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An Ostrowski Type Inequality for Double Integrals and Applications for Cubature Formulae

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AN OSTROWSKI TYPE INEQUALITY FOR DOUBLE INTEGRALS AND APPLICATIONS FOR CUBATURE FORMULAE

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ABSTRACT. An inequality of the Ostrowski type for double integrals and applications in Numerical Analysis in connection with cubature formulae are given.

1   INTRODUCTION

In 1938, A. Ostrowski proved the following integral inequality [5, p. 468]

Theorem 1.1. *Let* $f : [a, b] \to \mathbf{R}$ *be a differentiable mapping on* (a, b) *whose derivative* $f' : (a, b) \to \mathbf{R}$ *is bounded on* (a, b), *i.e.,* $\|f'\|_\infty := \sup\limits_{t \in (a,b)} |f'(t)| < \infty.$

Then we have the inequality

$$(1.1) \qquad \left| f(x) - \frac{1}{b-a} \int_a^b f(t)\, dt \right| \leq \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2} \right)}{(b-a)^2} \right] (b-a) \|f'\|_\infty$$

for all $x \in [a, b]$.
The constant $\frac{1}{4}$ *is the best possible.*

For some generalizations of this classic fact see the book [5, p. 468-484] by Mitrinović, Pečarić and Fink.

Some applications of the above results in Numerical Integration and for special means have been given in the recent paper [3] by S.S. Dragomir and S. Wang. For other results of Ostrowski's type see the papers [1], [2] and [4].

In 1975, G.N. Milovanović generalized Theorem 1.1 where f is a function of several variables [5, p. 468]

Theorem 1.2. *Let* $f : \mathbf{R}^m \to \mathbf{R}$ *be a differentiable function defined on* $D = \{(x_1, ..., x_m) \,|\, a_i \leq x_i \leq b_i \,(i = 1, ..., m)\}$ *and let* $\left| \frac{\partial f}{\partial x_i} \right| \leq M_i \; (M_i > 0, i = 1, ..., m)$ *in* D. *Furthermore, let function* $x \longmapsto p(x)$ *be integrable and* $p(x) > 0$ *for every* $x \in D$. *Then for every* $x \in D$, *we have the inequality:*

$$(1.2) \qquad \left| f(x) - \frac{\int\limits_D p(y) f(y)\, dy}{\int\limits_D p(y)\, dy} \right| \leq \frac{\sum\limits_{i=1}^m M_i \int\limits_D p(y) |x_i - y_i|\, dy}{\int\limits_D p(y)\, dy}.$$

In the present paper we point out an Ostrowski type inequality for double integrals and apply it in Numerical Integration obtaining a general cubature formula.

2   THE RESULTS

The following inequality of Ostrowski's type for mappings of two variables holds:

Theorem 2.1. *Let* $f : [a, b] \times [c, d] \to \mathbf{R}$ *continuous on* $[a, b] \times [c, d]$, $f''_{x,y} = \frac{\partial^2 f}{\partial x \partial y}$ *exists on* $(a, b) \times (c, d)$ *and is bounded, i.e.,*

$$\left\| f''_{s,t} \right\|_\infty := \sup_{(x,y) \in (a,b) \times (c,d)} \left| \frac{\partial^2 f(x,y)}{\partial x \partial y} \right| < \infty$$

then we have the inequality:

$$(2.1) \qquad | \int_a^b \int_c^d f(s,t)\, ds dt - [(b - a) \int_c^d f(x,t)\, dt + (d - c) \int_a^b f(s,y)\, ds$$

$$- (d - c)(b - a) f(x,y)] |$$

$$\leq \left[\frac{1}{4} (b - a)^2 + \left(x - \frac{a+b}{2} \right)^2 \right] \left[\frac{1}{4} (d - c)^2 + \left(y - \frac{c+d}{2} \right)^2 \right] \left\| f''_{s,t} \right\|_\infty$$

for all $(x, y) \in [a, b] \times [c, d]$.

Proof. We have the equality:

$$(2.2) \qquad \int_a^x \int_c^y (s - a)(t - c) f''_{s,t}(s,t)\, dt ds$$

$$= \int_a^x (s - a) [f'_s(s,y)(y - c) - \int_c^y f'_s(s,t)\, dt] ds$$

$$= (y - c) \int_a^x (s - a) f'_s(s,y)\, ds - \int_c^y \left(\int_a^x (s - a) f'_s(s,t)\, ds \right) dt$$

$$= (y - c) \left[(x - a) f(x,y) - \int_a^x f(s,y)\, ds \right] - \int_c^y \left[(x - a) f(x,t) - \int_a^x f(s,t)\, ds \right] dt$$

