Min\left\{C_{p1}, C_{p2}\right\}}{\frac{1}{3}\sqrt{-2\ln\delta}},$$

which can be taken as a measure of the process potential. Although this capability index is conservative by nature and thus must be carefully interpreted, it does provide some insight into the practical capability of the process. A value of 1 or greater can safely be interpreted as the process producing at a satisfactory level provided there is no serious departure from normality. However, if it has a value smaller than 1, it does not necessarily indicate that the expected proportion of usable items produced is less than $-\delta$, unless it is significantly different from 1. In this case, perhaps some simple

guidelines or ad hoc rules would help to determine if the process capability is adequate. Note the interesting fact that, although the covariance structure of the product characteristics is considered in the development of the above, the proposed bivariate capability index does not involve the correlation coefficient ρ of the two characteristics. It is also noted that the proposed index has some similarity to the bivariate capability vector proposed by Hubele et al.(1991). It differs from the latter, however, in that it incorporates both the process potential and the deviation of the process mean from target into a unitless measure. Hubele et al's capability index consists of three components, one for measuring the process location, one for process dispersion (potential) and the other for indicating whether any of the process limits is beyond its corresponding specification limit(s). Whilst it may be argued that using separate indicators for each of the above factors to reflect the process status may make the interpretation clearer, this process capability vector involves more calculation and does not have any clear advantages over the proposed index.

5.4.2 Bonferroni-Type Process Rectangular Region

According to the Bonferroni inequality, for a p-variate process for which the marginal distributions are normal, the p-dimensional centered process rectangle containing at least $100(1-\delta)\%$ of items is given by :

$$\mu_j - z_{\frac{\delta}{2p}}\sigma_j \le x_j \le \mu_j + z_{\frac{\delta}{2p}}\sigma_j, \qquad j = 1,2,\ldots\ldots,p.$$

where $z_{\delta/2p}$ denotes the upper $100(\delta/2p)$th percentile of a standard normal distribution. By replacing p by 2 in the above, the bivariate process limits are obtained. Proceeding as previously, another bivariate capability index is obtained and defined as,

$$ {}^{B}C_{pk}^{(2)} = Min\left\{ \frac{C_{p1}}{\frac{1}{3}z_{\delta/4} + \frac{|\mu_1 - T_1|}{3\sigma_1}}, \frac{C_{p2}}{\frac{1}{3}z_{\delta/4} + \frac{|\mu_2 - T_2|}{3\sigma_2}} \right\}, $$

which has a similar interpretation. If $p = 1$, $\delta = 0.0027$ and the process is on-target, this type of multivariate capability index reduces to the univariate C_p, C_{pk} and C_{pm} indices. It should be noted that this method of developing capability indices can be extended to non-normal processes by replacing $-z_{\delta/4}$ and $z_{\delta/4}$ by the appropriate quantiles of the process distribution.

5.4.3 Process Rectangular Region based on Sidak's Probability Inequality

As given by Sidak (1967), the multivariate normal probability inequality is :-

$$ \Pr\left\{ \bigcap_{j=1}^{p} |Z_j| \le c_j \right\} \ge \prod_{j=1}^{p} \Pr\left\{ |Z_j| \le c_j \right\}, $$

where Z_j's are standard normal variables and c_j's denote some specified constants. For $c_j = c \ \ (j = 1,2\ldots, p)$, this inequality becomes

$$ \Pr\left\{ \bigcap_{j=1}^{p} |Z_j| \le c \right\} \ge \left[2\Phi(c) - 1 \right]^p, $$

where $\Phi(\bullet)$ denotes the cumulative distribution function of the standard normal variable. Setting the lower bound, $\left[2\Phi(c) - 1\right]^p$ of the joint probability above, equal to $-\delta$, results in a p-dimensional process rectangle containing at least $100(-\delta)\%$ of items given by :-

$$ \mu_j - \sigma_j \Phi^{-1}\left(\tfrac{1}{2}\left[1 + (1-\delta)^{1/p} \right] \right) \le x_j \le \mu_j + \sigma_j \Phi^{-1}\left(\tfrac{1}{2}\left[1 + (1-\delta)^{1/p} \right] \right), \qquad j = 1,2,\ldots, p, $$

where $\Phi^{-1}(\bullet)$ represents the inverse of the standard normal distribution function. A bivariate capability index is obtainable by replacing p with 2 and comparing the resulting bivariate process limits with the corresponding specification limits. This index is defined as :-

$$ {}^{S}C_{pk}^{(2)} = Min\left\{ \frac{C_{p1}}{\frac{1}{3}\Phi^{-1}\left(\frac{1}{2}\left[1+\sqrt{1-\delta}\right]\right)+\frac{|\mu_1-T_1|}{3\sigma_1}}, \frac{C_{p2}}{\frac{1}{3}\Phi^{-1}\left(\frac{1}{2}\left[1+\sqrt{1-\delta}\right]\right)+\frac{|\mu_2-T_2|}{3\sigma_2}} \right\}. $$

In the development of the above indices, it has been assumed that the tolerances are bilateral and that the target or nominal specification is the midpoint of the specification band, this, of course, is not always the case in practice. Under these circumstances, redefinition of the indices using similar arguments is straightforward and will not be discussed further.

As all of the above are of similar form, it is preferable to choose the one which is least conservative or that best reflects the actual process capability. In the following section, some comparisons between the three are provided in order to resolve this issue.

5.5 Some Comparisons of the Projected, Bonferroni and Sidak-type Capability Indices

Following conventional practice, the relative merits of the proposed capability indices can be evaluated based on the following ratios :

$$I_{B:P} = \frac{\textit{Width of } 100(1-\delta)\% \textit{ projected Interval for } j\text{th characteristic}}{\textit{Width of } 100(1-\delta)\% \textit{ Bonferroni Interval for } j\text{th characteristic}}$$
$$= \frac{\sqrt{\chi^2_{p,1-\delta}}}{z_{\delta/2p}}$$

and

$$I_{S:P} = \frac{\textit{Width of } 100(1-\delta)\% \textit{ projected Interval for } j\text{th characteristic}}{\textit{Width of } 100(1-\delta)\% \textit{ Sidak Interval for } j\text{th characteristic}}$$
$$= \frac{\sqrt{\chi^2_{p,1-\delta}}}{\Phi^{-1}\left(\frac{1}{2}\left[1+(1-\delta)^{1/p}\right]\right)} .$$

Note that the expressions for $I_{B:P}$ and $I_{S:P}$ remain the same irrespective of the product characteristic being considered. One capability index is said to be less conservative than the other if its construction is based on a shorter interval for the same δ. Thus, according to the definitions above, if $I_{B:P}$ is greater than 1, the Bonferroni-type capability index is better (less conservative) than that which is based on projections. Similarly, a value of $I_{S:P}$ greater than 1 implies that the Sidak-type index is superior to the projected one. As for the relative effectiveness of the Bonferroni and Sidak-type indices, this is measured by the relative magnitude of their corresponding $I_{B:P}$ and $I_{S:P}$ values. The values of these indices are tabulated in Table 5.1 for some selected values of p and δ. It can be seen from this table that, in all the realistic cases considered, both the capability indices based on the Bonferroni and Sidak inequalities provide better measures than the

Projection-type capability index. The table also shows that, as the number of measured characteristics, p increases, the better the Bonferroni or Sidak-type capability indices become. Furthermore, as shown in the table, the Sidak-type capability index is marginally better than that based on the Bonferroni inequality. The following section is devoted to the development of a test concerning process capability based on the Sidak-type index.

Table 5.1. Relative Conservativeness of Projected, Bonferroni and Sidak-type Capability Indices.

P	δ	$I_{B:P} = \frac{\sqrt{\chi^2_{p,1-\delta}}}{z_{\delta/2p}}$	$I_{S:P} = \frac{\sqrt{\chi^2_{p,1-\delta}}}{\Phi^{-1}\left(\frac{1}{2}\left[1+(1-\delta)^{1/p}\right]\right)}$
2	0.0025	1.0726	1.0727
	0.005	1.0767	1.0768
	0.01	1.0811	1.0815
	0.02	1.0859	1.0867
	0.05	1.0921	1.0945
3	0.0025	1.1325	1.1326
	0.005	1.1397	1.1398
	0.01	1.1475	1.1479
	0.02	1.1561	1.1570
	0.05	1.1677	1.1708
5	0.0025	1.2319	1.2320
	0.005	1.2438	1.2440
	0.01	1.2569	1.2574
	0.02	1.2713	1.2724
	0.05	1.2917	1.2953
10	0.0025	1.4218	1.4219
	0.005	1.4419	1.4421
	0.01	1.4641	1.4646
	0.02	1.4886	1.4899
	0.05	1.5243	1.5283

5.6 Testing the Capability of a Bivariate Process

In practice, the assessment of process performance is often based on sample estimates of some capability indices which are subject to uncertainty. Unless the sample size is reasonably large, it is inappropriate to draw definite conclusions from these process capability estimates. Of course, the need for process stability before computation should also be emphasized, otherwise the interpretation of these indices is distorted, regardless of how large the sample is. If meaningful interpretation of the estimated capability is sought, it is important to take the sampling fluctuations of these estimates into consideration. A common approach is to employ confidence intervals. If point estimates are to be used, it is desirable that estimation is unbiased and that the minimum sample size required for an acceptable margin of estimation error is adhered to. Another approach is based on testing hypotheses. Either approach generally requires knowledge of the sampling distributions which are complicated. To circumvent this problem, we develop an *approximate* test for the Sidak-type index ($^{S}C_{pk}^{(2)}$).

Consider the problem of testing the following hypotheses :

$$H_0 : {}^{S}C_{pk}^{(2)} \geq 1 \quad vs. \quad H_a : {}^{S}C_{pk}^{(2)} < 1$$

Under the null hypothesis, H_0, the process is capable and the worst scenario is when both the Sidak-type process and specification rectangles coincide, in which case $^{S}C_{pk}^{(2)} = 1$. On the other hand, the alternative hypothesis, H_a corresponds to situations where at least one edge of the process rectangle is beyond its corresponding specification limit. The test proposed here is designed to capture such a situation.

A reasonable choice of the test statistic for this problem is,

$$ {}^{S}\hat{C}_{pk}^{(2)} = Min\left\{\frac{U_1 - L_1}{2cS_1 + 2\left|\bar{X}_1 - T_1\right|}, \frac{U_2 - L_2}{2cS_2 + 2\left|\bar{X}_2 - T_2\right|}\right\}, $$

where

$$ c = \Phi^{-1}\left(\tfrac{1}{2}\left[1 + \sqrt{1-\delta}\right]\right) $$

and, $\bar{X}_j$ and S_j denote respectively the mean and standard deviation of the jth product characteristic based on a sample of size n. The decision rule is to reject H_0 in favor of H_a if

$$ {}^{S}\hat{C}_{pk}^{(2)} < k, $$

where k is some positive constant depending on the significance level of the test (α) and is determined from

$$ Max\ \Pr\left\{{}^{S}\hat{C}_{pk}^{(2)} < k \mid H_0 \text{ is true}\right\} = \alpha. $$

The maximum value on the left-hand side of the above equation occurs when $\mu_j = T_j$ and $U_j - L_j = 2c\,\sigma_j$ for $j = 1,2$ (the worst situation under H_0). Thus, we have,

$$ \Pr\left\{Min\left\{\frac{c\sigma_1}{cS_1 + \left|\bar{X}_1 - \mu_1\right|}, \frac{c\sigma_2}{cS_2 + \left|\bar{X}_2 - \mu_2\right|}\right\} < k\right\} = \alpha $$

or

$$ \Pr\left\{\bigcap_{j=1}^{2}\left[\frac{1}{\sqrt{n-1}}\sqrt{\frac{(n-1)S_j^2}{\sigma_j^2}} + \frac{1}{c\sqrt{n}}\sqrt{\left(\frac{\bar{X}_j - \mu_j}{\sigma_j/\sqrt{n}}\right)^2} < \frac{1}{k}\right]\right\} = 1 - \alpha. \qquad (5.2) $$

According to the Bonferroni inequality,

$$ \Pr\left\{\bigcap_{j=1}^{2}\left[\frac{1}{\sqrt{n-1}}\sqrt{\frac{(n-1)S_j^2}{\sigma_j^2}} + \frac{1}{c\sqrt{n}}\sqrt{\left(\frac{\bar{X}_j - \mu_j}{\sigma_j/\sqrt{n}}\right)^2} < \frac{1}{k}\right]\right\} \geq 1 - \sum_{j=1}^{2}\Pr\left\{\frac{1}{\sqrt{n-1}}\sqrt{\frac{(n-1)S_j^2}{\sigma_j^2}} + \frac{1}{c\sqrt{n}}\sqrt{\left(\frac{\bar{X}_j - \mu_j}{\sigma_j/\sqrt{n}}\right)^2} \geq \frac{1}{k}\right\}. \qquad (5.3) $$

A *conservative* test of the hypotheses stipulated above may now be obtained by replacing the left-hand side of (5.2) by the right-hand side of (5.3) giving,

$$\sum_{j=1}^{2} \Pr\left\{ \frac{1}{\sqrt{n-1}} \sqrt{\frac{(n-1)S_j^2}{\sigma_j^2}} + \frac{1}{c\sqrt{n}} \sqrt{\left(\frac{\bar{X}_j - \mu_j}{\sigma_j / \sqrt{n}} \right)^2} \geq \frac{1}{k} \right\} = \alpha .$$

As

$$\frac{1}{\sqrt{n-1}} \sqrt{\frac{(n-1)S_j^2}{\sigma_j^2}} + \frac{1}{c\sqrt{n}} \sqrt{\left(\frac{\bar{X}_j - \mu_j}{\sigma_j / \sqrt{n}} \right)^2}, \qquad j = 1,2.$$

are identically distributed, it follows that,

$$\Pr\left\{ \frac{1}{\sqrt{n-1}} \sqrt{\frac{(n-1)S_j^2}{\sigma_j^2}} + \frac{1}{c\sqrt{n}} \sqrt{\left(\frac{\bar{X}_j - \mu_j}{\sigma_j / \sqrt{n}} \right)^2} \geq \frac{1}{k} \right\} = \frac{\alpha}{2} .$$

If $V = \dfrac{(n-1)S_j^2}{\sigma_j^2}$ and $W = \left(\dfrac{\bar{X}_j - \mu_j}{\sigma_j / \sqrt{n}} \right)^2$, the problem reduces to finding the $\left(1 - \frac{\alpha}{2}\right)$th quantile of

$$\frac{1}{\sqrt{n-1}} \sqrt{V} + \frac{1}{c\sqrt{n}} \sqrt{W},$$

which is a linear combination of the square root of two independent chi-square variables with n-1 and 1 degrees of freedom respectively. A closed-form representation of the probability density of this linear combination is not available. However, it is possible to obtain the approximate values of the required quantiles and thus the critical values, k, using Cornish-Fisher expansions. Johnson and Kotz (1970) outlined the method of obtaining these expansions and provided a formula which expresses the standardized quantiles of any distribution in terms of its standardized cumulants and the corresponding standard normal quantiles. However, it is found that there are some inconsistencies in the

results obtained by using the expression provided by these authors. As an alternative, numerical solutions are obtained from the following integral equation :-

$$\int_0^{\infty} F_{n-1}\left[(n-1)\left(1/k-\sqrt{w}/c\sqrt{n}\right)^2\right] f_1(w)dw = 1-\frac{\alpha}{2},$$

where $f_\nu(\bullet)$ and $F_\nu(\bullet)$ respectively denote the probability density and the cumulative distribution function of a chi-square variable with ν degrees of freedom. The approximate critical value, k, is obtained in this way using *Mathematica* version 2.2 (Wolfram (1991)) and given to 4 significant digits in Table 5.2 for various combinations of tolerable proportion of unusable items (δ), sample size (n) and significance level (α).

Table 5.2. Critical Values for Testing $^{S}C_{pk}^{(2)}$

	α			
n	0.01	0.025	0.05	0.1
10	* 0.5763	0.6093	0.6397	0.6770
	† (0.5636)	(0.5960)	(0.6258)	(0.6624)
15	0.6284	0.6590	0.6869	0.7206
	(0.6158)	(0.6461)	(0.6737)	(0.7070)
20	0.6630	0.6918	0.7178	0.7490
	(0.6507)	(0.6794)	(0.7052)	(0.7362)
25	0.6884	0.7158	0.7403	0.7695
	(0.6765)	(0.7038)	(0.7283)	(0.7574)
50	0.7594	0.7820	0.8020	0.8254
	(0.7489)	(0.7717)	(0.7918)	(0.8154)
100	0.8178	0.8359	0.8516	0.8698
	(0.8091)	(0.8275)	(0.8434)	(0.8619)

* unbracketed values correspond to $\delta = 0.01$.
† bracketed values correspond to $\delta = 0.05$.

5.7 Robustness to Departures From Normality --- Some Considerations

Various attempts have been made to extend the definitions of the standard univariate capability indices to situations where the process distribution is non-normal and corresponding estimation procedures have been proposed. These are intended to correctly reflect the proportion of items out of specification irrespective of the form of the process distribution. No attempts have appeared in the literature, however, to develop multivariate capability indices which are insensitive to departures from multivariate normality. Some robust univariate capability indices and procedures for assessing process performance currently available are briefly reviewed and an approach outlined for designing robust multivariate capability indices.

Chan, Cheng and Spiring (1988) suggested the use of a *tolerance interval* approach to estimate, with a certain level of confidence, the interval within which at least a specified proportion of items is contained. This estimated interval is then used in place of the normal-theory based interval (some multiple of σ) in the expressions for C_p, C_{pk} and C_{pm}. The 100(1-α)% confidence β-content tolerance interval is designed to capture at least 100β% of the process distribution, 100(1-α)% of the time by using appropriate order statistics. However, it was found by Chan et al. (1988) that the natural choice of β, 0.9973 and α, 0.05, results in the requirement of taking sample sizes, n of 1000 or larger. To circumvent this problem, they proposed the use of a tolerance interval with smaller β, specifically, with $\beta = 0.9546$ and $\beta = 0.6826$ in place of 4σ and 2σ respectively in the expressions for C_p, C_{pk} and C_{pm}, and provided the corresponding 95% confidence estimators for sample sizes less than 300. Although this modification

greatly reduces the minimum sample size required, Pearn et al. (1992) pointed out that 'it depends on the (somewhat doubtful) assumption that the ratios of distribution-free tolerance interval lengths for different β are always approximately the same as that for normal tolerance intervals'. Furthermore, the proposed extensions retain the process mean, μ in the original definitions of C_{pk} and C_{pm} rather than replacing it by the median. This complicates the interpretation of the resulting indices since the median may differ considerably from the process mean for heavily skewed distributions.

Another approach to analysing process capability for non-normal processes (especially unimodal and fairly smooth distributions) is based on systems or families of distributions. Having redefined the standard C_p and C_{pk} indices as

$$C_p = \frac{\mathrm{U} - \mathrm{L}}{P_{0.99865} - P_{0.00135}}$$

and

$$C_{pk} = Min\left\{\frac{\mathrm{U} - \mathrm{M}}{P_{0.99865} - \mathrm{M}}, \frac{\mathrm{M} - \mathrm{L}}{\mathrm{M} - P_{0.00135}}\right\}$$

$$= Min\left\{\frac{\mathrm{U} - P_{0.5}}{P_{0.99865} - P_{0.5}}, \frac{P_{0.5} - \mathrm{L}}{P_{0.5} - P_{0.00135}}\right\},$$

where P_δ denotes the 100δ th percentile of the distribution, Clement (1989) proposed fitting a Pearson-type curve to the observed data using the method of moments and the percentiles required for computation of these indices are then obtained from the fitted distribution. The required standardized percentiles were tabulated for various combinations of the coefficients of skewness and kurtosis. Some potential difficulties with this approach were given by Rodriguez (1992). In view of the complexity and difficulty of interpreting the equations for fitted Pearson and Johnson-type curves,

Rodriguez (1992) suggested the fitting of a particular parametric family of distributions such as the Gamma, Lognormal or Weibull distribution to the process data. For checking the adequacy of the distributional model, he recommended the use of statistical methods based on the empirical distribution function (EDF) including the Kolmogorov-Smirnov test, the Cramer Von Mises test and the Anderson-Darling test. As for the graphical checking of distributional adequacy, he stated that this can be accomplished by means of quantile-quantile plots or probability plots. In the same paper, he also briefly described the use of kernel density estimates for process capability analysis, especially for non-normal distributions.

Pearn et al. (1992) suggested a possible approach to obtain a robust capability index by defining an index

$$C_\theta = \frac{U - L}{\theta\sigma},$$

where θ is chosen such that

$$P_\theta = \Pr[\mu - \theta\sigma < X < \mu + \theta\sigma],$$

is as insensitive as possible to the form of the distribution of X. He showed that, for $P_\theta = 0.99$ the choice of $\theta = 5.15$ is quite adequate for a wide range of distributions.

For non-normal multivariate processes, it seems reasonable to use capability indices constructed based on multivariate Chebyshev-type inequalities (see Johnson and Kotz (1972), p.25) to reflect the process performance as no normality assumption is required. The most basic type of these inequalities is obtained by combining the Bonferroni and Chebyshev inequalities as follows :-

For our purpose here, the Bonferroni inequality is given by

$$\Pr\left\{\bigcap_{j=1}^{p}\left|\frac{X_j-\mu_j}{\sigma_j}\right|\le k\right\}\ge 1-\sum_{j=1}^{p}\Pr\left\{\left|\frac{X_j-\mu_j}{\sigma_j}\right|\ge k\right\} \tag{5.4}$$

Upon applying the Chebyshev inequality to each term in the summation on the right-hand side of (5.4), the following is obtained :-

$$\begin{aligned}\Pr\left\{\bigcap_{j=1}^{p}\left|\frac{X_j-\mu_j}{\sigma_j}\right|\le k\right\}&\ge 1-\sum_{j=1}^{p}\frac{1}{k^2}\\&\ge 1-\frac{p}{k^2}\end{aligned} \tag{5.5}$$

Note that, for the same k, the lower bound for (5.5) is smaller than that for (5.4). However, this does not imply that the capability index constructed based on inequality (5.5) is less conservative than that which is based on (5.4). For the same lower bound, 1-δ, the process rectangle based on the multivariate Chebyshev-type inequality (5.5) is always larger (as a result of larger k) than that of (5.4) irrespective of the underlying distribution. Note, however, that the Bonferroni-type capability index proposed in this chapter is obtained by imposing a normality condition on the marginal distributions of the process and thus it can be either too liberal or too stringent as a performance measure for non-normal processes. For instance, a value greater than 1 for this index does not guarantee that the expected proportion of non-defective items is more than 1-δ if the process distribution is heavy-tailed (such as a multivariate-t distribution) unless it is significantly different from 1.

There are some improvements to the above multivariate Chebyshev-type inequality, however, the expressions involved are complicated, causing the construction of multivariate capability indices based on them to be difficult except for situations where there are relatively few variables. It is also found that these capability indices are only

marginally better than that based on inequality (5.5). Thus, it is reasonable to use (5.5) whenever the use of distribution-free capability indices is warranted.

CHAPTER 6

A COMPARISON OF MEAN AND RANGE CHARTS WITH THE METHOD OF PRE-CONTROL [1]

6.1 Introduction

In 1924 Dr. Walter Shewhart first introduced the $\bar{X}$ *and* R charting technique for the statistical monitoring and control of industrial processes. Now, after many decades of use, they have become the core around which has been built a body of statistical techniques expressly designed for the purpose of controlling the quality of manufactured products.

A competing procedure, employing a different strategy and known as 'pre-control' (p.c.), was proposed by Shainin (1954) as a replacement for various special purpose plans for quality control and, in particular, as an improvement to the then 30 year old technique of $\bar{X}$ *and* R control. 'Pre-control' focuses directly on preventing non-conforming units from occurring, rather than on maintaining a process in a state of statistical control, which is the strategy underpinning the use of $\bar{X}$ *and* R charts.

When assessing the merits and shortcomings of competing industrial control procedures, the issue of statistical efficiency and more practical matters such as cost effectiveness, extent and ease of use should all be considered. In fact these factors, to varying degrees, play major roles in determining the overall success of quality monitoring, maintenance and improvement efforts.

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[1] This chapter is based on the paper entitled 'A comparison of mean and range charts with pre-control having particular reference to short-run production', *Quality and Reliability Engineering International*, Vol.10, pp.477-485, 1994.

After giving a brief outline of 'pre-control' and re-iterating its claimed practical benefits, this chapter provides a rationale for making a statistical comparison between the technique and that of traditional $\overline{X}$ *and* R charts. Special attention is drawn to the application of both techniques to the short run manufacturing environment where, for the use of $\overline{X}$ *and* R charts, parameter estimation is a problem. The total discussion in this chapter is in the context of the manufacture of discrete items.

6.2 A Review of Pre-Control

The basic principles underlying the 'pre-control' technique are illustrated in Figure 6.1.

