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AN OSTROWSKI'S TYPE INEQUALITY FOR A RANDOM VARIABLE WHOSE PROBABILITY DENSITY FUNCTION BELONGS TO $L_\infty[A, B]$

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ABSTRACT. An inequality of Ostrowski's type for a random variable whose probability density function is in $L_\infty[a, b]$ in terms of the cumulative distribution function and expectation is given. An application for a Beta random variable is also given.

1 INTRODUCTION

The following theorem contains the integral inequality which is known in the literature as Ostrowski's inequality [1, p. 469]

Theorem 1.1. *Let* $f : I \subseteq \mathbf{R} \to \mathbf{R}$ *be a differentiable mapping in* $I°$ *($I°$ is the interior of I), and let* $a, b \in I°$ *with* $a < b$. *If* $f' : (a, b) \to \mathbf{R}$ *is bounded on* (a, b), *i.e.,* $\|f'\|_\infty := \sup_{t \in (a,b)} |f'(t)| < \infty$, *then we have the inequality:*

$$(1.1) \qquad \left| f(x) - \frac{1}{b - a} \int_a^b f(t)\, dt \right| \le \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b - a)^2} \right] (b - a) \|f'\|_\infty$$

for all $x \in [a, b]$. *The constant* $\frac{1}{4}$ *is sharp in the sense that it can not be replaced by a smaller one.*

In [2], S.S. Dragomir and S. Wang applied Ostrowski's inequality in Numerical Analysis obtaining an estimation of the error bound for the quadrature rules of Riemann type in terms of the infinity norm $\|\cdot\|_\infty$. Application for special means : logarithmic mean, identric mean, p-logarithmic mean etc... were also given.

The main aim of this paper is to give an Ostrowski's type inequality for random variables whose probability density functions are in $L_\infty[a, b]$. An application for a Beta Random Variable is also given.

2 THE RESULTS

Let X be a random variable with the probability density function $f : [a, b] \subset \mathbf{R} \to \mathbf{R}_+$ and with cumulative distribution function $F(x) = \Pr(X \le x)$.

The following theorem holds

Theorem 2.1. *Let $f \in L_\infty [a, b]$ and put $\|f\|_\infty = \sup\limits_{t\in[a,b]} f(t) < \infty$. Then we have the inequality:*

$$(2.1) \qquad \left| \Pr\left(X \le x \right) - \frac{b - E\left(X\right)}{b - a} \right| \le \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2} \right)^2}{\left(b - a \right)^2} \right] \left(b - a \right) \|f\|_\infty$$

or equivalently,

$$(2.2) \qquad \left| \Pr\left(X \ge x \right) - \frac{E\left(X\right) - a}{b - a} \right| \le \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2} \right)^2}{\left(b - a \right)^2} \right] \left(b - a \right) \|f\|_\infty$$

for all $x \in [a, b]$. The constant $\frac{1}{4}$ in (2.1) and (2.2) is sharp.

Proof. Let $x, y \in [a, b]$. Then

$$\left| F\left(x\right) - F\left(y\right) \right| = \left| \int_x^y f\left(t\right) dt \right| \le \left| x - y \right| \|f\|_\infty$$

which shows that F is $\|f\|_\infty -$Lipschitzian on $[a, b]$.

Consider the kernel $p : [a, b]^2 \to \mathbf{R}$ given by

$$p\left(x, t\right) := \begin{cases} t - a & \text{if } t \in [a, x] \\[2mm] t - b & \text{if } t \in (x, b] \end{cases} .$$

