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*An Inequality of Ostrowski Type For Mappings Whose Second Derivatives are Bounded and Applications*

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# AN INEQUALITY OF OSTROWSKI TYPE FOR MAPPINGS WHOSE SECOND DERIVATIVES ARE BOUNDED AND APPLICATIONS

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ABSTRACT. An inequality of Ostrowski type for twice differentiable mappings whose derivatives are bounded and applications in Numerical Integration and for special means (logarithmic mean, identric mean, p-logarithmic mean etc...) are given.

## 1 INTRODUCTION

In 1938, Ostrowski (see for example [2, p. 468]) proved the following integral inequality:

**Theorem 1.1.** *Let $f : I \subseteq \mathbf{R} \to \mathbf{R}$ be a differentiable mapping on $I^\circ$ ($I^\circ$ is the interior of $I$), and let $a, b \in I^\circ$ with $a < b$. If $f' : (a, b) \to \mathbf{R}$ is bounded on $(a, b)$, i.e., $\|f'\|_\infty := \sup\limits_{t \in (a,b)} |f'(t)| < \infty$, then we have the inequality:*

$$(1.1) \qquad \left| f(x) - \frac{1}{b-a} \int_a^b f(t)\, dt \right| \leq \left[ \frac{1}{4} + \frac{\left( x - \frac{a+b}{2} \right)^2}{(b-a)^2} \right] (b-a) \|f'\|_\infty$$

*for all $x \in [a, b]$. The constant $\frac{1}{4}$ is sharp in the sense that it can not be replaced by a smaller one.*

For some applications of Ostrowski's inequality to some special means and numerical quadrature rules, we refer to the recent paper [1] by S.S. Dragomir and S. Wang.

In 1976, G.V. Milovanović and J.E. Pečarić proved a generalization of Ostrowski's inequality for $n$-time differentiable mappings (see for example [2, p.468] ) from which we would like to mention only the case of twice differentiable mappings [2, p. 470].

**Theorem 1.2.** *Let $f : [a, b] \to \mathbf{R}$ be a twice differentiable mapping such that $f'' : (a, b) \to \mathbf{R}$ is bounded on $(a, b)$, i.e., $\|f''\|_\infty = \sup\limits_{t \in (a,b)} |f''(t)| < \infty$. Then we have the inequality:*

$$(1.2) \qquad \left| \frac{1}{2} \left[ f(x) + \frac{(x-a) f(a) + (b-x) f(b)}{b-a} \right] - \frac{1}{b-a} \int_a^b f(t)\, dt \right|$$

$$\leq \frac{\|f''\|_\infty}{4} (b-a)^2 \left[ \frac{1}{12} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right]$$

*for all* $x \in [a, b]$.

In this paper we point out an inequality of Ostrowski's type which is similar, in a sense, to Milovanović-Pečarić result and apply it for Special Means and in Numerical Integration.

## 2   SOME INTEGRAL INEQUALITIES

The following result holds.

**Theorem 2.1.** *Let* $f : [a, b] \rightarrow \mathbf{R}$ *be a twice differentiable mapping on* $(a, b)$ *and* $f'' : (a, b) \rightarrow \mathbf{R}$ *is bounded, i.e.,* $\|f''\|_\infty = \sup\limits_{t \in (a,b)} |f''(t)| < \infty$. *Then we have the inequality:*

$$(2.1) \qquad \left| f(x) - \frac{1}{b-a} \int_a^b f(t)\, dt - \left(x - \frac{a+b}{2}\right) f'(x) \right|$$

$$\leq \left[ \frac{1}{24}(b-a)^2 + \frac{1}{2}\left(x - \frac{a+b}{2}\right)^2 \right] \|f''\|_\infty \leq \frac{(b-a)^2}{6} \|f''\|_\infty$$

*for all* $x \in [a, b]$.