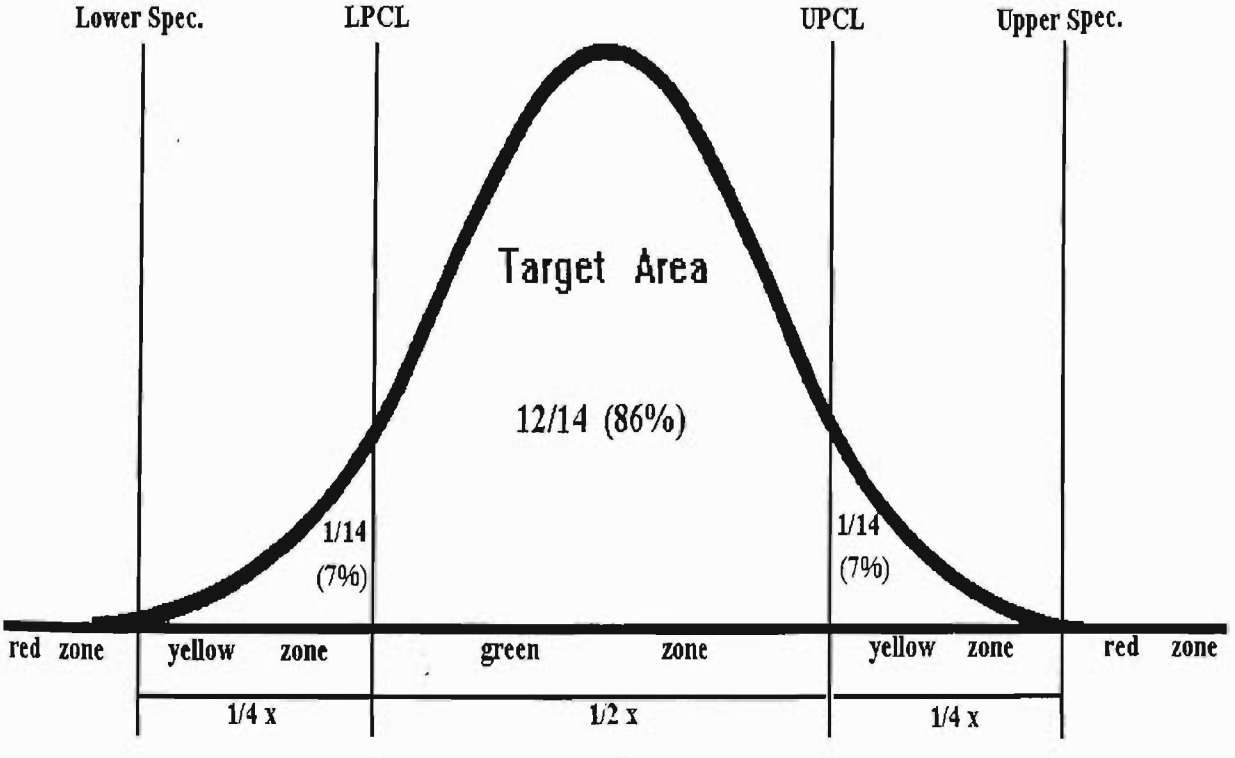

Figure 6.1. Pre-Control Scheme.

Suppose that the quality characteristic of interest is of the variable type such as a physical dimension. The tolerance (or specification band) is divided into four equal

sections and the boundaries of the outer two are called 'pre-control' lines. The area between these lines is described as the 'target area' or the green zone. The remaining areas between the two specification limits, L and U, are labelled the yellow zones and those beyond the specification limits, the red zones. Assuming that the measurements on the product characteristic are normally distributed, correctly centered and that the process is just capable of meeting the specifications, then approximately 1 in 14 times an observation will fall in either yellow zone by chance alone. A barely capable process is one having $C_p = 1$ where $C_p = (U - L)/6\sigma$, σ being the process standard deviation. A single observation falling in these zones is not deemed an indication of the presence of a process disruption. Two consecutive values in these zones, however, or one in the red zone, is considered adequate evidence of trouble and grounds for process adjustment.

'Pre-Control' operating rules are developed around these fundamental notions. In a slightly different version (Bhote (1980) and Logothetis (1990)), the decision for approval of set-up and resumption of a corrected process is based on the following rule :

'... If five consecutive units are within the target area before the occurrence of a red or a yellow, the set-up is qualified and full production can begin ...'.

The reason is that this occurrence indicates that the process is well centered and highly likely to be producing at a satisfactory quality level. The probabilities of approving a set-up which is centered at the nominal dimension, for various process capabilities, C_p, and using the above rule, are given in Table 6.1.

Table 6.1. Probability of Set-up Approval for Pre-Control

C_p	0.50	0.75	1.00	1.25	1.33	1.50
Prob.	0.0489	0.2210	0.4882	0.7308	0.7919	0.8838

If five consecutive greens prove difficult to obtain, then this is an indication that the process is either incorrectly centered and requires adjustment, or that the process is not capable of consistently meeting the specifications. This check rule is useful for short production runs for which the 'set-up' is a crucial factor affecting the quality of the subsequent process output.

Once the process has passed the initial set-up stage, periodically two consecutively produced items are examined to monitor performance. Having items in either of the yellow zones is acceptable, except when two occur consecutively. Two successive yellows on the same side of the target, signal the departure of the process mean from the target value. If they occur on different sides of the target, the process spread has most likely increased beyond its acceptable limit. In this manner, 'pre-control' enables corrective action to be taken usually before unacceptable work is produced and, hopefully, avoids repeated minor, and unnecessary corrections. In the event of getting an item in either part of the red zone, the process is stopped immediately, as it is already producing defectives. Variations in this 'pre-control' plan, applicable to less common situations are given by Shainin (1954) and Putnam (1962).

In order to justify his recommendation for 'pre-control', Shainin (1965, 1984) made some efforts to discuss its statistical power. These included consideration of the long run expected proportion of nonconforming units produced resulting from the ongoing use of 'pre-control' based on a particular sampling rule. He showed that the maximum value of this quality measure, termed the average produced quality limit (APQL), does not exceed 2% for normally distributed processes if 6 inspection checks, on average, are made between typical process adjustments (Shainin (1984)). Some very general discussion

about the sampling frequency appeared in Satterthwaite (1973), Shainin (1990) and Traver (1985). Without previous knowledge of the average time between process adjustments, Shainin (1984) suggested that a 20-minute sampling interval should first be used and adjusted subsequently.

As pointed out by Cook (1989), some of the expressed doubts about 'pre-control' relate to the normality assumption. The general consensus amongst practitioners is that even for stable processes it is doubtful that the fit to normality in the distribution tails is particularly close. Sinibaldi (1985) used simulation techniques to examine the effect of non-normality on the appropriateness of 'pre-control'. In addition, he evaluated the relative performance of 'pre-control' and $\overline{X}$ *and* R control on normal and skewed distributed processes with frequently changing means. The results of the comparison indicate that $\overline{X}$ control causes fewer incorrect mean shift signals and has better control to target (as measured by the overall average, $\overline{\overline{X}}$ and the average distance of all items produced from the process target) than 'pre-control'. However, using the R chart to detect deterioration in the process spread results in more false alarms than using 'pre-control' for the same purpose.

Bhote (1980) attempted to illustrate some 'weaknesses' in $\overline{X}$ *and* R control charting, using two case studies. Taking a more complete view, Logothetis (1990) argued effectively that, despite its simplicity, 'pre-control' cannot be considered a serious technique of statistical process control (SPC). He, in fact, used the same case studies as Bhote (who used them to illustrate the 'weaknesses' of $\overline{X}$ *and* R control charts) to demonstrate the usefulness of SPC as a whole and the weaknesses of 'pre-control'. However, no comparison has been made between 'pre-control' and $\overline{X}$ *and* R charts on the basis of average run length (ARL). This is due to the fact that, 'pre-control' lines are

derived from specification limits, causing the ARL for a given mean shift to vary according to the actual process capability (C_p).

6.3 The Practical Merits of Pre-Control

There is little mention of 'pre-control' in many standard text books on statistical process control, despite it having certain practical advantages over $\overline{X}$ *and* R charts. This could indicate a belief that a reasonable statistical comparison between the two techniques is not legitimate, that there is a reluctance to forsake $\overline{X}$ *and* R charting since the method has proven useful over many years and in many industries or indicate a view aligned with that of Logothetis (1990) that 'pre-control' is too limited in its perspective.

With the unique setup rule of 'pre-control', the first five consecutively manufactured units are all that is required to determine whether any process-tolerance incompatibilities exist before full production is allowed to commence. There is, of course, no definite knowledge of how many units will have to be checked before five consecutive good ones are obtained. In comparison, when using $\overline{X}$ *and* R charts, it is necessary to have fairly long process trial runs in order to collect sufficient sample data to establish the existence of a state of statistical control, and subsequently, to estimate the process standard deviation so as to correctly locate the control lines.

Following setup approval, 'pre-control' provides for the occasional sampling of two consecutively produced items in order to monitor on-going process performance, in contrast to the routine sample size of four or five often recommended for use of $\overline{X}$ *and* R charts. No calculations need to be performed for 'pre-control' operation except for the extremely simple initial setting of 'pre-control' lines, whereas continual routine

computations are involved with use of $\overline{X}$ *and* R charts. For these latter, it is not only necessary to calculate the control statistics for subgroups, but also necessary to estimate and revise the control limits from time to time.

For 'pre-control', measurements can be observed and compared to specification limits in a way that is easily understood by operators without much likelihood of misinterpretation. Additional worker participation in decision making and problem solving may be gained through operators having a better appreciation of the techniques in use.

Whilst, in practice, determination of the sampling interval for $\overline{X}$ *and* R charts is arbitrary, 'pre-control' provides a simple and flexible rule of six inspection checks per trouble indication which, on a long term basis, results in a maximum average fraction defective of less than 2% for a normally distributed process (Shainin (1984)). A successful application of 'pre-control' in a 'zero defects' environment has been reported by Brown (1966). Regulating sampling on the basis of recent process performance, seems a more reasonable and efficient approach to adopt than sampling at fixed intervals, as it entails more frequent sampling when the process is unsatisfactory.

Such eventualities as tool wear do not cause a premature reaction from 'pre-control'. It will only issue warning signals at times when the process is soon likely to produce defective products. What can be considered un-necessary process adjustments, which have the potential to make production performance worse, are, therefore, avoided.

Since 'pre-control' does not require exact measurements but only needs to note the zone into which the measurements fall, complex and expensive measuring equipment may be replaced by 'go/no-go' colour coded gauges. Furthermore, electronic gauging can

be considerably simplified if it is only required to distinguish between a few measurement bands. As a result, there can be a reduction in capital investment and calibration costs

Another important feature of 'pre-control' is its ready applicability to a variety of situations including the short production run manufacturing environment which has become increasingly prevalent following the general move into Just-In-Time (JIT) production and flexible manufacturing.

Despite its many years of existence and its apparent practical merits, given in brief here, 'pre-control' has not been widely adopted as a replacement for traditional $\overline{X}$ *and* R charts. Logothetis (1990) extensively criticised adoption of 'pre-control' over the use of $\overline{X}$ *and* R charts on a number of grounds. It is intended that the material contained in this chapter will provide some additional, statistically based arguments that will help facilitate a rational judgement on which of the two techniques to adopt in any given situation.

6.4 Short Runs and Pre-Control

There is no universally agreed definition of a 'short run', however, the term is often used to describe production processes with typically fewer than 50 items made within a single machine set-up. Short runs, therefore, at a first glance, do not readily lend themselves to the use of Shewhart $\overline{X}$ *and* R charts.

The essential problem that obstructs the application of standard control charting techniques in short production run situations is the inability to estimate the process variability, because of insufficient data. The problem is further aggravated by problems of process 'warm up'. Using data from the 'warm up' period to obtain control limits can

lead to erroneous conclusions regarding past, current and future states of the process (see Murray et al.(1988).

Unlike $\overline{X}$ *and* R charts, 'pre-control' is a control technique which predetermines its control limits by reference to product specifications only, rather than requiring an accumulation of data for computation of them. It is also capable of handling the problem of process 'warm up'. It is, therefore, highly suitable for application to short production runs.

6.5 A Statistical Comparison Between Pre-Control and X-bar and R Charts

For short production runs, when there is insufficient previous data available to obtain the control limits for $\overline{X}$ *and* R charts, a number of authors (see, for example, Sealy (1954) and Bayer (1957) have proposed setting control limits on the assumption that the process is just capable of meeting specifications (i.e. $C_p = 1$) and assuming that the mean level of the process is equal to the nominal or target value. This adaptation provides a basis for a statistical comparison between 'pre-control' and $\overline{X}$ *and* R charts.

In the following comparison, a subgroup size of four is chosen for the application of $\overline{X}$ *and* R charts because this is commonly recommended. It is also assumed that the quality characteristic under consideration is normally distributed or approximately so and that no supplementary run rules are used with the $\overline{X}$ chart. First, consider the probabilities of detection by the sample immediately following a process mean shift, using an $\overline{X}$ chart and a 'pre-control' chart. These probabilities are provided in Tables 6.2 and 6.3 for various combinations of process capability (C_p) and mean shifts in multiples (k) of the standard deviation (σ). In both tables, the entries are the probabilities of

issuing a correct signal of the mean shift except when k = 0, in which case the values tabulated are the probabilities of a false warning. Signals from 'pre-control' that we employ here as indication of a process mean shift, are 2 consecutive items in the same yellow zone, or 1 in the red zone and the other not falling beyond the 'pre-control' line on the opposite side of the nominal value. Furthermore, it should be noted that the control limits for the $\overline{X}$ *and* R charts are set using conventional control chart factors with the additional assumptions that,

$$\mu = \frac{U+L}{2} \quad and \quad \sigma = \frac{U-L}{6}.$$

The entries in Table 6.3, other than those corresponding to $C_p =$, are the probabilities of detecting a mean shift of the indicated magnitudes when the C_p has been assumed 1 but is in fact the value indicated. It has been adequately demonstrated in the literature that the $\overline{X}$ chart is tardy in registering small changes in the process mean. Where the 'speedy' detection of small mean shifts is required, additional control rules or alternative charting techniques are necessary. Thus the tables provide, for comparison, probabilities for a number of realistic mean shifts; realistic in the sense that they reflect situations where $\overline{X}$ (with no additional rules) and 'pre-control' can conceivably be considered competing techniques. Besides having a lower likelihood of a false signal, the $\overline{X}$ chart possesses a higher probability of 'picking up' the mean shift irrespective of the actual process capability, except where indicated by *, when the difference between corresponding entries in the two tables is marginal. In one sense, a more reasonable comparison can be accomplished through adjusting the control limits for the $\overline{X}$ chart in such a way that the resulting probability of issuing a false signal, when $C_p = 1$, is the same as that of 'pre-control'. This involves moving the control lines nearer to the nominal

or target value. Following such a modification, the corresponding probabilities of immediate detection are given in Table 6.4. Tables 6.2 and 6.4 clearly illustrate the superiority of the $\overline{X}$ chart in terms of sensitivity to process mean shifts.

Table 6.2. Power of Pre-Control-Mean Shift

C_p \ k	0	±1.0	±1.5	±2.0	±2.5
0.50	0.2488	0.5814	0.8125	0.9418	0.9879
0.75	0.0701	0.3153	0.5759	0.8072	0.9391
1.00	0.0136	0.1264	0.3166	0.5759	0.8057
1.25	0.0022	0.0412	0.1410	0.3383	0.5950
1.33	0.0012	0.0279	0.1051	0.2752	0.5223
1.50	0.0003	0.0116	0.0534	0.1685	0.3767

Table 6.3. Power of $\overline{X}$ Chart with 3σ Limits (assumed $C_p = 1$)

C_p \ k	0	±1.0	±1.5	±2.0	±2.5
0.50	0.1336	0.6915	0.9332	0.9938	0.9998
0.75	0.0244	0.4013	0.7734	0.9599	0.9970
1.00	0.0027	0.1587	0.5000	0.8413	0.9773
1.25	0.0002	0.0401*	0.2266	0.5987	0.8944
1.33	0.0001	0.0232*	0.1611	0.5040	0.8438
1.50	0.0000	0.0062*	0.0668	0.3085	0.6915

Table 6.4. Power of $\bar{X}$ Chart with Adjusted Limits (assumed $C_p = 1$)

C_p \ k	0	±1.0	±1.5	±2.0	±2.5
0.50	0.2173	0.7782	0.9613	0.9972	0.9999
0.75	0.0642	0.5594	0.8748	0.9842	0.9992
1.00	0.0136	0.3201	0.7028	0.9373	0.9943
1.25	0.0020	0.1391	0.4664	0.8201	0.9723
1.33	0.0010	0.0985	0.3890	0.7637	0.9571
1.50	0.0002	0.0444	0.2416	0.6174	0.9030

It is also of value to study the probabilistic behaviour of these two control techniques in relation to how quickly they respond to an increase in process dispersion. For 'pre-control', two successive measurements beyond different 'pre-control' lines constitute a warning signal that the process spread is worse than the one implicitly assumed. However, the occurrence of this event does not only depend upon the process capability, it is also affected by the deviation of the process mean from target. As reflected in Table 6.5, for a given level of process capability, the larger the deviation, the smaller the chance of getting such a signal. The corresponding probabilities of a signal from the R chart are given in Tables 6.6 and 6.7 for cases where conventional and adjusted control limits are used. Control lines are adjusted in the sense that they equate the probabilities of false alarms for the two methods. As shown in these tables, an R chart clearly provides better protection against a worsening process capability.

Table 6.5. Power of Pre-Control - Increase in Dispersion

C_p \	0	±1.0	±1.5	±2.0	±2.5
0.50	0.1027	0.0480	0.0189	0.0053	0.0011
0.65	0.0543	0.0246	0.0093	0.0025	0.0005
0.75	0.0340	0.0151	0.0056	0.0014	0.0003
0.85	0.0205	0.0090	0.0033	0.0008	0.0001
1.00	0.0089	0.0038	0.0014	0.0003	0.0001

Table 6.6. Power of *R* Chart-Conventional Limits (assumed $C_p = 1$)

C_p	0.50	0.65	0.75	0.85	1.00
Prob.	0.3445	0.1349	0.0613	0.0246	0.0049

Table 6.7. Power of *R* Chart-Adjusted Limits (assumed $C_p = 1$)

C_p	0.50	0.65	0.75	0.85	1.00
Prob.	0.3940	0.1715	0.0850	0.0376	0.0089

Tables 6.8 and 6.9 provide average run lengths for detection of a mean shift using 'pre-control' and an $\overline{X}$ chart respectively, based on the probabilities contained in Tables 6.2 and 6.3.

Table 6.8. Average Run Lengths for Pre-Control

C_p \ k	0	±1.0	±1.5	±2.0	±2.5
0.50	4.02	1.72	1.23	1.06	1.01
0.75	14.27	3.17	1.74	1.24	1.06
1.00	73.53	7.91	3.16	1.74	1.24
1.25	454.55	24.27	7.09	2.96	1.68
1.33	833.33	35.84	9.51	3.63	1.91
1.50	3333.33	86.21	18.73	5.93	2.65

Table 6.9. Average Run Lengths for $\overline{X}$ Chart - Adjusted Limits (assumed $C_p = 1$)

C_p \ k	0	±1.0	±1.5	±2.0	±2.5
0.50	7.49	1.45	1.07	1.00	1.00
0.75	40.98	2.49	1.29	1.04	1.00
1.00	370.37	6.30	2.00	1.19	1.02
1.25	5000	24.94	4.41	1.67	1.12
1.33	10000	42.92	6.21	1.98	1.19
1.50	-	161.29	14.97	3.24	1.45

It can be seen that if C_p is correctly taken to be 1, then the $\overline{X}$ chart is superior in terms of ARL. This is the case even if the true value of C_p is as low as 0.5 or as high as 1.25. Of course an ARL comparison is particularly meaningful if it is assumed that the sampling interval is common for the two methods. This further raises the matter of sampling effort, since 'pre-control' has an implied sample size of 2 and the $\overline{X}$ *and* R charts being used here for comparison, have a sample size of 4. This latter issue will be discussed later.

Tables 6.10, 6.11 and 6.12 are extensions to Table 6.3 where different C_p values are assumed at the outset. From these it can be seen that if C_p is taken to be 0.75 then

there is little difference in the two methods even if the C_p value is in fact 0.5. 'Pre-control' has less likelihood of false alarms, however. When C_p is assumed to be 1.25, even if the actual value is as low as 0.5 or as high as 1.50 the $\overline{X}$ chart is superior for detecting all the given mean shifts. Similarly for the assumption of $C_p = 1.50$, except here, 'pre-control' is marginally superior with respect to false alarms. Tables 6.13 and 6.14 are similar extensions to Table 6.6.

Table 6.10. Power of $\overline{X}$ Chart (assumed $C_p = 0.75$)

C_p \ k	0	±1.0	±1.5	±2.0	±2.5
0.50	0.0455	0.5000	0.8413	0.9773	0.9987
0.75	0.0027	0.1587	0.5000	0.8413	0.9773
1.00	0.0001	0.0228	0.1587	0.5000	0.8413
1.25	0.0000	0.0014	0.0228	0.1587	0.5000
1.33	0.0000	0.0005	0.0102	0.0934	0.3745
1.50	0.0000	0.0000	0.0014	0.0228	0.1587

Table 6.11. Power of $\overline{X}$ Chart (assumed $C_p = 1.25$)

C_p \ k	0	±1.0	±1.5	±2.0	±2.5
0.50	0.2301	0.7881	0.9641	0.9974	0.9999
0.75	0.0719	0.5793	0.8849	0.9861	0.9993
1.00	0.0164	0.3446	0.7258	0.9452	0.9953
1.25	0.0027	0.1587	0.5000	0.8413	0.9773
1.33	0.0014	0.1166	0.4239	0.7905	0.9647
1.50	0.0003	0.0548	0.2743	0.6554	0.9192

Table 6.12. Power of $\overline{X}$ Chart (assumed $C_p = 1.50$)

C_p \ k	0	±1.0	±1.5	±2.0	±2.5
0.50	0.3173	0.8413	0.9773	0.9987	1.0000
0.75	0.1336	0.6915	0.9332	0.9938	0.9998
1.00	0.0455	0.5000	0.8413	0.9773	0.9987
1.25	0.0124	0.3085	0.6915	0.9332	0.9938
1.33	0.0078	0.2546	0.6331	0.9099	0.9904
1.50	0.0027	0.1587	0.5000	0.8413	0.9773

Table 6.13. Power of R Chart (assumed $C_p = 1.25$)

C_p	0.50	0.65	0.75	0.85	1.00	1.25
Prob.	0.5445	0.3093	0.1904	0.1078	0.0393	0.0049

Table 6.14. Power of R Chart (assumed $C_p = 1.50$)

C_p	0.50	0.65	0.75	0.85	1.00	1.25	1.50
Prob.	0.6851	0.4745	0.3445	0.2355	0.1192	0.0289	0.0049

If the $\overline{X}$ *and* R charts are for use with short production runs, it may not make a great deal of sense to compare their effectiveness with 'pre-control' on the basis of average run length. This is the case when the total production time is less than the time taken to collect enough samples to match the ARL. As an alternative, we consider the probability of detection within 5 successive samples following a given mean shift. This probability is plotted against process mean shift in standard deviation units for 'pre-control' and $\overline{X}$ charts with both conventional and adjusted control limits in figures 6.2(a) to 6.2(f) where the $\overline{X}$ chart is constructed under the assumption that C_p is 1. As shown, there is no remarkable difference between 'pre-control' and $\overline{X}$ charts with either

conventional or adjusted control limits if $C_p = 0.5$ *or* 0.75 . However, considerable differences exist between these techniques if the process is more than capable, especially for mean shifts ranging from 1σ to 2σ.

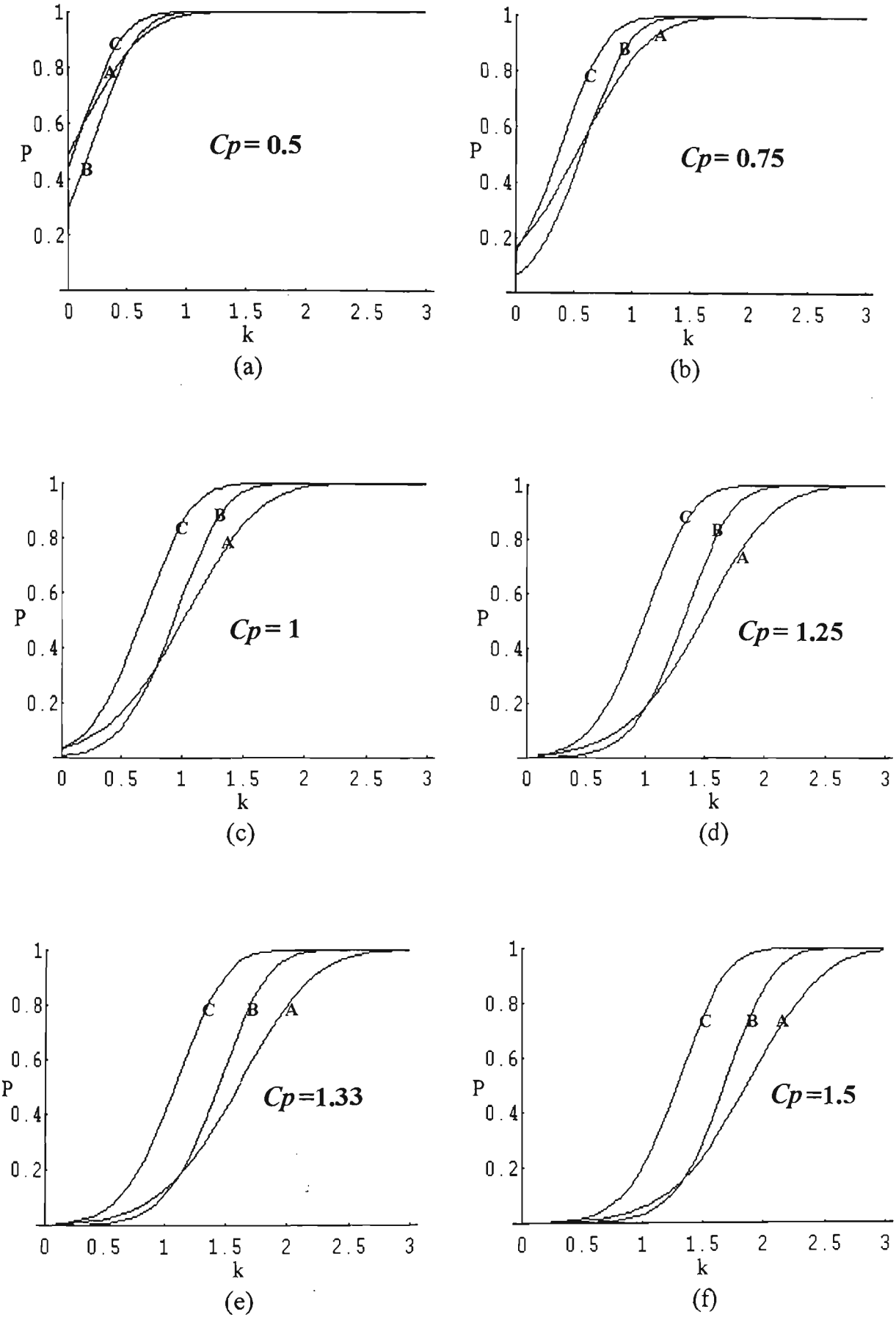

Figure 6.2. Probability of Detection Within 5 Successive Samples (P) vs Mean Shift in Multiples of Standard Deviation (k). A, B and C denote respectively Pre-Control, $\overline{X}$ Chart with Coventional and Adjusted Limits.