Then the Riemann-Stieltjes integral $\int_a^b p\left(x, t\right) dF\left(t\right)$ exists for any $x \in [a, b]$ and the formula of integration by parts for Riemann-Stieltjes integral gives:

$$(2.3) \qquad \int_a^b p\left(x, t\right) dF\left(t\right) = \int_a^x \left(t - a\right) dF\left(t\right) + \int_x^b \left(t - b\right) dF\left(t\right)$$

$$= \left(t - a\right) F\left(t\right)\big|_a^b - \int_a^x F\left(t\right) dt + \left(t - b\right) F\left(t\right)\big|_x^b - \int_x^b F\left(t\right) dt$$

$$= \left(b - a\right) F\left(x\right) - \int_a^b F\left(t\right) dt.$$

The integration by parts formula for Riemann-Stieltjes integral also gives

$$(2.4) \qquad E\left(X\right) = \int_a^b t\, dF\left(t\right) = t F\left(t\right)\big|_a^b - \int_a^b F\left(t\right) dt$$

$$= bF(b) - aF(a) - \int_a^b F(t)\, dt = b - \int_a^b F(t)\, dt.$$

Now, using (2.3) and (2.4), we get the equality

$$(2.5) \qquad (b-a)\, F(x) + E(X) - b = \int_a^b p(x,t)\, dF(t),$$

for all $x \in [a, b]$.

Now, assume that

$$\Delta_n := a = x_0^{(n)} < x_1^{(n)} < ... < x_{n-1}^{(n)} < x_n^{(n)} = b$$

is a sequence of divisions with $\nu(\Delta_n) \to 0$ as $n \to \infty$, where

$$\nu(\Delta_n) := \max\left\{ x_{i+1}^{(n)} - x_i^{(n)} : i = 0, ..., n-1 \right\}.$$

If $p : [a, b] \to \mathbf{R}$ is Riemann integrable on $[a, b]$ and $\nu : [a, b] \to \mathbf{R}$ is L-lipschitzian (lipschitzian with the constant L), then :

$$(2.6) \qquad \left| \int_a^b p(x)\, d\nu(x) \right| = \left| \lim_{\nu(\Delta_n) \to 0} \sum_{i=0}^{n-1} p\left(\xi_i^{(n)}\right) \left[\nu\left(x_{i+1}^{(n)}\right) - \nu\left(x_i^{(n)}\right) \right] \right|$$

$$\leq \lim_{\nu(\Delta_n) \to 0} \sum_{i=0}^{n-1} \left| p\left(\xi_i^{(n)}\right) \right| \left(x_{i+1}^{(n)} - x_i^{(n)} \right) \left| \frac{\nu\left(x_{i+1}^{(n)}\right) - \nu\left(x_i^{(n)}\right)}{x_{i+1}^{(n)} - x_i^{(n)}} \right|$$

$$\leq L \lim_{\nu(\Delta_n) \to 0} \sum_{i=0}^{n-1} \left| p\left(\xi_i^{(n)}\right) \right| \left(x_{i+1}^{(n)} - x_i^{(n)} \right) = L \int_a^b |p(x)|\, dx.$$

Applying the inequality (2.6) for the mappings $p(x, \cdot)$ and $F(\cdot)$, we get :

$$\left| \int_a^b p(x,t)\, dF(t) \right| \leq \|f\|_\infty \int_a^b |p(x,t)|\, dt$$

$$= \|f\|_\infty \left[\int_a^x (t-a)\, dt + \int_x^b (b-t)\, dt \right] = \|f\|_\infty \left[\frac{(x-a)^2 + (b-x)^2}{2} \right]$$

$$= \left[\frac{1}{4}(b-a)^2 + \left(x - \frac{a+b}{2} \right)^2 \right] \|f\|_\infty$$

for all $x \in [a, b]$.

Finally, by the identity (2.5) we deduce:

$$\left| F\left(x\right) - \frac{b - E\left(X\right)}{b - a} \right| \leq \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{\left(b - a\right)^2} \right] \left(b - a\right) \left\| f \right\|_\infty$$

for all $x \in [a, b]$ which is (2.1).

Now, taking into account the fact that

$$\Pr\left(X \geq x\right) = 1 - \Pr\left(X \leq x\right)$$

the inequality (2.2) is also obtained.

To prove the sharpness of the constant $k = \frac{1}{4}$, assume that the inequality (2.1) holds with a constant $c > 0$, i.e.,

$$(2.7) \qquad \left| \Pr\left(X \leq x\right) - \frac{b - E\left(X\right)}{b - a} \right| \leq \left[c + \frac{\left(x - \frac{a+b}{2}\right)^2}{\left(b - a\right)^2} \right] \left(b - a\right) \left\| f \right\|_\infty$$

for all $x \in [a, b]$.

