On Weighted Simpson Type Inequalities and Applications

ON WEIGHTED SIMPSON TYPE INEQUALITIES AND APPLICATIONS

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Abstract. In this paper we establish some weighted Simpson type inequalities and give several applications for the $r - moments$ and the expectation of a continuous random variable. An approximation for Euler's Beta mapping is given as well.

1. Introduction

The *Simpson's inequality*, states that if $f^{(4)}$ exists and is bounded on (a,b), then

$$\left|\int_a^b f(t)dt - \frac{b-a}{3}\left[\frac{f(a)+f(b)}{2} + 2f\left(\frac{a+b}{2}\right)\right]\right| \leq \frac{(b-a)^5}{2880}\left\|f^{(4)}\right\|_\infty, \tag{1.1}$$

where

$$\left\|f^{(4)}\right\|_\infty := \sup_{t\in(a,b)}\left|f^{(4)}(t)\right| < \infty.$$

Now if we assume that $I_n : a = x_0 < x_1 < \cdots < x_n = b$ is a partition of the interval $[a,b]$ and f is as above, then we can approximate the integral $\int_a^b f(t)\,dt$ by the *Simpson's quadrature formula* $A_S(f, I_n)$, having an error given by $R_S(f, I_n)$, where

$$A_S(f, I_n) := \sum_{i=0}^{n-1} \frac{l_i}{3}\left[\frac{f(x_i)+f(x_{i+1})}{2} + 2f\left(\frac{x_i+x_{i+1}}{2}\right)\right], \tag{1.2}$$

and the remainder $R_S(f, I_n) = \int_a^b f(t)\,dt - A_S(f, I_n)$ satisfies the estimation

$$|R_S(f, I_n)| \leq \frac{1}{2880}\left\|f^{(4)}\right\|_\infty \sum_{i=0}^{n-1} l_i^5, \tag{1.3}$$

with $l_i := x_{i+1} - x_i$ for $i = 0, 1, \ldots, n-1$.

For some recent results which generalize, improve and extend this classic inequality (1.1), see the papers [2] – [7] and [9] – [12].

Recently, Dragomir [6], (see also the survey paper authored by Dragomir, Agarwal and Cerone [7]) has proved the following two Simpson type inequalities for functions of bounded variation:

Theorem 1. *Let $f : [a,b] \to \mathbb{R}$ be a mapping of bounded variation. Then*

$$\left|\int_a^b f(t)dt - \frac{b-a}{3}\left[\frac{f(a)+f(b)}{2} + 2f\left(\frac{a+b}{2}\right)\right]\right| \leq \frac{1}{3}(b-a)\bigvee_a^b(f), \tag{1.4}$$

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where $\bigvee_a^b(f)$ *denotes the total variation of* f *on the interval* $[a,b]$. *The constant* $\frac{1}{3}$ *is the best possible.*

Let I_n, l_i $(i=0,1,\ldots,n-1)$, $A_S(f,I_n)$ and $R_S(f,I_n)$ be as above. We have the following result concerning the approximation of the integral $\int_a^b f(t)dt$ in terms of $A_S(f,I_n)$.

Theorem 2. *Let* f *be defined as in Theorem 1. Then the remainder*

$$R_S(f,I_n)=\int_a^b f(x)dx-A_S(f,I_n) \tag{1.5}$$

satisfies the estimate

$$|R_S(f,I_n)|\le \frac{1}{3}\nu(l)\bigvee_a^b(f), \tag{1.6}$$

where $\nu(l):=\max\{l_i\,|i=0,1,\ldots,n-1\}$. *The constant* $\frac{1}{3}$ *is best posible in (1.6).*

In this paper, we establish some generalizations of Theorems 1 – 2, and give several applications for the $r-$*moments* and expectation of a continuous random variable. Approximations for Euler's Beta mapping are also provided.