6.6 Equating Sampling Effort

In our discussion so far, the total sampling effort has not been taken into consideration; the assumption being made that the time and cost of sampling, measurement or testing are not significant. This may, however, be unrealistic in certain circumstances. If it is, no useful comparison can be made unless the relative sampling frequency of 'pre-control' and $\bar{X}$ and R control is first set in such a way that use of both methods involves the same sampling effort.

To compensate for the smaller sample size of 2 for 'pre-control', we assume that process checks are made twice as often as $\bar{X}$ and R control with a sample size of 4. This being the case, we focus on the average number of items required to detect a change in the process mean or process dispersion, using the two methods.

In Tables 6.15 and 6.16, the average number of units sampled before 'picking up' various mean shifts under different process capability levels are given for 'pre-control' and $\bar{X}$ chart control (control lines fixed on the basis that $C_p = 1$). To facilitate the comparison, we have computed the following index :

$$I_{MS} = \begin{cases} \dfrac{\text{ANII}_{MS}(\text{PC})}{\text{ANII}(\bar{X})} & \text{if } k = 0 \\ \dfrac{\text{ANII}(\bar{X})}{\text{ANII}_{MS}(\text{PC})} & \text{if } k \neq 0 \end{cases}$$

where $\text{ANII}_{MS}(\text{PC})$ and $\text{ANII}(\bar{X})$ denote the average number of items inspected before detecting the mean shift using 'pre-control' and conventional $\bar{X}$ chart (control based on $C_p = 1$) respectively except when $k = 0$, in which case they are the average number of items inspected prior to the occurrence of a false signal.

If $I_{MS} > 1$ when $k \neq 0$ then 'pre-control' performs better than the $\bar{X}$ chart in the sense that, on average, it 'picks up' the mean shift with fewer sampled items. Similarly, when $k = 0$ and $I_{MS} > 1$ then 'pre-control' takes longer, on average, before issuing a false signal. The values of this index for various combinations of mean shift (in multiples of σ) and actual process capability are tabulated in Table 6.17. As shown, although it is superior in detecting mean shifts of all the given magnitudes, irrespective of the actual process capability, 'pre-control' is far worse than the $\bar{X}$ chart with regard to false alarms, a factor alluded to by Logothetis (1990).

Table 6.15. $ANII_{MS}$(PC)

C_p \ k	0	±0.5	±1.0	±1.5	±2.0	±2.5	±3.0
0.50	8.04	6.30	3.44	2.46	2.12	2.02	2.00
0.75	28.53	15.95	6.34	3.47	2.48	2.13	2.03
1.00	147.01	56.60	15.83	6.32	3.47	2.48	2.13
1.25	917.36	243.97	48.53	14.19	5.91	3.36	2.45
1.33	1686.06	400.16	71.69	19.03	7.27	3.83	2.63
1.50	6407.56	1200.90	172.74	37.42	11.87	5.31	3.18

Table 6.16. $ANII(\bar{X})$ (sample size 4, Control based on $C_p = 1$)

C_p \ k	0	±0.5	±1.0	±1.5	±2.0	±2.5	±3.0
0.50	30	13.0	5.79	4.29	4.03	4.00	4.00
0.75	164	37.9	9.97	5.17	4.17	4.01	4.00
1.00	1481	175.8	25.21	8.00	4.75	4.09	4.00
1.25	22611	1342.4	99.85	17.65	6.68	4.47	4.05
1.33	60458	2867.5	171.71	24.83	7.94	4.74	4.09
1.50	588674	17189.8	644.16	59.87	12.96	5.78	4.29

Table 6.17. I_{MS}

C_p \ k	0	±0.5	±1.0	±1.5	±2.0	±2.5	±3.0
0.50	0.27	2.06	1.68	1.74	1.90	1.98	2.00
0.75	0.17	2.37	1.57	1.49	1.68	1.88	1.97
1.00	0.10	3.11	1.59	1.27	1.37	1.65	1.88
1.25	0.04	5.50	2.06	1.24	1.13	1.33	1.65
1.33	0.03	7.17	2.40	1.30	1.09	1.24	1.56
1.50	0.01	14.31	3.73	1.60	1.09	1.09	1.35

A similar comparison between 'pre-control' and use of the R chart (control based on $C_p = 1$) with respect to detection of increase in process variance can also be made. Let

$$I_{VI} = \begin{cases} \dfrac{\text{ANII}_{VI}(\text{PC})}{\text{ANII}(R)} & \text{if } C_p = 1 \\ \dfrac{\text{ANII}(R)}{\text{ANII}_{VI}(\text{PC})} & \text{if } C_p < 1 \end{cases}$$

where $\text{ANII}_{VI}(\text{PC})$ and ANII(R) denote the average number of items inspected prior to a signal from 'pre-control' and a conventional R chart (control based on $C_p = 1$) respectively.

The values of $\text{ANII}_{VI}(\text{PC})$, $\text{ANII}(R)$ and I_{VI} are provided in Tables 6.18, 6.19 and 6.20 respectively. The R chart can be seen to be more sensitive to increase in process spread except where marked by *.

Table 6.18. ANII$_{VI}$(PC)

C_p \ k	0	±0.5	±1.0	±1.5	±2.0	±2.5	±3.0
0.50	19.47	23.59	41.70	105.77	375.23	1805.10	11445
0.65	36.83	44.94	81.25	214.38	805.58	4180.40	29010
0.75	58.90	72.19	132.25	357.20	1389.97	7557.50	55637
0.85	97.73	120.23	222.93	615.09	2471.39	14040.90	109485
1.00	224.05	277.05	521.94	1481.50	6215.00	37483.00	315409

Table 6.19. ANII(R) (sample size 4, control based on $C_p = 1$)

C_p	0.50	0.65	0.75	0.85	1.00
ANII(R)	11.61	29.68	65.39	162.94	811.69

Table 6.20. I_{VI}

C_p \ k	0	±0.5	±1.0	±1.5	±2.0	±2.5	±3.0
0.50	0.5964	0.4923	0.2785	0.1098	0.0309	0.0064	0.0010
0.65	0.8060	0.6605	0.3653	0.1385	0.0369	0.0071	0.0010
0.75	1.1102*	0.9059	0.4945	0.1831	0.0471	0.0087	0.0010
0.85	1.6673*	1.3552*	0.7309	0.2649	0.0659	0.0116	0.0010
1.00	0.2760	0.3413	0.6430	1.8252	7.6569	46.1790	388.5830

In many applications, the cost, effort or time expended to investigate a trouble-free process only to conclude subsequently that no change has occurred, is high. Under such circumstances, it seems appropriate to evaluate the relative effectiveness of competing control procedures having equated, for the two methods, the average number tested to produce a false signal. For this reason, the control limits of the $\overline{X}$ and R charts were adjusted so that both lead to the same average time elapsed or average number of items inspected prior to occurrence of a false signal as 'pre-control', when $C_p = 1$. The

resulting ANII($\overline{X}$) and ANII(R) values are shown in Tables 6.21 and 6.22 respectively. As illustrated in Tables 6.15 and 6.21, if the process capability is correctly assumed (i.e $C_p = 1$) or underestimated (i.e $C_p > 1$), the adjusted $\overline{X}$ chart requires a much smaller number of units, on average, to 'pick up' the given mean shift except when $C_p = 1$ and k = 2, in which case the difference is marginal. For $C_p = 0.5$ or $C_p = 0.75$, it is found that in most cases (marked with *), 'pre-control' is marginally better than the adjusted $\overline{X}$ chart. It can also be seen that false signals from the adjusted $\overline{X}$ chart tend to occur after a longer period of time when the process is incapable. However, if the process is more than capable, the adjusted $\overline{X}$ chart will tend to issue a false signal sooner than 'pre-control' although, due to the large magnitudes, this is of little practical consequence. It should be noted that we have delibrately omitted those cases where k = 2.5 and k = 3.0 in Table 6.21 because mean shifts as large as these are likely to be detected early anyway irrespective of method. The R chart is found to be far superior to 'pre-control' in reacting to worsening process capability (refer to Tables 6.18 and 6.22). This is especially true when the process is not on target.

Table 6.21. ANII($\overline{X}$) (sample size 4, adjusted limits, control based on $C_p = 1$)

C_p \ k	0	±0.5	±1.0	±1.5	±2.0
0.50	14.84	8.72*	4.91*	4.12*	4.01*
0.75	40.96	15.64	6.30	4.39*	4.04*
1.00	147.01	35.27	9.58	5.09	4.15*
1.25	693.33†	102.17	17.90	6.73	4.48
1.33	1208.12†	151.78	22.95	7.62	4.67
1.50	4329.05†	385.80	42.27	10.60	5.30

* these are slightly larger than the corresponding figures for 'pre-control' in table 6.15.
† these are smaller than the corresponding figures for 'pre-control' in table 6.15.

Table 6.22. ANII(R) (sample size 4, adjusted limits, control based on $C_p = 1$)

C_p	0.50	0.65	0.75	0.85	1.00
ANII(Adjusted R)	8.72	17.69	32.28	65.13	224.05

6.7 Concluding Remarks

On practical considerations and from the perspective of monitoring and control, proponents of 'pre-control' state the method to be superior to $\overline{X}$ and R charts. Its simplicity and versatility make it a useful tool for a large variety of applications. Nevertheless, as shown in the previous sections, $\overline{X}$ and R control charting still have merits on statistical grounds.

It is clear that, if sampling effort is of little importance, C_p is known and provides a value of 1, 1.25 or 1.50, then the $\overline{X}$ chart is superior in 'picking up' mean shifts greater than 1σ. When σ is not known, as is frequently the case in short production runs, and therefore its value has to be estimated or assumed for use of Shewhart charts, in many instances $\overline{X}$ control is seen to still be superior. Specifically, if

the C_p is assumed to be 1.25 but is actually between 0.5 and 1.5 then $\overline{X}$ control is superior to 'pre-control' in 'picking up' shifts in the process mean. If C_p is assumed to be 1, yet actually has a value somewhere between 0.5 and 1.50, then an $\overline{X}$ chart is as good as or significantly better for detecting mean shifts than 'pre-control'. For detection of a deterioration in the process capability we have observed the standard R chart to be more sensitive than 'pre-control'. It would seem, therefore, that the advocacy of Maxwell (1953) and others is sound, that in the absence of knowledge of σ we can use, for construction of standard $\overline{X}$ and R charts, an assumed C_p value of 1. Certainly this is so in regard to providing a more sensitive instrument for process control than 'pre-control'.

When sampling effort is important, a comparison that fairly compares the two techniques when the sampling effort is the same, reveals that for capable processes, $\overline{X}$ and R charts are superior to 'pre-control'.

The material presented in this chapter has taken a perspective that focuses narrowly on monitoring and control. Broadening the perspective and perceiving charting techniques as merely a part in the effort of continuous improvement underscores further the value of traditional Shewhart charts.

We have sought to create some common ground for $\overline{X}$ and R control and 'pre-control' in order to examine their performance for monitoring and control on a statistical basis. It is hoped that the material contained herein will provide more complete grounds on which to base a comparison between the two techniques and thus to facilitate more rational judgements on which of the two to use in any given situation.

CHAPTER 7

CONCLUSIONS AND SOME SUGGESTIONS FOR FUTURE INVESTIGATION

7.1 Summary and Conclusions

In this thesis, some techniques for monitoring the mean vector of a multivariate normal process have been presented. As the proposed techniques involve sequences of independent or approximately independent standard normal variables, the resulting control charts can all be plotted in the same scale and with the same control limits irrespective of product types, thus simplyfying charting administration. Any non-random patterns on such standardized charts which suggest various process instabilities can also be readily detected by the use of additional sequence rules. Those techniques presented for the case with no prior knowledge of process parameters are particularly attractive for short production runs and low volume manufacturing environments since control can be initiated essentially with the first units or samples of production. When used in the context of new or long run processes, this charting approach eliminates the need for a separate calibration study. Simulation results indicate that the techniques presented for use with subgroup data have desirable performance whether or not the process parameters are assumed known in advance of the production run. As for individual values control techniques, those which do not assume known values of the process covariance matrix Σ or both the process parameters, are found to be insensitive to sustained mean shift. However, the two alternative EWMA procedures (EWMAZ1 and EWMAZ1U) specifically designed for detecting this type of process change have been demonstrated to be very effective. In addition, they are found to be far superior to some

competing techniques including the M charts of Chan et al.(1994) for 'picking up' linear trends.

In practice, it is also of value to monitor the process dispersion as measured by the variance-covariance matrix Σ which may be subject to occasional changes. The procedures presented for this purpose (which are based on subgroup data) involve use of independent statistics resulting from the decomposition of the covariance matrix. A simulation study indicates that the proposed techniques outperform previously proposed procedures for many sustained shifts in Σ. It has also been demonstrated that the technique presented for the known Σ case is more powerful in 'picking up' certain shifts than that which involves the separate charting of the standardized variances of the principal components or the individual components resulting from the decomposition of the covariance matrices. In addition, the proposed methods have some practical advantages over the existing procedures. Besides providing better control over the false alarm rate and the ease of locating the control limits, the proposed techniques can help identify the nature of change in the process dispersion parameters.

In order to satisfactorily describe the capability of multivariate processes, three multivariate capability indices have been presented. These are based on the relative area and position of a conservative process rectangle containing at least a specified proportion of items, and the specification rectangle. The development of the first involves the projection of a process ellipse containing a specified percentage of products on to its component axes whereas the other two are based on the Bonferroni and Sidak inequalities respectively. Although this work is devoted to two-sided rectangular specifications, it can be extended to unilateral specification situations in a straightforward manner. Some calculations that fairly compare the three reveal that the latter two are

superior to the former and that the Sidak-type capability index is marginally better than that based on the Bonferroni inequality. A reasonable test for the Sidak-type index has also been proposed and critical values provided for some chosen levels of significance, sample sizes and acceptable percentages of nonconforming items. The computation of these indices is easier than other proposed indices and capability analysis methods. However, as with other multivariate capability indices, it has not yet been possible to obtain the unbiased estimators and appropriate confidence intervals for the proposed indices except to note that for large sample sizes, it seems appropriate to replace the parameters involved with the usual sample estimates. A conservative type of distribution-free capability index has also been considered. This is obtained by use of multivariate Chebyshev-type probability inequalities. Although this is no better (more conservative) than the Bonferroni-type capability index, the process rectangle containing at least a specified proportion of items used for defining the index can be constructed easily for any type of process distribution. If the underlying distribution for each quality characteristic is known to belong to some well-known system or family of distributions and hence appropriate quantiles may be obtained, it is advisable to consider the use of the capability index constructed based on the Bonferroni inequality although in some cases, this might not be practical.

A rationale for making a statistical comparison between the techniques of 'pre-control' and traditional $\overline{X}$ *and* R charts has also been provided in this thesis. Special attention was drawn to the application of both techniques to the short run manufacturing environment where, for the use of $\overline{X}$ *and* R charts, parameter estimation is a problem. Despite its many touted practical attributes, the results show that 'pre-control' is not as good as the $\overline{X}$ *and* R charting techniques in many circumstances.

7.2 Suggestions for Future Work

In the following, some suggestions are made for future investigation, building on the work of this thesis.

1. A study of the robustness of the proposed mean and dispersion control procedures to departures from the multivariate normality assumptions using run length distributions. Convenient process models recommended for this purpose include the *Elliptically Contoured* distributions (see Johnson (1987), p.106-124) which have the multivariate normal distributions as special cases.
2. A comparison of the statistical performance of the proposed procedures, including EWMAZ1 and EWMAZ1U, with the corresponding nonparametric techniques of Hawkins (1992) for various distributional models and types of process change. Performance criteria recommended include the run length probability, $\Pr(RL \leq k)$. A comparison could also be made between EWMAZ1 and the MEWMA technique of Lowry et al.(1992).
3. A study of the properties of applying multiple univariate 'Q' charts to the principal components and the individual quality characteristics for the cases with known and unknown Σ respectively, and similar charts constructed based on independent variables that result from the decomposition of the χ^2 and T^2 statistics of (3.3) to (3.11) in a manner similar to that of Mason et al.(1995).
4. Analysis of $\Pr(RL \leq k)$ for the multivariate mean control techniques based on statistics (3.4), (3.5), (3.6), (3.8), (3.9), (3.10) and (3.11) and the proposed dispersion control procedures for sustained shifts in μ and Σ respectively.

5. Develop dispersion control methods based on *individual* observations for the case where Σ is unknown and study its RL performance relative to that of the nonparametric procedure presented by Hawkins (1992). For the case with specified or known Σ, consideration should be made of the use of separate univariate charts based on the variability of the principal components or some aggregate-type statistic like those of Chapter 4 with a comparison of the resulting RL performance with that of Hawkins's method.
6. Develop exact multivariate capability indices which accurately reflect the process status (i.e the expected proportion of usable items produced) and the expected costs incurred. As not all the measured characteristics are equally important in determining the product quality in some situations, indices which take this factor into consideration should also be designed.

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Appendices

A.1 Dependence of Statistical Performance on the Noncentrality Parameter, λ for a Step Shift in Process Mean Vector

Assume that the process variance-covariance matrix, Σ is *constant* but that the process mean vector may change from μ to μ_{new} at an arbitrary point in time. It is shown below that for the same *change point* i.e the observation or subgroup number after which the shift occurs, the joint distribution of the T_k's (or equivalently, Z_k's) and hence the *statistical performance* of the control techniques presented in section 3.3 of Chapter 3 depends on μ, μ_{new} and Σ only through the value of the *noncentrality parameter*,

$$\lambda = \sqrt{(\mu_{new} - \mu)^T \Sigma^{-1} (\mu_{new} - \mu)}$$

To establish the proof for the above, use is made of the following lemmas and theorems which are adapted from Crosier (1988) and Lowry (1989).

Lemma 1

If $\mathbf{X}_k^* = \mathbf{M}\mathbf{X}_k$, $\mathbf{X}_k^{**} = \mathbf{M}(\mathbf{X}_k - \mu)$, $\mathbf{X}_{kj}^* = \mathbf{M}\mathbf{X}_{kj}$ and $\mathbf{X}_{kj}^{**} = \mathbf{M}\left(\mathbf{X}_{kj} - \mu\right)$, $k = 1,2,\ldots,$ $j = 1,2,\ldots,n$ where $\mathbf{M}$ is a $p \times p$ matrix of full rank, then the relevent T_k statistics have the same values whether they are computed from $\mathbf{X}_k$, $(\mathbf{X}_k - \mu)$, $\mathbf{X}_{kj}$ and $(\mathbf{X}_{kj} - \mu)$ or the corresponding transformed vectors $\mathbf{X}_k^*$, $\mathbf{X}_k^{**}$, $\mathbf{X}_{kj}^*$ and $\mathbf{X}_{kj}^{**}$, i.e, the T_k statistics are *invariant* with respect to these transformations. In other words, a full rank linear transformation of the observation vectors or their deviations from target (known mean vector) has no effect on the T_k statistics.

Proof : Immediate.

Lemma 2

If $\mathbf{X}_k^* = \mathbf{M}\mathbf{X}_k$, $\mathbf{X}_k^{**} = \mathbf{M}(\mathbf{X}_k - \mu)$, $\mathbf{X}_{kj}^* = \mathbf{M}\mathbf{X}_{kj}$ and $\mathbf{X}_{kj}^{**} = \mathbf{M}\left(\mathbf{X}_{kj} - \mu\right)$, $k = 1,2,\ldots,$

$j = 1,2,\ldots,n$ where $\mathbf{M}$ is a $p \times p$ matrix of full rank, then

$$\mu^* = E\left(\mathbf{X}_k^*\right) = E\left(\mathbf{X}_{kj}^*\right) = \begin{cases} \mu_{old}^* = \mathbf{M}\mu & , k \le r \\ \mu_{new}^* = \mathbf{M}\mu_{new} & , k > r \end{cases}$$

$$\mu^{**} = E\left(\mathbf{X}_k^{**}\right) = E\left(\mathbf{X}_{kj}^{**}\right) = \begin{cases} \mu_{old}^{**} = \mathbf{0} & , k \le r \\ \mu_{new}^{**} = \mathbf{M}\left(\mu_{new} - \mu\right) & , k > r \end{cases}$$

$$\Sigma^* = \Sigma_{\mathbf{M}\mathbf{X}_k} = \Sigma_{\mathbf{M}\mathbf{X}_{kj}} = \Sigma^{**} = \Sigma_{\mathbf{M}(\mathbf{X}_k - \mu)} = \Sigma_{\mathbf{M}(\mathbf{X}_{kj} - \mu)} = \mathbf{M}\Sigma\mathbf{M}^{\mathrm{T}}$$

where r is the observation or subgroup number after which the process mean vector changes from μ to μ_{new}. Thus,

$$\left(\mu_{new}^* - \mu_{old}^*\right)^{\mathrm{T}} \Sigma^{*-1} \left(\mu_{new}^* - \mu_{old}^*\right) = \left(\mu_{new} - \mu\right)^{\mathrm{T}} \Sigma^{-1} \left(\mu_{new} - \mu\right)$$

$$\left(\mu_{new}^{**} - \mu_{old}^{**}\right)^{\mathrm{T}} \Sigma^{**-1} \left(\mu_{new}^{**} - \mu_{old}^{**}\right) = \left(\mu_{new} - \mu\right)^{\mathrm{T}} \Sigma^{-1} \left(\mu_{new} - \mu\right)$$

Proof : Immediate.

This result implies that the noncentrality parameter has the same value whether computed from the original dependent variables or from some linearly independent combinations of them (or their deviations from targets or known means).

Lemma 3

If $(\mu_{1new}-\mu_1)^T\Sigma^{-1}(\mu_{1new}-\mu_1)=(\mu_{2new}-\mu_2)^T\Sigma^{-1}(\mu_{2new}-\mu_2)$, there exists a nonsingular matrix $\mathbf{M}$ such that

$$(\mu_{1new}-\mu_1)=\mathbf{M}(\mu_{2new}-\mu_2)$$

$$\mathbf{M}\Sigma\mathbf{M}^T=\Sigma$$

Proof :

First, transform each variable of the form $(\mathbf{X}_{After}-\mathbf{X}_{Before})$ to $\mathbf{Y}$, principal components scaled to have unit variances where $\mathbf{X}_{Before}$ and $\mathbf{X}_{After}$ respectively denote observation vectors *before* and *after* the change in the process mean vector. Let $E(\mathbf{Y})=\mathbf{V}$, by lemma 2,

$$\mathbf{V}_1^T\mathbf{V}_1=\mathbf{V}_2^T\mathbf{V}_2 \quad \text{where } \mathbf{V}_1=\mathbf{D}^{-\frac{1}{2}}\mathbf{P}(\mu_{1new}-\mu_1)$$
$$\mathbf{V}_2=\mathbf{D}^{-\frac{1}{2}}\mathbf{P}(\mu_{2new}-\mu_2)$$

and $\mathbf{P}$ is an orthogonal matrix that diagonalizes $\Sigma_{(\mathbf{X}_{After}-\mathbf{X}_{Before})}$ giving

$$\mathbf{P}\Sigma_{(\mathbf{X}_{After}-\mathbf{X}_{Before})}\mathbf{P}^T=\mathbf{D}.$$

Hence, there exists an orthogonal matrix $\mathbf{Q}$ such that

$$\mathbf{V}_1=\mathbf{Q}\mathbf{V}_2.$$

Upon substituting $\mathbf{V}_1=\mathbf{D}^{-\frac{1}{2}}\mathbf{P}(\mu_{1new}-\mu_1)$ and $\mathbf{V}_2=\mathbf{D}^{-\frac{1}{2}}\mathbf{P}(\mu_{2new}-\mu_2)$ into the equation above, $\mathbf{M}$ is obtainable as follows :

$$\mathbf{D}^{-\frac{1}{2}}\mathbf{P}\left(\mu_{1new}-\mu_1\right)=\mathbf{Q}\mathbf{D}^{-\frac{1}{2}}\mathbf{P}\left(\mu_{2new}-\mu_2\right)$$

$$\left(\mu_{1new}-\mu_1\right)=\mathbf{P}^{-1}\mathbf{D}^{\frac{1}{2}}\mathbf{Q}\mathbf{D}^{-\frac{1}{2}}\mathbf{P}\left(\mu_{2new}-\mu_2\right)$$

$$=\mathbf{P}^{\mathrm{T}}\mathbf{D}^{\frac{1}{2}}\mathbf{Q}\mathbf{D}^{-\frac{1}{2}}\mathbf{P}\left(\mu_{2new}-\mu_2\right)$$

$$\therefore \quad \mathbf{M}=\mathbf{P}^{\mathrm{T}}\mathbf{D}^{\frac{1}{2}}\mathbf{Q}\mathbf{D}^{-\frac{1}{2}}\mathbf{P}.$$

Now, it is shown that $\mathbf{M}\Sigma\mathbf{M}^{\mathrm{T}}=\Sigma$. We have

$$\mathbf{M}\Sigma_{(\mathrm{X}_{After}-\mathrm{X}_{Before})}\mathbf{M}^{\mathrm{T}}=\mathbf{P}^{\mathrm{T}}\mathbf{D}^{\frac{1}{2}}\mathbf{Q}\mathbf{D}^{-\frac{1}{2}}\mathbf{P}\Sigma_{(\mathrm{X}_{After}-\mathrm{X}_{Before})}\mathbf{P}^{\mathrm{T}}\mathbf{D}^{-\frac{1}{2}}\mathbf{Q}^{\mathrm{T}}\mathbf{D}^{\frac{1}{2}}\mathbf{P}$$

$$=\mathbf{P}^{\mathrm{T}}\mathbf{D}^{\frac{1}{2}}\mathbf{Q}\mathbf{D}^{-\frac{1}{2}}\mathbf{D}\mathbf{D}^{-\frac{1}{2}}\mathbf{Q}^{\mathrm{T}}\mathbf{D}^{\frac{1}{2}}\mathbf{P}$$

$$=\mathbf{P}^{\mathrm{T}}\mathbf{D}\mathbf{P}$$

$$=\Sigma_{(\mathrm{X}_{After}-\mathrm{X}_{Before})}$$

$$\Rightarrow \quad \mathbf{M}(2\Sigma)\mathbf{M}^{\mathrm{T}}=2\Sigma$$

$$\Rightarrow \quad \mathbf{M}\Sigma\mathbf{M}^{\mathrm{T}}=\Sigma\ .$$

(Q.E.D.)