*Proof.* Let us define the mapping $K(\cdot, \cdot) : [a, b]^2 \rightarrow \mathbf{R}$ given by

$$K(x, t) := \begin{cases} \dfrac{(t-a)^2}{2} & \text{if } t \in [a, x] \\[4mm] \dfrac{(t-b)^2}{2} & \text{if } t \in (x, b] \end{cases} .$$

Integrating by parts, we have successively

$$\int_a^b K(x, t) f''(t)\, dt = \int_a^x \frac{(t-a)^2}{2} f''(t)\, dt + \int_x^b \frac{(t-b)^2}{2} f''(t)\, dt$$

$$= \frac{(t-a)^2}{2} f'(t)\Big|_a^x - \int_a^x (t-a) f'(t)\, dt + \frac{(t-b)^2}{2} f'(t)\Big|_x^b - \int_x^b (t-b) f'(t)\, dt$$

$$= \frac{(x-a)^2}{2} f'(x) - \left[ (t-a) f(t)\Big|_a^x - \int_a^x f(t)\, dt \right]$$

$$-\frac{(b-x)^2}{2}f'(x) - \left[(t-b)f(t)|_x^b - \int_x^b f(t)\,dt\right]$$

$$= \frac{1}{2}\left[(x-a)^2 - (b-x)^2\right]f'(x)$$

$$-(x-a)f(x) + \int_a^x f(t)\,dt + (x-b)f(x) + \int_x^b f(t)\,dt$$

$$= (b-a)\left(x-\frac{a+b}{2}\right)f'(x) - (b-a)f(x) + \int_a^b f(t)\,dt$$

from which we get the integral identity:

(2.2)
$$\int_a^b f(t)\,dt = (b-a)f(x) - (b-a)\left(x-\frac{a+b}{2}\right)f'(x) + \int_a^b K(x,t)f''(t)\,dt$$

for all $x \in [a,b]$.

Using the identity $(2.2)$, we have

(2.3)
$$\left|f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt - \left(x-\frac{a+b}{2}\right)f'(x)\right|$$

$$= \frac{1}{b-a}\left|\int_a^b K(x,t)f''(t)\,dt\right| \le \frac{1}{b-a}\|f''\|_\infty \int_a^b |K(x,t)|\,dt$$

$$= \frac{1}{b-a}\|f''\|_\infty \left[\int_a^x \frac{(t-a)^2}{2}\,dt + \int_x^b \frac{(t-b)^2}{2}\,dt\right]$$

$$= \frac{1}{b-a}\|f''\|_\infty \left[\frac{(t-a)^3}{6}\bigg|_a^x + \frac{(t-b)^3}{6}\bigg|_x^b\right]$$

$$= \frac{1}{b-a}\|f''\|_\infty \left[\frac{(x-a)^3 + (b-x)^3}{6}\right].$$

Now, observe that

$$(x-a)^3 + (b-x)^3 = (b-a)\left[(x-a)^2 + (b-x)^2 - (x-a)(b-x)\right]$$

$$= (b-a)[(x-a+b-x)^2 - 3(x-a)(b-x)]$$

$$= (b-a)[(b-a)^2 + 3[x^2 - (a+b)x + ab]]$$

$$= (b-a)\left[(b-a)^2 + 3\left[\left(x - \frac{a+b}{2}\right)^2 - \left(\frac{b-a}{2}\right)^2\right]\right]$$

$$= (b-a)\left[\left(\frac{b-a}{2}\right)^2 + 3\left(x - \frac{a+b}{2}\right)^2\right].$$

Using $(2.3)$, we get the desired inequality $(2.1)$. ∎

**Corollary 2.2.** *Under the above assumptions, we have the mid-point inequality:*

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \leq \frac{(b-a)^2}{24}\|f''\|_\infty.$$

This follows by Theorem 2.1, choosing $x = \frac{a+b}{2}$.

**Corollary 2.3.** *Under the above assumptions we have the following trapezoid like inequality:*

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)\,dt - \frac{b-a}{4}(f'(b) - f'(a)) \right|$$

$$\leq \frac{(b-a)^2}{6}\|f''\|_\infty.$$

This follows using Theorem 2.1 with $x = a,\quad x = b$, adding the results and using the triangle inequality for the modulus.

