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AN INEQUALITY OF GRÜSS TYPE FOR RIEMANN-STIELTJES INTEGRAL AND APPLICATIONS FOR SPECIAL MEANS

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ABSTRACT. In this paper we derive a new inequality of Grüss' type for Riemann-Stieltjes integral and apply it for special means (logarithmic mean, identric mean, etc...).

1   INTRODUCTION

In 1935, G. Grüss proved the following inequality which establishes a connection between the integral of the product of two functions and the product of the integrals:

$$\left| \frac{1}{b-a} \int_a^b f(x)g(x)dx - \frac{1}{b-a} \int_a^b f(x)dx \cdot \frac{1}{b-a} \int_a^b g(x)dx \right| \leq \frac{1}{4}(\Phi - \phi)(\Gamma - \gamma)$$

provided that f and g are two integrable functions on $[a,b]$ and satisfying the condition

$$\phi \leq f(x) \leq \Phi \quad \text{and } \gamma \leq g(x) \leq \Gamma \quad \text{for all } x \in [a,b].$$

The constant $\frac{1}{4}$ is the *best possible* and is achieved for

$$f(x) = g(x) = sgn\left(x - \frac{a+b}{2}\right).$$

For other similar results, generalizations for positive linear functionals, discrete versions, determinantal versions etc. see the Chapter X of the book [1] due to Mitrinović, Pečarić and Fink where further references are given.

In this paper we point out a Grüss' type inequality for Riemann-Stieltjes integral and apply it for special means, i.e., logarithmic mean, identric mean, etc...

2   THE RESULTS

The following result of Grüss' type holds:

Theorem 2.1. *Let $f, u : [a, b] \to \mathbf{R}$ be so that u is L-lipschitzian on $[a, b]$, i.e.,*

$$(2.1) \qquad\qquad |u(x) - u(y)| \leq L\,|x - y|$$

for all $x, y \in [a, b]$, f is Riemann integrable on $[a, b]$ and there exists the real numbers m, M so that

$$(2.2) \qquad\qquad m \leq f(x) \leq M \quad \text{for all } x \in [a, b].$$

Then we have the inequality

$$(2.3) \qquad \left| \int_a^b f(x)du(x) - \frac{u(b) - u(a)}{b - a} \int_a^b f(t)dt \right| \leq \frac{1}{2} L\,(M - m)\,(b - a),$$

and the constant $\frac{1}{2}$ is sharp, in the sense that it can not be replaced by a smaller one.

Proof. First of all let observe that if v is an L-lipschitzian mapping and q is Riemann integrable on $[a, b]$, then

$$(2.4) \qquad \left| \int_a^b q(x)dv(x) \right| \leq L \int_a^b |q(t)|\,dt.$$

Indeed, if $\Delta_n := x_0^n < x_1^n < ... < x_{n-1}^n < x_n^n = b$ is a sequence of partitions of $[a, b]$ with $\nu(\Delta_n) := \max_{i=0,n-1} \left(x_{i+1}^n - x_i^n \right) \to 0$ (for $n \to \infty$) and $\xi_i^n \in \left[x_i^n, x_{i+1}^n \right]$ then

$$\left| \int_a^b q(x)dv(x) \right| = \left| \lim_{\nu(\Delta_n)\to 0} \sum_{i=0}^{n-1} q\left(\xi_i^n\right) \left(v\left(x_{i+1}^n\right) - v\left(x_i^n\right) \right) \right|$$

$$\leq \lim_{\nu(\Delta_n)\to 0} \sum_{i=0}^{n-1} |q(\xi_i^n)| \left| \frac{\left(v(x_{i+1}^n) - v(x_i^n)\right)}{x_{i+1}^n - x_i^n} \right| \left(x_{i+1}^n - x_i^n \right)$$

$$\leq L \lim_{\nu(\Delta_n)\to 0} \sum_{i=0}^{n-1} |q\left(\xi_i^n\right)| \left(x_{i+1}^n - x_i^n \right) = L \int_a^b |q(t)|\,dt.$$

Now, let observe that

$$(2.5) \qquad \left| \int_a^b f(x)du(x) - \frac{u(b) - u(a)}{b - a} \int_a^b f(t)dt \right|$$

$$= \left| \int_a^b (f(x) - \frac{1}{b - a} \int_a^b f(t)dt)du(x) \right|$$

$$\leq L \int_a^b \left| f(x) - \frac{1}{b-a} \int_a^b f(t)dt \right| dx.$$

Now, define

$$I := \frac{1}{b-a} \int_a^b \left(f(x) - \frac{1}{b-a} \int_a^b f(t)dt \right)^2 dx.$$

