An Inequality in Metric Spaces

AN INEQUALITY IN METRIC SPACES

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Abstract. In this note we establish a general inequality valid in metric spaces that is related to the polygonal inequality and admits also a natural geometrical interpretation. Particular instances of interest holding in normed linear spaces and inner product spaces are pointed out as well.

1. Introduction

Let X be a nonempty set. A function $d : X \times X \to [0, \infty)$ is called a *distance* on X if the following properties are satisfied:

(d) $d(x, y) = 0$ if and only if $x = y$;

(dd) $d(x, y) = d(y, x)$ for any $x, y \in X$ (the *symmetry* of the distance);

(ddd) $d(x, y) \leq d(x, z) + d(z, y)$ for any $x, y, z \in X$ (the *triangle inequality*).

The pair (X, d) is called in the literature a *metric space.*

Important examples of metric spaces are normed linear spaces. We recall that, a linear space E over the real or complex number field $\mathbb{K}$ endowed with a function $\|\cdot\| : E \to [0, \infty)$, is called a *normed space* if $\|\cdot\|$, the *norm*, satisfies the properties

(n) $\|x\| = 0$ if and only if $x = 0$;

(nn) $\|\alpha x\| = |\alpha| \|x\|$ for any scalar $\alpha \in \mathbb{K}$ and any vector $x \in E$;

(nnn) $\|x + y\| \leq \|x\| + \|y\|$ for each $x, y \in E$ (the triangle inequality).

Further, we recall that, the linear space H over the real or complex number field $\mathbb{K}$ endowed with an application $\langle \cdot, \cdot \rangle : H \times H \to \mathbb{K}$ is called an *inner product space*, if the function $\langle \cdot, \cdot \rangle$, called the *inner product*, satisfies the following properties:

(i) $\langle x, x \rangle \geq 0$ for any $x \in H$ and $\langle x, x \rangle = 0$ if and only if $x = 0$;

(ii) $\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle$ for any scalars α, β and any vectors x, y, z;

(iii) $\langle y, x \rangle = \overline{\langle x, y \rangle}$ for any $x, y \in H$.

It is well know that the function $\|x\| := \sqrt{\langle x, x \rangle}$ defines a norm on H and thus an important example of normed spaces are the inner product spaces.

A fundamental inequality in metric spaces, which obviously follows by the triangle inequality and mathematical induction, is the *generalised triangle inequality*, or the *polygonal inequality* which states that: for any points $x_1, x_2, ..., x_{n-1}, x_n$ $(n \geq 3)$ in a metric space (X, d), we have the inequality

$$d(x_1, x_n) \leq d(x_1, x_2) + ... + d(x_{n-1}, x_n). \tag{1.1}$$

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The main aim of this note is to point out a general inequality valid in metric spaces that is related to the polygonal inequality and admits also a natural geometrical interpretation. Particular instances of interest holding in normed linear spaces and inner product spaces are pointed out as well.

2. The Results

The following result in the general setting of metric spaces holds.

Theorem 1. *Let (X,d) be a metric space and $x_i \in X, p_i \geq 0$ $(i \in \{1,...,n\})$ with $\sum_{i=1}^{n} p_i = 1$. Then we have the inequality*

$$\sum_{1\leq i<j\leq n} p_i p_j d(x_i,x_j) \leq \inf_{x\in X}\left[\sum_{i=1}^{n} p_i d(x_i,x)\right]. \tag{2.1}$$

The inequality is sharp in the sense that the multiplicative constant $c=1$ in front of " inf " cannot be replaced by a smaller quantity.