## 3   APPLICATIONS IN NUMERICAL INTEGRATION

Let $I_n : a = x_0 < x_1 < ... < x_{n-1} < x_n = b$ be a division of the interval $[a,b]$, $\xi_i \in [x_i, x_{i+1}]\quad(i = 0, ..., n-1)$. We have the following quadrature formula:

**Theorem 3.1.** *Let $f : [a,b] \to \mathbf{R}$ be a twice differentiable mapping on $(a,b)$ whose second derivative $f'' : (a,b) \to \mathbf{R}$ is bounded, i.e., $\|f''\|_\infty < \infty$. Then we have the following :*

$$(3.1)\qquad \int_a^b f(x)\,dx = A(f, f', \boldsymbol{\xi}, I_n) + R(f, f', \boldsymbol{\xi}, I_n)$$

*where*

$$A\left(f, f', \boldsymbol{\xi}, I_n\right) = \sum_{i=0}^{n-1} h_i f\left(\xi_i\right) - \sum_{i=0}^{n-1} f'\left(\xi_i\right)\left(\xi_i - \frac{x_i + x_{i+1}}{2}\right) h_i$$

*and the remainder satisfies the estimation:*

$$(3.2) \quad \left|R\left(f, f', \boldsymbol{\xi}, I_n\right)\right| \le \left[\frac{1}{24}\sum_{i=0}^{n-1} h_i^3 + \frac{1}{2}\sum_{i=0}^{n-1} h_i\left(\xi_i - \frac{x_i + x_{i+1}}{2}\right)^2\right]\|f''\|_\infty$$

$$\le \frac{\|f''\|}{6}\sum_{i=0}^{n-1} h_i^3.$$

*for all $\xi_i$ as above, where $h_i := x_{i+1} - x_i \quad (i = 0, ..., n-1)$.*

*Proof.* Apply Theorem 2.1 on the interval $[x_i, x_{i+1}] \quad (i = 0, ..., n-1)$ to get

$$\left|\int_{x_i}^{x_{i+1}} f\left(t\right) dt - h_i f\left(\xi_i\right) + \left(\xi_i - \frac{x_i + x_{i+1}}{2}\right) h_i f'\left(\xi_i\right)\right|$$

$$\le \left[\frac{1}{24} h_i^3 + \frac{1}{2} h_i\left(\xi_i - \frac{x_i + x_{i+1}}{2}\right)^2\right]\|f''\|_\infty \le \frac{\|f''\|_\infty}{6} h_i^3.$$

Summing over $i$ from 0 to $n-1$ and using the generalized triangle inequality we deduce the desired estimation. ▮

**Remark 3.1.** *Choosing $\xi_i = \frac{x_i + x_{i+1}}{2}$, we recapture the midpoint quadrature formula*

$$\int_a^b f\left(x\right) dx = A_M\left(f, I_n\right) + R_M\left(f, I_n\right)$$

*where the remainder $R_M\left(f, I_n\right)$ satisfies the estimation*

$$\left|R_M\left(f, I_n\right)\right| \le \frac{\|f''\|_\infty}{24}\sum_{i=0}^{n-1} h_i^3.$$

## 4   APPLICATIONS FOR SPECIAL MEANS

Let us recall the following means :

a) The arithmetic mean

$$A = A\left(a, b\right) := \frac{a + b}{2}, \qquad a, b \ge 0;$$

b) The geometric mean:

$$G = G(a, b) := \sqrt{ab}, \qquad a, b \geq 0;$$

c) The harmonic mean:

$$H = H(a, b) := \frac{2}{\dfrac{1}{a} + \dfrac{1}{b}}, \qquad a, b \geq 0;$$

d) The logarithmic mean:

$$L = L(a, b) := \begin{cases} a & \text{if } a = b, \qquad a, b > 0; \\[2mm] \dfrac{b - a}{\ln b - \ln a} & \text{if } a \neq b \end{cases}$$

e) The identric mean:

$$I = I(a, b) := \begin{cases} a & \text{if } a = b, \qquad a, b > 0; \\[2mm] \dfrac{1}{e} \left( \dfrac{b^b}{a^a} \right)^{\frac{1}{b-a}} & \text{if } a \neq b \end{cases}$$

f) The $p$-logarithmic mean:

$$L_p = L_p(a, b) := \begin{cases} \left[ \dfrac{b^{p+1} - a^{p+1}}{(p+1)(b-a)} \right]^{\frac{1}{p}} & \text{if } a \neq b; \\[3mm] a & \text{if } a = b \end{cases}$$

where $p \in \mathbf{R} \backslash \{-1, 0\}, \quad a, b > 0.$

The following simple relationships are known in the literature

$$H \leq G \leq L \leq I \leq A.$$

It is also known that $L_p$ is monotonically increasing in $p \in \mathbf{R}$ with $L_0 = I$ and $L_{-1} = L$.