This lemma is also applicable to cases with known μ, namely, if $\left(\mu_{1new}-\mu\right)^{\mathrm{T}}\Sigma^{-1}\left(\mu_{1new}-\mu\right)=\left(\mu_{2new}-\mu\right)^{\mathrm{T}}\Sigma^{-1}\left(\mu_{2new}-\mu\right)$, there exists a nonsingular matrix $\mathbf{M}$ such that

$$\left(\mu_{1new}-\mu\right)=\mathbf{M}\left(\mu_{2new}-\mu\right)$$

$$\mathbf{M}\Sigma\mathbf{M}^{\mathrm{T}}=\Sigma\ .$$

The proof for this is similar to the above except that μ_1 and μ_2 should both be replaced by μ.

Theorem 1

For the cases with unknown μ, if

$$(\mu_{1new}-\mu_1)^T\Sigma^{-1}(\mu_{1new}-\mu_1)=(\mu_{2new}-\mu_2)^T\Sigma^{-1}(\mu_{2new}-\mu_2),$$

then $f\left(T_k's\middle|E(\mathbf{X}_k \text{ or } \mathbf{X}_{kj})=\begin{cases}\mu_1 & ,k\le r\\ \mu_{1new} & ,k>r\end{cases}\right)=f\left(T_k's\middle|E(\mathbf{X}_k \text{ or } \mathbf{X}_{kj})=\begin{cases}\mu_2 & ,k\le r\\ \mu_{2new} & ,k>r\end{cases}\right)$

where $f\left(T_k's\middle|E(\mathbf{X}_k \text{ or } \mathbf{X}_{kj})=\begin{cases}\mu_1 & ,k\le r\\ \mu_{1new} & ,k>r\end{cases}\right)$ denotes the joint pdf of $T_k's$ given

$$E(\mathbf{X}_k)=E(\mathbf{X}_{kj})=\begin{cases}\mu_1 & ,k\le r\\ \mu_{1new} & ,k>r\end{cases}$$

and $f\left(T_k's\middle|E(\mathbf{X}_k \text{ or } \mathbf{X}_{kj})=\begin{cases}\mu_2 & ,k\le r\\ \mu_{2new} & ,k>r\end{cases}\right)$ denotes the joint pdf of $T_k's$ given

$$E(\mathbf{X}_k)=E(\mathbf{X}_{kj})=\begin{cases}\mu_2 & ,k\le r\\ \mu_{2new} & ,k>r\end{cases}$$

where r is the change point. This theorem implies equivalent performance of the control techniques under the two alternative probability densities.

Proof :

Note that the $T_k's$ of (3.3), (3.6), (3.8) and (3.11) are expressible either in terms of random vectors of the form $(\mathbf{X}_p-\mathbf{X}_q)$ for $p\neq q$ or $(\mathbf{X}_{pi}-\mathbf{X}_{qj})$ for $p\neq q$, $i,j=1,2,\ldots,n$ and $p=q$, $i\neq j$, $i,j=1,2,\ldots,n$. Let *pdf* 1 refer to the multivariate normal density specified by

$$E(\mathbf{X}_k)=E(\mathbf{X}_{kj})=\begin{cases}\mu_1 & ,k\le r\\ \mu_{1new} & ,k>r\end{cases}$$

and *pdf* 2 refer to the multivariate normal density specified by

$$E(\mathbf{X}_k) = E(\mathbf{X}_{kj}) = \begin{cases} \mu_2 & ,k \le r \\ \mu_{2new} & ,k > r \end{cases}.$$

If *pdf* 2 is expressed in the transformed variates $\mathbf{Z}_k = \mathbf{M}\mathbf{X}_k$ $\left(\text{or } \mathbf{Z}_{kj} = \mathbf{M}\mathbf{X}_{kj}\right)$ where $\mathbf{M} = \mathbf{P}^{\mathrm{T}}\mathbf{D}^{\frac{1}{2}}\mathbf{Q}\mathbf{D}^{-\frac{1}{2}}\mathbf{P}$ is as given in lemma 3, then $Var(\mathbf{Z}_k) = Var(\mathbf{Z}_{kj}) = \mathbf{M}\Sigma\mathbf{M}^{\mathrm{T}} = \Sigma$ and

$$E(\mathbf{Z}_p - \mathbf{Z}_q) = E(\mathbf{Z}_{pi} - \mathbf{Z}_{qj})$$
$$= \begin{cases} \mathbf{0} & \text{if } p,q \le r \text{ or } p,q > r \\ \mu_{1new} - \mu_1 & \text{if } p > r,\ q \le r \\ \mu_1 - \mu_{1new} & \text{if } p \le r,\ q > r \end{cases}$$

It can also easily be shown that the covariance of the transformed variates $(\mathbf{Z}_p - \mathbf{Z}_q)$ and $(\mathbf{Z}_s - \mathbf{Z}_t)$ under *pdf* 2 is the same as the covariance of $(\mathbf{X}_p - \mathbf{X}_q)$ $\left[(\mathbf{X}_{pi} - \mathbf{X}_{qj})\right]$and $(\mathbf{X}_s - \mathbf{X}_t)$ $\left[(\mathbf{X}_{su} - \mathbf{X}_{tv})\right]$ under *pdf* 1, i.e

$$\underset{\text{under } pdf\,2}{Cov\left[(\mathbf{Z}_p - \mathbf{Z}_q),(\mathbf{Z}_s - \mathbf{Z}_t)\right]} = \underset{\text{under } pdf\,1}{Cov\left[(\mathbf{X}_p - \mathbf{X}_q),(\mathbf{X}_s - \mathbf{X}_t)\right]}$$

$$\left\{\underset{\text{under } pdf\,2}{Cov\left[(\mathbf{Z}_{pi} - \mathbf{Z}_{qj}),(\mathbf{Z}_{su} - \mathbf{Z}_{tv})\right]} = \underset{\text{under } pdf\,1}{Cov\left[(\mathbf{X}_{pi} - \mathbf{X}_{qj}),(\mathbf{X}_{su} - \mathbf{X}_{tv})\right]}\right\}$$

for all p, q, s and t (p, q, s, t, i, j, u and v). Hence, *pdf* 2 expressed in the transformed variates $\mathbf{Z}_k$ ($\mathbf{Z}_{kj}$) is the same as *pdf* 1 expressed in $\mathbf{X}_k$ ($\mathbf{X}_{kj}$) variates, i.e

$$f\left(T_k's \middle| E(\mathbf{X}_k \text{ or } \mathbf{X}_{kj}) = \begin{cases} \mu_1 & ,k \le r \\ \mu_{1new} & ,k > r \end{cases}\right) = f\left(T_k's \middle| E(\mathbf{Z}_k \text{ or } \mathbf{Z}_{kj}) = \begin{cases} \mathbf{M}\mu_2 & ,k \le r \\ \mathbf{M}\mu_{2new} & ,k > r \end{cases}\right) \quad \text{(A.1.1)}$$

By lemma 1, the values of the $T_k's$ are invariant with respect to the transformation from $\mathbf{X}_k$ ($\mathbf{X}_{kj}$) to $\mathbf{Z}_k$ ($\mathbf{Z}_{kj}$) so that

$$f\left(T_k's \middle| E(\mathbf{X}_k \text{ or } \mathbf{X}_{kj}) = \begin{cases} \mu_2 & ,k \le r \\ \mu_{2new} & ,k > r \end{cases}\right) = f\left(T_k's \middle| E(\mathbf{Z}_k \text{ or } \mathbf{Z}_{kj}) = \begin{cases} \mathbf{M}\mu_2 & ,k \le r \\ \mathbf{M}\mu_{2new} & ,k > r \end{cases}\right) \quad \text{(A.1.2)}$$

Combining (A.1.1) and (A.1.2) gives

$$f\left(T_k's \middle| E\left(\mathbf{X}_k \text{ or } \mathbf{X}_{kj}\right)=\begin{cases}\mu_1 & ,k\le r\\ \mu_{1new} & ,k>r\end{cases}\right)=f\left(T_k's \middle| E\left(\mathbf{X}_k \text{ or } \mathbf{X}_{kj}\right)=\begin{cases}\mu_2 & ,k\le r\\ \mu_{2new} & ,k>r\end{cases}\right)$$

Therefore, the joint distribution of $T_k's$ given μ_1 and μ_{1new} (and Σ) is the same as the joint distribution of $T_k's$ given μ_2 and μ_{2new} (and Σ) if

$$\left(\mu_{1new}-\mu_1\right)^{\mathrm{T}}\Sigma^{-1}\left(\mu_{1new}-\mu_1\right)=\left(\mu_{2new}-\mu_2\right)^{\mathrm{T}}\Sigma^{-1}\left(\mu_{2new}-\mu_2\right).$$

(Q.E.D.)

Theorem 1 can be adapted to cases with known μ, namely, if $\left(\mu_{1new}-\mu\right)^{\mathrm{T}}\Sigma^{-1}\left(\mu_{1new}-\mu\right)=\left(\mu_{2new}-\mu\right)^{\mathrm{T}}\Sigma^{-1}\left(\mu_{2new}-\mu\right)$, then

$$f\left(T_k's \middle| E\left(\mathbf{X}_k \text{ or } \mathbf{X}_{kj}\right)=\begin{cases}\mu & ,k\le r\\ \mu_{1new} & ,k>r\end{cases}\right)=f\left(T_k's \middle| E\left(\mathbf{X}_k \text{ or } \mathbf{X}_{kj}\right)=\begin{cases}\mu & ,k\le r\\ \mu_{2new} & ,k>r\end{cases}\right)$$

The proof for this is similar to the above except that $\mathbf{X}_k$ $(\mathbf{X}_{kj})$ and $\mu_{inew}-\mu_i$, $i=1,2$ should be replaced by $\mathbf{X}_k-\mu$ $(\mathbf{X}_{kj}-\mu)$ and $\mu_{inew}-\mu$ respectively.

Theorem 2

For the cases with unknown μ, let $\mathbf{X}_{1k}$'s $(\mathbf{X}_{1kj}$'$s)$ be independent observation vectors from *pdf* 1 and $\mathbf{X}_{2k}$'s $(\mathbf{X}_{2kj}$'$s)$ be independent observation vectors from *pdf* 2. Let *pdf* 1 be multivariate normal with mean vector μ_1 and μ_{1new} *before* and *after* the change and variance-covariance matrix Σ_1. Let *pdf* 2 be multivariate normal with mean vector μ_2 and μ_{2new} *before* and *after* the change and variance-covariance matrix Σ_2. Denote the *change point* by *r*. If $(\mu_{1new}-\mu_1)^T\Sigma_1^{-1}(\mu_{1new}-\mu_1)=(\mu_{2new}-\mu_2)^T\Sigma_2^{-1}(\mu_{2new}-\mu_2)$, then $f_1(T_k$'$s)=f_2(T_k$'$s)$ where $f_1(T_k$'$s)$ and $f_2(T_k$'$s)$ represent the joint distribution of T_k's given *pdf* 1 and *pdf* 2 respectively.

Proof

Let $\mathbf{X}_{1k}^*=\mathbf{D}_1^{-\frac{1}{2}}\mathbf{P}_1\mathbf{X}_{1k}\left(\mathbf{X}_{1kj}^*=\mathbf{D}_1^{-\frac{1}{2}}\mathbf{P}_1\mathbf{X}_{1kj}\right)$ and $\mathbf{X}_{2k}^*=\mathbf{D}_2^{-\frac{1}{2}}\mathbf{P}_2\mathbf{X}_{2k}\left(\mathbf{X}_{2kj}^*=\mathbf{D}_2^{-\frac{1}{2}}\mathbf{P}_2\mathbf{X}_{2kj}\right)$

where $\mathbf{P}_1$ and $\mathbf{P}_2$ are matrices that diagonalize $\Sigma_{(\mathbf{X}_{1p}-\mathbf{X}_{1q})}$ (or $\Sigma_{(\mathbf{X}_{1pi}-\mathbf{X}_{1qj})}$) and $\Sigma_{(\mathbf{X}_{2p}-\mathbf{X}_{2q})}$ (or $\Sigma_{(\mathbf{X}_{2pi}-\mathbf{X}_{2qj})}$), $p\neq q$ respectively, i.e

$$\mathbf{P}_1\Sigma_{(\mathbf{X}_{1p}-\mathbf{X}_{1q})}\mathbf{P}_1^T=\mathbf{D}_1$$

$$\mathbf{P}_2\Sigma_{(\mathbf{X}_{2p}-\mathbf{X}_{2q})}\mathbf{P}_2^T=\mathbf{D}_2.$$

Then,

$$E\left(\mathbf{X}_{1p}^*-\mathbf{X}_{1q}^*\right)=\mu_{1p}^*-\mu_{1q}^*=\begin{cases}\mathbf{D}_1^{-\frac{1}{2}}\mathbf{P}_1(\mu_{1new}-\mu_1) & ,p>r,\ q\leq r\\ \mathbf{D}_1^{-\frac{1}{2}}\mathbf{P}_1(\mu_1-\mu_{1new}) & ,p\leq r,\ q>r\\ \mathbf{0} & \textit{elsewhere}\end{cases}$$

$$E\left(\mathbf{X}_{2p}^{*}-\mathbf{X}_{2q}^{*}\right)=\mu_{2p}^{*}-\mu_{2q}^{*}=\begin{cases}\mathbf{D}_{2}^{-\frac{1}{2}}\mathbf{P}_{2}\left(\mu_{2new}-\mu_{2}\right) & ,p>r,\ q\le r\\ \mathbf{D}_{2}^{-\frac{1}{2}}\mathbf{P}_{2}\left(\mu_{2}-\mu_{2new}\right) & ,p\le r,\ q>r\\ \mathbf{0} & \textit{elsewhere}\end{cases}$$

and $\Sigma_{(\mathbf{X}_{1p}^{*}-\mathbf{X}_{1q}^{*})}=\Sigma_{(\mathbf{X}_{2p}^{*}-\mathbf{X}_{2q}^{*})}=\mathbf{I}$.

By lemma 2,

$$\left(\mu_{1p}^{*}-\mu_{1q}^{*}\right)^{\mathrm{T}}\Sigma_{(\mathbf{X}_{1p}^{*}-\mathbf{X}_{1q}^{*})}^{-1}\left(\mu_{1p}^{*}-\mu_{1q}^{*}\right)=\left(\mu_{1new}-\mu_{1}\right)^{\mathrm{T}}\Sigma_{(\mathbf{X}_{1p}-\mathbf{X}_{1q})}^{-1}\left(\mu_{1new}-\mu_{1}\right)$$

$$\Rightarrow\left(\mu_{1p}^{*}-\mu_{1q}^{*}\right)^{\mathrm{T}}\mathbf{I}^{-1}\left(\mu_{1p}^{*}-\mu_{1q}^{*}\right)=\tfrac{1}{2}\left(\mu_{1new}-\mu_{1}\right)^{\mathrm{T}}\Sigma_{1}^{-1}\left(\mu_{1new}-\mu_{1}\right)$$

and

$$\left(\mu_{2p}^{*}-\mu_{2q}^{*}\right)^{\mathrm{T}}\Sigma_{(\mathbf{X}_{2p}^{*}-\mathbf{X}_{2q}^{*})}^{-1}\left(\mu_{2p}^{*}-\mu_{2q}^{*}\right)=\left(\mu_{2new}-\mu_{2}\right)^{\mathrm{T}}\Sigma_{(\mathbf{X}_{2p}-\mathbf{X}_{2q})}^{-1}\left(\mu_{2new}-\mu_{2}\right)$$

$$\Rightarrow\left(\mu_{2p}^{*}-\mu_{2q}^{*}\right)^{\mathrm{T}}\mathbf{I}^{-1}\left(\mu_{2p}^{*}-\mu_{2q}^{*}\right)=\tfrac{1}{2}\left(\mu_{2new}-\mu_{2}\right)^{\mathrm{T}}\Sigma_{2}^{-1}\left(\mu_{2new}-\mu_{2}\right)$$

for $p>r,\ q\le r$ or $p\le r,\ q>r$. Because the values of T_k's are invariant with respect to the transformations, the condition

$$\left(\mu_{1new}-\mu_{1}\right)^{\mathrm{T}}\Sigma_{1}^{-1}\left(\mu_{1new}-\mu_{1}\right)=\left(\mu_{2new}-\mu_{2}\right)^{\mathrm{T}}\Sigma_{2}^{-1}\left(\mu_{2new}-\mu_{2}\right)$$

is equivalent to

$$\left(\mu_{1p}^{*}-\mu_{1q}^{*}\right)^{\mathrm{T}}\mathbf{I}^{-1}\left(\mu_{1p}^{*}-\mu_{1q}^{*}\right)=\left(\mu_{2p}^{*}-\mu_{2q}^{*}\right)^{\mathrm{T}}\mathbf{I}^{-1}\left(\mu_{2p}^{*}-\mu_{2q}^{*}\right)$$

for $p>r,\ q\le r$ or $p\le r,\ q>r$. Since $\left(\mathbf{X}_{1p}^{*}-\mathbf{X}_{1q}^{*}\right)$ and $\left(\mathbf{X}_{2p}^{*}-\mathbf{X}_{2q}^{*}\right)$ are principal components of $\left(\mathbf{X}_{1p}-\mathbf{X}_{1q}\right)$ and $\left(\mathbf{X}_{2p}-\mathbf{X}_{2q}\right)$ scaled to have the same variance-covariance matrix, the identity matrix $\mathbf{I}$, by theorem 1, the joint distribution of T_k's given *pdf* 1 is the same as the joint distribution of T_k's given *pdf* 2.

(Q.E.D.)

This theorem can also be adapted to the cases where μ is known, namely, if

$$(\mu_{1new} - \mu)^T \Sigma_1^{-1} (\mu_{1new} - \mu) = (\mu_{2new} - \mu)^T \Sigma_2^{-1} (\mu_{2new} - \mu),$$

then

$$f\left(T_k's \middle| E(\mathbf{X}_k \text{ or } \mathbf{X}_{kj}) = \begin{cases} \mu & ,k \le r \\ \mu_{1new} & ,k > r \end{cases}, \Sigma_1 \right) = f\left(T_k's \middle| E(\mathbf{X}_k \text{ or } \mathbf{X}_{kj}) = \begin{cases} \mu & ,k \le r \\ \mu_{2new} & ,k > r \end{cases}, \Sigma_2 \right)$$

The proof for this is along the same line of argument as above except that $\mathbf{X}_{ik}$ $(\mathbf{X}_{ikj})$ and $\mu_{inew} - \mu_i$ should be replaced by $\mathbf{X}_{ik} - \mu$ $(\mathbf{X}_{ikj} - \mu)$ and $\mu_{inew} - \mu$ respectively for $i = 1, 2$.

A.2 Dependence of Statistical Performance on $\lambda_1,\ldots,\lambda_p$ for a Change in Process Covariance Matrix

Suppose that the process under consideration changes in covariance matrix from Σ to Σ_{new} after the rth observation, whilst the mean vector μ remains constant. It is shown below that the joint distribution of the $T_k's$ (or equivalently, $Z_k's$) for each of the control techniques (3.3) to (3.11) depends on μ, Σ and Σ_{new} through the eigenvalues, $\lambda_1,\ldots,\lambda_p$, of $\Sigma^{-1}\Sigma_{new}$ or equivalently, of $\Sigma_{new}\Sigma^{-1}$.

Lemma 4

If $\Sigma_{1new}^{\frac{1}{2}}\Sigma_1^{-1}\Sigma_{1new}^{\frac{1}{2}} = \Sigma_{2new}^{\frac{1}{2}}\Sigma_2^{-1}\Sigma_{2new}^{\frac{1}{2}}$, then there exists a full rank matrix $\mathbf{D}$ such that $\mathbf{D}\Sigma_2\mathbf{D}^{\mathrm{T}} = \Sigma_1$ and $\mathbf{D}\Sigma_{2new}\mathbf{D}^{\mathrm{T}} = \Sigma_{1new}$.

Proof

From $\Sigma_{1new}^{\frac{1}{2}}\Sigma_1^{-1}\Sigma_{1new}^{\frac{1}{2}} = \Sigma_{2new}^{\frac{1}{2}}\Sigma_2^{-1}\Sigma_{2new}^{\frac{1}{2}}$, we have $\Sigma_1 = \left(\Sigma_{1new}^{\frac{1}{2}}\Sigma_{2new}^{-\frac{1}{2}}\right)\Sigma_2\left(\Sigma_{1new}^{\frac{1}{2}}\Sigma_{2new}^{-\frac{1}{2}}\right)^{\mathrm{T}}$.

Since $\mathbf{D}\Sigma_2\mathbf{D}^{\mathrm{T}} = \Sigma_1$, $\mathbf{D} = \Sigma_{1new}^{\frac{1}{2}}\Sigma_{2new}^{-\frac{1}{2}}$. It can easily be verified that $\mathbf{D} = \Sigma_{1new}^{\frac{1}{2}}\Sigma_{2new}^{-\frac{1}{2}}$ also satisfies the equation $\mathbf{D}\Sigma_{2new}\mathbf{D}^{\mathrm{T}} = \Sigma_{1new}$. Furthermore, it can readily be seen that $\mathbf{D} = \Sigma_{1new}^{\frac{1}{2}}\Sigma_{2new}^{-\frac{1}{2}}$ is of full rank.

(Q.E.D.)

Theorem 3

Let *pdf*1 be the probability density function of the independent multivariate normal observation vectors $\mathbf{X}_k$'s (or $\mathbf{X}_{kj}$'s) with mean vector μ_1, covariance matrix Σ_1 and Σ_{1new} *before* and *after* the change. Let *pdf*2 be the probability density function of the independent multivariate normal observation vectors $\mathbf{X}_k$'s (or $\mathbf{X}_{kj}$'s) with mean vector μ_2, covariance matrix Σ_2 and Σ_{2new} *before* and *after* the change. If $\Sigma_{1new}^{\frac{1}{2}} \Sigma_1^{-1} \Sigma_{1new}^{\frac{1}{2}} = \Sigma_{2new}^{\frac{1}{2}} \Sigma_2^{-1} \Sigma_{2new}^{\frac{1}{2}}$, then $f_1(T_k's) = f_2(T_k's)$ where $f_1(T_k's)$ and $f_2(T_k's)$ represent the joint distribution of T_k's given *pdf*1 and *pdf*2 respectively.

Proof

The proof here is similar to that of theorem 1 except that $\mathbf{M}$ should be replaced by $\mathbf{D} = \Sigma_{1new}^{\frac{1}{2}} \Sigma_{2new}^{-\frac{1}{2}}$ as given in lemma 4.

Theorem 4

If $\mathbf{X}_k$'s (or $\mathbf{X}_{kj}$'s) are independent random vectors from a multivariate normal distribution with constant mean μ and a step shift in the covariance matrix from Σ to Σ_{new}, after the rth observation (or subgroup), then the joint distribution of T_k's depend on the eigenvalues of $\Sigma^{-1}\Sigma_{new}$ or equivalently, of $\Sigma_{new}\Sigma^{-1}$.

Proof

Theorem 3 implies that the joint distribution of T_k's depend on $\Sigma_{new}^{\frac{1}{2}} \Sigma^{-1} \Sigma_{new}^{\frac{1}{2}}$. Since T_k's are invariant w.r.t any orthogonal transformation, it is readily seen that T_k's

depend on the eigenvalues of $\Sigma_{new}^{\frac{1}{2}} \Sigma^{-1} \Sigma_{new}^{\frac{1}{2}}$. Using the characteristic equation, these eigenvalues can in turn be shown to be the eigenvalues of $\Sigma^{-1}\Sigma_{new}$ or $\Sigma_{new}\Sigma^{-1}$.

(Q.E.D.)

A.3 Dependence of Statistical Performance on η and Ω for a Change in Mean vector and Covariance Matrix

Suppose that the process under consideration changes in mean vector from μ to μ_{new} and covariance matrix from Σ to Σ_{new} after the rth observation or subgroup. It is shown below that the joint distribution of the $T_k's$ (or equivalently $Z_k's$) for each of the control techniques (3.3) - (3.11) depends on μ, μ_{new}, Σ and Σ_{new} through

$$\eta = \Sigma^{-\frac{1}{2}}(\mu_{new} - \mu)$$

, which can be interpreted as the vector of noncentrality parameter in the directions of the principal axes, and the symmetric matrix

$$\Omega = \Sigma^{-\frac{1}{2}} \Sigma_{new} \Sigma^{-\frac{1}{2}}.$$

Theorem 5

Let *pdf*1 be the probability density of the independent multivariate normal observation vectors $\mathbf{X}_k's$ (or $\mathbf{X}_{kj}'s$) with parameters (μ_1, Σ_1) and $(\mu_{1new}, \Sigma_{1new})$ *before* and *after* the change. Let *pdf*2 be the probability density of the independent multivariate normal observation vectors $\mathbf{X}_k's$ (or $\mathbf{X}_{kj}'s$) with parameters (μ_2, Σ_2) and $(\mu_{2new}, \Sigma_{2new})$ *before* and *after* the change. If

$$\Sigma_1^{-\frac{1}{2}}(\mu_{1new} - \mu_1) = \Sigma_2^{-\frac{1}{2}}(\mu_{2new} - \mu_2) \quad \text{and} \quad \Sigma_1^{-\frac{1}{2}} \Sigma_{1new} \Sigma_1^{-\frac{1}{2}} = \Sigma_2^{-\frac{1}{2}} \Sigma_{2new} \Sigma_2^{-\frac{1}{2}}$$

then $f_1(T_k's) = f_2(T_k's)$ where $f_1(T_k's)$ and $f_2(T_k's)$ denote the joint distribution of $T_k's$ given *pdf*1 and *pdf*2 respectively. This theorem implies equivalent performance of the control techniques under the two alternative probability densities.

Proof

The proof here is similar to that of theorem 2.

