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SOME OSTROWSKI TYPE INEQUALITIES FOR N -TIME DIFFERENTIABLE MAPPINGS AND APPLICATIONS

P. CERONE, S.S. DRAGOMIR AND J. ROUMELIOTIS

ABSTRACT. Some generalizations of the Ostrowski inequality for n -time differentiable mappings are given. Applications in Numerical Integration and for power series expansions are also presented.

1 INTRODUCTION

In 1938, Ostrowski (see for example [3, p.468]) proved the following integral inequality:

Let $f : I \subseteq \mathbf{R} \rightarrow \mathbf{R}$ be a differentiable mapping on I° (I° is the interior of I), and let $a, b \in I^\circ$ with $a < b$. If $f' : (a, b) \rightarrow \mathbf{R}$ is bounded on (a, b) , i.e., $\|f'\|_\infty := \sup_{t \in (a, b)} |f'(t)| < \infty$, then we have the inequality:

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] (b-a) \|f'\|_\infty$$

for all $x \in [a, b]$.

The constant $\frac{1}{4}$ is sharp in the sense that it can not be replaced by a smaller one.

For applications of Ostrowski's inequality to some special means and some numerical quadrature rules, we refer the reader to the recent paper [2] by S.S. Dragomir and S. Wang.

In 1976, G.V. Milovanović and J.E. Pečarić (see for example [3, p. 468]), proved the following generalization of Ostrowski's result:

Let $f : [a, b] \rightarrow \mathbf{R}$ be an n -times differentiable function, $n \geq 1$, and such that $\|f^{(n)}\|_\infty := \sup_{t \in (a, b)} |f^{(n)}(t)| < \infty$. Then

$$\left| \frac{1}{n} \left(f(x) + \sum_{k=1}^{n-1} \frac{n-k}{k!} \cdot \frac{f^{(k-1)}(a)(x-a)^k - f^{(k-1)}(b)(x-b)^k}{b-a} \right) - \frac{1}{b-a} \int_a^b f(t) dt \right|$$

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$$\leq \frac{\|f^{(n)}\|_{\infty}}{n(n+1)!} \cdot \frac{(x-a)^{n+1} + (b-x)^{n+1}}{b-a}$$

for all $x \in [a, b]$.

In [1], P. Cerone, S.S. Dragomir and J. Roumeliotis proved the following Ostrowski type inequality for twice differentiable mappings:

Let $f : [a, b] \rightarrow \mathbf{R}$ be a twice differentiable mapping on (a, b) and $f'' : (a, b) \rightarrow \mathbf{R}$ is bounded, i.e., $\|f''\|_{\infty} = \sup_{t \in (a, b)} |f''(t)| < \infty$. Then we have the inequality:

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt - \left(x - \frac{a+b}{2}\right) f'(x) \right| \leq \left[\frac{1}{24} (b-a)^2 + \frac{1}{2} \left(x - \frac{a+b}{2}\right)^2 \right] \|f''\|_{\infty} \leq \frac{(b-a)^2}{6} \|f''\|_{\infty}$$

for all $x \in [a, b]$.

In this paper we establish another generalization of Ostrowski inequality for n -time differentiable mappings which naturally generalizes the result from [1] and apply it in Numerical Integration and for power series expansions of functions on an interval.

2 INTEGRAL IDENTITIES

The following lemma holds:

Lemma 2.1. *Let $f : [a, b] \rightarrow \mathbf{R}$ be a mapping such that $f^{(n-1)}$ is absolutely continuous on $[a, b]$. Then for all $x \in [a, b]$ we have the identity:*

$$(2.1) \quad \int_a^b f(t) dt = \sum_{k=0}^{n-1} \left[\frac{(b-x)^{k+1} + (-1)^k (x-a)^{k+1}}{(k+1)!} \right] f^{(k)}(x) + (-1)^n \int_a^b K_n(x, t) f^{(n)}(t) dt$$

where the kernel $K_n : [a, b]^2 \rightarrow \mathbf{R}$ is given by

$$(2.2) \quad K_n(x, t) := \begin{cases} \frac{(t-a)^n}{n!} & \text{if } t \in [a, x] \\ \frac{(t-b)^n}{n!} & \text{if } t \in (x, b] \end{cases}, \quad x \in [a, b]$$

and n is a natural number, $n \geq 1$.

Proof. The proof is by mathematical induction.

For $n = 1$, we have to prove the equality

$$(2.3) \quad \int_a^b f(t) dt = (b-a)f(x) - \int_a^b K_1(x, t) f^{(1)}(t) dt$$

where

$$K_1(x, t) := \begin{cases} t-a & \text{if } t \in [a, x] \\ t-b & \text{if } t \in (x, b] \end{cases}.$$

Integrating by parts, we have:

$$\begin{aligned} \int_a^b K_1(x, t) f^{(1)}(t) dt &= \int_a^x (t-a) f'(t) dt + \int_x^b (t-b) f'(t) dt \\ &= (t-a)f(t)|_a^x - \int_a^x f(t) dt + (t-b)f(t)|_x^b - \int_x^b f(t) dt \\ &= (x-a)f(x) + (b-x)f(x) - \int_a^b f(t) dt = (b-a)f(x) - \int_a^b f(t) dt \end{aligned}$$

and the identity (2.3) is proved.