Assume that X_0 is a random variable having the probability density function $f_0 : [0, 1] \to \mathbf{R}$, $f_0\left(t\right) = 1$. Then

$$\Pr\left(X_0 \geq x\right) = x, \qquad x \in [0, 1],$$

$$E\left(X_0\right) = \frac{1}{2},$$

and

$$\left\| f_0 \right\|_\infty = 1.$$

Consequently, (2.7) becomes

$$\left| x - \frac{1}{2} \right| \leq c + \left(x - \frac{1}{2} \right)^2 \qquad \text{for all } x \in [0, 1].$$

Choosing $x = 0$, we get $c \geq \frac{1}{4}$ and the sharpness of the constant is thus proved. ∎

The above theorem has some interesting corollaries for the expectation of X as follows:

Corollary 2.2. *Under the above assumptions, we have the double inequality:*

$$(2.8) \qquad b - \frac{1}{2}\left(b - a\right)^2 \left\| f \right\|_\infty \leq E\left(X\right) \leq a + \frac{1}{2}\left(b - a\right)^2 \left\| f \right\|_\infty.$$

Proof. We know that

$$a \leq E\left(X\right) \leq b.$$

Now, choose $x = a$ in (2.1) to obtain

$$\left| \frac{b - E\left(X\right)}{b - a} \right| \leq \frac{1}{2}\left(b - a\right) \left\| f \right\|_\infty$$

i.e.,

$$b - E(X) \le \frac{1}{2}(b-a)^2 \|f\|_\infty$$

which is equivalent to the first inequality in (2.8).

Also, choose $x = b$ in (2.1) to get

$$\left| 1 - \frac{b - E(X)}{b - a} \right| \le \frac{1}{2}(b-a)\|f\|_\infty$$

i.e.,

$$E(X) - a \le \frac{1}{2}(b-a)^2 \|f\|_\infty$$

and the inequality (2.8) is proved. ∎

Remark 2.1. *We know that*

$$1 = \int_a^b f(x)\,dx \le (b-a)\|f\|_\infty$$

which gives us

$$\|f\|_\infty \ge \frac{1}{b-a}.$$

Now, if we assume that $\|f\|_\infty$ is not too large, i.e.,

$$(2.9) \qquad \|f\|_\infty \le \frac{2}{b-a}$$

then

$$a + \frac{1}{2}(b-a)^2 \|f\|_\infty \le b$$

and

$$b - \frac{1}{2}(b-a)^2 \|f\|_\infty \ge a$$

which shows that the inequality (2.8) is a tighter inequality than $a \le E(X) \le b$ when (2.9) holds.

Another equivalent inequality to (2.8) which can be more useful in practice is the following one:

Corollary 2.3. *With the above assumptions, we have the inequality*

$$(2.10) \qquad \left| E(X) - \frac{a+b}{2} \right| \le \frac{(b-a)^2}{2}\left(\|f\|_\infty - \frac{1}{b-a} \right).$$

Proof. From the inequality (2.8) we have :

$$b - \frac{a-b}{2} - \frac{1}{2}(b-a)^2 \|f\|_\infty \leq E(X) - \frac{a+b}{2}$$

$$\leq a - \frac{a+b}{2} + \frac{1}{2}(b-a)^2 \|f\|_\infty$$

i.e.,

$$-\frac{(b-a)^2}{2}\left(\|f\|_\infty - \frac{1}{b-a}\right) \leq E(X) - \frac{a+b}{2}$$

$$\leq \frac{(b-a)^2}{2}\left(\|f\|_\infty - \frac{1}{b-a}\right)$$

which is exactly (2.10). ∎

This corollary provides the possibility of finding a sufficient condition in terms of $\|f\|_\infty$ for the expectation $E(X)$ to be close to the mean value $\frac{a+b}{2}$.

Corollary 2.4. *Let X and f be as above and $\varepsilon > 0$. If*

(2.11) $$\|f\|_\infty \leq \frac{1}{b-a} + \frac{2\varepsilon}{(b-a)^2}$$

then

$$\left| E(X) - \frac{a+b}{2} \right| \leq \varepsilon.$$

The proof is obvious and we shall omit the details.