A.4 The Distributional Properties of W^* Depend on the Eigenvalues of $\Sigma_0^{-\frac{1}{2}}\Sigma_1\Sigma_0^{-\frac{1}{2}}$

Note that W^* may be written as

$$W^* = -p(n-1)-(n-1)\ln\left|\Sigma_0^{-\frac{1}{2}}\mathbf{S}\Sigma_0^{-\frac{1}{2}}\right|+(n-1)\mathrm{tr}\left(\Sigma_0^{-\frac{1}{2}}\mathbf{S}\Sigma_0^{-\frac{1}{2}}\right)$$

where $(n-1)\Sigma_0^{-\frac{1}{2}}\mathbf{S}\Sigma_0^{-\frac{1}{2}} \sim W_p\left(n-1,\ \Sigma_0^{-\frac{1}{2}}\Sigma_1\Sigma_0^{-\frac{1}{2}}\right)$ when $\Sigma=\Sigma_1$.

Since $\mathbf{A}=\Sigma_0^{-\frac{1}{2}}\Sigma_1\Sigma_0^{-\frac{1}{2}}$ is positive definite, $\exists$ an orthogonal matrix Γ $(\Gamma\Gamma^{\mathrm{T}}=\Gamma^{\mathrm{T}}\Gamma=\mathrm{I})$ s.t.

$$\Gamma\mathbf{A}\Gamma^{\mathrm{T}}=\Lambda=diag\left(\lambda_1,\ldots,\lambda_p\right)=\begin{pmatrix}\lambda_1 & 0 & \cdots & 0\\ 0 & \lambda_2 & \ddots & \vdots\\ \vdots & \ddots & \ddots & 0\\ 0 & \cdots & 0 & \lambda_p\end{pmatrix}$$

where $\lambda_1,\ldots,\lambda_p$ are the eigenvalues of $\mathbf{A}$. Due to invariance w.r.t. any orthonormal transformation (*upon* $\Sigma_0^{-\frac{1}{2}}\mathbf{X}$),

$$\left|\Sigma_0^{-\frac{1}{2}}\mathbf{S}\Sigma_0^{-\frac{1}{2}}\right| \sim \frac{1}{(n-1)^p}\prod_{i=1}^{p}\lambda_i V_i$$

where $V_i's$ are independently distributed as χ^2_{n-i} variables (see Theorem 3.3.8, p.82, Srivastava et al.(1979)). Similarly, $tr\left(\Sigma_0^{-\frac{1}{2}}\mathbf{S}\Sigma_0^{-\frac{1}{2}}\right)$ is distributed as

$$\frac{1}{(n-1)}\sum_{i=1}^{p}\lambda_i U_i$$

where $U_i's$ are i.i.d χ^2_{n-1} variables. Combining these, the distribution of W^* therefore depends on the *combination* of $\lambda_1,\ldots,\lambda_p$. In addition, it is readily shown that the

eigenvalues of $\Sigma_0^{-\frac{1}{2}}\Sigma_1\Sigma_0^{-\frac{1}{2}}$ are the same as that of $\Sigma_0^{-1}\Sigma_1$ or $\Sigma_1\Sigma_0^{-1}$ by using the characteristic equation. Note also that the result is not affected by the rotation of the coordinate axes.

A.5 The Statistical Performance of $|\mathbf{S}|^{1/2}$ Chart Depends on the Product of the Eigenvalues of $\Sigma_0^{-\frac{1}{2}}\Sigma_1\Sigma_0^{-\frac{1}{2}}$

Note that the use of $|\mathbf{S}|^{1/2}$ charts with control limits of the form $k_1|\Sigma_0|$ and $k_2|\Sigma_0|$ is equivalent to charting and comparing the statistic $\left|\Sigma_0^{-\frac{1}{2}}\mathbf{S}\Sigma_0^{-\frac{1}{2}}\right|$ with the constants k_1 and k_2. By using similar arguments to those in A.4, it is readily shown that the statistical performance of this control technique depends on the product of the eigenvalues of $\Sigma_0^{-\frac{1}{2}}\Sigma_1\Sigma_0^{-\frac{1}{2}}$, i.e. $\lambda_1\lambda_2\ldots\lambda_p$.

A.6 The Distributional Properties of The Sum of Standardized Variances of The Principal Components (SSVPC) Depend on The Eigenvalues of $\Sigma_0^{-\frac{1}{2}}\Sigma_1\Sigma_0^{-\frac{1}{2}}$

The sum of the standardized variances (multiplied by $n - 1$) of the principal components is given by

$$tr\left[(n-1)\left(\Lambda_0^{-\frac{1}{2}}\Gamma_0\right)\mathbf{S}\left(\Lambda_0^{-\frac{1}{2}}\Gamma_0\right)^{\mathrm{T}}\right]$$
$$= tr\left[(n-1)\Lambda_0^{-\frac{1}{2}}\Gamma_0\mathbf{S}\Gamma_0^{\mathrm{T}}\Lambda_0^{-\frac{1}{2}}\right].$$

where Λ_0 and Γ_0^{T} denote respectively the diagonal matrix of the eigenvalues of Σ_0 and the matrix with the corresponding normalized eigenvectors. This statistic is invariant

w.r.t. *any* $p \times p$ orthonormal transformation (*upon* $\Lambda_0^{-\frac{1}{2}}\Gamma_0\mathbf{X}$). In particular, it is expressible as

$$
\begin{aligned}
& tr\left[(n-1)\Gamma_0^{\mathrm{T}}\left(\Lambda_0^{-\frac{1}{2}}\Gamma_0\mathbf{S}\Gamma_0^{\mathrm{T}}\Lambda_0^{-\frac{1}{2}}\right)\Gamma_0\right] \\
&= tr\left[(n-1)\left(\Gamma_0^{\mathrm{T}}\Lambda_0^{-\frac{1}{2}}\Gamma_0\right)\mathbf{S}\left(\Gamma_0^{\mathrm{T}}\Lambda_0^{-\frac{1}{2}}\Gamma_0\right)^{\mathrm{T}}\right] \\
&= tr\left[(n-1)\Sigma_0^{-\frac{1}{2}}\mathbf{S}\left(\Sigma_0^{-\frac{1}{2}}\right)^{\mathrm{T}}\right] && \text{(see Johnson et al.(1988), p.51)} \\
&= tr\left[(n-1)\Sigma_0^{-\frac{1}{2}}\mathbf{S}\Sigma_0^{-\frac{1}{2}}\right] && \because\ \Sigma_0^{-\frac{1}{2}} \text{ is symmetric} \\
&= (n-1)tr\left[\Sigma_0^{-\frac{1}{2}}\mathbf{S}\Sigma_0^{-\frac{1}{2}}\right].
\end{aligned}
$$

Note that this implies that the value of the plot statistic remains the same whether it is computed from the principal components or any other set of linearly independent combinations of the individual variables obtained in the above manner.

Now, let Γ be an orthogonal matrix such that

$$\Gamma\left(\Sigma_0^{-\frac{1}{2}}\Sigma_1\Sigma_0^{-\frac{1}{2}}\right)\Gamma^{\mathrm{T}} = \Lambda = diag(\lambda_1,\ldots,\lambda_p)$$

where $\lambda_1,\ldots,\lambda_p$ represent the eigenvalues of $\Sigma_0^{-\frac{1}{2}}\Sigma_1\Sigma_0^{-\frac{1}{2}}$. Since

$$
\begin{aligned}
& (n-1)tr\left[\Sigma_0^{-\frac{1}{2}}\mathbf{S}\Sigma_0^{-\frac{1}{2}}\right] \\
&= (n-1)tr\left[\Gamma\left(\Sigma_0^{-\frac{1}{2}}\mathbf{S}\Sigma_0^{-\frac{1}{2}}\right)\Gamma^{\mathrm{T}}\right]
\end{aligned}
$$

where $\Gamma\left(\Sigma_0^{-\frac{1}{2}}\mathbf{S}\Sigma_0^{-\frac{1}{2}}\right)\Gamma^{\mathrm{T}} \sim W_p(n-1,\Lambda)$ when $\Sigma = \Sigma_1$, it is readily seen that the sum of the standardized variances of the principal components is distributed as a linear combination of p independent chi-square variables with n-1 degrees of freedom each and with coefficients $\lambda_1,\ldots,\lambda_p$. Thus, the statistical performance of this technique depends on $\lambda_1,\ldots,\lambda_p$.

A.7 Statistical Performance of MLRTECM Depends on the Eigenvalues of $\Sigma_0^{-1/2}\Sigma_1\Sigma_0^{-1/2}$

The MLRTECM criterion is $W = -2\ln\lambda$ where

$$\lambda = \frac{\prod_{i=1}^{q}\left|\mathbf{S}_{(i)}\right|^{\frac{1}{2}(n_i-1)}}{\left|\mathbf{S}_{pooled}\right|^{\frac{1}{2}(n_1+\ldots+n_q-q)}}$$

$$= C\frac{\prod_{i=1}^{q}\left|(n_i-1)\mathbf{S}_{(i)}\right|^{\frac{1}{2}(n_i-1)}}{\left|\sum_{i=1}^{q}(n_i-1)\mathbf{S}_{(i)}\right|^{\frac{1}{2}(n_1+\ldots+n_q-q)}}$$

and C is a constant depending on the n_i's. Due to invariance, λ may be characterized by

$$\lambda = \frac{C\prod_{j=1}^{q}\left|\mathbf{V}_j\right|^{\frac{1}{2}(n_j-1)}}{\left|\sum_{j=1}^{q}\mathbf{V}_j\right|^{\frac{1}{2}(n_1+\ldots+n_q-q)}},$$

where

$$\mathbf{V}_j \sim W_p\left(n_j-1,\ \mathbf{I}\right), \qquad j=1,\ldots,r$$
$$\sim W_p\left(n_j-1,\ \Lambda\right), \qquad j=r+1,\ldots,q$$

r is the *change point* and Λ denotes a diagonal matrix containing the eigenvalues $\lambda_1,\ldots,\lambda_p$ of $\Sigma_0^{-1/2}\Sigma_1\Sigma_0^{-1/2}$. Thus, it is readily seen that the statistical performance of MLRTECM depends on these eigenvalues.

A.8 Scale Invariance of Fisher, Tippett and The Proposed Statistic (Known Σ Case)

Let $\mathbf{G} = diag\left(\frac{1}{\sigma_1},\ldots,\frac{1}{\sigma_p}\right)$ be the diagonal matrix with ith element being the reciprocal of the standard deviation of the ith variable. The sample covariance matrix computed from the variables scaled to have unit variances (i.e the sample correlation matrix) is then given by

$$
\begin{aligned}
\mathbf{S}^* &= \mathbf{GSG}^{\mathrm{T}} \\
&= \begin{pmatrix} \frac{1}{\sigma_1} & \underset{\sim}{\mathbf{d}}^{\mathrm{T}} \\ \underset{\sim}{\mathbf{d}} & \mathbf{D} \end{pmatrix} \begin{pmatrix} S_1^2 & \mathbf{S}_{\sim 12}^{\mathrm{T}} \\ \mathbf{S}_{\sim 12} & \mathbf{S}_{22} \end{pmatrix} \begin{pmatrix} \frac{1}{\sigma_1} & \underset{\sim}{\mathbf{d}}^{\mathrm{T}} \\ \underset{\sim}{\mathbf{d}} & \mathbf{D}^{\mathrm{T}} \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{\sigma_1} S_1^2 + \underset{\sim}{\mathbf{d}}^{\mathrm{T}} \mathbf{S}_{\sim 12} & \frac{1}{\sigma_1} \mathbf{S}_{\sim 12}^{\mathrm{T}} + \underset{\sim}{\mathbf{d}}^{\mathrm{T}} \mathbf{S}_{22} \\ \underset{\sim}{\mathbf{d}} S_1^2 + \mathbf{D}\mathbf{S}_{\sim 12} & \underset{\sim}{\mathbf{d}} \mathbf{S}_{\sim 12}^{\mathrm{T}} + \mathbf{D}\mathbf{S}_{22} \end{pmatrix} \begin{pmatrix} \frac{1}{\sigma_1} & \underset{\sim}{\mathbf{d}}^{\mathrm{T}} \\ \underset{\sim}{\mathbf{d}} & \mathbf{D}^{\mathrm{T}} \end{pmatrix} \\
&= \begin{pmatrix} S_1^2/\sigma_1 & \mathbf{S}_{\sim 12}^{\mathrm{T}}/\sigma_1 \\ \mathbf{D}\mathbf{S}_{\sim 12} & \mathbf{D}\mathbf{S}_{22} \end{pmatrix} \begin{pmatrix} \frac{1}{\sigma_1} & \underset{\sim}{\mathbf{d}}^{\mathrm{T}} \\ \underset{\sim}{\mathbf{d}} & \mathbf{D}^{\mathrm{T}} \end{pmatrix} \\
&= \begin{pmatrix} S_1^2/\sigma_1^2 + \mathbf{S}_{\sim 12}^{\mathrm{T}} \underset{\sim}{\mathbf{d}}/\sigma_1 & \left(S_1^2/\sigma_1\right)\underset{\sim}{\mathbf{d}}^{\mathrm{T}} + \mathbf{S}_{\sim 12}^{\mathrm{T}} \mathbf{D}^{\mathrm{T}}/\sigma_1 \\ \mathbf{D}\mathbf{S}_{\sim 12}/\sigma_1 + \mathbf{D}\mathbf{S}_{22}\underset{\sim}{\mathbf{d}} & \mathbf{D}\mathbf{S}_{22}\mathbf{D}^{\mathrm{T}} \end{pmatrix} \\
&= \begin{pmatrix} S_1^2/\sigma_1^2 & \mathbf{S}_{\sim 12}^{\mathrm{T}} \mathbf{D}^{\mathrm{T}}/\sigma_1 \\ \mathbf{D}\mathbf{S}_{\sim 12}/\sigma_1 & \mathbf{D}\mathbf{S}_{22}\mathbf{D}^{\mathrm{T}} \end{pmatrix} \qquad \text{(see p.591, Anderson(1984))}
\end{aligned}
$$

where $\underset{\sim}{\mathbf{d}}$ denotes the zero vector of $(p-1)$ elements, $\mathbf{S}_{22}$ is the sample covariance matrix of the last (p-1) variables, $\mathbf{D} = diag\left(\frac{1}{\sigma_2},\ldots,\frac{1}{\sigma_p}\right)$ and $\mathbf{S}_{\sim 12}^{\mathrm{T}} = \left(R_{12}S_1S_2 \quad R_{13}S_1S_3 \quad \cdots \quad R_{1p}S_1S_p\right)$. Similarly, the population correlation matrix is

$$
\begin{aligned}
\Sigma^* &= \mathbf{G}\Sigma\mathbf{G}^{\mathrm{T}} \\
&= \begin{pmatrix} 1 & \frac{1}{\sigma_1}\underset{\sim}{\Sigma}_{12}^{\mathrm{T}} \mathbf{D}^{\mathrm{T}} \\ \frac{1}{\sigma_1}\mathbf{D}\underset{\sim}{\Sigma}_{12} & \mathbf{D}\Sigma_{22}\mathbf{D}^{\mathrm{T}} \end{pmatrix}
\end{aligned}
$$

where Σ_{22} represents the population covariance matrix of the last (p-1) variables and $\underset{\sim}{\Sigma}_{12}^{T} = \left(\rho_{12}\sigma_1\sigma_2 \quad \rho_{13}\sigma_1\sigma_3 \quad \ldots \quad \rho_{1p}\sigma_1\sigma_p\right)$. Furthermore, the conditional correlation matrix of the last (p-1) variables, given the first one, is :-

$$\begin{aligned}
\Sigma^{*}_{22\bullet 1} &= \mathbf{D}\Sigma_{22}\mathbf{D}^{T} - \left(\frac{1}{\sigma_1}\mathbf{D}\underset{\sim}{\Sigma}_{12}\right)(1)^{-1}\left(\frac{1}{\sigma_1}\underset{\sim}{\Sigma}_{12}^{T}\mathbf{D}^{T}\right) \\
&= \mathbf{D}\Sigma_{22}\mathbf{D}^{T} - \mathbf{D}\underset{\sim}{\Sigma}_{12}\left(\sigma_1^2\right)^{-1}\underset{\sim}{\Sigma}_{12}^{T}\mathbf{D}^{T} \\
&= \mathbf{D}\left(\Sigma_{22} - \underset{\sim}{\Sigma}_{12}\left(\sigma_1^2\right)^{-1}\underset{\sim}{\Sigma}_{12}^{T}\right)\mathbf{D}^{T} \\
&= \mathbf{D}\Sigma_{22\bullet 1}\mathbf{D}^{T},
\end{aligned}$$

where $\Sigma_{22\bullet 1}$ denotes the corresponding quantity calculated from the covariance matrix. Note that the charting variables of Fisher, Tippett and the proposed procedures are functions of, amongst others, the variance ratio of the first variable and the Hotelling chi-square statistic based on the vector of regression coefficients when each of the last (p-1) variables is regressed on the first variable (1st off-diagonal vector divided by 1st diagonal element of the sample covariance matrix). When calculated from $\mathbf{S}^{*}$ and Σ^{*}, these are respectively

$$\frac{S_1^{*2}}{\sigma_1^{*2}} = \frac{S_1^2/\sigma_1^2}{1} = \frac{S_1^2}{\sigma_1^2}$$

and

$$\left(\frac{\mathbf{S}^{*\mathrm{T}}_{\sim 12}}{S_1^{*2}} - \frac{\Sigma^{*\mathrm{T}}_{12}}{\sigma_1^{*2}}\right)\left(\frac{\Sigma^{*}_{22\bullet 1}}{(n-1)S_1^{*2}}\right)^{-1}\left(\frac{\mathbf{S}^{*}_{\sim 12}}{S_1^{*2}} - \frac{\Sigma^{*}_{12}}{\sigma_1^{*2}}\right)$$

$$= \left(\frac{\frac{1}{\sigma_1}\mathbf{S}^{\mathrm{T}}_{\sim 12}\mathbf{D}^{\mathrm{T}}}{S_1^2/\sigma_1^2} - \frac{\frac{1}{\sigma_1}\Sigma^{\mathrm{T}}_{12}\mathbf{D}^{\mathrm{T}}}{1}\right)\left(\frac{\mathbf{D}\Sigma_{22\bullet 1}\mathbf{D}^{\mathrm{T}}}{(n-1)S_1^2/\sigma_1^2}\right)^{-1}\left(\frac{\frac{1}{\sigma_1}\mathbf{D}\mathbf{S}_{\sim 12}}{S_1^2/\sigma_1^2} - \frac{\frac{1}{\sigma_1}\mathbf{D}\Sigma_{12}}{1}\right)$$

$$= \left(\frac{\mathbf{S}^{\mathrm{T}}_{\sim 12}}{S_1^2} - \frac{\Sigma^{\mathrm{T}}_{12}}{\sigma_1^2}\right)\left(\frac{\Sigma_{22\bullet 1}}{(n-1)S_1^2}\right)^{-1}\left(\frac{\mathbf{S}_{\sim 12}}{S_1^2} - \frac{\Sigma_{12}}{\sigma_1^2}\right)$$

i.e their values remain the same whether they are calculated from the original or standardized variables. Note also that the other statistics incorporated into the charting variables are obtained from $\mathbf{S}_{j,\ldots,p\bullet 1,\ldots,j-1}$ and $\Sigma_{j,\ldots,p\bullet 1,\ldots,j-1}$ $(j = 2,\ldots,p)$ in exactly the same manner. Thus, to complete the proof, it is only required to show that $\mathbf{S}^{*}_{j,\ldots,p\bullet 1,\ldots,j-1}$ and $\Sigma^{*}_{j,\ldots,p\bullet 1,\ldots,j-1}$ are of the same form as $\mathbf{S}^{*}$ and Σ^{*}. For this, let

$$\mathbf{G} = \begin{pmatrix} \mathbf{G}_{11} & \mathbf{G}_{12} \\ \mathbf{G}_{21} & \mathbf{G}_{22} \end{pmatrix} \quad \text{and} \quad \mathbf{S} = \begin{pmatrix} \mathbf{S}_{11} & \mathbf{S}_{12} \\ \mathbf{S}_{21} & \mathbf{S}_{22} \end{pmatrix},$$

where $\mathbf{G}_{11}$ and $\mathbf{S}_{11}$ denotes the $(j-1)\times(j-1)$ submatrices. Note that $\mathbf{G}_{12}$ here represents the $(j-1)\times(p-j+1)$ *zero* matrix. Thus,

$$\mathbf{S}^{*} = \mathbf{G}\mathbf{S}\mathbf{G}^{\mathrm{T}} = \begin{pmatrix} \mathbf{G}_{11}\mathbf{S}_{11}\mathbf{G}_{11}^{\mathrm{T}} & \mathbf{G}_{11}\mathbf{S}_{12}\mathbf{G}_{22}^{\mathrm{T}} \\ \mathbf{G}_{22}\mathbf{S}_{21}\mathbf{G}_{11}^{\mathrm{T}} & \mathbf{G}_{22}\mathbf{S}_{22}\mathbf{G}_{22}^{\mathrm{T}} \end{pmatrix}$$

and

$$\begin{aligned}
\mathbf{S}^{*}_{j,\ldots,p\bullet 1,\ldots,j-1} &= \mathbf{G}_{22}\mathbf{S}_{22}\mathbf{G}_{22}^{\mathrm{T}} - \left(\mathbf{G}_{22}\mathbf{S}_{21}\mathbf{G}_{11}^{\mathrm{T}}\right)\left(\mathbf{G}_{11}\mathbf{S}_{11}\mathbf{G}_{11}^{\mathrm{T}}\right)^{-1}\left(\mathbf{G}_{11}\mathbf{S}_{12}\mathbf{G}_{22}^{\mathrm{T}}\right) \\
&= \mathbf{G}_{22}\mathbf{S}_{22}\mathbf{G}_{22}^{\mathrm{T}} - \mathbf{G}_{22}\mathbf{S}_{21}\mathbf{G}_{11}^{\mathrm{T}}\left(\mathbf{G}_{11}^{\mathrm{T}}\right)^{-1}\mathbf{S}_{11}^{-1}\mathbf{G}_{11}^{-1}\mathbf{G}_{11}\mathbf{S}_{12}\mathbf{G}_{22}^{\mathrm{T}} \\
&= \mathbf{G}_{22}\mathbf{S}_{22}\mathbf{G}_{22}^{\mathrm{T}} - \mathbf{G}_{22}\mathbf{S}_{21}\mathbf{S}_{11}^{-1}\mathbf{S}_{12}\mathbf{G}_{22}^{\mathrm{T}} \\
&= \mathbf{G}_{22}\left(\mathbf{S}_{22} - \mathbf{S}_{21}\mathbf{S}_{11}^{-1}\mathbf{S}_{12}\right)\mathbf{G}_{22}^{\mathrm{T}} \\
&= \mathbf{G}_{22}\mathbf{S}_{j,\ldots,p\bullet 1,\ldots,j-1}\mathbf{G}_{22}^{\mathrm{T}}
\end{aligned}$$

which is clearly of the same form as $\mathbf{S}^* = \mathbf{GSG}^T$. Likewise, it can be shown that $\Sigma^*_{j,\ldots,p\bullet 1,\ldots,j-1}$ is of the same form as Σ^*.

A.9 Scale Invariance of Fisher, Tippett and The Proposed Statistic (Unknown Σ Case)

The proof here is similar to those in A.8.

A.10 Statistical Performance of MLRT, SSVPC, MLRTECM and $|\mathbf{S}|^{1/2}$ Charting Technique when Σ Shifts Along The Principal Axes

The eigenvalues of $\Sigma_0^{-\frac{1}{2}}\Sigma_1\Sigma_0^{-\frac{1}{2}}$ are the solutions $(\lambda's)$ to the following characteristic equation :

$$\left|\Sigma_0^{-\frac{1}{2}}\Sigma_1\Sigma_0^{-\frac{1}{2}} - \lambda\mathbf{I}\right| = 0$$

$$\Rightarrow \left|\Sigma_1 - \lambda\Sigma_0\right| = 0$$

$$\Rightarrow \left|\Gamma_1^{\mathrm{T}}\Lambda_1\Gamma_1 - \lambda\left(\Gamma_0^{\mathrm{T}}\Lambda_0\Gamma_0\right)\right| = 0,$$

where $\Lambda_0 = diag\left(\lambda_{01},\ldots,\lambda_{0p}\right)$ and $\Lambda_1 = diag\left(\lambda_{11},\ldots,\lambda_{1p}\right)$ denote the diagonal matrices containing the eigenvalues of Σ_0 and Σ_1 respectively, Γ_0^{T} and Γ_1^{T} are the orthogonal matrices with the corresponding normalized eigenvectors.

If Σ changes from Σ_0 to Σ_1 in the directions of the principal axes, Σ_0 and Σ_1 can be diagonalized by the same orthogonal matrix i.e. $\Gamma_1^{\mathrm{T}} = \Gamma_0^{\mathrm{T}}$. Thus,

$$\left|\Gamma_0^{\mathrm{T}}\right|\left|\Lambda_1 - \lambda\Lambda_0\right|\left|\Gamma_0\right| = 0$$

$$\Rightarrow \left|\Lambda_1 - \lambda\Lambda_0\right| = 0$$

$$\Rightarrow \lambda = \frac{\lambda_{11}}{\lambda_{01}}, \frac{\lambda_{12}}{\lambda_{02}}, \ldots, \frac{\lambda_{1p}}{\lambda_{0p}}.$$

Combining this with A.4, A.5, A.6 and A.7 yields the following results :

(i) The statistical performance of MLRT, SSVPC and MLRTECM depend on the ratios of variances of the principal components when Σ changes along the principal axes.

(ii) The statistical performance of the $|\mathbf{S}|^{1/2}$ charts depends on the product of the ratios of variances of the principal components when Σ changes along the principal axes.