2. Some Integral Inequalities

We may state and prove the following main result:

Theorem 3. *Let* $g:[a,b]\to\mathbb{R}$ *be positive and continuous and let* $h(x)=\int_a^x g(t)dt$, $x\in[a,b]$. *Let* f *be as in Theorem 3. Then*

$$\left|\int_a^b f(t)g(t)\,dt-\frac{1}{3}\left[\frac{f(a)+f(b)}{2}+2f\left(h^{-1}(x)\right)\right]\int_a^b g(t)\,dt\right| \le\left[\frac{1}{3}h(b)+\left|x-\frac{h(b)}{2}\right|\right]\cdot\bigvee_a^b(f), \tag{2.1}$$

for all $x\in\left[\frac{h(b)}{6},\frac{5h(b)}{6}\right]$, *where* $\bigvee_a^b(f)$ *denotes the total variation of* f *on the interval* $[a,b]$. *The constant* $\frac{1}{3}$ *is the best possible.*

Proof. Fix $x\in\left[\frac{h(b)}{6},\frac{5h(b)}{6}\right]$. Define

$$s(t):=\begin{cases} h(t)-\frac{h(b)}{6}, & t\in\left[a,h^{-1}(x)\right) \\ h(t)-\frac{5h(b)}{6}, & t\in\left[h^{-1}(x),b\right] \end{cases}.$$

By integration by parts, we have the following identity

$$\begin{aligned} &\int_a^b s(t)\,df(t) \\ &=\left[\left(h(t)-\frac{h(b)}{6}\right)f(t)\,|_a^{h^{-1}(x)}-\int_a^{h^{-1}(x)}f(t)g(t)\,dt\right] \\ &\quad+\left[\left(h(t)-\frac{5h(b)}{6}\right)f(t)\,|_{h^{-1}(x)}^b-\int_{h^{-1}(x)}^b f(t)g(t)\,dt\right] \end{aligned} \tag{2.2}$$

$$= \frac{1}{3}h(b)\left[\frac{f(a)+f(b)}{2} + 2f\left(h^{-1}(x)\right)\right] - \int_a^b f(t)g(t)\,dt$$
$$= \frac{1}{3}\left[\frac{f(a)+f(b)}{2} + 2f\left(h^{-1}(x)\right)\right]\int_a^b g(t)\,dt - \int_a^b f(t)g(t)\,dt.$$

It is well known (see for instance [1, p. 159]) that, if $\mu, \nu : [a,b] \to \mathbb{R}$ are such that μ is continuous on $[a,b]$ and ν is of bounded variation on $[a,b]$, then $\int_a^b \mu(t)\,d\nu(t)$ exists and [1, p. 177]

$$\left|\int_a^b \mu(t)\,d\nu(t)\right| \le \sup_{t\in[a,b]} |\mu(t)| \bigvee_a^b (\nu). \tag{2.3}$$

Now, using (2.2) and (2.3), we have

$$\left|\int_a^b f(t)g(t)\,dt - \frac{1}{3}\left[\frac{f(a)+f(b)}{2} + 2f\left(h^{-1}(x)\right)\right]\int_a^b g(t)\,dt\right| \le \sup_{t\in[a,b]} |s(t)| \bigvee_a^b (f). \tag{2.4}$$

Since $h(t)-\frac{h(b)}{6}$ is increasing on $\left[a, h^{-1}(x)\right)$, $h(t)-\frac{5h(b)}{6}$ is increasing on $\left[h^{-1}(x), b\right]$ and the fact that $\max\{c,d\} = \frac{c+d}{2} + \frac{1}{2}|c-d|$ for any real c and d, hence we have

$$\sup_{t\in[a,b]} |s(t)| = \max\left\{\frac{h(b)}{6}, x - \frac{h(b)}{6}, \frac{5h(b)}{6} - x\right\}$$

and

$$\begin{aligned}
\sup_{t\in[a,b]} |s(t)| &= \max\left\{\frac{h(b)}{6}, x - \frac{h(b)}{6}, \frac{5h(b)}{6} - x\right\} \\
&= \max\left\{x - \frac{h(b)}{6}, \frac{5h(b)}{6} - x\right\} \\
&= \frac{1}{2}\left[\left(x - \frac{h(b)}{6}\right) + \left(\frac{5h(b)}{6} - x\right)\right] \\
&\quad + \frac{1}{2}\left|\left(x - \frac{h(b)}{6}\right) - \left(\frac{5h(b)}{6} - x\right)\right| \\
&= \frac{h(b)}{3} + \left|x - \frac{h(b)}{2}\right| \\
&= \frac{1}{3}\int_a^b g(t)\,dt + \left|x - \frac{1}{2}\int_a^b g(t)dt\right|.
\end{aligned} \tag{2.5}$$

Thus, by (2.4) and (2.5), we obtain the desired inequality (2.1).