Assume that (2.1) holds for " n " and let us prove it for " $n+1$ ". That is, we have to prove the equality

$$(2.4) \quad \begin{aligned} \int_a^b f(t) dt &= \sum_{k=0}^n \left[\frac{(b-x)^{k+1} + (-1)^k (x-a)^{k+1}}{(k+1)!} \right] f^{(k)}(x) \\ &\quad + (-1)^{n+1} \int_a^b K_{n+1}(x, t) f^{(n+1)}(t) dt \end{aligned}$$

where, obviously,

$$K_{n+1}(t) := \begin{cases} \frac{(t-a)^{n+1}}{(n+1)!} & \text{if } t \in [a, x] \\ \frac{(t-b)^{n+1}}{(n+1)!} & \text{if } t \in (x, b] \end{cases}.$$

We have

$$\int_a^b K_{n+1}(x, t) f^{(n+1)}(t) dt = \int_a^x \frac{(t-a)^{n+1}}{(n+1)!} f^{(n+1)}(t) dt + \int_x^b \frac{(t-b)^{n+1}}{(n+1)!} f^{(n+1)}(t) dt$$

and integrating by parts gives

$$\begin{aligned}
 \int_a^b K_{n+1}(x, t) f^{(n+1)}(t) dt &= \frac{(t-a)^{n+1}}{(n+1)!} f^{(n)}(t) \Big|_a^x - \frac{1}{n!} \int_a^x (t-a)^n f^{(n)}(t) dt \\
 &\quad + \frac{(t-b)^{n+1}}{(n+1)!} f^{(n)}(t) \Big|_x^b - \frac{1}{n!} \int_x^b (t-b)^n f^{(n)}(t) dt \\
 &= \frac{(x-a)^{n+1} + (-1)^{n+2} (b-x)^{n+1}}{(n+1)!} f^{(n)}(x) - \int_a^b K_n(x, t) f^{(n)}(t) dt.
 \end{aligned}$$

That is

$$\begin{aligned}
 \int_a^b K_n(x, t) f^{(n)}(t) dt &= \frac{(x-a)^{n+1} + (-1)^{n+2} (b-x)^{n+1}}{(n+1)!} f^{(n)}(x) \\
 &\quad - \int_a^b K_{n+1}(x, t) f^{(n+1)}(t) dt.
 \end{aligned}$$

Now, using the mathematical induction hypothesis, we get

$$\begin{aligned}
 \int_a^b f(t) dt &= \sum_{k=0}^{n-1} \left[\frac{(b-x)^{k+1} + (-1)^k (x-a)^{k+1}}{(k+1)!} \right] f^{(k)}(x) \\
 &\quad + \frac{(b-x)^{n+1} + (-1)^n (x-a)^{n+1}}{(n+1)!} f^{(n)}(x) - (-1)^n \int_a^b K_{n+1}(x, t) f^{(n+1)}(t) dt \\
 &= \sum_{k=0}^n \left[\frac{(b-x)^{k+1} + (-1)^k (x-a)^{k+1}}{(k+1)!} \right] f^{(k)}(x) \\
 &\quad + (-1)^{n+1} \int_a^b K_{n+1}(x, t) f^{(n+1)}(t) dt.
 \end{aligned}$$

That is, the identity (2.4); and the theorem is thus proved. ■

Corollary 2.2. *With the above assumptions, we have the representation*

$$(2.5) \quad \int_a^b f(t) dt = \sum_{k=0}^{n-1} \left[\frac{1 + (-1)^k}{(k+1)!} \right] \frac{(b-a)^{k+1}}{2^{k+1}} f^{(k)} \left(\frac{a+b}{2} \right) \\ + (-1)^n \int_a^b M_n(t) f^{(n)}(t) dt$$

where

$$M_n(t) := \begin{cases} \frac{(t-a)^n}{n!} & \text{if } t \in [a, \frac{a+b}{2}] \\ \frac{(t-b)^n}{n!} & \text{if } t \in (\frac{a+b}{2}, b] \end{cases}.$$

The proof follows by Lemma 2.1 by choosing $x = \frac{a+b}{2}$.