$$= (y - c)(x - a) f(x,y) - (y - c) \int_a^x f(s,y)\, ds$$

$$- (x - a) \int_c^y f(x, t) \, dt + \int_a^x \int_c^y f(s, t) \, dsdt.$$

Also, by similar computations we have

(2.3)
$$\int_a^x \int_y^d (s - a)(t - d) f''_{s,t}(s, t) \, dsdt$$

$$= \int_a^x (s - a) \left[(d - y) f'_s(s, y) - \int_y^d f'_s(s, t) \, dt \right] ds$$

$$= (d - y) \int_a^x (s - a) f'_s(s, y) \, ds - \int_y^d \left(\int_a^x (s - a) f'_s(s, t) \, ds \right) dt$$

$$= (d - y) \left[(x - a) f(x, y) - \int_a^x f(s, y) \, ds \right] - \int_y^d \left[(x - a) f(x, t) - \int_a^x f(s, t) \, ds \right] dt$$

$$= (x - a)(d - y) f(x, y) - (d - y) \int_a^x f(s, y) \, ds$$

$$- (x - a) \int_y^d f(x, t) \, dt + \int_a^x \int_c^y f(s, t) \, dsdt.$$

Now,

(2.4)
$$\int_x^b \int_y^d (s - b)(t - d) f''_{s,t}(s, t) \, dsdt$$

$$= \int_x^b (s - b) \left[(d - y) f'_s(s, y) - \int_y^d f'_s(s, t) \, dt \right] ds$$

$$= (d - y) \int_x^b (s - b) f'_s(s, y) \, ds - \int_y^d \left(\int_x^b (s - b) f'_s(s, t) \, ds \right) dt$$

$$= (d - y) \left[(b - x) f(x, y) - \int_x^b f(s, y)\, ds \right] - \int_y^d \left[(b - x) f(x, t) - \int_x^b f(s, t)\, ds \right] dt$$

$$= (d - y)(b - x) f(x, y) - (d - y) \int_x^b f(s, y)\, ds$$

$$- (b - x) \int_y^d f(x, t)\, dt + \int_x^b \int_y^d f(s, t)\, dsdt$$

and finally

$$(2.5) \qquad \int_x^b \int_c^y (s - b)(t - c) f_{s,t}''(s, t)\, dsdt$$

$$= \int_x^b (s - b) \left[(y - c) f_s'(s, y) - \int_c^y f_s'(s, t)\, dt \right] ds$$

$$= (y - c) \int_x^b (s - b) f_s'(s, y)\, ds - \int_c^y \left(\int_x^b (s - b) f_s'(s, t)\, ds \right) dt$$

$$= (y - c) \left[(b - x) f(x, y) - \int_x^b f(s, y)\, ds \right] - \int_c^y \left[(b - x) f(x, t) - \int_x^b f(s, t)\, ds \right] dt$$

$$= (y - c)(b - x) f(x, y) - (y - c) \int_x^b f(s, y)\, ds$$

$$- (b - x) \int_c^y f(x, t)\, dt + \int_x^b \int_c^y f(s, t)\, dsdt.$$

If we add the equalities $(2.2) - (2.5)$ we get in the right membership:

$$[(y - c)(x - a) + (x - a)(d - y) + (d - y)(b - x) + (y - c)(b - x)] f(x, y)$$

$$- (d - c) \int\limits_{a}^{x} f(s, y)\, ds - (d - c) \int\limits_{x}^{b} f(s, y)\, ds - (b - a) \int\limits_{c}^{y} f(x, t)\, dt$$

$$- (b - a) \int\limits_{y}^{d} f(x, t)\, dt + \int\limits_{a}^{x} \int\limits_{c}^{y} f(s, t)\, ds dt + \int\limits_{a}^{x} \int\limits_{y}^{d} f(s, t)\, ds dt$$

$$+ \int\limits_{x}^{b} \int\limits_{y}^{d} f(s, t)\, ds dt + \int\limits_{x}^{b} \int\limits_{c}^{y} f(s, t)\, ds dt$$

$$= (d - c)(b - a) f(x, y) - (d - c) \int\limits_{a}^{b} f(s, y)\, ds$$

$$- (b - a) \int\limits_{c}^{b} f(x, t)\, dt + \int\limits_{a}^{b} \int\limits_{c}^{d} f(s, t)\, ds dt.$$

For the first membership, let us define the kernels: $p : [a, b]^2 \to \mathbf{R}$, $q : [c, d]^2 \to \mathbf{R}$ given by:

$$p(x, s) := \begin{cases} s - a & \text{if } s \in [a, x] \\ \\ s - b & \text{if } s \in (x, b] \end{cases}$$

and

$$q(y, t) := \begin{cases} t - c & \text{if } t \in [c, y] \\ \\ t - d & \text{if } t \in (y, d] \end{cases} .$$

Now, using these notations, we deduce that the left membership can be represented as :

$$\int\limits_{a}^{b} \int\limits_{c}^{d} p(x, s)\, q(y, t)\, f''_{s,t}(s, t)\, ds dt.$$

Consequently, we get the identity

$$(2.6) \qquad \int\limits_{a}^{b} \int\limits_{c}^{d} p(x, s)\, q(y, t)\, f''_{s,t}(s, t)\, ds dt$$

$$= (d - c)(b - a) f(x, y) - (d - c) \int\limits_{a}^{b} f(x, y)\, ds - (b - a) \int\limits_{c}^{d} f(x, t)\, dt + \int\limits_{a}^{b} \int\limits_{c}^{d} f(s, t)\, ds dt$$

for all $(x, y) \in [a, b] \times [c, d]$.