A.11 The Distributional Properties of Hotelling χ^2-type Statistic When the Mean vector and Covariance Matrix are Not As Specified

The Hotelling χ^2-type statistic is of the form

$$Q = (\mathbf{Y} - \mu_0)^{\mathrm{T}} \Sigma_0^{-1} (\mathbf{Y} - \mu_0)$$

where $\mathbf{Y}$ denotes a p-dimensional random vector, μ_0 and Σ_0 are respectively the specified population mean vector and covariance matrix of the same dimension. Due to invariance, Q is expressible as

$$Q = \left[\Gamma \Sigma_0^{-1/2} (\mathbf{Y} - \mu_0)\right]^{\mathrm{T}} \left[\Gamma \Sigma_0^{-1/2} (\mathbf{Y} - \mu_0)\right]$$

where Γ is any (conformable) orthonormal matrix. Thus, Q is a function of $\Gamma \Sigma_0^{-1/2} (\mathbf{Y} - \mu_0)$ which is distributed as $N_p\left(\Gamma \Sigma_0^{-1/2} (\mu_1 - \mu_0),\ \Gamma \Sigma_0^{-1/2} \Sigma_1 \Sigma_0^{-1/2} \Gamma^{\mathrm{T}}\right)$ when $\mathbf{Y} \sim N_p(\mu_1,\ \Sigma_1)$. By letting Γ to be the orthonormal matrix that diagonalizes $\Sigma_0^{-1/2} \Sigma_1 \Sigma_0^{-1/2}$, it is readily seen that Q is distributed as

$$\sum_{j=1}^{p} \lambda_j U_j$$

where λ_j's denote the eigenvalues of $\Sigma_0^{-1/2} \Sigma_1 \Sigma_0^{-1/2}$ and U_j's are independent noncentral chi-square variables with one degree of freedom and noncentrality parameters ν_j's given by

$$\nu_j = \frac{\left[\Gamma \Sigma_0^{-1/2} (\mu_1 - \mu_0)\right]_j^2}{\lambda_j},$$

where the subscript j in the numerator indicates the jth component of $\Gamma \Sigma_0^{-1/2} (\mu_1 - \mu_0)$.