Corollary 2.3. *With the above assumptions, we have the representation:*

$$(2.6) \quad \int_a^b f(t) dt = \sum_{k=0}^{n-1} \frac{(b-a)^{k+1}}{(k+1)!} \left[\frac{f^{(k)}(a) + (-1)^k f^{(k)}(b)}{2} \right] \\ + \int_a^b T_n(t) f^{(n)}(t) dt$$

where

$$T_n(t) := \frac{1}{n!} \left[\frac{(b-t)^n + (-1)^n (t-a)^n}{2} \right], \quad t \in [a, b].$$

Proof. Firstly, choose in (2.1) $x = a$ to get

$$\int_a^b f(t) dt = \sum_{k=0}^{n-1} \frac{(b-a)^{k+1}}{(k+1)!} f^{(k)}(a) + (-1)^n \int_a^b \frac{(t-b)^n}{n!} f^{(n)}(t) dt \\ = \sum_{k=0}^{n-1} \frac{(b-a)^{k+1}}{(k+1)!} f^{(k)}(a) + \int_a^b \frac{(b-t)^n}{n!} f^{(n)}(t) dt.$$

Also, if we put $x = b$ in (2.1), we get

$$\int_a^b f(t) dt = \sum_{k=0}^{n-1} (-1)^k \frac{(b-a)^{k+1}}{(k+1)!} f^{(k)}(b) + (-1)^n \int_a^b \frac{(t-a)^n}{n!} f^{(n)}(t) dt.$$

Summing the above two identities and dividing by 2, we get

$$\int_a^b f(t) dt = \sum_{k=0}^n \frac{(b-a)^{k+1}}{(k+1)!} \left[\frac{f^{(k)}(a) + (-1)^k f^{(k)}(b)}{2} \right] \\ + \int_a^b T_n(t) f^{(n)}(t) dt$$

and the corollary is proved. ■

The following Taylor-like formula with integral remainder also holds:

Corollary 2.4. *Let $g : [a, y] \rightarrow \mathbf{R}$ be a mapping such that $g^{(n)}$ is absolutely continuous on $[a, y]$. Then for all $x \in [a, y]$, we have the identity*

$$(2.7) \quad g(y) = g(a) + \sum_{k=0}^{n-1} \frac{[(y-x)^{k+1} + (-1)^k (x-a)^{k+1}]}{(k+1)!} g^{(k+1)}(x) \\ + (-1)^n \int_a^y K_n(y, t) g^{(n+1)}(t) dt.$$

The proof is obvious by Lemma 2.1 choosing $f = g'$, and $b = y$.

Remark 2.1. *If we choose $n = 1$ in (2.1), we get the identity*

$$(2.8) \quad \int_a^b f(t) dt = (b-a) f(x) - \int_a^b K_1(x, t) f'(t) dt$$

for all $x \in [a, b]$, where

$$K_1(x, t) := \begin{cases} t-a & \text{if } t \in [a, x] \\ t-b & \text{if } t \in (x, b] \end{cases}$$

which is the identity employed by S.S. Dragomir and S. Wang to prove an Ostrowski type inequality in paper [2].

If in (2.5) we choose $n = 1$, then we get

$$(2.9) \quad \int_a^b f(t) dt = (b-a) f\left(\frac{a+b}{2}\right) - \int_a^b M_1(t) f'(t) dt$$

where

$$M_1(t) = \begin{cases} t-a & \text{if } t \in [a, \frac{a+b}{2}] \\ t-b & \text{if } t \in (\frac{a+b}{2}, b] \end{cases}$$

which gives the mid-point type inequality useful in Numerical Analysis.

Also, if we put $n = 1$ in (2.6), we get the trapezoid identity

$$(2.10) \quad \int_a^b f(t) dt = \frac{f(a) + f(b)}{2} (b - a) + \int_a^b T_1(t) f'(t) dt$$

where

$$T_1(t) = \frac{a+b}{2} - t, \quad t \in [a, b].$$

Finally, if in the Taylor-like formula (2.7) we put $n = 1$, we get

$$g(y) = g(a) + (y - a) g'(x) - \int_a^y K_1(y, t) g^{(2)}(t) dt$$

where $x \in [a, y]$.

Remark 2.2. If we choose $n = 2$ in (2.1), we get the identity:

$$(2.11) \quad \int_a^b f(t) dt = (b - a) f(x) - \left(x - \frac{a+b}{2}\right) f'(x) + \int_a^b K_2(x, t) f''(t) dt$$

where

$$K_2(x, t) := \begin{cases} \frac{(t-a)^2}{2} & \text{if } t \in [a, x] \\ \frac{(t-b)^2}{2} & \text{if } t \in (x, b] \end{cases}$$

and $x \in [a, b]$, which is the identity employed by P. Cerone, S.S. Dragomir and J. Roumeliotis to prove some Ostrowski type inequalities for twice differentiable mappings in the paper [1].