Now, using the identity (2.6) we get

$$(2.7) \qquad | \int_a^b \int_c^d f(s,t)\, dsdt - [(b-a) \int_c^d f(x,t)\, dt + (d-c) \int_a^b f(x,y)\, ds$$

$$- (d-c)(b-a) f(x,y)] |$$

$$\leq \int_a^b \int_c^d |p(x,s)|\, |q(y,t)|\, \left| f''_{s,t}(s,t) \right|\, dsdt \leq \left\| f''_{s,t} \right\|_\infty \int_a^b \int_c^d |p(x,s)|\, |q(y,t)|\, dsdt.$$

Now, observe that

$$\int_a^b |p(x,s)|\, ds = \int_a^x (s-a)\, ds + \int_x^b (b-s)\, ds$$

$$= \frac{(x-a)^2 + (b-x)^2}{2} = \frac{1}{4} (b-a)^2 + \left(x - \frac{a+b}{2} \right)^2$$

and, similarly,

$$\int_c^d |q(y,t)|\, dt = \frac{1}{4} (d-c)^2 + \left(y - \frac{c+d}{2} \right)^2.$$

Finally, using (2.7), we get the desired inequality (2.1). ∎

Corollary 2.2. *Under the above assumptions, we have the inequality:*

$$(2.8) \qquad \left| \int_a^b \int_c^d f(s,t)\, dsdt - \left[(b-a) \int_c^d f\left(\frac{a+b}{2}, t \right) dt \right. \right.$$

$$\left. \left. + (d-c) \int_a^b f\left(s, \frac{c+d}{2} \right) ds - (d-c)(b-a) f\left(\frac{a+b}{2}, \frac{c+d}{2} \right) \right] \right|$$

$$\leq \frac{1}{16} (b-a)^2 (d-c)^2 \left\| f''_{s,t} \right\|_\infty.$$

Remark 2.1. *The constants $\frac{1}{4}$ from the first and the second bracket are optimal in the sense that not both of them can be less than $\frac{1}{4}$.*

Indeed, if we had assumed that there exists $c_1, c_2 \in \left(0, \frac{1}{4}\right)$ so that

$$(2.9) \qquad \left| \int_a^b \int_c^d f(s,t)\, ds dt - \left[(b-a) \int_c^d f(x,t)\, dt + (d-c) \int_a^b f(s,y)\, ds \right. \right.$$

$$\left. \left. - (d-c)(b-a) f(x,y) \right] \right|$$

$$\leq \left[c_1 (b-a)^2 + \left(x - \frac{a+b}{2} \right)^2 \right] \left[c_2 (d-c)^2 + \left(y - \frac{c+d}{2} \right)^2 \right] \left\| f''_{s,t} \right\|_\infty$$

for all f as in Theorem 2.1 and $(x,y) \in [a,b] \times [c,d]$, then we would have had for $f(s,t) = st$ and $x = a, y = c$ that:

$$\int_a^b \int_c^d f(s,t)\, ds dt = \frac{(b^2 - a^2)(d^2 - c^2)}{4},$$

$$\int_c^d f(x,t)\, dt = a \cdot \frac{d^2 - c^2}{2}, \int_a^b f(s,y)\, ds = c \cdot \frac{b^2 - a^2}{2},$$

$$\left\| f''_{s,t} \right\|_\infty = 1$$

and by (2.9), the inequality:

$$\left| \frac{(b^2 - a^2)(d^2 - c^2)}{4} - (b-a) a \cdot \frac{d^2 - c^2}{2} - (d-c) c \cdot \frac{b^2 - a^2}{2} + (d-c)(b-a) ac \right|$$

$$\leq (b-a)^2 \left(c_1 + \frac{1}{4} \right) (d-c)^2 \left(c_2 + \frac{1}{4} \right)$$

i.e.

$$\frac{(b-a)^2 (d-c)^2}{4} \leq (b-a)^2 (d-c)^2 \left(c_1 + \frac{1}{4} \right) \left(c_2 + \frac{1}{4} \right)$$

i.e.