Let us consider the particular functions:

$$g(t) \equiv 1, \ t \in [a,b],$$
$$h(t) = t - a, \ t \in [a,b],$$
$$f(t) = \begin{cases} 1 & \text{as } t \in \left[a, \frac{a+b}{2}\right) \cup \left(\frac{a+b}{2}, b\right] \\ -1 & \text{as } t = \frac{a+b}{2} \end{cases}$$

and $x=\frac{b-a}{2}$. Since for these choices we get equality in (2.1), it is easy to see that the constant $\frac{1}{3}$ is the best possible constant in (2.1). This completes the proof. ∎

Remark 1. (1) *If we choose $g(t)\equiv 1$, $h(t)=t-a$ on $[a,b]$ and $x=\frac{b-a}{2}$, then the inequality (2.1) reduces to (1.4).*
(2) *If we choose $x=\frac{h(b)}{2}$, then we get*

$$\left|\int_a^b f(t)g(t)\,dt-\frac{1}{3}\left[\frac{f(a)+f(b)}{2}+2f\left(h^{-1}\left(\frac{h(b)}{2}\right)\right)\right]\int_a^b g(t)\,dt\right| \leq \frac{1}{3}\int_a^b g(t)\,dt\cdot\bigvee_a^b(f)\,. \tag{2.6}$$

Under the conditions of Theorem 3, we have the following corollaries.

Corollary 1. *Let $f\in C^{(1)}[a,b]$. Then we have the inequality*

$$\left|\int_a^b f(t)g(t)\,dt-\frac{1}{3}\left[\frac{f(a)+f(b)}{2}+2f\left(h^{-1}(x)\right)\right]\int_a^b g(t)\,dt\right| \leq \left[\frac{1}{3}\int_a^b g(t)\,dt+\left|x-\frac{h(b)}{2}\right|\right]\left\|f'\right\|_1, \tag{2.7}$$

for all $x\in\left[\frac{h(b)}{6},\frac{5h(b)}{6}\right]$, where $\|\cdot\|_1$ is the L_1-norm, namely

$$\left\|f'\right\|_1:=\int_a^b\left|f'(t)\right|dt.$$

Corollary 2. *Let $f:[a,b]\to\mathbb{R}$ be a Lipschitzian mapping with the constant $M>0$. Then we have the inequality*

$$\left|\int_a^b f(t)g(t)\,dt-\frac{1}{3}\left[\frac{f(a)+f(b)}{2}+2f\left(h^{-1}(x)\right)\right]\int_a^b g(t)\,dt\right| \leq \left[\frac{1}{3}\int_a^b g(t)\,dt+\left|x-\frac{h(b)}{2}\right|\right](b-a)\,M, \tag{2.8}$$

for all $x\in\left[\frac{h(b)}{6},\frac{5h(b)}{6}\right]$.

Corollary 3. *Let $f:[a,b]\to\mathbb{R}$ be a monotonic mapping. Then we have the inequality*

$$\left|\int_a^b f(t)g(t)\,dt-\frac{1}{3}\left[\frac{f(a)+f(b)}{2}+2f\left(h^{-1}(x)\right)\right]\int_a^b g(t)\,dt\right| \leq \left[\frac{1}{3}\int_a^b g(t)\,dt+\left|x-\frac{h(b)}{2}\right|\right]\cdot\left|f(b)-f(a)\right| \tag{2.9}$$

for all $x\in\left[\frac{h(b)}{6},\frac{5h(b)}{6}\right]$.