If in (2.5) we choose $n = 2$, then we get

$$(2.12) \quad \int_a^b f(t) dt = (b - a) f\left(\frac{a+b}{2}\right) + \int_a^b M_2(t) f''(t) dt$$

where

$$M_2(t) = \begin{cases} \frac{(t-a)^2}{2} & \text{if } t \in [a, \frac{a+b}{2}] \\ \frac{(t-b)^2}{2} & \text{if } t \in (\frac{a+b}{2}, b] \end{cases}$$

which is the classical mid-point identity.

Also, if we put $n = 2$ in (2.6), we get the identity

$$(2.13) \quad \int_a^b f(t) dt = \frac{f(a) + f(b)}{2} (b - a) + \frac{(b - a)^2}{2} \cdot \frac{f'(a) - f'(b)}{2} \\ + \int_a^b T_2(t) f''(t) dt$$

where

$$T_2(t) = \frac{1}{2} \cdot \frac{(b - t)^2 + (t - a)^2}{2}, \quad t \in [a, b].$$

Finally, if we put $n = 2$ in (2.7), we get

$$(2.14) \quad g(y) = g(a) + (y - a) g'(x) - (y - a) \left(x - \frac{a + y}{2} \right) g''(x) \\ + \int_a^y K_2(y, t) g^{(3)}(t) dt,$$

where K_2 is as above and $a \leq x \leq y$.

3 SOME INTEGRAL INEQUALITIES FOR $\|\cdot\|_\infty$ -NORM

The following theorem holds:

Theorem 3.1. *Let $f : [a, b] \rightarrow \mathbf{R}$ be a mapping such that $f^{(n-1)}$ is absolutely continuous on $[a, b]$ and $f^{(n)} \in L_\infty[a, b]$. Then for all $x \in [a, b]$, we have the inequality:*

$$(3.1) \quad \left| \int_a^b f(t) dt - \sum_{k=0}^{n-1} \left[\frac{(b-x)^{k+1} + (-1)^k (x-a)^{k+1}}{(k+1)!} \right] f^{(k)}(x) \right| \\ \leq \frac{\|f^{(n)}\|_\infty}{(n+1)!} \left[(x-a)^{n+1} + (b-x)^{n+1} \right] \leq \frac{\|f^{(n)}\|_\infty (b-a)^{n+1}}{(n+1)!}$$

where

$$\|f^{(n)}\|_\infty := \sup_{t \in [a, b]} |f^{(n)}(t)| < \infty.$$

Proof. Using the identity (2.1), we have:

$$\left| \int_a^b f(t) dt - \sum_{k=0}^{n-1} \left[\frac{(b-x)^{k+1} + (-1)^k (x-a)^{k+1}}{(k+1)!} \right] f^{(k)}(x) \right|$$

$$\begin{aligned}
&= \left| \int_a^b K_n(x, t) f^{(n)}(t) dt \right| \leq \|f^{(n)}\|_{\infty} \int_a^b |K_n(x, t)| dt \\
&= \|f^{(n)}\|_{\infty} \left[\int_a^x \frac{(t-a)^n}{n!} dt + \int_x^b \frac{(b-t)^n}{n!} dt \right] \\
&= \frac{\|f^{(n)}\|_{\infty}}{(n+1)!} \left[(x-a)^{n+1} + (b-x)^{n+1} \right],
\end{aligned}$$

and the first part of inequality (3.1) is proved.

To prove the second inequality in (3.1), we observe that

$$(3.2) \quad (x-a)^{n+1} + (b-x)^{n+1} \leq (b-a)^{n+1},$$

for all $x \in [a, b]$. ■

Taking into account the fact that the mapping $h_n : [a, b] \rightarrow \mathbf{R}$, $h_n(x) := (x-a)^{n+1} + (b-x)^{n+1}$, has the property

$$\inf_{x \in [a, b]} h_n(x) = h_n\left(\frac{a+b}{2}\right) = \frac{(b-a)^{n+1}}{2^n},$$

then the best inequality we can get from (3.1) is the one for which $x = \frac{a+b}{2}$.

In this way, we can state the following corollary.

Corollary 3.2. *Assume that f is as in Theorem 3.1. Then we have the inequality:*

$$\begin{aligned}
(3.3) \quad &\left| \int_a^b f(t) dt - \sum_{k=0}^{n-1} \left[\frac{1 + (-1)^k}{(k+1)!} \right] \frac{(b-a)^{k+1}}{2^{k+1}} f^{(k)}\left(\frac{a+b}{2}\right) \right| \\
&\leq \frac{\|f^{(n)}\|_{\infty} (b-a)^{n+1}}{2^n (n+1)!}.
\end{aligned}$$

Another result which generalizes the trapezoid inequality, is the following one:

Corollary 3.3. *With the above assumptions, we have the inequality:*

$$\begin{aligned}
(3.4) \quad &\left| \int_a^b f(t) dt - \sum_{k=0}^{n-1} \frac{(b-a)^{k+1}}{(k+1)!} \left[\frac{f^{(k)}(a) + (-1)^k f^{(k)}(b)}{2} \right] \right| \\
&\leq \frac{(b-a)^{n+1}}{(n+1)!} \|f^{(n)}\|_{\infty} \times \begin{cases} 1 & \text{if } n = 2r \\ \frac{2^{2r+1} - 1}{2^{2r}} & \text{if } n = 2r + 1 \end{cases}.
\end{aligned}$$

Proof. Using the identity (2.6), we get

$$\begin{aligned} & \left| \int_a^b f(t) dt - \sum_{k=0}^{n-1} \frac{(b-a)^{k+1}}{(k+1)!} \left[\frac{f^{(k)}(a) + (-1)^k f^{(k)}(b)}{2} \right] \right| \\ &= \left| \int_a^b T_n(t) f^{(n)}(t) dt \right| \leq \|f^{(n)}\|_{\infty} \int_a^b |T_n(t)| dt. \end{aligned}$$

If $n = 2r$, then

$$\begin{aligned} (3.5) \quad & \int_a^b |T_n(t)| dt = \frac{1}{(2r)!} \int_a^b \left[\frac{(b-t)^{2r} + (t-a)^{2r}}{2} \right] dt \\ &= \frac{1}{(2r)!} \cdot \frac{1}{2} \left[\frac{(b-a)^{2r+1}}{(2r+1)} + \frac{(b-a)^{2r+1}}{(2r+1)} \right] = \frac{(b-a)^{2r+1}}{(2r+1)!}. \end{aligned}$$

For $n = 2r+1$, put $h_{2r+1}(t) := (b-t)^{2r+1} - (t-a)^{2r+1}$, $t \in [a, b]$. Observe that $h_{2r+1}(t) = 0$ iff $t = \frac{a+b}{2}$ and $h_{2r+1}(t) > 0$ if $t \in [a, \frac{a+b}{2})$ and $h_{2r+1}(t) < 0$ if $t \in (\frac{a+b}{2}, b]$.

Then

$$\begin{aligned} & \int_a^b |T_{2r+1}(t)| dt \\ &= \int_a^{\frac{a+b}{2}} \left[(b-t)^{2r+1} - (t-a)^{2r+1} \right] dt + \int_{\frac{a+b}{2}}^b \left[(t-a)^{2r+1} - (b-t)^{2r+1} \right] dt \\ &= 2 \frac{(b-a)^{2r+2}}{2r+2} - \frac{4 \left(\frac{b-a}{2} \right)^{2r+2}}{2r+2} \\ &= \frac{1}{2r+2} \left[2(b-a)^{2r+2} - \frac{(b-a)^{2r+2}}{2^{2r}} \right] = \frac{(b-a)^{2r+2}}{2r+2} \left(2 - \frac{1}{2^{2r}} \right) \\ &= \frac{(b-a)^{2r+2}}{2r+2} \cdot \frac{2^{2r+1} - 1}{2^{2r}}. \end{aligned}$$

Using (3.4) we get the desired inequality (3.3). ■

The following inequality in terms of $\|\cdot\|_{\infty}$ – norm for the Taylor like expansion (2.7) also holds:

Corollary 3.4. *Let g be as in Corollary 2.4. Then we have the inequality:*

$$(3.6) \quad \left| g(y) - g(a) - \sum_{k=0}^{n-1} \frac{[(y-x)^{k+1} + (-1)^k (x-a)^{k+1}]}{(k+1)!} g^{(k+1)}(x) \right|$$

$$\leq \frac{\|g^{(n+1)}\|_{\infty}}{(n+1)!} [(y-x)^{n+1} + (x-a)^{n+1}] \leq \frac{\|g^{(n+1)}\|_{\infty}}{(n+1)!} (y-a)^{n+1}$$

for all $a \leq x \leq y$.

Remark 3.1. *It is well known that for the classical Taylor expansion around a we have the inequality*

$$(3.7) \quad \left| g(y) - \sum_{k=0}^n \frac{(y-a)^k}{k!} g^{(k)}(a) \right| \leq \frac{\|g^{(n+1)}\|_{\infty}}{(n+1)!} (y-a)^{n+1}$$

for all $y \geq a$. It is clear now that the above approximation (3.6) around the arbitrary point $x \in [a, y]$ provides a better approximation for the mapping g at the point y than the classical Taylor expansion around the point a .

If in (3.6) we choose $x = \frac{a+y}{2}$, then we get

$$(3.8) \quad \left| g(y) - g(a) - \sum_{k=1}^n \frac{[1 + (-1)^{k-1}]}{k!} \frac{(y-a)^k}{2^k} g^{(k)}\left(\frac{a+y}{2}\right) \right|$$

$$\leq \frac{\|g^{(n+1)}\|_{\infty}}{(n+1)! 2^n} (y-a)^{n+1}.$$

The above inequality (3.8) shows that for $g \in C^{\infty}[a, b]$ the series

$$g(a) + \sum_{k=0}^{\infty} \frac{[1 + (-1)^k]}{(k+1)!} \frac{(y-a)^{k+1}}{2^{k+1}} g^{(k+1)}\left(\frac{a+y}{2}\right)$$

converges more rapidly to $g(y)$ than the usual one

$$\sum_{k=0}^{\infty} \frac{(y-a)^k}{k!} g^{(k)}(a)$$

which comes from Taylor's expansion. Further, it should be noted that (3.8) only involves odd derivatives of g .