Proof. Using the triangle inequality, we have for any $x \in X$ and $i,j \in \{1,...,n\}$, that

$$d(x_i,x_j) \leq d(x_i,x) + d(x,x_j). \tag{2.2}$$

If we multiply (2.2) with $p_i p_j \geq 0$ and sum over i and j from 1 to n, then we deduce

$$\sum_{i,j=1}^{n} p_i p_j d(x_i,x_j) \leq \sum_{i,j=1}^{n} p_i p_j \left[d(x_i,x) + d(x,x_j)\right]. \tag{2.3}$$

However, by the symmetry of distance,

$$\sum_{i,j=1}^{n} p_i p_j d(x_i,x_j) = 2\sum_{1\leq i<j\leq n} p_i p_j d(x_i,x_j)$$

and

$$\sum_{i,j=1}^{n} p_i p_j \left[d(x_i,x) + d(x,x_j)\right] = 2\left[\sum_{i=1}^{n} p_i d(x_i,x)\right]$$

therefore, by (2.3), we deduce

$$\sum_{1\leq i<j\leq n} p_i p_j d(x_i,x_j) \leq \sum_{i=1}^{n} p_i d(x_i,x), \tag{2.4}$$

for any $x \in X$.

Taking the infimum over x in (2.4), we deduce the desired inequality (2.1).

Now, suppose that (2.1) holds with a constant $c>0$, i.e.,

$$\sum_{1\leq i<j\leq n} p_i p_j d(x_i,x_j) \leq c \inf_{x\in X}\left[\sum_{i=1}^{n} p_i d(x_i,x)\right]. \tag{2.5}$$

Then, on choosing $n=2, p_1=p, p_2=1-p, p\in(0,1)$, we deduce

$$p(1-p)d(x_1,x_2) \leq c\left[pd(x_1,x) + (1-p)d(x,x_2)\right] \tag{2.6}$$

for any $x \in X$ and $p \in (0,1)$. If in this inequality we let $x = x_1$, then we get

$$pd(x_1,x_2) \leq cd(x_1,x_2)$$

for any $x_1, x_2 \in X$ and $p \in (0,1)$ which implies that $c \geq 1$, and the proof is complete. ∎

The following particular case holds.

Corollary 1. *Let (X,d) be a metric space and $x_i \in X$ $(i \in \{1,...,n\})$. Then we have the inequality*

$$\sum_{1\leq i<j\leq n} d(x_i,x_j) \leq n \inf_{x\in X}\left[\sum_{i=1}^{n} d(x_i,x)\right]. \tag{2.7}$$

The proof is obvious from the above theorem on choosing $p_i = \frac{1}{n}, i \in \{1,...,n\}$.

The above corollary has an interesting geometrical interpretation:

Proposition 1. *The sum of all edges and diagonals of a polygon with n vertices in a metric space is less than n-times the sum of the distances from any arbitrary point in the space to its vertices.*

The following corollary holds as well.

Corollary 2. *Let (X,d) be a metric space and $x_i \in X$, $(i \in \{1,...,n\})$. If there exists a closed ball of radius $r > 0$ centered in a point x containing all the points x_i, i.e., $x_i \in \overline{B}(x,r) := \{y \in X : d(x,y) \leq r\}$, then for any $p_i \geq 0\,(i \in \{1,...,n\})$ with $\sum_{i=1}^{n} p_i = 1$ we have the inequality*

$$\sum_{1\leq i<j\leq n} p_i p_j d(x_i,x_j) \leq r. \tag{2.8}$$

The proof is obvious from the above Theorem 1 and we omit the details.

3. Applications

If $(E,\|\cdot\|)$ is a normed linear space and $x_i \in E$, $(i \in \{1,...,n\})$, $p_i \geq 0\,(i \in \{1,...,n\})$ with $\sum_{i=1}^{n} p_i = 1$, then by (2.1) we have the inequality

$$\sum_{1\leq i<j\leq n} p_i p_j \|x_i - x_j\| \leq \inf_{x\in X}\left[\sum_{i=1}^{n} p_i \|x_i - x\|\right]. \tag{3.1}$$

In particular, for the uniform distribution $p_i = \frac{1}{n}$, we have

$$\sum_{1\leq i<j\leq n} \|x_i - x_j\| \leq n \inf_{x\in X}\left[\sum_{i=1}^{n} \|x_i - x\|\right]. \tag{3.2}$$

We can state the following results as well.