Then we have

$$I = \frac{1}{b-a} \int_a^b \left[f^2(x) - 2f(x) \frac{1}{b-a} \int_a^b f(t)dt + \left(\frac{1}{b-a} \int_a^b f(t)dt \right)^2 \right] dx$$

$$= \frac{1}{b-a} \int_a^b f^2(x)dx - \left(\frac{1}{b-a} \int_a^b f(t)dt \right)^2$$

and

$$I = \left(M - \frac{1}{b-a} \int_a^b f(t)dt \right) \left(\frac{1}{b-a} \int_a^b f(t)dt - m \right)$$

$$- \frac{1}{b-a} \int_a^b \left(M - f(t) \right) \left(f(t) - m \right) dt.$$

As $m \leq f(x) \leq M$ for all $x \in [a,b]$, then

$$\int_a^b \left(M - f(t) \right) \left(f(t) - m \right) dt \geq 0$$

which implies

$$I \leq \left(M - \frac{1}{b-a} \int_a^b f(t)dt \right) \left(\frac{1}{b-a} \int_a^b f(t)dt - m \right).$$

Using the elementary inequality

$$(M - k)(k - m) \leq \frac{1}{4} \left[(M - k) + (k - m) \right]^2 = \frac{1}{4}(M - m)^2$$

which holds for $k, m, M \in R$, we get

(2.6) $$I \leq \frac{1}{4}(M - m)^2.$$

Using Cauchy-Buniakowski-Schwarz's integral inequality we have

$$I \geq \left[\frac{1}{b-a} \int_a^b \left| f(x) - \frac{1}{b-a} \int_a^b f(t)dt \right| dx \right]^2.$$

Now, by (2.6), we get

$$\int_a^b \left| f(x) - \frac{1}{b-a} \int_a^b f(t)dt \right| dx \leq \frac{1}{2}(M - m)(b - a)$$

and then by (2.5) we obtain the desired inequality (2.3).

To prove the sharpness of the constant $\frac{1}{2}$, let choose

$$u(x) := \left| x - \frac{a+b}{2} \right|, \ f(x) := sgn\left(x - \frac{a+b}{2} \right), x \in [a, b].$$

Then

$$|u(x) - u(y)| = \left| \left| x - \frac{a+b}{2} \right| - \left| y - \frac{a+b}{2} \right| \right| \leq |x - y|$$

for all $x, y \in [a, b]$, which shows that u is L-lipschitzian with the constant $L = 1$.

Also, because $-1 \leq f(x) \leq 1$, for all $x \in [a, b]$, then $M - m = 2$ and

$$\frac{L}{2}(M - m)(b - a) = b - a.$$

On the other hand,

$$\int_a^b f(x)du(x) - \frac{u(b) - u(a)}{b - a} \int_a^b f(t)dt = \int_a^b sgn\left(x - \frac{a+b}{2} \right) du(x)$$

$$= -\int_a^{\frac{a+b}{2}} du(x) + \int_{\frac{a+b}{2}}^b du(x) = -u\left(\frac{a+b}{2} \right) + u(a) + u(b) - u\left(\frac{a+b}{2} \right)$$

$$= b - a$$

which shows that the equality is realized in (2.3). ∎

Corollary 2.2. *Let $f : [a, b] \to \mathbf{R}$ be as above and $u : [a, b] \to \mathbf{R}$ a differentiable mapping whose derivative $u' : (a, b) \to \mathbf{R}$ is bounded on (a, b). Denote $\|u'\|_\infty := \sup_{t \in (a,b)} |u'(t)| < \infty$. Then we have the inequality*

$$\left| \int_a^b f(t)u'(t)dt - \frac{u(b) - u(a)}{b - a} \int_a^b f(t)dt \right| \leq \frac{\|u'\|_\infty}{2}(M - m)(b - a).$$

Corollary 2.3. *Let* $f : [a,b] \to \mathbf{R}$ *be as above and* $g : [a,b] \to \mathbf{R}$ *a continuous mapping. Denote* $\|g\|_\infty := \sup_{t \in [a,b]} |g(t)| < \infty$. *Then we have the inequality*

$$\left| \int_a^b f(t)g(t)dt - \frac{1}{b-a} \int_a^b g(t)dt \int_a^b f(t)dt \right| \le \frac{\|g\|_\infty}{2} (M-m)(b-a).$$

Corollary 2.4. *Let* $f : [a,b] \to \mathbf{R}$ *be a differentiable mapping whose derivative* $f' : (a,b) \to \mathbf{R}$ *is bounded on* (a,b) *and* $f(a) \neq f(b)$. *Denote* $\|f'\|_\infty := \sup_{t \in (a,b)} |f'(t)| < \infty$. *Then we have the inequality*

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(t)dt \right| \le \frac{\|f'\|_\infty}{2 \, (f(b) - f(a))} (M-m)(b-a).$$

The proof is obvious from the above corollary choosing $u = f$.