Remark 3.2. *If in the inequality (3.1) we choose $n = 1$ we get*

$$\left| \int_a^b f(t) dt - (b-a) f(x) \right| \leq \frac{(x-a)^2 + (b-x)^2}{2} \|f'\|_{\infty}.$$

As a simple calculation shows that

$$\frac{1}{2} \left[(x-a)^2 + (b-x)^2 \right] = \frac{1}{4} (b-a)^2 + \left(x - \frac{a+b}{2} \right)^2$$

consequently we obtain the Ostrowski inequality:

$$(3.9) \quad \left| \int_a^b f(t) dt - (b-a) f(x) \right| \leq \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2} \right)^2}{(b-a)^2} \right] (b-a)^2 \|f'\|_\infty$$

for all $x \in [a, b]$.

If in (3.2) we put $n = 1$, we get the mid-point inequality

$$(3.10) \quad \left| \int_a^b f(t) dt - f\left(\frac{a+b}{2}\right) (b-a) \right| \leq \frac{1}{4} (b-a)^2 \|f'\|_\infty.$$

From the inequality (3.4), for $n = 1$, we get the trapezoid inequality

$$(3.11) \quad \left| \int_a^b f(t) dt - \frac{f(a) + f(b)}{2} (b-a) \right| \leq \frac{1}{2} (b-a)^2 \|f'\|_\infty.$$

Also, from (3.6) we deduce

$$(3.12) \quad |g(y) - g(a) - (y-a)g'(x)| \leq \left[\frac{1}{4} + \frac{\left(x - \frac{a+y}{2} \right)^2}{(y-a)^2} \right] \|g''\|_\infty$$

for all $a \leq x \leq y$.

Remark 3.3. If in the inequality (3.1) we choose $n = 2$, then we get

$$\begin{aligned} & \left| \int_a^b f(t) dt - (b-a) f(x) + \left(x - \frac{a+b}{2} \right) (b-a) f'(x) \right| \\ & \leq \frac{1}{6} \left[(x-a)^3 + (b-x)^3 \right] \|f''\|_\infty \end{aligned}$$

for all $x \in [a, b]$.

Now, observe that

$$\begin{aligned} (x-a)^3 + (b-x)^3 &= (b-a) \left[(x-a)^2 + (b-x)^2 - (x-a)(b-x) \right] \\ &= (b-a) \left[(x-a+b-x)^2 - 3(x-a)(b-x) \right] \\ &= (b-a) \left[(b-a)^2 + 3[x^2 - (a+b)x + ab] \right] \\ &= (b-a) \left[(b-a)^2 + 3 \left[\left(x - \frac{a+b}{2} \right)^2 - \left(\frac{b-a}{2} \right)^2 \right] \right] \\ &= (b-a) \left[\left(\frac{b-a}{2} \right)^2 + 3 \left(x - \frac{a+b}{2} \right)^2 \right] \end{aligned}$$

and then we recapture the result obtained in [1], namely

$$(3.13) \quad \left| \int_a^b f(t) dt - (b-a) f(x) + \left(x - \frac{a+b}{2}\right) (b-a) f'(x) \right| \\ \leq \left[\frac{1}{24} + \frac{1}{2} \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] (b-a)^3 \|f''\|_\infty.$$

If we put $n = 2$ in (3.3), we get the classical mid-point inequality

$$(3.14) \quad \left| \int_a^b f(t) dt - (b-a) f\left(\frac{a+b}{2}\right) \right| \leq \frac{1}{24} (b-a)^3 \|f''\|_\infty.$$

Now, if in (3.4) we put $n = 2$, then we get the inequality:

$$(3.15) \quad \left| \int_a^b f(t) dt - \frac{f(a) + f(b)}{2} (b-a) - \frac{(b-a)^2}{2} \cdot \frac{f'(a) - f'(b)}{2} \right| \\ \leq \frac{(b-a)^3}{6} \|f''\|_\infty.$$

Finally, if we put $n = 2$ in (3.6), then we get the inequality:

$$\left| g(y) - g(a) - (y-a) g'(x) + (y-a) \left(x - \frac{a+y}{2}\right) g''(x) \right| \\ \leq \left[\frac{1}{24} + \frac{1}{2} \cdot \frac{\left(x - \frac{a+y}{2}\right)^2}{(y-a)^2} \right] (y-a)^3 \|g'''\|_\infty.$$