Proposition 2. *Let $(E,\|\cdot\|)$ be a normed linear space and $x_i \in E$, $(i \in \{1,...,n\})$, $p_i \geq 0\,(i \in \{1,...,n\})$ with $\sum_{i=1}^{n} p_i = 1$. Denote $x_p := \sum_{i=1}^{n} p_i x_i$. Then we have the inequalities*

$$\frac{1}{2}\sum_{i=1}^{n} p_i \|x_i - x_p\| \leq \sum_{1\leq i<j\leq n} p_i p_j \|x_i - x_j\| \leq \sum_{i=1}^{n} p_i \|x_i - x_p\|. \tag{3.3}$$

The constant $\frac{1}{2}$ is best possible in the sense that it cannot be replaced by a larger quantity.

Proof. The second inequality is obvious by (3.1).

By the generalised triangle inequality we have

$$\begin{aligned}\sum_{1\le i<j\le n} p_i p_j \|x_i - x_j\| &= \frac{1}{2}\sum_{i,j=1}^{n} p_i p_j \|x_i - x_j\| \\ &\ge \frac{1}{2}\sum_{i=1}^{n} p_i \left\| x_i - \sum_{j=1}^{n} p_j x_j \right\| = \frac{1}{2}\sum_{i=1}^{n} p_i \|x_i - x_p\|,\end{aligned}$$

proving the first part of (3.3).

Now, assume that the first inequality holds with a constant $k > 0$, i.e.,

$$k\sum_{i=1}^{n} p_i \|x_i - x_p\| \le \sum_{1\le i<j\le n} p_i p_j \|x_i - x_j\| \tag{3.4}$$

under the hypothesis of the proposition stated above. Then, by (3.4) for $n = 2$ and $p_1 = p_2 = \frac{1}{2}$ we deduce

$$\frac{1}{2}k\|x_1 - x_2\| \le \frac{1}{4}\|x_1 - x_2\|,$$

for any $x_1, x_2 \in E$, implying $k \le \frac{1}{2}$, and the proposition is proved. ∎

Remark 1. *It is an open question whether the multiplicative constant $c = 1$ in the second part of (3.3) is sharp or not in the general setting of normed linear spaces.*

The following particular case with a simple geometric interpretation holds.

Corollary 3. *Let $(E, \|\cdot\|)$ be a normed linear space and $x_i \in E$, $(i \in \{1, ..., n\})$. If*

$$\overline{x} := \frac{x_1 + ... + x_n}{n}$$

denotes the gravity center of the vectors $x_i, i \in \{1, ..., n\}$, then we have the inequality

$$\frac{1}{2}n\sum_{i=1}^{n}\|x_i - \overline{x}\| \le \sum_{1\le i<j\le n}\|x_i - x_j\| \le \sum_{i=1}^{n}\|x_i - \overline{x}\|. \tag{3.5}$$

The constant $\frac{1}{2}$ in the first inequality is sharp.

Remark 2. *Geometrically, the inequality (3.5) means that: the sum of all edges and diagonals of a polygon with n vertices in a normed linear space is less than n-times the sum of the distances from the gravity center to its vertices and greater than $\frac{n}{2}$-times the same quantity.*

Finally, in the case of inner product spaces, we may point out an upper bound as follows.

Proposition 3. *Let $(H, \langle\cdot,\cdot\rangle)$ be an inner product space, $x_i \in H, (i \in \{1, ..., n\})$ and assume that there exists the vectors $a, A \in H$ so that either*

$$\operatorname{Re}\langle A - x_i, x_i - a\rangle \ge 0, \text{ for } i \in \{1, ..., n\},$$

or, equivalently,

$$\left\| x_i - \frac{a + A}{2} \right\| \le \frac{1}{2}\|A - a\|, \text{ for } i \in \{1, ..., n\}.$$

Then for any $p_i \geq 0\,(i \in \{1,...,n\})$ *with* $\sum_{i=1}^{n} p_i = 1$ *one has the inequality*

$$\sum_{1 \leq i < j \leq n} p_i p_j \left\| x_i - x_j \right\| \leq \frac{1}{2} \left\| A - a \right\| . \tag{3.6}$$

The proof is obvious by Corollary 2 and we omit the details.

Remark 3. *It is an open problem if* $\frac{1}{2}$ *in (3.6) is the best possible constant in the general case of inner product spaces.*

For other classical and recent results related to the triangle and polygonal inequality, see the papers [1]- [3], [5], Chapter XVII of the book [4] and the references therein.

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