3. Applications for Quadrature Formulae

Throughout this section, let g, h be as in Theorem 3, $f:[a,b]\to\mathbb{R}$, and let $I_n : a = x_0 < x_1 < \cdots < x_n = b$ be a partition of $[a,b]$, and $h_i(x) = \int_{x_i}^{x} g(t)dt$, $x\in[x_i,x_{i+1}]$, $\xi_i \in \left[\frac{h(x_{i+1})}{6}, \frac{5h(x_{i+1})}{6}\right]$ $(i = 0,1,\ldots,n-1)$ are intermediate points. Put $L_i := h_i(x_{i+1}) = \int_{x_i}^{x_{i+1}} g(t)\,dt$ and define the sum

$$A_S(f,g,I_n,\xi) := \sum_{i=0}^{n-1} \frac{L_i}{3}\left[\frac{f(x_i)+f(x_{i+1})}{2} + 2f\left(h^{-1}(\xi_i)\right)\right]$$

and

$$R_S(f,g,I_n,\xi) = \int_a^b f(t)g(t)dx - A_S(f,g,I_n,\xi).$$

We have the following approximation of the integral $\int_a^b f(t)g(t)\,dt$.

Theorem 4. *Let f be defined as in Theorem 3 and let*

$$\int_a^b f(t)g(t)\,dt = A_S(f,g,I_n,\xi) + R_S(f,g,I_n,\xi). \tag{3.1}$$

Then, the remainder term $R_S(f,g,h,I_n,\xi)$ satisfies the estimate

$$\begin{aligned}
&|R_S(f,g,h,I_n,\xi)| \\
&\le \left[\frac{1}{3}\nu(L) + \max_{i=0,1,\ldots,n-1}\left|\xi_i - \frac{h_i(x_{i+1})}{2}\right|\right]\bigvee_a^b(f) \\
&\le \frac{2}{3}\nu(L)\bigvee_a^b(f),
\end{aligned} \tag{3.2}$$

where $\nu(L) := \max\{L_i\,|i = 0,1,\ldots,n-1\}$. The constant $\frac{1}{3}$ in the first inequality of (3.2) is the best possible.

Proof. Apply Theorem 3 on the intervals $[x_i,x_{i+1}]$ $(i = 0,1,\ldots,n-1)$ to get

$$\left|\int_{x_i}^{x_{i+1}} f(t)g(t)\,dt - \frac{l_i}{3}\left[\frac{f(x_i)+f(x_{i+1})}{2} + 2f\left(h_i^{-1}(\xi_i)\right)\right]\right| \le \left[\frac{1}{3}L_i + \left|\xi_i - \frac{h_i(x_{i+1})}{2}\right|\right]\bigvee_{x_i}^{x_{i+1}}(f),$$

for all $i = 0, 1, \ldots, n-1$. Using this and the generalized triangle inequality, we have

$$\begin{aligned}
&\left|R_S\left(f, g, I_n, \xi\right)\right| \\
&\leq \sum_{i=0}^{n-1}\left|\int_{x_i}^{x_{i+1}} f(t) g\left(t\right) dt - \frac{L_i}{3}\left[\frac{f\left(x_i\right)+f\left(x_{i+1}\right)}{2}+2 f\left(h_i^{-1}\left(\xi_i\right)\right)\right]\right| \\
&\leq \sum_{i=0}^{n-1}\left[\frac{1}{3} L_i+\left|\xi_i-\frac{h_i(x_{i+1})}{2}\right|\right] \bigvee_{x_i}^{x_{i+1}}(f) \\
&\leq \max_{i=0,1,\ldots,n-1}\left[\frac{1}{3} L_i+\left|\xi_i-\frac{h_i(x_{i+1})}{2}\right|\right] \sum_{i=0}^{n-1} \bigvee_{x_i}^{x_{i+1}}(f) \\
&\leq\left[\frac{1}{3} \nu\left(L\right)+\max_{i=0,1,\ldots,n-1}\left|\xi_i-\frac{h_i(x_{i+1})}{2}\right|\right] \bigvee_a^b(f)
\end{aligned}$$

and the first inequality in (3.2) is proved.