The following corollary of Theorem 2.1 also holds:

Corollary 2.5. *Let X and f be as above. Then we have the inequality:*

(2.12) $$\left| \Pr\left(X \leq \frac{a+b}{2} \right) - \frac{1}{2} \right|$$

$$\leq \frac{1}{4}(b-a)\|f\|_\infty + \frac{1}{b-a}\left| E(X) - \frac{a+b}{2} \right|$$

$$\leq \frac{3}{4}(b-a)\|f\|_\infty - \frac{1}{2}.$$

Proof. If we choose in (2.1) $x = \frac{a+b}{2}$, we get

$$\left| \Pr\left(X \leq \frac{a+b}{2} \right) - \frac{b - E(X)}{b-a} \right| \leq \frac{1}{4}(b-a)\|f\|_\infty$$

which is clearly equivalent to

$$\left| \Pr \left(X \le \frac{a+b}{2} \right) - \frac{1}{2} + \frac{1}{b-a} \left(E\left(X \right) - \frac{a+b}{2} \right) \right|$$

$$\le \frac{1}{4} \left(b-a \right) \left\| f \right\|_{\infty}.$$

Now, using the triangle inequality, we get

$$\left| \Pr \left(X \le \frac{a+b}{2} \right) - \frac{1}{2} \right|$$

$$= \left| \Pr \left(X \le \frac{a+b}{2} \right) - \frac{1}{2} + \frac{1}{b-a} \left(E\left(X \right) - \frac{a+b}{2} \right) - \frac{1}{b-a} \left(E\left(X \right) - \frac{a+b}{2} \right) \right|$$

$$\le \left| \Pr \left(X \le \frac{a+b}{2} \right) - \frac{1}{2} + \frac{1}{b-a} \left(E\left(X \right) - \frac{a+b}{2} \right) \right| + \left| \frac{1}{b-a} \left(E\left(X \right) - \frac{a+b}{2} \right) \right|$$

$$\le \frac{1}{4} \left(b-a \right) \left\| f \right\|_{\infty} + \frac{1}{b-a} \left| E\left(X \right) - \frac{a+b}{2} \right| \le \frac{3}{4} \left(b-a \right) \left\| f \right\|_{\infty} - \frac{1}{2}$$

and the desired inequality is proved. ∎

Remark 2.2. *A similar result applies for*

$$\Pr \left(X \ge \frac{a+b}{2} \right).$$

We shall omit the details.
Finally, the following result holds:

Corollary 2.6. *Let X and f be as above. Then we have the inequality:*

$$(2.13) \quad \left| E\left(X \right) - \frac{a+b}{2} \right| \le \frac{1}{4} \left(b-a \right)^2 \left\| f \right\|_{\infty} + \left(b-a \right) \left| \Pr \left(X \le \frac{a+b}{2} \right) - \frac{1}{2} \right|.$$

Proof. As in the above corollary, we have:

$$\frac{1}{b-a} \left| E\left(X \right) - \frac{a+b}{2} \right|$$

$$\le \left| \Pr \left(X \le \frac{a+b}{2} \right) - \frac{1}{2} + \frac{1}{b-a} \left(E\left(X \right) - \frac{a+b}{2} \right) \right| + \left| \Pr \left(X \le \frac{a+b}{2} \right) - \frac{1}{2} \right|$$

$$\le \frac{1}{4} \left(b-a \right) \left\| f \right\|_{\infty} + \left| \Pr \left(X \le \frac{a+b}{2} \right) - \frac{1}{2} \right|$$

from whence we get (2.13). ∎

Remark 2.3. *If we assume that $f \in C\left[a, b\right]$, then F is differentiable on (a, b) and by Ostrowski's inequality (1.1) applied for F we get*

$$\left| F\left(x\right) - \frac{1}{b-a} \int_a^b F\left(t\right) dt \right| \leq \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{\left(b-a\right)^2} \right] \left(b - a\right) \left\| f \right\|_\infty$$

for all $x \in \left[a, b\right]$.