(1) Consider the mapping $f : (0, \infty) \to \mathbf{R}, \quad f(x) = x^r, \quad r \in \mathbf{R} \backslash \{-1, 0\}.$

Then, we have, for $0 < a < b$:

$$\frac{1}{b-a} \int_a^b f(x)\, dx = L_r^r(a, b)$$

and

$$\|f''\|_\infty = |r(r-1)| \, \delta_r(a, b), \qquad r \in \mathbf{R} \backslash \{-1, 0\};$$

where

$$\delta_r(a,b) := \begin{cases} b^{r-1} & \text{if } r \in (1, \infty) \\ \\ a^{r-1} & \text{if } r \in (-\infty, 1) \setminus \{-1, 0\} \end{cases}.$$

Using the inequality (2.1) we have the result:

(4.1) $$\left| x^r - L_r^r(a,b) - r(x-A)x^{r-1} \right|$$

$$\leq \frac{|r(r-1)|}{6} \left[ \frac{1}{4}(b-a)^2 + 3(x-A)^2 \right] \delta_r(a,b)$$

$$\leq \frac{|r(r-1)|(b-a)^2}{6} \delta_r(a,b)$$

for all $x \in [a,b]$. If in (4.1) we choose $x = A$, we get

(4.2) $$|A^r - L_r^r| \leq \frac{|r(r-1)|(b-a)^2}{24} \delta_r(a,b).$$

(2) Consider the mapping $f(x) = \frac{1}{x}, \quad x \in [a,b] \subset (0, \infty)$. Then we have :

$$\frac{1}{b-a} \int_a^b f(x)\,dx = L_{-1}^{-1}(a,b)$$

and

$$\|f''\|_\infty = \frac{2}{a^3}.$$

Applying the inequality (2.1) for the above mapping, we get

$$\left| \frac{1}{x} - \frac{1}{L} + \frac{x-A}{x^2} \right| \leq \frac{1}{3a^3} \left[ \frac{1}{4}(b-a)^2 + 3(x-A)^2 \right]$$

$$\leq \frac{(b-a)^2}{3a^3}$$

which is equivalent to

(4.3) $$|x(L-x) - L(A-x)| \leq \frac{x^2 L}{3a^3} \left[ \frac{1}{4}(b-a)^2 + 3(x-A)^2 \right]$$

$$\leq \frac{x^2 L(b-a)^2}{3a^3}$$

for all $x \in [a,b]$. Now, if we choose in (4.3), $x = A$, we get

(4.4) $$0 \leq A - L \leq \frac{(b-a)^2 AL}{12a^3}.$$

If in (4.3) we choose $x = L$, we get

(4.5) $$0 \leq A - L \leq \frac{L^2}{3a^3} \left[ \frac{1}{4}(b-a)^2 + 3(L-A)^2 \right].$$

(3) Let us consider the mapping

$$f(x) = \ln x, \qquad x \in [a,b] \subset (0, \infty).$$

Then we have :

$$\frac{1}{b-a} \int_a^b f(x)\, dx = \ln I(a,b),$$

and

$$\|f''\|_\infty = \frac{1}{a^2}.$$

Inequality (2.1) gives us

(4.6)
$$\left| \ln x - \ln I - \frac{x-A}{x} \right|$$

$$\leq \frac{1}{6a^2} \left[ \frac{1}{4}(b-a)^2 + 3(x-A)^2 \right] \leq \frac{(b-a)^2}{6a^2}.$$

Now, if in (4.6) we choose $x = A$, we get

(4.7)
$$1 \leq \frac{A}{I} \leq \exp\left[ \frac{1}{24a^2}(b-a)^2 \right].$$

If in (4.6) we choose $x = I$, we get

(4.8)
$$0 \leq A - I \leq \frac{I}{6a^2}\left[ \frac{1}{4}(b-a)^2 + 3(A-I)^2 \right].$$

### References

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