Simulation Programs

```
"PROG1"<-
function(p, r, m, λ , noiter)
{
#############################################################################
# This program, which is written in Splus, simulates the probabilities of detecting a sustained shift in  #
# the mean vector within m observations by the multivariate control charts (3.3), (3.4), (3.5) and (3.6). #
# Only upper control limit is used. This is set at 0.27% level of significance. p - dimension, r - change #
# point,  λ  - noncentrality parameter, noiter - number of replications.                                    #
#############################################################################
  noobs <- r + m
  result1 <- 0
  result2 <- 0
  result3 <- 0
  result4 <- 0
  stat3sing <- 0
  stat4sing <- 0
  temp1 <- rep(0,p)
  temp2 <- c( λ ,rep(0,p-1))
  temp3 <- diag(p)
  for(i in 1:noiter){
    samp <- rbind(GENERATE(temp1,temp3, r), GENERATE(temp2,temp3,m))
    meanvec2 <- apply(samp[(1:r),  ], 2, mean)
    meanvec4 <- meanvec2
    covmat4 <- var(samp[(1:r),  ])
    devmat3 <- samp - matrix(temp1, noobs, p, T)
    sumcrspd3 <- matrix(0, p, p)
    for(g in 1:r){
            sumcrspd3 <- sumcrspd3 + outer(devmat3[g,  ], devmat3[g,  ])
    }
    j <- r + 1
    repeat {
        stat1 <- samp[j,  ] %*% samp[j,  ]
        stat1 <- pchisq(stat1, p)
        if(stat1 >= 0.9973) {
            result1 <- result1 + 1
            break
        }
        j <- j + 1
        if(j > noobs)
            break
        }
    j <- r + 1
    repeat {
        stat2 <- ((j - 1)/j) * (samp[j,  ] - meanvec2) %*% (samp[j,  ] - meanvec2)
        stat2 <- pchisq(stat2, p)
        if((stat2 >= 0.9973)) {
                result2 <- result2 + 1
                break
        }
        meanvec2 <- (((j - 1) * meanvec2) + samp[j,  ])/j
        j <- j + 1
        if(j > noobs){
            break
        }
    }
    j <- r + 1
    repeat {
```

```
        if (qr(sumcrspd3)$rank < p){
            stat3sing <- stat3sing + 1
            break
        }
        stat3 <- (((j - p) * (j - 1))/(p * (j - 1))) * devmat3[j, ] %*% solve(sumcrspd3) %*%
                devmat3[j, ]
        stat3 <- pf(stat3, p, j - p)
        if(stat3 >= 0.9973) {
            result3 <- result3 + 1
            break
        }
        sumcrspd3 <- sumcrspd3 + outer(devmat3[j, ], devmat3[j, ])
        j <- j + 1
        if(j > noobs){
            break
        }
    }
    j <- r + 1
    repeat {
        if (qr(covmat4)$rank < p){
            stat4sing <- stat4sing + 1
            break
        }
        stat4 <- (((j - 1 - p) * (j - 1))/(j * (j - 2) * p)) * (samp[j, ] - meanvec4) %*%
                solve(covmat4) %*% (samp[j, ] - meanvec4)
        stat4 <- pf(stat4, p, j - 1 - p)
        if(stat4 >= 0.9973){
            result4 <- result4 + 1
            break
        }
        meanvec4 <- (((j - 1) * meanvec4) + samp[j, ])/j
        covmat4 <- var(samp[(1:j), ])
        j <- j + 1
        if(j > noobs){
            break
        }
    }
    result3 <- result3 / (noiter - stat3sing)
    result4 <- result4 / (noiter - stat4sing)
    c(result1,result2,result3,result4)
}
```

```
"PROG2"<-
function(p, n, r, m, λ , noiter)
{
#############################################################################
# This program, which is written in Splus, simulates the probabilities of detecting a sustained shift in #
# the mean vector within m subgroups of size n each by the multivariate control charts (3.7), (3.8),    #
# (3.9), (3.10) and (3.11). Only upper control limit is used. This is set at 0.27% level of significance. #
# p - dimension, r - change point, λ - noncentrality parameter, noiter - number of replications.        #
#############################################################################
        nosub <- r + m
        result1 <- 0
        result2 <- 0
        result3 <- 0
        result4 <- 0
        result5 <- 0
        stat3sing <- 0
        stat4sing <- 0
        stat5sing <- 0
        temp1 <- rep(0,p)
        temp2 <- c( λ ,rep(0,p-1))
        temp3 <- diag(p)
        temp4 <- matrix(0,p,p)
        for(i in 1:noiter){
                samp <- rbind(GENERATE(temp1, temp3, r*n), GENERATE(temp2, temp3, m*n))
                sumcrspd3 <- crossprod(samp[(1:(n * r)),  ], samp[(1:(n * r)), ])
                meanvec2 <- apply(samp[1:(n * r),  ], 2, mean)
                meanvec5 <- meanvec2
                cov5 <- temp4
                for(g in 1:(r + 1)) {
                        cov5 <- cov5 + var(samp[(((g - 1) * n) + 1):(g * n),  ])
                }
                cov5 <- cov5/(n + 1)
                cov4 <- cov5
                j <- r + 1
                repeat {
                        stat1 <- n * apply(samp[(((j - 1) * n) + 1):(j * n),  ], 2,mean) %*%
                                apply(samp[(((j - 1) * n) + 1):(j * n),  ], 2, mean)
                        if(stat1 >= qchisq(0.9973,p)) {
                                result1 <- result1 + 1
                                break
                        }
                        j <- j + 1
                        if(j > nosub){
                                break
                        }
                }
                j <- r + 1
                repeat {
                        stat2 <- (n/j) * (j - 1) * (apply(samp[(((j - 1) * n) + 1):(j * n),  ], 2, mean) -
                                meanvec2) %*% (apply(samp[(((j - 1) * n) + 1):(j * n),  ], 2, mean) -
                                meanvec2)
                        if(stat2 >= qchisq(0.9973,p)) {
                                result2 <- result2 + 1
                                break
                        }
                        j <- j + 1
                        if(j > nosub){
```

```
                break
        }
        meanvec2 <- (((j - 2) * meanvec2) + apply(samp[(((j - 2) * n) + 1):((j - 1) *
                        n), ], 2, mean))/(j - 1)
}
j <- r + 1
repeat {
        if( qr(sumcrspd3)$rank < p){
                stat3sing <- stat3sing + 1
                break
        }
        stat3 <- (n/p) * (n * (j - 1) - p + 1) * apply(samp[(((j - 1) * n) + 1):(j * n),  ],
                2, mean) %*% solve(sumcrspd3) %*% apply(samp[(((j - 1) * n)
                +1):(j * n),  ], 2, mean)
        if(stat3 >= qf(0.9973, p, n * (j-1) - p + 1)) {
                result3 <- result3 + 1
                break
        }
        j <- j + 1
        if(j > nosub){
                break
        }
        sumcrspd3 <- sumcrspd3 + crossprod(samp[(((j - 2) * n) + 1):((j - 1) * n),  ],
                        samp[(((j - 2) * n) + 1):((j - 1) * n),  ])
}
j <- r + 1
repeat {
        if( qr(cov4)$rank < p ){
                stat4sing <- stat4sing + 1
                break
        }
        stat4 <- (n/(p * j * (n - 1))) * ((j * (n - 1)) - p + 1) * apply(samp[(((j - 1) * n)
                + 1):(j * n),  ], 2, mean) %*% solve(cov4) %*% apply(samp[(((j - 1)
                * n) + 1):(j * n),  ], 2, mean)
        if(stat4 >= qf(0.9973, p, (j * (n - 1)) - p + 1)){
                result4 <- result4 + 1
                break
        }
        j <- j + 1
        if(j > nosub){
                break
        }
        cov4 <- ((j - 1) * cov4 + var(samp[(((j - 1) * n) + 1):(j * n),  ]))/j
}
j <- r + 1
repeat {
        if( qr(cov5)$rank < p){
                stat5sing <- stat5sing + 1
                break
        }
        stat5 <- (n/(j * j * p * (n - 1))) * (j - 1) * ((j * (n - 1)) - p + 1) *
                (apply(samp[(((j - 1) * n) + 1):(j * n),  ], 2, mean) - meanvec5) %*%
                solve(cov5) %*% (apply(samp[(((j - 1) * n) + 1):(j * n),  ], 2, mean) -
                meanvec5)
        if(stat5 >= qf(0.9973, p, j * (n - 1) - p + 1)){
                result5 <- result5 + 1
                break
        }
```

```
                    j <- j + 1
                    if(j > nosub){
                            break
                    }
                    meanvec5 <- (((j - 2) * n * meanvec5) + apply(samp[(((j - 2) * n) + 1):((j - 1)
                                    * n),  ], 2, sum))/((j - 1) * n)
                    cov5 <- (((j - 1) * cov5) + var(samp[(((j - 1) * n) + 1):(j * n),  ]))/j
            }
    }
    result1 <- result1 / noiter
    result2 <- result2 / noiter
    result3 <- result3 / (noiter - stat3sing)
    result4 <- result4 / (noiter - stat4sing)
    result5 <- result5 / (noiter - stat5sing)
    c(result1,result2,result3,result4,result5)
}
```

```
"PROG3"<-
function(p, γ , h, d, r, k, noiter, α , λ trend )
{
#############################################################################
# This program, which is written in Splus, simulates the run length probabilities of EWMAZ1, #
# EWMAZ2, EWMAZ3 and M chart for a linear trend. p - dimension, γ - EWMA smoothing constant #
# h - control chart factor of EWMA, d - moving sample size for M chart,r - change point,k - maximum #
# run length for which the probability is simulated, noiter- number of iterations, α - significance level #
# used with the M chart, λ trend - trend parameter.                                          #
#############################################################################
        if(p == 2) {
                a <- 2
        }
        else {
                a <- ceiling((3 * (p - 1) + sqrt((p - 1) * (9 * p - 17)))/4)
        }
        EWMAZ1RL <- numeric(k)
        MRL <- numeric(k)
        EWMAZ2RL <- numeric(k)
        EWMAZ3RL <- numeric(k)
        EWMAZ1CP <- numeric(k)
        MCP <- numeric(k)
        EWMAZ2CP <- numeric(k)
        EWMAZ3CP <- numeric(k)
        EWMAZ1sing <- 0
        Msing <- 0
        EWMAZ2sing <- 0
        EWMAZ3sing <- 0
        temp <- matrix(0, p, p, T)
        temp1 <- matrix(seq(1, k), k, p, F) * matrix(c( λ trend , rep(0, p - 1)), k, p, T)
        temp2 <- h * sqrt( γ /(2 - γ ))
        temp3 <- k + r - a - 1
        temp4 <- qf(1 - α , p, d - p)
        temp5 <- k + r - d + 1
        temp6 <- r + 2 - d
        for(i in 1:noiter) {
                cov <- temp
                samp <- rbind(GENERATE(rep(0, p), diag(p), r), GENERATE(rep(0, p), diag(p), k)
                        + temp1)
                for(g in 1:a) {
                        cov <- cov + outer(samp[g + 1, ] - samp[g, ], samp[g +1, ] - samp[g, ])
                }
                EWMAZ1 <- 0
                if((a + 2) > r) {
                        j <- 1
                        repeat {
                                if(qr(cov)$rank < p) {
                                 EWMAZ1sing <- EWMAZ1sing + 1
                                 break
                                }
                                f <- (2 * ((j + a - 1)^2))/(3 * (j + a) - 4)
                                Z1 <- ((f - p + 1) * (3 * (j + a) - 4) * (samp[j + a + 1, ]) %*%
                                                solve(cov) %*% (samp[j + a + 1, ]))/(p * (j + a -
                                                1))
                                Z1 <- qnorm(pf(Z1, p, f - p + 1), 0, 1)
                                EWMAZ1 <- γ  * Z1 + (1 - γ ) * EWMAZ1
```

```
                if(abs(EWMAZ1) > temp2) {
                 EWMAZ1RL[j + a + 1 - r] <- EWMAZ1RL[j + a + 1 - r]  + 1
                 break
                }
                cov <- cov + outer(samp[j + a + 1,  ] - samp[j + a,  ], samp[j + a + 1,
                        ] - samp[j + a,  ])
                j <- j + 1
                if(j > temp3) {
                 break
                }
        }
}
else if((a + 2) < r) {
        Z1flag <- 0
        for(j in 1:(r - a - 1)) {
                if(qr(cov)$rank < p) {
                        EWMAZ1sing <- EWMAZ1sing + 1
                        Z1flag <- 1
                        break
                }
                f <- (2 * ((j + a - 1)^2))/(3 * (j + a) - 4)
                Z1 <- ((f - p + 1) * (3 * (j + a) - 4) * (samp[j + a + 1,  ]) %*%
                                solve(cov) %*% (samp[j + a + 1,  ]))/(p * (j + a -
                                1))
                Z1 <- qnorm(pf(Z1, p, f - p + 1), 0, 1)
                EWMAZ1 <-  γ  * Z1 + (1 - γ ) * EWMAZ1
                cov <- cov + outer(samp[j + a + 1,  ] - samp[j + a,  ], samp[j + a + 1,
                        ] - samp[j + a,  ])
        }
        if(Z1flag == 0) {
                j <- r - a
                repeat {
                 if(qr(cov)$rank < p) {
                        EWMAZ1sing <- EWMAZ1sing + 1
                        break
                 }
                 f <- (2 * ((j + a - 2)^2))/(3 * (j + a) - 4)
                 Z1 <- ((f - p + 1) * (3 * (j + a) - 4) * (samp[j + a + 1,  ]) %*%
                        solve(cov) %*% (samp[j + a + 1,  ]))/(p * (j + a - 1))
                 Z1 <- qnorm(pf(Z1, p, f - p + 1), 0, 1)
                 EWMAZ1 <- γ  * Z1 + (1 - γ ) * EWMAZ1
                 if(abs(EWMAZ1) > temp2) {
                  EWMAZ1RL[j - r + a + 1] <- EWMAZ1RL[j - r + a + 1] + 1
                  break
                 }
                 cov <- cov + outer(samp[j + a + 1,  ] - samp[j + a,  ], samp[j + a +
                        1,] - samp[j + a, ])
                 j <- j + 1
                 if(j > temp3) {
                  break
                 }
                }
        }
 }
 else {
        if(qr(cov)$rank >= p) {
                f <- (2 * (a^2))/(3 * a - 1)
```

```
                Z1 <- ((f - p + 1) * (3 * a - 1) * samp[a + 2, ] %*% solve(cov) %*%
                        samp[a + 2, ])/(p * a)
                Z1 <- qnorm(pf(Z1, p, f - p + 1), 0, 1)
                EWMAZ1 <-  γ  * Z1 + (1 - γ ) * EWMAZ1
                cov <- cov + outer(samp[a + 2, ] - samp[a + 1,], samp[a + 2, ] -
                        samp[a + 1, ])
                j <- 1
                repeat {
                 if(qr(cov)$rank < p) {
                  EWMAZ1sing <- EWMAZ1sing + 1
                  break
                 }
                 f <- (2 * ((j + a)^2))/(3 * (j + a) - 1)
                 Z1 <- ((f - p + 1) * (3 * (j + a) - 1) * samp[j + a + 2, ] %*%
                                solve(cov) %*% samp[j + a + 2, ])/(p * (j + a))
                 Z1 <- qnorm(pf(Z1, p, f - p + 1), 0, 1)
                 EWMAZ1 <- γ  * Z1 + (1 - γ ) * EWMAZ1
                 if(abs(EWMAZ1) > temp2) {
                  EWMAZ1RL[j] <- EWMAZ1RL[j] + 1
                  break
                 }
                 cov <- cov + outer(samp[j + a + 2, ] - samp[j + a + 1, ], samp[j + a
                        + 2, ] - samp[j + a + 1, ])
                 j <- j + 1
                 if(j > k) {
                  break
                 }
                }
        }
        else {
                EWMAZ1sing <- EWMAZ1sing + 1
        }
}
if(d > r) {
        j <- 1
        repeat {
                x <- samp[(j:(j + d - 1)), ]
                w <- seq(j, j + d - 1) - rep((2 * j + d - 1)/2, d)
                z <- (t(x) %*% w)/sqrt(as.numeric(w %*% w))
                M <- t(x) %*% x
                if(qr(M)$rank >= p) {
                 M <- t(solve(M) %*% z) %*% z
                 M <- M/(1 - M)
                 if(M > temp4) {
                  MRL[j + d - 1 - r] <- MRL[j + d - 1 - r] + 1
                  break
                 }
                 j <- j + 1
                 if(j > temp5) {
                  break
                 }
                }
                else {
                 Msing <- Msing + 1
                 break
                }
         }
```

```
}
else {
        j <- temp6
        repeat {
                x <- samp[(j:(j + d - 1)), ]
                w <- seq(j, j + d - 1) - rep((2 * j + d - 1)/2, d)
                z <- (t(x) %*% w)/sqrt(as.numeric(w %*% w))
                M <- t(x) %*% x
                if(qr(M)$rank >= p) {
                 M <- t(solve(M) %*% z) %*% z
                 M <- M/(1 - M)
                 if(M > temp4) {
                  MRL[j + d - 1 - r] <- MRL[j + d - 1 - r] + 1
                  break
                 }
                 j <- j + 1
                 if(j > temp5) {
                  break
                 }
                }
                else {
                 Msing <- Msing + 1
                 break
                }
        }
}
if((p + 1) <= r) {
        EWMAZ2 <- 0
        Z2flag <- 0
        for(j in (p + 1):r) {
                Z2 <- apply(samp[(1:(j - 1)), ], 2, mean)
                w <- ((j - 2)/(j - 1)) * var(samp[(1:(j - 1)), ]) + outer(Z2, Z2)
                if(qr(w)$rank < p) {
                 Z2flag <- 1
                 EWMAZ2sing <- EWMAZ2sing + 1
                 break
                }
                Z2 <- ((j - p) * samp[j, ] %*% solve(w) %*% samp[j, ])/(p * (j - 1))
                Z2 <- qnorm(pf(Z2, p, j - p), 0, 1)
                EWMAZ2 <- γ * Z2 + (1 - γ ) * EWMAZ2
        }
        if(Z2flag == 0) {
                j <- r + 1
                repeat {
                 Z2 <- apply(samp[(1:(j - 1)), ], 2, mean)
                 w <- ((j - 2)/(j - 1)) * var(samp[(1:(j - 1)), ]) + outer(Z2, Z2)
                 if(qr(w)$rank < p) {
                  EWMAZ2sing <- EWMAZ2sing + 1
                  break
                 }
                 Z2 <- ((j - p) * samp[j, ] %*% solve(w) %*% samp[j, ])/(p * (j -
                        1))
                 Z2 <- qnorm(pf(Z2, p, j - p), 0, 1)
                 EWMAZ2 <- γ * Z2 + (1 - γ ) * EWMAZ2
                 if(abs(EWMAZ2) > temp2) {
                  EWMAZ2RL[j - r] <- EWMAZ2RL[j - r] + 1
                  break
```

```
                }
                j <- j + 1
                if(j > (r + k)) {
                 break
                }
               }
        }
}
else {
        EWMAZ2 <- 0
        j <- p + 1
        repeat {
                Z2 <- apply(samp[(1:(j - 1)),  ], 2, mean)
                w <- ((j - 2)/(j - 1)) * var(samp[(1:(j - 1)), ]) + outer(Z2, Z2)
                if(qr(w)$rank < p) {
                 EWMAZ2sing <- EWMAZ2sing + 1
                 break
                }
                Z2 <- ((j - p) * samp[j,  ] %*% solve(w) %*% samp[j,  ])/(p * (j - 1))
                Z2 <- qnorm(pf(Z2, p, j - p), 0, 1)
                EWMAZ2 <- γ  * Z2 + (1 - γ ) * EWMAZ2
                if(abs(EWMAZ2) > temp2) {
                 EWMAZ2RL[j - r] <- EWMAZ2RL[j - r] + 1
                 break
                }
                j <- j + 1
                if(j > r + k) {
                 break
                }
        }
}
if((p + 2) <= r) {
        EWMAZ3 <- 0
        Z3flag <- 0
        for(j in (p + 2):r) {
                w <- var(samp[(1:(j - 1)),  ])
                if(qr(w)$rank < p) {
                 EWMAZ3sing <- EWMAZ3sing + 1
                 Z3flag <- 1
                 break
                }
                Z3 <- ((j - p - 1) * samp[j,  ] %*% solve(w) %*% samp[j,  ])/(p * (j -
                        2))
                Z3 <- qnorm(pf(Z3, p, j - p - 1), 0, 1)
                EWMAZ3 <- γ  * Z3 + (1 - γ ) * EWMAZ3
        }
        if(Z3flag == 0) {
                j <- r + 1
                repeat {
                 w <- var(samp[(1:(j - 1)),  ])
                 if(qr(w)$rank < p) {
                  EWMAZ3sing <- EWMAZ3sing + 1
                  break
                 }
                 Z3 <- ((j - p - 1) * samp[j,  ] %*% solve(w) %*% samp[j,  ])/(p * (j
                         - 2))
                 Z3 <- qnorm(pf(Z3, p, j - p - 1), 0, 1)
```

```
                        EWMAZ3 <- γ * Z3 + (1 - γ ) * EWMAZ3
                        if(abs(EWMAZ3) > temp2) {
                          EWMAZ3RL[j - r] <- EWMAZ3RL[j - r] + 1
                          break
                        }
                        j <- j + 1
                        if(j > (k + r)) {
                          break
                        }
                      }
                }
        }
        else {
                EWMAZ3 <- 0
                j <- p + 2
                repeat {
                        w <- var(samp[(1:(j - 1)), ])
                        if(qr(w)$rank < p) {
                          EWMAZ3sing <- EWMAZ3sing + 1
                          break
                        }
                        Z3 <- ((j - p - 1) * samp[j, ] %*% solve(w) %*% samp[j, ])/(p * (j -
                                2))
                        Z3 <- qnorm(pf(Z3, p, j - p - 1), 0, 1)
                        EWMAZ3 <- γ * Z3 + (1 - γ ) * EWMAZ3
                        if(abs(EWMAZ3) > temp2) {
                          EWMAZ3RL[j - r] <- EWMAZ3RL[j - r] + 1
                          break
                        }
                        j <- j + 1
                        if(j > (k + r)) {
                          break
                        }
                }
        }
}
for(i in 1:k) {
        EWMAZ1CP[i] <- sum(EWMAZ1RL[1:i])/(noiter - EWMAZ1sing)
        MCP[i] <- sum(MRL[1:i])/(noiter - Msing)
        EWMAZ2CP[i] <- sum(EWMAZ2RL[1:i])/(noiter - EWMAZ2sing)
        EWMAZ3CP[i] <- sum(EWMAZ3RL[1:i])/(noiter - EWMAZ3sing)
}
results <- matrix(c(EWMAZ1CP, MCP, EWMAZ2CP, EWMAZ3CP), k, 4, F)
if((r == 1) && (λ_trend == 0)){
        results <- rbind(rep(0,4),results[(1:(k-1)),])
}
results
}
```

```
"PROG4"<-
function(p, γ , h, d, r, k, noiter, α , λ_trend)
{
#############################################################################
# This program, which is written in Splus, simulates the run length probabilities of EWMAZ1U,  #
# EWMAZ2U, MCHARTU for a linear trend. p - dimension,  γ - EWMA smoothing constant,  #
# h - control chart factor of EWMA, d - moving sample size for MCHARTU, r - change point, k - max #
# run length for which the probability is simulated, noiter- number of iterations, α - significance level #
# used with the MCHARTU, λ_trend - trend parameter.                                  #
#############################################################################
        if(p == 2) {
                a <- 2
        }
        else {
                a <- ceiling((3 * (p - 1) + sqrt((p - 1) * (9 * p - 17)))/4)
        }
        EWMAZ1URL <- numeric(k)
        MURL <- numeric(k)
        EWMAZ2URL <- numeric(k)
        EWMAZ1UCP <- numeric(k)
        MUCP <- numeric(k)
        EWMAZ2UCP <- numeric(k)
        EWMAZ1Using <- 0
        MUsing <- 0
        EWMAZ2Using <- 0
        temp1 <- matrix(0, p, p, T)
        temp2 <- matrix(seq(1, k), k, p, F) * matrix(c( λ_trend , rep(0, p - 1)
                ), k, p, T)
        temp3 <- h * sqrt( γ /(2 - γ ))
        temp4 <- k + r - a - 1
        temp5 <- qf(1 - α , p, d - p - 1)
        temp6 <- k + r - d + 1
        temp7 <- r + 2 - d
        for(i in 1:noiter) {
                cov <- temp1
                samp <- rbind(GENERATE(rep(0, p), diag(p), r), GENERATE(rep(0, p), diag(p), k)
                        + temp2)
                meanvec <- apply(samp[(1:(a + 1)),  ], 2, mean)
                for(g in 1:a) {
                        cov <- cov + outer(samp[g + 1,  ] - samp[g,  ], samp[g + 1,  ] - samp[g,  ])
                }
                EWMAZ1U <- 0
                if((a + 2) > r) {
                        j <- 1
                        repeat {
                                if(qr(cov)$rank < p) {
                                 EWMAZ1Using <- EWMAZ1Using + 1
                                 break
                                }
                                f <- (2 * ((j + a - 1)^2))/(3 * (j + a) - 4)
                                Z1U <- ((j + a) * (f - p + 1) * (3 * (j + a) - 4) * (samp[j + a + 1,  ] -
                                        meanvec) %*% solve(cov) %*% (samp[j + a + 1,  ] -
                                                meanvec))/(p * (j + a - 1) * (j + a + 1))
                                Z1U <- qnorm(pf(Z1U, p, f - p + 1), 0, 1)
                                EWMAZ1U <- γ  * Z1U + (1 - γ ) * EWMAZ1U
                                if(abs(EWMAZ1U) > temp3) {
```

```
                    EWMAZ1URL[j + a + 1 - r] <- EWMAZ1URL[j + a + 1 - r] + 1
                    break
                  }
                  meanvec <- apply(samp[(1:(j + a + 1)), ], 2, mean)
                  cov <- cov + outer(samp[j + a + 1, ] - samp[j + a, ], samp[j + a + 1,
                        ] - samp[j + a, ])
                  j <- j + 1
                  if(j > temp4) {
                   break
                  }
          }
}
else if((a + 2) < r) {
          j <- 1
          while((j <= (r - a - 1)) && (qr(cov)$rank >= p)) {
                  f <- (2 * ((j + a - 1)^2))/(3 * (j + a) - 4)
                  Z1U <- ((j + a) * (f - p + 1) * (3 * (j + a) - 4) * (samp[j + a + 1, ] -
                        meanvec) %*% solve(cov) %*% (samp[j + a + 1, ] -
                        meanvec))/(p * (j + a - 1) * (j + a + 1))
                  Z1U <- qnorm(pf(Z1U, p, f - p + 1), 0, 1)
                  EWMAZ1U <- γ * Z1U + (1 - γ ) * EWMAZ1U
                  meanvec <- apply(samp[(1:(j + a + 1)), ], 2, mean)
                  cov <- cov + outer(samp[j + a + 1, ] - samp[j + a, ], samp[j + a + 1,
                        ] - samp[j + a, ])
                  j <- j + 1
          }
          if(j == (r - a)) {
                  repeat {
                   if(qr(cov)$rank < p) {
                    EWMAZ1Using <- EWMAZ1Using + 1
                    break
                   }
                   f <- (2 * ((j + a - 1)^2))/(3 * (j + a) - 4)
                   Z1U <- ((j + a) * (f - p + 1) * (3 * (j + a) - 4) * (samp[j + a + 1, ] -
                         meanvec) %*% solve(cov) %*% (samp[j + a + 1, ] -
                         meanvec))/(p * (j + a - 1) * (j + a + 1))
                   Z1U <- qnorm(pf(Z1U, p, f - p + 1), 0, 1)
                   EWMAZ1U <- γ * Z1U + (1 - γ ) * EWMAZ1U
                   if(abs(EWMAZ1U) > temp3) {
                    EWMAZ1URL[j + a + 1 - r] <- EWMAZ1URL[j + a + 1 - r] + 1
                    break
                   }
                   meanvec <- apply(samp[(1:(j + a + 1)), ], 2, mean)
                   cov <- cov + outer(samp[j + a + 1, ] - samp[j + a, ],samp[j + a +
                         1, ] - samp[j + a, ])
                   j <- j + 1
                   if(j > temp4) {
                    break
                   }
                  }
          }
          else {
                  EWMAZ1Using <- EWMAZ1Using + 1
          }
}
else {
          if(qr(cov)$rank >= p) {
```

```
                f <- (2 * (a^2))/(3 * a - 1)
                Z1U <- ((a + 1) * (f - p + 1) * (3 * a - 1) * (samp[a + 2, ] -
                        meanvec) %*% solve(cov) %*% (samp[a + 2, ] -
                        meanvec))/(p * a * (a + 2))
                Z1U <- qnorm(pf(Z1U, p, f - p + 1), 0, 1)
                EWMAZ1U <- γ * Z1U + (1 - γ ) * EWMAZ1U
                meanvec <- apply(samp[(1:(a + 2)), ], 2, mean)
                cov <- cov + outer(samp[a + 2, ] - samp[a + 1,], samp[a + 2, ] -
                        samp[a + 1, ])
                j <- 1
                repeat {
                 if(qr(cov)$rank < p) {
                  EWMAZ1Using <- EWMAZ1Using + 1
                  break
                 }
                 f <- (2 * ((j + a)^2))/(3 * (j + a) - 1)
                 Z1U <- ((j + a + 1) * (f - p + 1) * (3 * (j +a) - 1) * (samp[j + a + 2,
                        ] - meanvec) %*% solve(cov) %*% (samp[j + a + 2, ] -
                         meanvec))/(p * (j + a) * (j + a + 2))
                 Z1U <- qnorm(pf(Z1U, p, f - p + 1), 0, 1)
                 EWMAZ1U <- γ * Z1U + (1 - γ ) * EWMAZ1U
                 if(abs(EWMAZ1U) > temp3) {
                  EWMAZ1URL[j] <- EWMAZ1URL[j] + 1
                  break
                 }
                 meanvec <- apply(samp[(1:(j + a + 2)), ], 2, mean)
                 cov <- cov + outer(samp[j + a + 2, ] - samp[j + a + 1, ], samp[j + a
                        + 2, ] - samp[j + a + 1, ])
                 j <- j + 1
                 if(j > k) {
                  break
                 }
                }
        }
        else {
                EWMAZ1Using <- EWMAZ1Using + 1
        }
}
if(d > r) {
        j <- 1
        repeat {
                x <- samp[(j:(j + d - 1)), ] - matrix(apply(samp[(j:(j + d - 1)), ], 2,
                     mean), d, p, T)
                w <- seq(j, j + d - 1) - rep((2 * j + d - 1)/2, d)
                z <- (t(x) %*% w)/sqrt(as.numeric(w %*% w))
                MU <- t(x) %*% x
                if(qr(MU)$rank >= p) {
                 MU <- t(solve(MU) %*% z) %*% z
                 MU <- MU/(1 - MU)
                 if(MU > temp5) {
                  MURL[j + d - 1 - r] <- MURL[j + d - 1 - r] + 1
                  break
                 }
                 j <- j + 1
                 if(j > temp6) {
                  break
                 }
```

```
                }
                else {
                 MUsing <- MUsing + 1
                 break
                }
        }
}
else {
        j <- temp7
        repeat {
                x <- samp[(j:(j + d - 1)),  ] - matrix(apply(samp[(j:(j + d - 1)),  ], 2,
                     mean), d, p, T)
                w <- seq(j, j + d - 1) - rep((2 * j + d - 1)/2, d)
                z <- (t(x) %*% w)/sqrt(as.numeric(w %*% w))
                MU <- t(x) %*% x
                if(qr(MU)$rank >= p) {
                 MU <- t(solve(MU) %*% z) %*% z
                 MU <- MU/(1 - MU)
                 if(MU > temp5) {
                  MURL[j + d - 1 - r] <- MURL[j + d - 1 - r] +1
                  break
                 }
                 j <- j + 1
                 if(j > temp6) {
                  break
                 }
                }
                else {
                 MUsing <- MUsing + 1
                 break
                }
        }
}
if((p + 2) <= r) {
        EWMAZ2U <- 0
        j <- p + 2
        repeat {
                Z2U <- apply(samp[(1:(j - 1)),  ], 2, mean)
                w <- var(samp[(1:(j - 1)),  ])
                if(qr(w)$rank >= p) {
                 Z2U <- ((j - 1) * (j - p - 1) * (samp[j,  ] - Z2U) %*% solve(w) %*%
                        (samp[j,  ] - Z2U))/(j * p * (j - 2))
                 Z2U <- qnorm(pf(Z2U, p, j - p - 1), 0, 1)
                 EWMAZ2U <- γ  * Z2U + (1 - γ ) * EWMAZ2U
                 j <- j + 1
                 if(j > r) {
                  break
                 }
                }
                else {
                 EWMAZ2Using <- EWMAZ2Using + 1
                 break
                }
        }
        if(j == (r + 1)) {
                repeat {
                 Z2U <- apply(samp[(1:(j - 1)),  ], 2, mean)
```

```
                        w <- var(samp[(1:(j - 1)),  ])
                        if(qr(w)$rank < p) {
                         EWMAZ2Using <- EWMAZ2Using + 1
                         break
                        }
                        Z2U <- ((j - 1) * (j - p - 1) * (samp[j,  ] - Z2U) %*% solve(w) %*%
                                (samp[j,  ] - Z2U))/(j * p * (j - 2))
                        Z2U <- qnorm(pf(Z2U, p, j - p - 1), 0, 1)
                        EWMAZ2U <- γ  * Z2U + (1 - γ ) * EWMAZ2U
                        if(abs(EWMAZ2U) > temp3) {
                         EWMAZ2URL[j - r] <- EWMAZ2URL[j - r] + 1
                         break
                        }
                        j <- j + 1
                        if(j > (k + r)) {
                         break
                        }
                       }
                }
        }
        else {
                EWMAZ2U <- 0
                j <- p + 2
                repeat {
                        Z2U <- apply(samp[(1:(j - 1)),  ], 2, mean)
                        w <- var(samp[(1:(j - 1)),  ])
                        if(qr(w)$rank < p) {
                         EWMAZ2Using <- EWMAZ2Using + 1
                         break
                        }
                        Z2U <- ((j - 1) * (j - p - 1) * (samp[j,  ] - Z2U) %*% solve(w) %*%
                                (samp[j,  ] - Z2U))/(j * p * (j - 2))
                        Z2U <- qnorm(pf(Z2U, p, j - p - 1), 0, 1)
                        EWMAZ2U <- γ  * Z2U + (1 - γ ) * EWMAZ2U
                        if(abs(EWMAZ2U) > temp3) {
                         EWMAZ2URL[j - r] <- EWMAZ2URL[j - r] + 1
                         break
                        }
                        j <- j + 1
                        if(j > (k + r)) {
                         break
                        }
                }
        }
}
cat("", fill = T)
for(i in 1:k) {
        EWMAZ1UCP[i] <- sum(EWMAZ1URL[1:i])/(noiter - EWMAZ1Using)
        MUCP[i] <- sum(MURL[1:i])/(noiter - MUsing)
        EWMAZ2UCP[i] <- sum(EWMAZ2URL[1:i])/(noiter - EWMAZ2Using)
}
results <- matrix(c(EWMAZ1UCP, MUCP, EWMAZ2UCP), k, 3, F)
if((r == 1) && (λ_trend == 0)){
        results <- rbind(rep(0,3),results[(1:(k-1)),])
}
results
}
```

```
"PROG5"<-
function(p, γ , h, r, k, noiter, λ )
{
#############################################################################
# This program, which is written in Splus, simulates the run length probabilities of EWMAZ1,      #
# EWMAZ2 and EWMAZ3 for a step shift in the mean vector. p - dimension, γ - EWMA smoothing  #
# constant, h - control chart factor for EWMA,r - change point,k - maximum run length for which the  #
# probability is simulated, noiter - number of iterations, λ - noncentrality parameter.          #
#############################################################################
        if(p == 2) {
                a <- 2
        }
        else {
                a <- ceiling((3 * (p - 1) + sqrt((p - 1) * (9 * p - 17)))/4)
        }
        EWMAZ1RL <- numeric(k)
        EWMAZ2RL <- numeric(k)
        EWMAZ3RL <- numeric(k)
        EWMAZ1CP <- numeric(k)
        EWMAZ2CP <- numeric(k)
        EWMAZ3CP <- numeric(k)
        EWMAZ1sing <- 0
        EWMAZ2sing <- 0
        EWMAZ3sing <- 0
        temp1 <- matrix(0, p, p, T)
        temp2 <- h * sqrt( γ /(2 - γ ))
        temp3 <- k + r - a - 1
        for(i in 1:noiter) {
                cov <- temp1
                samp <- rbind(GENERATE(rep(0, p), diag(p), r), GENERATE(c( λ ,rep(0, p-1)),
                        diag(p), k))
                for(g in 1:a) {
                        cov <- cov + outer(samp[g + 1,  ] - samp[g,  ], samp[g +1,  ] - samp[g,  ])
                }
                EWMAZ1 <- 0
                if((a + 2) > r) {
                        j <- 1
                        repeat {
                                if(qr(cov)$rank < p) {
                                 EWMAZ1sing <- EWMAZ1sing + 1
                                 break
                                }
                                f <- (2 * ((j + a - 1)^2))/(3 * (j + a) - 4)
                                Z1 <- ((f - p + 1) * (3 * (j + a) - 4) * (samp[j + a + 1,  ]) %*%
                                                solve(cov) %*% (samp[j + a + 1,  ]))/(p * (j + a -
                                                1))
                                Z1 <- qnorm(pf(Z1, p, f - p + 1), 0, 1)
                                EWMAZ1 <- γ  * Z1 + (1 - γ ) * EWMAZ1
                                if(abs(EWMAZ1) > temp2) {
                                 EWMAZ1RL[j + a + 1 - r] <- EWMAZ1RL[j + a + 1 - r] + 1
                                 break
                                }
                                cov <- cov + outer(samp[j + a + 1,  ] - samp[j + a,  ], samp[j + a + 1,
                                        ] - samp[j + a,  ])
                                j <- j + 1
                                if(j > temp3) {
```

```
                    break
                    }
            }
    }
    else if((a + 2) < r) {
            Z1flag <- 0
            for(j in 1:(r - a - 1)) {
                    if(qr(cov)$rank < p) {
                     EWMAZ1sing <- EWMAZ1sing + 1
                     Z1flag <- 1
                     break
                    }
                    f <- (2 * ((j + a - 1)^2))/(3 * (j + a) - 4)
                    Z1 <- ((f - p + 1) * (3 * (j + a) - 4) * (samp[j + a + 1, ]) %*%
                                    solve(cov) %*% (samp[j + a + 1, ]))/(p * (j + a -
                                    1))
                    Z1 <- qnorm(pf(Z1, p, f - p + 1), 0, 1)
                    EWMAZ1 <- γ  * Z1 + (1 - γ ) * EWMAZ1
                    cov <- cov + outer(samp[j + a + 1, ] - samp[j + a, ], samp[j + a + 1,
                            ] - samp[j + a, ])
            }
            if(Z1flag == 0) {
                    j <- r - a
                    repeat {
                     if(qr(cov)$rank < p) {
                      EWMAZ1sing <- EWMAZ1sing + 1
                      break
                     }
                     f <- (2 * ((j + a - 2)^2))/(3 * (j + a) - 4)
                     Z1 <- ((f - p + 1) * (3 * (j + a) - 4) * (samp[j + a + 1, ]) %*%