Now, using the identity (2.4) we recapture the inequality (2.1) and (2.2) for random variables whose probability density functions are continuous on $\left[a, b\right]$.

3   APPLICATION FOR A BETA RANDOM VARIABLE

A Beta random variable X, with parameters (p, q) has the probability density function

$$f\left(x : p, q\right) := \frac{x^{p-1}\left(1-x\right)^{q-1}}{B\left(p, q\right)}; \qquad 0 < x < 1$$

where

$$\Omega = \left\{ (p, q) : p, q > 0 \right\}$$

and

$$B\left(p, q\right) := \int_0^1 t^{p-1}\left(1-t\right)^{q-1} dt.$$

We observe that for $0 < p < 1$,

$$\left\| f\left(\cdot, p, q\right) \right\|_\infty = \sup_{x \in (0,1)} \left[\frac{x^{p-1}\left(1-x\right)^{q-1}}{B\left(p, q\right)} \right] = \infty.$$

Assume that $p, q \geq 1$. Then

$$\frac{df\left(x, p, q\right)}{dx} = \frac{1}{B\left(p, q\right)} \left[\left(p-1\right) x^{p-2}\left(1-x\right)^{q-1} - \left(q-1\right) x^{p-1}\left(1-x\right)^{q-2} \right]$$

$$= \frac{x^{p-2}\left(1-x\right)^{q-2}}{B\left(p, q\right)} \left[\left(p-1\right)\left(1-x\right) - \left(q-1\right) x \right]$$

$$= \frac{x^{p-2}\left(1-x\right)^{q-2}}{B\left(p, q\right)} \left[-\left(p+q-2\right) x + \left(p-1\right) \right].$$

We observe that

$$\frac{df\left(x, p, q\right)}{dx} = 0$$

iff $x_0 = \frac{p-1}{p+q-2} \in (0,1)$ (for $p, q > 1$) and then $\frac{df(x,p,q)}{dx} > 0$ on $(0, x_0)$ and $\frac{df(x,p,q)}{dx} < 0$ on $(x_0, 1)$. Consequently,

$$\|f(\cdot, p, q)\|_{\infty} = f(x_0; p, q) = \frac{(p-1)^{p-1}(q-1)^{q-1}}{B(p,q)(p+q-2)^{p+q-2}}.$$

On the other hand we have

$$E(X) = \frac{1}{B(p,q)} \int_0^1 x \cdot x^{p-1}(1-x)^{q-1}\, dx = \frac{B(p+1,q)}{B(p,q)} = \frac{p}{p+q}.$$

Now, using Theorem 2.1 we can state the following proposition:

Proposition 3.1. *Let X be a Beta random variable with the parameters (p, q), $p, q \geq 1$. Then we have the inequality :*

$$\left| \Pr(X \leq x) - \frac{q}{p+q} \right|$$
$$\leq \left[\frac{1}{4} + \left(x - \frac{1}{2} \right)^2 \right] \times \frac{(p-1)^{p-1}(q-1)^{q-1}}{B(p,q)(p+q-2)^{p+q-2}}$$

and

$$\left| \Pr(X \geq x) - \frac{p}{p+q} \right|$$
$$\leq \left[\frac{1}{4} + \left(x - \frac{1}{2} \right)^2 \right] \times \frac{(p-1)^{p-1}(q-1)^{q-1}}{B(p,q)(p+q-2)^{p+q-2}}$$

where $x \in [0, 1]$. Particularly, we have

$$\left| \Pr\left(X \leq \frac{1}{2} \right) - \frac{q}{p+q} \right| \leq \frac{1}{4} \cdot \frac{(p-1)^{p-1}(q-1)^{q-1}}{B(p,q)(p+q-2)^{p+q-2}}$$

and

$$\left| \Pr\left(X \geq \frac{1}{2} \right) - \frac{p}{p+q} \right| \leq \frac{1}{4} \cdot \frac{(p-1)^{p-1}(q-1)^{q-1}}{B(p,q)(p+q-2)^{p+q-2}}.$$

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