For the second inequality in (3.2), we observe that

$$\left|\xi_i-\frac{h_i(x_{i+1})}{2}\right| \leq \frac{1}{3} L_i \; (i=0,1,\ldots,n-1);$$

and then

$$\max_{i=0,1,\ldots,n-1}\left|\xi_i-\frac{h(x_i)+h(x_{i+1})}{2}\right| \leq \frac{1}{3} \nu\left(L\right).$$

Thus the theorem is proved. ∎

Remark 2. *If we choose $g\left(t\right) \equiv 1$, then $h\left(t\right)=t-a$ on $\left[a, b\right]$, $\xi_i=\frac{x_{i+1}-x_i}{2}$ $(i=0,1,\ldots,n-1)$, and the first inequality in (3.2) reduces to (1.6).*

The following corollaries are useful in practice.

Corollary 4. *Let $f:[a, b] \rightarrow \mathbb{R}$ be a Lipschitzian mapping with the constant $M>0$, I_n be defined as above and choose $\xi_i=\frac{h_i(x_{i+1})}{2}$ $(i=0,1,\ldots,n-1)$. Then we have the formula*

$$\begin{aligned}
\text{(3.3)} \quad \int_a^b f(t) g\left(t\right) dt &= A_S\left(f, g, I_n, \xi\right)+R_S\left(f, g, I_n, \xi\right) \\
&= \sum_{i=0}^{n-1} \frac{L_i}{3}\left[\frac{f\left(x_i\right)+f\left(x_{i+1}\right)}{2}+2 f\left(h_i^{-1}\left(\xi_i\right)\right)\right]+R_S\left(f, g, I_n, \xi\right)
\end{aligned}$$

and the remainder satisfies the estimate

$$\text{(3.4)} \qquad \left|R_S\left(f, g, I_n, \xi\right)\right| \leq \frac{\nu\left(L\right) \cdot M \cdot(b-a)}{3}.$$

Corollary 5. *Let $f:[a, b] \rightarrow \mathbb{R}$ be a monotonic mapping and let ξ_i $(i=0,1,\ldots,n-1)$ be defined as in Corollary 4. Then we have the formula (3.3) and the remainder satisfies the estimate*

$$\text{(3.5)} \qquad \left|R_S\left(f, g, I_n, \xi\right)\right| \leq \frac{\nu\left(L\right)}{3} \cdot\left|f\left(b\right)-f\left(a\right)\right|.$$

The case of equidistant division is embodied in the following corollary and remark:

Corollary 6. *Suppose that* $G(x)=\int_a^x g(t)dt, x\in[a,b]$,

$$x_i = G^{-1}\left(\frac{i}{n}\int_a^b g(t)dt\right)\ \ (i=0,1,\ldots,n),$$

$$h_i(x) = \int_{x_i}^x g(t)dt, x\in[x_i,x_{i+1}], (i=0,1,\ldots,n-1),$$

and

$$L_i := h_i(x_{i+1}) = G(x_{i+1})-G(x_i) = \frac{1}{n}\int_a^b g(t)\,dt \quad (i=0,1,\ldots,n-1).$$

Let f *be defined as in Theorem 4 and choose* $\xi_i = \frac{h_i(x_{i+1})}{2}$ $(i=0,1,\ldots,n-1)$. *Then we have the formula*

$$\int_a^b f(t)g(t)\,dt = A_S(f,g,h,I_n,\xi)+R_S(f,g,h,I_n,\xi) \tag{3.6}$$
$$= \frac{1}{3n}\sum_{i=0}^{n-1}\left[\frac{f(x_i)+f(x_{i+1})}{2}+2f\left(h_i^{-1}\left(\frac{h_i(x_{i+1})}{2}\right)\right)\right]\int_a^b g(t)\,dt + R_S(f,g,h,I_n,\xi)$$

and the remainder satisfies the estimate

$$|R_S(f,g,h,I_n,\xi)| \le \frac{1}{3n}\bigvee_a^b(f)\int_a^b g(t)\,dt. \tag{3.7}$$

Remark 3. *If we want to approximate the integral* $\int_a^b f(t)g(t)\,dt$ *by* $A_S(f,g,h,I_n,\xi)$ *with an error less that* $\varepsilon>0$, *then we need at least* $n_\varepsilon\in N$ *points for the partition* I_n, *where*

$$n_\varepsilon := \left[\frac{1}{3\varepsilon}\int_a^b g(t)\,dt\cdot\bigvee_a^b(f)\right]+1$$

and $[r]$ *denotes the Gaussian integer of* $r\in\mathbb{R}$.