Remark 2.1. *If in Corollary 2.2 we put* $u = f^p, u = \ln f, u = \sin f$ *etc..., we can obtain some other interesting inequalities . We shall omit the details.*

3   APPLICATIONS FOR SPECIAL MEANS

We first discuss the application of the results in the previous section to lower and upper bounds estimation of some important relationships between the following means:

The arithmetic mean:

$$A = A(a,b) := (a+b)/2, \quad a,b \ge 0.$$

The geometric mean:

$$G = G(a,b) := \sqrt{ab}, \quad a,b \ge 0.$$

The harmonic mean:

$$H = H(a,b) := \frac{2}{\dfrac{1}{a} + \dfrac{1}{b}}, \quad a,b > 0.$$

The logarithmic mean:

$$L = L(a,b) := \begin{cases} \dfrac{b-a}{\ln b - \ln a} & \text{if } a \neq b \\[2mm] a & \text{if } a = b \end{cases}, \quad a,b > 0.$$

Note that for the convex mapping $f : (0, \infty) \to \mathbf{R}$, $f(t) = \frac{1}{t}$ we have

$$\frac{1}{b-a} \int_a^b f(t)dt = L^{-1}(a,b)$$

for $a \neq b$.

The p-logarithmic mean

$$L_p = L_p(a,b) := \begin{cases} \left[\dfrac{b^{p+1} - a^{p+1}}{(p+1)(b-a)} \right]^{1/p} & \text{if } a \neq b \\ \\ a & \text{if } a = b \end{cases}, \quad p \in \mathbf{R} \setminus \{-1, 0\}, a, b > 0.$$

For the convex (or concave) mapping $f(t) = t^p$, $p \in (-\infty, 0) \cup [1, \infty) \setminus \{-1\}$ (or $p \in (0,1)$) we have

$$\frac{1}{b-a} \int_a^b f(t)dt = L_p^p(a,b)$$

for $a \neq b$.

The identric mean:

$$I = I(a,b) := \begin{cases} \dfrac{1}{e} \left(\dfrac{b^b}{a^a} \right)^{\frac{1}{b-a}} & \text{if } a \neq b \\ \\ a & \text{if } a = b \end{cases}, \quad a, b > 0.$$

For the convex mapping $f(t) = \ln t, t > 0$ we have

$$\frac{1}{b-a} \int_a^b f(t)dt = \ln I(a,b)$$

if $a \neq b$.

These means are often used in numerical approximation and in other areas. However, the following simple relationships are known in the literature

$$H \leq G \leq L \leq I \leq A.$$

It is also known that L_p is monotonically increasing in $p \in \mathbf{R}$ with $L_0 = I$ and $L_{-1} = L$.

We now derive various sophisticated bounds for some differences and products of the above special means using the results obtained in the previous section. These bounds are useful in applications since the special means are often used in numerical approximations.

    1 . If in Corollary 2.2 we choose $f(x) = x^q$, $u(x) = x^{p+1}$ $(p, q > 0)$, then we get

$$(3.1) \qquad\qquad \left| L_{p+q}^{p+q} - L_p^p L_q^q \right| \leq \frac{qb^p}{2}(b-a)L_{q-1}^{q-1}.$$

    2 . If in Corollary 2.2 we choose $f(x) = x^q$ $(q > 0)$, $u(x) = \frac{1}{x}$, then we get

$$(3.2) \qquad\qquad \left| L_q^q - G^2 L_{q-2}^{q-2} \right| \leq \frac{q}{2a^2}(b-a)G^2 L_{q-1}^{q-1}.$$

3 . If in the same corollary we choose

$$f(x) = x^q (q > 0), u(x) = \ln x,$$

then we get

(3.3) $$\left| L_q^q - LL_{q-1}^{q-1} \right| \le \frac{q}{2a^2}(b-a)LL_{q-1}^{q-1}.$$

We remark that if in the Corollary 2.2 we choose f and u in other appropriate ways, we get some other interesting inequalities for special means.

We omit the details.

References

[1] D.S. Mitrinović, J.E. Pečarić and A.M. Fink, *Classical and New Inequalities in Analysis*, Kluwer Academic Publishers, Dordrecht, 1993.

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