4 APPLICATIONS FOR NUMERICAL INTEGRATION

Consider the partition $I_m : a = x_0 < x_1 < \dots < x_{m-1} < x_m = b$ of the interval $[a, b]$ and the intermediate points $\xi = (\xi_0, \dots, \xi_{m-1})$ where $\xi_j \in [x_j, x_{j+1}]$ $j = 0, \dots, m-1$. Define the formula

$$F_{m,k}(f, I_m, \xi) := \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} \frac{[(x_{j+1} - \xi_j)^{k+1} + (-1)^k (\xi_j - x_j)^{k+1}]}{(k+1)!} f^{(k)}(\xi_j)$$

which can be regarded as a perturbation of Riemann's sum

$$\Gamma(f, I_m, \xi) = \sum_{j=0}^{m-1} f(\xi_j) h_j$$

where $h_j := x_{j+1} - x_j$, $j = 0, \dots, m-1$.

The following theorem holds.

Theorem 4.1. *Let $f : [a, b] \rightarrow \mathbf{R}$ be a mapping such that $f^{(n-1)}$ is absolutely continuous on $[a, b]$ and I_m a partitioning of $[a, b]$ as above. Then we have the quadrature formula*

$$(4.1) \quad \int_a^b f(x) dx = F_{m,k}(f, I_m, \xi) + R_{m,k}(f, I_m, \xi)$$

where $F_{m,k}$ is defined above and the remainder $R_{m,k}$ satisfies the estimation:

$$(4.2) \quad |R_{m,k}(f, I_m, \xi)| \leq \frac{\|f^{(n)}\|_\infty}{(n+1)!} \sum_{j=0}^{m-1} [(\xi_j - x_j)^{n+1} + (x_{j+1} - \xi_j)^{n+1}]$$

$$\leq \frac{\|f^{(n)}\|_\infty}{(n+1)!} \sum_{j=0}^{m-1} h_j^{n+1}$$

for all ξ as above.

Proof. Apply Theorem 3.1 on the interval $[x_j, x_{j+1}]$ to get

$$\left| \int_{x_j}^{x_{j+1}} f(t) dt - \sum_{k=0}^{n-1} \left[\frac{[(x_{j+1} - \xi_j)^{k+1} + (-1)^k (\xi_j - x_j)^{k+1}]}{(k+1)!} \right] f^{(k)}(\xi_j) \right|$$

$$\leq \frac{\|f^{(n)}\|_\infty}{(n+1)!} [(\xi_j - x_j)^{n+1} + (x_{j+1} - \xi_j)^{n+1}]$$

$$\leq \frac{\|f^{(n)}\|_\infty}{(n+1)!} h_j^{n+1}$$

for all $j \in \{0, \dots, m-1\}$.

Summing over j from 0 to $m-1$ and using the generalized triangle inequality, we deduce the desired estimation (4.2). ■

As an interesting particular case, we can consider the following perturbed mid-point formula

$$M_{m,k}(f, I_m) := \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} \left[\frac{1 + (-1)^k}{(k+1)!} \right] \frac{h_j^{k+1}}{2^{k+1}} f^{(k)}\left(\frac{x_j + x_{j+1}}{2}\right),$$

which in effect involves only even k .

We state the following result concerning the estimation of the remainder term.

Corollary 4.2. *Let f and I_m be as in Theorem 4.1. Then we have*

$$(4.3) \quad \int_a^b f(t) dt = M_{m,k}(f, I_m) + R_{m,k}(f, I_m)$$

and the remainder term $R_{m,k}$ satisfies the estimation

$$(4.4) \quad |R_{m,k}(f, I_m)| \leq \frac{\|f^{(n)}\|_{\infty}}{2^n (n+1)!} \sum_{j=0}^{m-1} h_j^{n+1}.$$

We can consider the following perturbed version of the trapezoid formula:

$$T_{m,k}(f, I_m) := \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} \frac{h_j^{k+1}}{(k+1)!} \left[\frac{f^{(k)}(x_j) + (-1)^k f^{(k)}(x_{j+1})}{2} \right].$$

By the use of Corollary 3.3, we have the following approximation of the integral $\int_a^b f(t) dt$ in terms of $T_{m,k}(f, I_m)$:

Corollary 4.3. *Let f and I_m be as in Theorem 4.1. Then we have*

$$(4.5) \quad \int_a^b f(t) dt = T_{m,k}(f, I_m) + \tilde{R}_{m,k}(f, I_m)$$

and the remainder $\tilde{R}_{m,k}(f, I_m)$ satisfies the inequality

$$|\tilde{R}_{m,k}(f, I_m)| \leq \frac{C_n}{(n+1)!} \|f^{(n)}\|_{\infty} \sum_{j=0}^{m-1} h_j^{n+1}$$

where

$$C_n := \begin{cases} 1 & \text{if } n = 2r \\ \frac{2^{2r+1} - 1}{2^{2r}} & \text{if } n = 2r + 1 \end{cases}.$$

Remark 4.1. a). If we choose $n = 1$ in the above quadrature formulae (4.1) and (4.3), we recapture some results from the paper [2].

b). If we put $n = 2$, then by the above Theorem 4.1 and Corollary 4.2, we recover some results from the paper [1].