4. Some Inequalities for Random Variables

Throughout this section, let $0<a<b$, $r\in\mathbb{R}$, and let X be a continuous random variable having the continuous probability density function $g:[a,b]\to[0,\infty)$ and assume the $r-$*moment*, defined by

$$E_r(X) := \int_a^b t^r g(t)\,dt,$$

is finite.

Theorem 5. *The inequality*

$$\left|E_r(X)-\frac{1}{6}\left[a^r+4\left(h^{-1}\left(\frac{1}{2}\right)\right)^r+b^r\right]\right| \le \frac{1}{3}|b^r-a^r| \tag{4.1}$$

holds, where $h(t)=\int_a^t g(x)\,dx$ $(t\in[a,b])$.

Proof. If we put $f(t) = t^r$ and $x = \frac{h(b)}{2} = \frac{1}{2}$ in Corollary 3, then we obtain the inequality

$$\left|\int_a^b f(t)g(t)\,dt - \frac{1}{3}\left[\frac{f(a)+f(b)}{2} + 2f\left(h^{-1}\left(\frac{1}{2}\right)\right)\right]\int_a^b g(t)\,dt\right| \le \frac{1}{3}\left|f(b) - f(a)\right|\int_a^b g(t)\,dt. \tag{4.2}$$

Since

$$\int_a^b f(t)g(t)\,dt = E_r(X), \qquad \int_a^b g(t)\,dt = 1,$$
$$\frac{f(a)+f(b)}{2} = \frac{a^r+b^r}{2}, \text{ and } \left|f(b)-f(a)\right| = \left|b^r - a^r\right|,$$

(4.1) follows from (4.2). ∎

If we choose $r = 1$ in Theorem 5, then we have the following remark:

Remark 4. *If $E(X)$ is the expectation of random variable X, then*

$$\left|E(X) - \frac{1}{6}\left[a + 4h^{-1}\left(\frac{1}{2}\right) + b\right]\right| \le \frac{b-a}{3}. \tag{4.3}$$

5. Inequality for the Beta Mapping

The following mapping is well-known in the literature as the *Beta mapping*:

$$\beta(p,q) := \int_0^1 t^{p-1}(1-t)^{q-1}\,dt, \quad p > 0,\ q > 0.$$

The following result may be stated:

Theorem 6. *Let $p > 0$, $q > 1$. Then the inequality*

$$\left|\beta(p,q) - \frac{1}{np}\sum_{i=0}^{n-1}\left\{\frac{1}{6}\left(\left[1-\left(\frac{i}{n}\right)^{\frac{1}{p}}\right]^{q-1} + \left[1-\left(\frac{i+1}{n}\right)^{\frac{1}{p}}\right]^{q-1}\right) + \frac{2}{3}\left[1-\left(\frac{2i+1}{2n}\right)^{\frac{1}{p}}\right]^{q-1}\right\}\right| \le \frac{1}{3np} \tag{5.1}$$

holds for any positive integer n.

Proof. If we put $a = 0$, $b = 1$, $f(t) = (1-t)^{q-1}$, $g(t) = t^{p-1}$ and $G(t) = \frac{t^p}{p}$ $(t \in [0,1])$ in Corollary 6, then,

$$\int_a^b g(t)dt = \frac{1}{p}, x_i = \left(\frac{i}{n}\right)^{\frac{1}{p}} \quad (i = 0,1,\ldots,n),$$
$$h_i(x) = \frac{nx^p - i}{np} \quad (x \in [x_i, x_{i+1}],\ i = 0,1,\ldots,n-1),$$
$$h_i^{-1}\left(\frac{h_i(x_{i+1})}{2}\right) = \left(\frac{2i+1}{2n}\right)^{\frac{1}{p}} \quad (i = 0,1,\ldots,n-1)$$

and $\bigvee_a^b(f) = 1$, so that the inequality (5.1) holds. ∎

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