                             solve(cov) %*% (samp[j + a + 1, ]))/(p * (j + a - 1))
                     Z1 <- qnorm(pf(Z1, p, f - p + 1), 0, 1)
                     EWMAZ1 <- γ  * Z1 + (1 - γ ) * EWMAZ1
                     if(abs(EWMAZ1) > temp2) {
                      EWMAZ1RL[j - r + a + 1] <- EWMAZ1RL[j - r + a + 1] + 1
                      break
                     }
                     cov <- cov + outer(samp[j + a + 1, ] - samp[j + a, ],samp[j + a +
                             1,]- samp[j + a, ])
                     j <- j + 1
                     if(j > temp3) {
                      break
                     }
                    }
            }
    }
    else {
            if(qr(cov)$rank >= p) {
                    f <- (2 * (a^2))/(3 * a - 1)
                    Z1 <- ((f - p + 1) * (3 * a - 1) * samp[a + 2, ] %*% solve(cov) %*%
                            samp[a + 2, ])/(p * a)
                    Z1 <- qnorm(pf(Z1, p, f - p + 1), 0, 1)
                    EWMAZ1 <- γ  * Z1 + (1 - γ ) * EWMAZ1
                    cov <- cov + outer(samp[a + 2, ] - samp[a + 1, ], samp[a + 2, ] -
                            samp[a + 1, ])
                    j <- 1
                    repeat {
```

```
                    if(qr(cov)$rank < p) {
                     EWMAZ1sing <- EWMAZ1sing + 1
                     break
                    }
                    f <- (2 * ((j + a)^2))/(3 * (j + a) - 1)
                    Z1 <- ((f - p + 1) * (3 * (j + a) - 1) * samp[j + a + 2, ] %*%
                                solve(cov) %*% samp[j + a + 2, ])/(p * (j + a))
                    Z1 <- qnorm(pf(Z1, p, f - p + 1), 0, 1)
                    EWMAZ1 <- γ  * Z1 + (1 - γ ) * EWMAZ1
                    if(abs(EWMAZ1) > temp2) {
                     EWMAZ1RL[j] <- EWMAZ1RL[j] + 1
                     break
                    }
                    cov <- cov + outer(samp[j + a + 2, ] - samp[j + a + 1, ], samp[j + a
                          + 2, ] - samp[j + a + 1, ])
                    j <- j + 1
                    if(j > k) {
                     break
                    }
                   }
          }
          else {
                   EWMAZ1sing <- EWMAZ1sing + 1
          }
}
if((p + 1) <= r) {
          EWMAZ2 <- 0
          Z2flag <- 0
          for(j in (p + 1):r) {
                   Z2 <- apply(samp[(1:(j - 1)), ], 2, mean)
                   w <- ((j - 2)/(j - 1)) * var(samp[(1:(j - 1)), ]) + outer(Z2, Z2)
                   if(qr(w)$rank < p) {
                    Z2flag <- 1
                    EWMAZ2sing <- EWMAZ2sing + 1
                    break
                   }
                   Z2 <- ((j - p) * samp[j, ] %*% solve(w) %*% samp[j, ])/(p * (j - 1))
                   Z2 <- qnorm(pf(Z2, p, j - p), 0, 1)
                   EWMAZ2 <- γ  * Z2 + (1 - γ ) * EWMAZ2
          }
          if(Z2flag == 0) {
                   j <- r + 1
                   repeat {
                    Z2 <- apply(samp[(1:(j - 1)), ], 2, mean)
                    w <- ((j - 2)/(j - 1)) * var(samp[(1:(j - 1)), ]) + outer(Z2, Z2)
                    if(qr(w)$rank < p) {
                     EWMAZ2sing <- EWMAZ2sing + 1
                     break
                    }
                    Z2 <- ((j - p) * samp[j, ] %*% solve(w) %*% samp[j, ])/(p * (j -
                          1))
                    Z2 <- qnorm(pf(Z2, p, j - p), 0, 1)
                    EWMAZ2 <- γ  * Z2 + (1 - γ ) * EWMAZ2
                    if(abs(EWMAZ2) > temp2) {
                     EWMAZ2RL[j - r] <- EWMAZ2RL[j - r] + 1
                     break
                    }
```

```
                j <- j + 1
                if(j > (r + k)) {
                  break
                }
              }
        }
}
else {
        EWMAZ2 <- 0
        j <- p + 1
        repeat {
                Z2 <- apply(samp[(1:(j - 1)), ], 2, mean)
                w <- ((j - 2)/(j - 1)) * var(samp[(1:(j - 1)), ]) + outer(Z2, Z2)
                if(qr(w)$rank < p) {
                  EWMAZ2sing <- EWMAZ2sing + 1
                  break
                }
                Z2 <- ((j - p) * samp[j, ] %*% solve(w) %*% samp[j, ])/(p * (j - 1))
                Z2 <- qnorm(pf(Z2, p, j - p), 0, 1)
                EWMAZ2 <- γ * Z2 + (1 - γ ) * EWMAZ2
                if(abs(EWMAZ2) > temp2) {
                  EWMAZ2RL[j - r] <- EWMAZ2RL[j - r] + 1
                  break
                }
                j <- j + 1
                if(j > r + k) {
                  break
                }
        }
}
if((p + 2) <= r) {
        EWMAZ3 <- 0
        Z3flag <- 0
        for(j in (p + 2):r) {
                w <- var(samp[(1:(j - 1)), ])
                if(qr(w)$rank < p) {
                  EWMAZ3sing <- EWMAZ3sing + 1
                  Z3flag <- 1
                  break
                }
                Z3 <- ((j - p - 1) * samp[j, ] %*% solve(w) %*% samp[j, ])/(p * (j -
                        2))
                Z3 <- qnorm(pf(Z3, p, j - p - 1), 0, 1)
                EWMAZ3 <- γ * Z3 + (1 - γ ) * EWMAZ3
        }
        if(Z3flag == 0) {
                j <- r + 1
                repeat {
                  w <- var(samp[(1:(j - 1)), ])
                  if(qr(w)$rank < p) {
                    EWMAZ3sing <- EWMAZ3sing + 1
                    break
                  }
                  Z3 <- ((j - p - 1) * samp[j, ] %*% solve(w) %*% samp[j, ])/(p * (j
                          - 2))
                  Z3 <- qnorm(pf(Z3, p, j - p - 1), 0, 1)
                  EWMAZ3 <- γ * Z3 + (1 - γ ) * EWMAZ3
```

```
                        if(abs(EWMAZ3) > temp2) {
                         EWMAZ3RL[j - r] <- EWMAZ3RL[j - r] + 1
                         break
                        }
                        j <- j + 1
                        if(j > (k + r)) {
                         break
                        }
                       }
                }
        }
        else {
                EWMAZ3 <- 0
                j <- p + 2
                repeat {
                        w <- var(samp[(1:(j - 1)), ])
                        if(qr(w)$rank < p) {
                         EWMAZ3sing <- EWMAZ3sing + 1
                         break
                        }
                        Z3 <- ((j - p - 1) * samp[j, ] %*% solve(w) %*% samp[j, ])/(p * (j -
                                2))
                        Z3 <- qnorm(pf(Z3, p, j - p - 1), 0, 1)
                        EWMAZ3 <- γ * Z3 + (1 - γ ) * EWMAZ3
                        if(abs(EWMAZ3) > temp2) {
                         EWMAZ3RL[j - r] <- EWMAZ3RL[j - r] + 1
                         break
                        }
                        j <- j + 1
                        if(j > (k + r)) {
                         break
                        }
                }
        }
}
for(i in 1:k) {
        EWMAZ1CP[i] <- sum(EWMAZ1RL[1:i])/(noiter - EWMAZ1sing)
        EWMAZ2CP[i] <- sum(EWMAZ2RL[1:i])/(noiter - EWMAZ2sing)
        EWMAZ3CP[i] <- sum(EWMAZ3RL[1:i])/(noiter - EWMAZ3sing)
}
results <- matrix(c(EWMAZ1CP, EWMAZ2CP, EWMAZ3CP), k, 3, F)
results
}
```

```
"PROG6"<-
function(p, γ , h, r, k, noiter, λ )
{
#############################################################################
# This program, which is wirtten in Splus, simulates the run length probabilities of EWMAZ1U and  #
# EWMAZ2U for a step shift in the mean vector. p - dimension, γ  - EWMA smoothing constant,      #
# h - control chart factor for EWMA,r - change point,k - maximum run length for which the probability#
# is simulated, noiter - number of iterations, λ  - noncentrality parameter.                     #
#############################################################################
        if(p == 2) {
                a <- 2
        }
        else {
                a <- ceiling((3 * (p - 1) + sqrt((p - 1) * (9 * p - 17)))/4)
        }
        EWMAZ1URL <- numeric(k)
        EWMAZ2URL <- numeric(k)
        EWMAZ1UCP <- numeric(k)
        EWMAZ2UCP <- numeric(k)
        EWMAZ1Using <- 0
        EWMAZ2Using <- 0
        temp1 <- matrix(0, p, p, T)
        temp2 <- h * sqrt( γ /(2 - γ ))
        temp3 <- k + r - a - 1
        for(i in 1:noiter) {
                cov <- temp1
                samp <- rbind(GENERATE(rep(0, p), diag(p), r), GENERATE(c( λ ,rep(0, p-1)),
                        diag(p), k))
                meanvec <- apply(samp[(1:(a + 1)),  ], 2, mean)
                for(g in 1:a) {
                        cov <- cov + outer(samp[g + 1,  ] - samp[g,  ], samp[g + 1,  ] - samp[g,  ])
                }
                EWMAZ1U <- 0
                if((a + 2) > r) {
                        j <- 1
                        repeat {
                                if(qr(cov)$rank < p) {
                                 EWMAZ1Using <- EWMAZ1Using + 1
                                 break
                                }
                                f <- (2 * ((j + a - 1)^2))/(3 * (j + a) - 4)
                                Z1U <- ((j + a) * (f - p + 1) * (3 * (j + a) - 4) * (samp[j + a + 1,  ] -
                                        meanvec) %*% solve(cov) %*% (samp[j + a + 1,  ] -
                                                meanvec))/(p * (j + a - 1) * (j + a + 1))
                                Z1U <- qnorm(pf(Z1U, p, f - p + 1), 0, 1)
                                EWMAZ1U <- γ  * Z1U + (1 - γ ) * EWMAZ1U
                                if(abs(EWMAZ1U) > temp2) {
                                 EWMAZ1URL[j + a + 1 - r] <- EWMAZ1URL[j + a + 1 - r] + 1
                                 break
                                }
                                meanvec <- apply(samp[(1:(j + a + 1)),  ], 2, mean)
                                cov <- cov + outer(samp[j + a + 1,  ] - samp[j + a,  ], samp[j + a + 1,
                                        ] - samp[j + a,  ])
                                j <- j + 1
                                if(j > temp3) {
                                 break
```

```
                }
        }
}
else if((a + 2) < r) {
        j <- 1
        while((j <= (r - a - 1)) && (qr(cov)$rank >= p)) {
                f <- (2 * ((j + a - 1)^2))/(3 * (j + a) - 4)
                Z1U <- ((j + a) * (f - p + 1) * (3 * (j + a) - 4) * (samp[j + a + 1, ] -
                        meanvec) %*% solve(cov) %*% (samp[j + a + 1, ] -
                                meanvec))/(p * (j + a - 1) * (j + a + 1))
                Z1U <- qnorm(pf(Z1U, p, f - p + 1), 0, 1)
                EWMAZ1U <- γ * Z1U + (1 - γ ) * EWMAZ1U
                meanvec <- apply(samp[(1:(j + a + 1)), ], 2, mean)
                cov <- cov + outer(samp[j + a + 1, ] - samp[j + a, ], samp[j + a + 1,
                        ] - samp[j + a, ])
                j <- j + 1
        }
        if(j == (r - a)) {
                repeat {
                 if(qr(cov)$rank < p) {
                  EWMAZ1Using <- EWMAZ1Using + 1
                  break
                 }
                 f <- (2 * ((j + a - 1)^2))/(3 * (j + a) - 4)
                 Z1U <- ((j + a) * (f - p + 1) * (3 * (j + a) - 4) * (samp[j + a + 1, ] -
                         meanvec) %*% solve(cov) %*% (samp[j + a + 1, ] -
                         meanvec))/(p * (j + a - 1) * (j + a + 1))
                 Z1U <- qnorm(pf(Z1U, p, f - p + 1), 0, 1)
                 EWMAZ1U <- γ * Z1U + (1 - γ ) * EWMAZ1U
                 if(abs(EWMAZ1U) > temp2) {
                  EWMAZ1URL[j + a + 1 - r] <- EWMAZ1URL[j + a + 1 - r] + 1
                  break
                 }
                 meanvec <- apply(samp[(1:(j + a + 1)), ], 2, mean)
                 cov <- cov + outer(samp[j + a + 1, ] - samp[j + a, ],samp[j + a +
                         1,] - samp[j + a, ])
                 j <- j + 1
                 if(j > temp3) {
                  break
                 }
                }
        }
        else {
                EWMAZ1Using <- EWMAZ1Using + 1
        }
}
else {
        if(qr(cov)$rank >= p) {
                f <- (2 * (a^2))/(3 * a - 1)
                Z1U <- ((a + 1) * (f - p + 1) * (3 * a - 1) * (samp[a + 2, ] -
                        meanvec) %*% solve(cov) %*% (samp[a + 2, ] -
                        meanvec))/(p * a * (a + 2))
                Z1U <- qnorm(pf(Z1U, p, f - p + 1), 0, 1)
                EWMAZ1U <- γ * Z1U + (1 - γ ) * EWMAZ1U
                meanvec <- apply(samp[(1:(a + 2)), ], 2, mean)
                cov <- cov + outer(samp[a + 2, ] - samp[a + 1, ], samp[a + 2, ] -
                        samp[a + 1, ])
```

```
                j <- 1
                repeat {
                 if(qr(cov)$rank < p) {
                  EWMAZ1Using <- EWMAZ1Using + 1
                  break
                 }
                 f <- (2 * ((j + a)^2))/(3 * (j + a) - 1)
                 Z1U <- ((j + a + 1) * (f - p + 1) * (3 * (j + a) - 1) * (samp[j + a + 2,
                        ] - meanvec) %*% solve(cov) %*% (samp[j + a + 2, ] -
                        meanvec))/(p * (j + a) * (j + a + 2))
                 Z1U <- qnorm(pf(Z1U, p, f - p + 1), 0, 1)
                 EWMAZ1U <- γ  * Z1U + (1 - γ ) * EWMAZ1U
                 if(abs(EWMAZ1U) > temp2) {
                  EWMAZ1URL[j] <- EWMAZ1URL[j] + 1
                  break
                 }
                 meanvec <- apply(samp[(1:(j + a + 2)), ], 2, mean)
                 cov <- cov + outer(samp[j + a + 2, ] - samp[j + a + 1, ], samp[j + a
                        + 2, ] - samp[j + a + 1, ])
                 j <- j + 1
                 if(j > k) {
                  break
                 }
                }
        }
        else {
                EWMAZ1Using <- EWMAZ1Using + 1
        }
}
if((p + 2) <= r) {
        EWMAZ2U <- 0
        j <- p + 2
        repeat {
                Z2U <- apply(samp[(1:(j - 1)), ], 2, mean)
                w <- var(samp[(1:(j - 1)), ])
                if(qr(w)$rank >= p) {
                 Z2U <- ((j - 1) * (j - p - 1) * (samp[j, ] - Z2U) %*% solve(w) %*%
                        (samp[j, ] - Z2U))/(j * p * (j - 2))
                 Z2U <- qnorm(pf(Z2U, p, j - p - 1), 0, 1)
                 EWMAZ2U <- γ  * Z2U + (1 - γ ) * EWMAZ2U
                 j <- j + 1
                 if(j > r) {
                  break
                 }
                }
                else {
                 EWMAZ2Using <- EWMAZ2Using + 1
                 break
                }
        }
        if(j == (r + 1)) {
                repeat {
                 Z2U <- apply(samp[(1:(j - 1)), ], 2, mean)
                 w <- var(samp[(1:(j - 1)), ])
                 if(qr(w)$rank < p) {
                  EWMAZ2Using <- EWMAZ2Using + 1
                  break
```

```
                        }
                        Z2U <- ((j - 1) * (j - p - 1) * (samp[j, ] - Z2U) %*% solve(w) %*%
                                (samp[j, ] - Z2U))/(j * p * (j - 2))
                        Z2U <- qnorm(pf(Z2U, p, j - p - 1), 0, 1)
                        EWMAZ2U <- γ * Z2U + (1 - γ ) * EWMAZ2U
                        if(abs(EWMAZ2U) > temp2) {
                          EWMAZ2URL[j - r] <- EWMAZ2URL[j - r] + 1
                          break
                        }
                        j <- j + 1
                        if(j > (k + r)) {
                          break
                        }
                      }
                }
        }
        else {
                EWMAZ2U <- 0
                j <- p + 2
                repeat {
                        Z2U <- apply(samp[(1:(j - 1)), ], 2, mean)
                        w <- var(samp[(1:(j - 1)), ])
                        if(qr(w)$rank < p) {
                          EWMAZ2Using <- EWMAZ2Using + 1
                          break
                        }
                        Z2U <- ((j - 1) * (j - p - 1) * (samp[j, ] - Z2U) %*% solve(w) %*%
                                (samp[j, ] - Z2U))/(j * p * (j - 2))
                        Z2U <- qnorm(pf(Z2U, p, j - p - 1), 0, 1)
                        EWMAZ2U <- γ * Z2U + (1 - γ ) * EWMAZ2U
                        if(abs(EWMAZ2U) > temp2) {
                          EWMAZ2URL[j - r] <- EWMAZ2URL[j - r] + 1
                          break
                        }
                        j <- j + 1
                        if(j > (k + r)) {
                          break
                        }
                }
        }
    }
    cat("", fill = T)
    for(i in 1:k) {
        EWMAZ1UCP[i] <- sum(EWMAZ1URL[1:i])/(noiter - EWMAZ1Using)
        EWMAZ2UCP[i] <- sum(EWMAZ2URL[1:i])/(noiter - EWMAZ2Using)
    }
    results <- matrix(c(EWMAZ1UCP, EWMAZ2UCP), k, 2, F)
    results
}
```

```
"PROG7"<-
function(Σ0, Σ1, n, α, noiter)
{
#
##########################################################################
# This Splus program simulates the probability of a signal from the proposed dispersion control  #
# technique (process covariance matrix assumed known) and the associated Fisher and Tippett   #
# procedures for a change in the process covariance matrix from Σ0 to Σ1.                      #
# n - subgroup size, α - significance level, noiter - number of iterations                     #
##########################################################################
#
        p <- nrow(Σ0)
        gsize <- noiter * n
        result <- 0
        popmat <- vector("list", p)
        popmat[[1]] <- Σ0
        for(i in 2:(p - 1)) {
                if(qr(Σ0[(1:(i - 1)), (1:(i - 1))])$rank < (i - 1)) {
                        stop("One of the principal minors of Σ0 is not of full rank !")
                }
                else {
                        popmat[[i]] <- Σ0[(i:p), (i:p)] - Σ0[(i:p), (1:(i - 1))] %*% solve(Σ0[(1:(i -
                                1)), (1:(i - 1))]) %*% t(Σ0[(i:p), (1:(i - 1))])
                        if(qr(popmat[[i]])$rank < (p - i + 1)){
                                stop("One of the conditional Σ0 is not of full rank")
                        }
                }
        }
        if(qr(Σ0[(1:(p - 1)), (1:(p - 1))])$rank < (p - 1)) {
                stop("The (p-1) by (p-1) principal minor of Σ0 is not of full rank !")
        }
        else {
                popmat[[p]] <- Σ0[p, p] - Σ0[p, (1:(p - 1))] %*% solve(Σ0[(1:(p - 1)), (1:(p - 1))])
                                %*% Σ0[p, (1:(p - 1))]
        }
        samp <- GENERATE(rep(0,p), Σ1, gsize)
        results <- DISPER(samp, n, popmat, α)
        results
}
```

```
"PROG8"<-
function(Σ0, Σ1, n, MLRTlim, gvL, gvU, α, noiter)
{
#
#####################################################################
# This Splus program simulates the probability of a signal from the MLRT, generalized  #
# variance chart and SSVPC for a change in the process variance-covariance matrix from #
# Σ0 to Σ1. n - subgroup size, MLRTlim - control limit for MLRT, gvL and gvU are       #
# respectively the lower and upper control limit factors for the generalized variance chart, #
# α - significance level, noiter - number of iterations.                               #
#####################################################################
#
        p <- nrow(Σ0)
        if(qr(sigma)$rank < p) {
                stop("Σ0 is not of full rank !")
        }
        else {
                invsigma <- solve(Σ0)
                LCL <- sqrt(prod(eigen(Σ0)$values)) * gvL
                UCL <- sqrt(prod(eigen(Σ0)$values)) * gvU
        }
        result1 <- 0
        result2 <- 0
        result3 <- 0
        samp <- generate6(rep(0,p), Σ1, n * noiter)
        for(j in 1:noiter) {
                sampcov <- var(samp[(((j - 1) * n) + 1):(j * n), ])
                temp <- invsigma %*% sampcov
                temp <- (n - 1) * ( - p - log(prod(eigen(temp)$values)) + sum(diag(temp)))
                if(temp > MLRTlim) {
                        result1 <- result1 + 1
                }
                temp <- sqrt(prod(eigen(sampcov)$values))
                if((temp > UCL) || (temp < LCL)) {
                        result2 <- result2 + 1
                }
                temp <- eigen(Σ0, T)$vectors %*% diag(sqrt(eigen(Σ0,T)$values)) %*%
                        t(eigen(Σ0,T)$vectors)
                temp <- solve(temp) %*% sampcov %*% solve(temp)
                temp3 <- (n - 1) * sum(diag(temp))
                if((temp3 > qchisq(1 - (α/2), p * (n - 1))) || (temp3 < qchisq(α/2, p * (n - 1)))){
                        result3 <- result3 + 1
                }
        }
        result1 <- result1 / noiter
        result2 <- result2 / noiter
        result3 <- result3 / noiter
        c(result1, result2, result3)
}
```

```
"PROG9"<-
function(Σ0, Σ1, n, r, k, noiter, α)
{
#
##########################################################################
# This Splus program simulates the probability of detection by the proposed dispersion control      #
# technique (for the case with unknown process covariance matrix) and the associated Fisher and     #
# Tippett procedures, within k subgroups of size n each, following a change in the process covariance#
# matrix from Σ0 to Σ1. r - change point,  noiter - number of iterations, α -  significance level.  #
##########################################################################
        p <- ncol(Σ0)
        mvec <- rep(0, p)
        convar <- numeric(p)
        v <- vector("list", p - 1)
        u <- vector("list", p - 1)
        propres <- 0
        tippettres <- 0
        fisherres <- 0
        nosing <- 0
        for(i in 1:noiter) {
                flag <- 0
                flag1 <- 0
                flag2 <- 0
                flag3 <- 0
                samp <- rbind(GENERATE(mvec, Σ0, r * n), GENERATE (mvec, Σ1, k * n))
                cov <- var(samp[(1:n),  ])
                convar[1] <- cov[1, 1]
                for(j in 2:p) {
                        if(qr(cov[(1:(j - 1)), (1:(j - 1))])$rank < (j - 1)) {
                                nosing <- nosing + 1
                                flag <- 1
                                break
                        }
                        convar[j] <- cov[j, j] - cov[j, (1:(j - 1))] %*% solve(cov[(1:(j - 1)), (1:(j -1))])
                                        %*% cov[(1:(j - 1)), j]
                        v[[j - 1]] <- solve(cov[(1:(j - 1)), (1:(j - 1))]) %*% cov[(1:(j - 1)), j]
                        u[[j - 1]] <- solve(cov[(1:(j - 1)), (1:(j - 1))]) / (n - 1)
                }
                if(flag == 1){
                        next
                }
                for(g in 2:r) {
                        cov <- var(samp[((((g - 1) * n) + 1):(g * n)),  ])
                        convar[1] <- ((g - 1) * convar[1] + cov[1, 1])/g
                        for(j in 2:p) {
                                if(qr(cov[(1:(j - 1)), (1:(j - 1))])$rank < (j - 1)) {
                                        nosing <- nosing + 1
                                        flag <- 1
                                        break
                                }
                                temp <- cov[j, j] - cov[j, (1:(j - 1))] %*% solve(cov[(1:(j - 1)), (1: (j -
                                        1))]) %*% cov[(1:(j - 1)), j]
                                convar[j] <- ((g - 1) * convar[j] +  temp)/g
                                temp <- solve(cov[(1:(j - 1)), (1:(j - 1))]) %*% cov[(1:(j - 1)), j]
                                v[[j - 1]] <- ((g - 1) * v[[j - 1]] + temp)/g
```

```
                u[[j - 1]] <- (((g - 1)^2) * u[[j - 1]] + solve(cov[(1:(j - 1)), (1:(j -
                        1))])/(n - 1))/(g^2)
        }
        if(flag == 1){
                break
        }
}
if(flag == 1){
        next
}
g <- r + 1
repeat {
        cov <- var(samp[((((g - 1) * n) + 1):(g * n)),  ])
        prop <- qnorm(pf(cov[1, 1]/convar[1], n - 1, (g - 1) * (n -
                1)),0, 1)^2
        tippett <- 1 - pchisq(prop,1)
        fisher <- -2 * log(tippett)
        convar[1] <- ((g - 1) * convar[1] + cov[1, 1])/g
        for(j in 2:p) {
                if(qr(cov[(1:(j - 1)), (1:(j - 1))])$rank < (j - 1)) {
                        nosing <- nosing + 1
                        flag <- 1
                        break
                }
                temp <- cov[j, j] - cov[j, (1:(j - 1))] %*% solve(cov[(1:(j - 1)), (1: (j -
                        1))]) %*% cov[(1:(j - 1)), j]
                dummy1 <- qnorm(pf(temp/convar[j], n - j,(g - 1) * (n - j)), 0,1)^2
                dummy2 <- 1-pchisq(dummy1,1)
                prop <- prop + dummy1
                tippett <- min(tippett, dummy2)
                fisher <- fisher + (-2 * log(dummy2))
                convar[j] <- ((g - 1) * convar[j] + temp)/g
                temp <- solve(cov[(1:(j - 1)), (1:(j - 1))]) %*% cov[(1:(j - 1)), j]
                dummy1 <- qnorm(pf(t(temp - v[[j - 1]]) %*% solve
                        (solve(cov[(1:(j - 1)), (1:(j - 1))])/(n - 1) + u[[j - 1]]) %*%
                        (temp - v[[j - 1]])/((j - 1) * convar[j]*(n-1)/(n-j)), j - 1, (g*(n
                        - j))), 0,1)^2
                dummy2 <- 1 - pchisq(dummy1,1)
                prop <- prop + dummy1
                tippett <- min(tippett,dummy2)
                fisher <- fisher + (-2 * log(dummy2))
                v[[j - 1]] <- ((g - 1) * v[[j - 1]] + temp)/g
                u[[j - 1]] <- (((g - 1)^2) * u[[j - 1]] + solve(cov[(1:(j - 1)), (1:(j -
                                1))])/(n - 1))/(g^2)
        }
        if(flag == 1){
                break
        }
        if(flag1 == 0){
                if(tippett <= (1 - (1 - alp)^(1/(2*p - 1)))){
                        tippettres <- tippettres + 1
                        flag1 <- 1
                        if((flag2 == 1) && (flag3 == 1)){
                                break
                        }
                }
        }
        if(flag2 == 0){
```

```
                        if(fisher  > qchisq(1 - alp, 2 * (2 * p - 1))) {
                                fisherres <- fisherres + 1
                                flag2 <- 1
                                if((flag1 == 1) && (flag3 == 1)){
                                        break
                                }
                        }
                }
                if(flag3 == 0){
                        if(prop > qchisq(1 - alp, 2*p -1)){
                                propres <- propres + 1
                                flag3 <- 1
                                if((flag1 == 1) && (flag2 == 1)){
                                        break
                                }
                        }
                }
                g <- g + 1
                if(g > r + k){
                        break
                }
        }
    }
    results <- c(propres, fisherres, tippettres) / (noiter - nosing)
    results
}
```

```
"PROG10"<-
function(p, n, r, k, Σ0, Σ1, cvvec, noiter)
{
#
##############################################################################
# This Splus program simulates the probability of detection by MLRTECM within k subgroups of size  #
# n each, following a change in the process variance-covariance matrix from Σ0 to Σ1. p -dimension,#
# r - change point, noiter - number of iterations, cvvec - vector of critical values at 0.27% significance  #
# level.                                                                                          #
##############################################################################
#
        mvec <- rep(0, p)
        result <- 0
        for(i in 1:m) {
                samp <- rbind(GENERATE(mvec, Σ0, r * n), GENERATE(mvec, Σ1, k * n))
                cov <- var(samp[(1:n), ])
                covavg <- cov
                gvprod <- prod(eigen(cov)$values)
                for(g in 2:r) {
                        cov <- var(samp[((((g - 1) * n) + 1):(g * n)), ])
                        covavg <- ((g - 1) * covavg + cov)/g
                        gvprod <- gvprod * prod(eigen(cov)$values)
                }
                g <- r + 1
                repeat {
                        cov <- var(samp[((((g - 1) * n) + 1):(g * n)), ])
                        covavg <- ((g - 1) * covavg + cov)/g
                        gvprod <- gvprod * prod(eigen(cov)$values)
                        stat1 <- -2 * log((gvprod^(0.5 * (n - 1)))/(prod(eigen(covavg)$values)^(0.5 *
                                g * (n - 1))))
                        if(stat1 > cvvec[g - 1]) {
                                result <- result + 1
                                break
                        }
                        g <- g + 1
                        if(g > r + k){
                                break
                        }
                }
        }
        result <- result / noiter
        result
}
```

```
"PROG11"<-
function(p, n, noiter)
{
#########################################################################
# This Splus program simulates the false signal rate of |S|^1/2 chart with '3-sigma' limits #
# p - dimension, n - subgroup size, noiter - number of replications.                        #
#########################################################################
        gsize <- noiter * n
        result <- 0
        prod1 <- 1
        prod2 <- 1
        for(i in 1:p) {
                prod1 <- prod1 * (n - i)
                prod2 <- prod2 * (gamma((n - i + 1)/2)/gamma((n - i)/2))
        }
        b3 <- (2/(n - 1))^(p/2) * prod2
        b4 <- (n - 1)^( - p) * (prod1 - ((2^p) * (prod2^2)))
        LCL <- b3 - (3 * sqrt(b4))
        UCL <- b3 + (3 * sqrt(b4))
        samp <- GENERATE(rep(0,p),diag(p),gsize)
        for(j in 1:noiter) {
                temp <- sqrt(prod(eigen(var(samp[(((j - 1) * n) + 1):(j * n), ]))$values))
                if((temp > UCL) || (temp < LCL)) {
                        result <- result + 1
                }
        }
        result <- result / noiter
        result
}
```

```
"PROG12"<-
function(p, n, α , noiter)
{
#
##########################################################################
# This Splus program simulates the lower and upper 100α th percentiles of |S|^1/2 for a multivariate #
# normal distribution. p - dimension, n - subgroup size, noiter - number of replications.          #
##########################################################################
#
        result <- numeric(noiter)
        for(j in 1:noiter) {
                randsamp <- GENERATE(rep(0,p),diag(p),n)
                result[j] <- sqrt(prod(eigen(var(randsamp))$values))
        }
        result <- sort(result)
        L <- (result[floor(0.5 * a * noiter)] + result[floor(0.5 * a * noiter) + 1])/2
        U <- (result[ceiling((1 - 0.5 * α )*noiter)] + result[ceiling((1-0.5* α )*noiter)-1])/2
        c(L,U)
}
```

```
"GENERATE"<-
function( μ , Σ , n)
{
#
##############################################################################
# This Splus subroutine, which is called by PROG1, PROG2, PROG3, PROG4, PROG5, PROG6, #
# PROG7, PROG8, PROG9 and PROG10, generates n random vectors from multivariate normal  #
# distribution with mean vector μ and variance-covariance matrix Σ.                    #
##############################################################################
#
  randsamp <- matrix(rnorm(n*length( μ )),n,length( μ ),T) %*%
                  chol( Σ ) + matrix( μ ,n,length( μ ),T)
  randsamp
}
```

```
"DISPER"<-
function(samp, n, popmat, α )
{
#
##############################################################################
# This Splus subroutine, which is called by PROG7, computes the number of samples in samp that  #
# result in a signal by the proposed dispersion control technique (process variance-covariance matrix #
# assumed known) and the associated Fisher and Tippett procedures when the control limit is set at  #
# 100α % sig. level.                                                                              #
##############################################################################
#
        p <- ncol(samp)
        m <- nrow(samp)/n
        result <- matrix(0, m, (2 * p - 1))
        nosing <- 0
        flag <- 0
        propres <- numeric(m)
        fisherres <- numeric(m)
        tippettres <- numeric(m)
        sampmat <- vector("list", m)
        for(j in 1:m) {
                sampmat[[j]] <- vector("list", p)
        }
        cov <- var(samp[1:n,])
        sampmat[[1]][[1]] <- cov
        if(qr(cov[(1:(p-1)),(1:(p-1))])$rank < (p-1)){
                sampmat[[1]][[p]] <- -99
                flag <- 1
        }
        else {
                sampmat[[1]][[p]] <- cov[p, p] - cov[p, (1:(p - 1))] %*% solve(cov[(1:(p - 1)), (1:(p -
                                        1) )]) %*% cov[p, (1:(p - 1))]
        }
        if(flag == 0){
           for(i in 2:(p - 1)) {
                if(qr(cov[(1:(i-1)),(1:(i-1))])$rank < (i-1)){
                        sampmat[[1]][[p]] <- -99
                        break
                }
                sampmat[[1]][[i]] <- cov[(i:p), (i:p)] - cov[(i:p), (1:(i - 1))] %*% solve(cov[(1:(i - 1)),
                                        (1:(i - 1))]) %*% t(cov[(i:p), (1:(i - 1))])
           }
        }
        for(j in 2:m) {
                cov <- var(samp[(((j-1)*n)+1):(j*n),])
                sampmat[[j]][[1]] <- cov
                sampmat[[j]] <- DISPERinner1(cov,sampmat,j,p)
        }
        for(j in 1:m) {
                if(sampmat[[j]][[p]] == -99){
                        result[j,] <- rep(0,2*p-1)
                        nosing <- nosing + 1
                }
                else{
                        result[j, ] <- DISPERinner2(sampmat, popmat, result, j, n, p)
                        result[j, ] <- DISPERinner3(sampmat, popmat, result, j, n, p)
                }
```

```
        }
        for(i in 1:(2*p-1)){
                propres[1:m] <- propres[1:m] + result[(1:m),i]^2
                fisherres[1:m] <- fisherres[1:m] + (-2 * log(rep(1,m)-pchisq(result[(1:m),i]^2,1)))
        }
        for(i in 1:m){
                tippettres[i] <- min(rep(1,2*p - 1) - pchisq(result[i,(1:(2*p - 1))]^2,1))
        }
        propres <- length(propres[propres > qchisq(1-α,2*p-1)])
        fisherres <- length(fisherres[fisherres > qchisq(1-α, 2*(2*p - 1))])
        tippettres <- length(tippettres[tippettres <= (1-(1 - α)^(1/(2*p - 1)))])
        results <- c(propres,fisherres,tippettres) / (m - nosing)
        results
}
```

```
"DISPERinner1"<-
function(cov,sampmat,j,p)
{
#
#########################################################################
#                          This Splus subroutine is called by DISPER.                          #
#########################################################################
#
        flag <- 0
        if(qr(cov[(1:(p-1)),(1:(p-1))])$rank < (p-1)){
                sampmat[[j]][[p]] <- -99
                flag <- 1
        }
        else{
                sampmat[[j]][[p]] <- cov[p,p] - cov[p,(1:(p-1))] %*% solve(cov[(1:(p-1)),(1:(p-1))])
                                        %*% cov[p,(1:(p-1))]
        }
        if(flag == 0){
           for(i in 2:(p - 1)) {
                if(qr(cov[(1:(i-1)),(1:(i-1))])$rank < (i-1)){
                        sampmat[[j]][[p]] <- -99
                        break
                }
                sampmat[[j]][[i]] <- cov[(i:p), (i:p)] - cov[(i:p), (1:(i - 1))] %*% solve(cov[(1:(i - 1)),
                                        (1:(i - 1))]) %*% t(cov[(i:p), (1:(i - 1))])
           }
        }
        sampmat[[j]]
}
```

```
"DISPERinner2"<-
function(sampmat, popmat, result, j, n, p)
{
#
#########################################################################
#                          This Splus subroutine is called by DISPER.                          #
#########################################################################
#
        for(i in 1:p) {
                result[j, i] <- ((n - 1) * sampmat[[j]][[i]][1, 1])/popmat[[i]][1, 1]
                result[j, i] <- qnorm(pchisq(result[j, i], n - i), 0, 1)
        }
        result[j, ]
}
```

```
"DISPERinner3"<-
function(sampmat, popmat, result, j, n, p)
{
#
##########################################################################
#                        This Splus subroutine is called by DISPER.                        #
##########################################################################
#
        for(i in (p + 1):(2 * p - 1)) {
                temp <- (sampmat[[j]][[i - p]][1, (2:(2 * p - i + 1))]/sampmat[[j]][[i - p]][1, 1] -
                        popmat[[i - p]][1, (2:(2 * p - i + 1))]/popmat[[i - p]][1, 1])
                result[j, i] <- (n - 1) * sampmat[[j]][[i - p]][1, 1] * temp %*% solve(popmat[[i - p +
                                1]]) %*% temp
                result[j, i] <- qnorm(pchisq(result[j, i], (2 * p - i)), 0, 1)
        }
        result[j,  ]
}
```