We omit the details.

5 APPLICATIONS FOR SOME PARTICULAR MAPPINGS

a) Consider $g: \mathbf{R} \rightarrow \mathbf{R}$, $g(x) = e^x$. Then $g^{(n)}(x) = e^x$, $n \in \mathbf{N}$ and

$$\|g^{(n+1)}\|_{\infty} = \sup_{t \in [a, y]} |g^{(n+1)}(t)| = e^y.$$

Using inequality (3.6), we have

$$(5.1) \quad \left| e^y - e^a - e^x \sum_{k=0}^{n-1} \frac{[(y-x)^{k+1} + (-1)^k (x-a)^{k+1}]}{(k+1)!} \right|$$

$$\leq \frac{e^y}{(n+1)!} \left[(y-x)^{n+1} + (x-a)^{n+1} \right] \leq \frac{e^y}{(n+1)!} (y-a)^{n+1}$$

for all $a \leq x \leq y$.

Particularly, if we choose $a = 0$, then we get

$$(5.2) \quad \left| e^y - 1 - e^x \sum_{k=0}^{n-1} \frac{[(y-x)^{k+1} + (-1)^k x^{k+1}]}{(k+1)!} \right|$$

$$\leq \frac{e^y}{(n+1)!} \left[(y-x)^{n+1} + x^{n+1} \right] \leq \frac{e^y}{(n+1)!} y^{n+1}.$$

Moreover, if we choose $x = \frac{y}{2}$, then we get

$$(5.3) \quad \left| e^y - 1 - e^{\frac{y}{2}} \sum_{k=0}^{n-1} \frac{1 + (-1)^k}{(k+1)!} \cdot \frac{y^{k+1}}{2^{k+1}} \right| \leq \frac{e^y y^{n+1}}{2^n (n+1)!}$$

for all $y \geq 0$.

b) Consider $g : (0, \infty) \rightarrow \mathbf{R}$, $g(x) = \ln x$. Then

$$g^{(n)}(x) = \frac{(-1)^{n-1} (n-1)!}{x^n}, \quad n \geq 1, x > 0$$

and

$$\|g^{(n+1)}\|_{\infty} = \sup_{t \in [a, y]} \left| \frac{(-1)^n n!}{t^{n+1}} \right| = \frac{n!}{a^{n+1}}, \quad a > 0.$$

Using the inequality (3.6) we can state:

$$\left| \ln y - \ln a - \sum_{k=0}^{n-1} \frac{(y-x)^{k+1} + (-1)^k (x-a)^{k+1}}{(k+1)!} \cdot \frac{(-1)^k k!}{x^{k+1}} \right|$$

$$\leq \frac{n!}{(n+1)! a^{n+1}} \left[(y-x)^{n+1} + (x-a)^{n+1} \right] \leq \frac{n!}{(n+1)! a^{n+1}} (y-a)^{n+1}$$

which is equivalent to

$$(5.4) \quad \left| \ln\left(\frac{y}{a}\right) - \sum_{k=0}^{n-1} \frac{1}{k+1} \cdot \frac{(x-a)^{k+1} + (-1)^k (y-x)^{k+1}}{x^{k+1}} \right|$$

$$\leq \frac{(y-x)^{n+1} + (x-a)^{n+1}}{(n+1) a^{n+1}} \leq \frac{1}{(n+1) a^{n+1}} (y-a)^{n+1}.$$

Now, if we choose in (5.4) $y = z+1$, $x = w+1$, $a = 1$, $z \geq w \geq 0$, then we get

$$(5.5) \quad \left| \ln(z+1) - \sum_{k=0}^{n-1} \frac{1}{k+1} \cdot \frac{w^{k+1} + (-1)^k (z-w)^{k+1}}{(w+1)^{k+1}} \right|$$

$$\leq \frac{(z-w)^{n+1} + w^{n+1}}{n+1} \leq \frac{1}{(n+1)} z^{n+1}.$$

Finally, if we choose in (5.4), $y = ua, x = wa$ with $u \geq w > 1$, then we have

$$\left| \ln u - \sum_{k=0}^{n-1} \frac{1}{k+1} \frac{(w-1)^{k+1} + (-1)^k (u-w)^{k+1}}{w^{k+1}} \right|$$

$$\leq \frac{(u-w)^{n+1} + (w-1)^{n+1}}{n+1} \leq \frac{(u-1)^{n+1}}{(n+1)}.$$

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