$$(2.10) \qquad \frac{1}{4} \leq \left(c_1 + \frac{1}{4} \right) \left(c_2 + \frac{1}{4} \right).$$

Now, as we have assumed that $c_1, c_2 \in \left(0, \frac{1}{4}\right)$, we get

$$c_1 + \frac{1}{4} < \frac{1}{2}, c_2 + \frac{1}{4} < \frac{1}{2}$$

and then $\left(c_1 + \frac{1}{4} \right) \left(c_2 + \frac{1}{4} \right) < \frac{1}{4}$ which contradicts the inequality (2.10), and the statement is proved. ∎

Remark 2.2. *Now, if we assume that $f(s,t) = h(s)h(t)$, $h : [a,b] \to \mathbf{R}$, h is continuous and suppose that $\|h'\|_\infty < \infty$, then from (2.1) we get (for $x = y$)*

$$\left| \int_a^b h(s)\,ds \int_a^b h(s)\,ds - h(x)(b-a) \int_a^b h(s)\,ds \right.$$

$$\left. -h(x)(b-a) \int_a^b h(s)\,ds + (b-a)^2 h^2(x) \right|$$

$$\leq \left[\frac{1}{4}(b-a)^2 + \left(x - \frac{a+b}{2} \right)^2 \right]^2 \|h\|_\infty^2$$

i.e.

$$\left[\int_a^b h(s)\,ds - h(x)(b-a) \right]^2 \leq \left[\frac{1}{4}(b-a)^2 + \left(x - \frac{a+b}{2} \right) \right]^2 \|h\|_\infty^2$$

which is clearly equivalent to Ostrowski's inequality.

Consequently (2.1) can be also regarded as a generalization for double integrals of the classical result due to Ostrowski.

3   APPLICATIONS FOR CUBATURE FORMULAE

Let us consider the arbitrary division $I_n : a = x_1 < x_1 < ... < x_{n-1} < x_n = b$ and $J_m : c = y_0 < y_1 < ... < y_{m-1} < y_m = b$ and $\xi_i \in [x_i, x_{i+1}]$ $(i = 0, ..., n-1)$, $\eta_j \in [y_j, y_{j+1}]$ $(j = 0, ..., m-1)$ be intermediate points. Consider the sum

(3.1) $$C(f, I_n, J_m, \boldsymbol{\xi}, \boldsymbol{\eta}) := \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} h_i \int_{y_j}^{y_{j+1}} f(\xi_i, t)\,dt$$

$$+ \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} l_j \int_{x_i}^{x_{i+1}} f(s, \eta_j)\,ds - \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} h_i l_j f(\xi_i, \eta_j)$$

for which we assume that the involved integrals can more easily be computed than the original double integral

$$D := \int_a^b \int_c^d f(s,t)\,ds\,dt,$$

and

$$h_i := x_{i+1} - x_i \ (i = 0, ..., n-1), \qquad l_j := y_{j+1} - y_j \quad (j = 0, ..., m-1).$$

With this assumption, we can state the following cubature formula:

Theorem 3.1. *Let* $f : [a,b] \times [c,d] \to \mathbf{R}$ *be as in Theorem 2.1 and* $I_n, J_m, \boldsymbol{\xi}$ *and* $\boldsymbol{\eta}$ *be as above. Then we have the cubature formula:*

$$(3.2) \qquad \int\limits_a^b \int\limits_c^d f(s,t)\,dsdt = C(f, I_n, J_m, \boldsymbol{\xi}, \boldsymbol{\eta}) + R(f, I_n, J_m, \boldsymbol{\xi}, \boldsymbol{\eta})$$

where the remainder term $R(f, I_n, J_m, \boldsymbol{\xi}, \boldsymbol{\eta})$ *satisfies the estimation:*

$$(3.3) \qquad\qquad |R(f, I_n, J_m, \boldsymbol{\xi}, \boldsymbol{\eta})|$$

$$\leq \|f''_{s,t}\|_\infty \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} \left[\frac{1}{4} h_i^2 + \left(\xi_i - \frac{x_i + x_{i+1}}{2} \right)^2 \right] \left[\frac{1}{4} l_j^2 + \left(\eta_j - \frac{y_j + y_{j+1}}{2} \right)^2 \right]$$

$$\leq \frac{1}{4} \|f''_{s,t}\|_\infty \sum_{i=0}^{n-1} h_i^2 \sum_{j=0}^{m-1} l_j^2.$$

Proof. Apply Theorem 2.1 on the interval $[x_i, x_{i+1}] \times [y_j, y_{j+1}]$
$(i = 0, ..., n-1; j = 0, ..., m-1)$ to get:

$$\left| \int\limits_{x_i}^{x_{i+1}} \int\limits_{y_j}^{y_{j+1}} f(s,t)\,dsdt - \left[h_i \int\limits_{y_j}^{y_{j+1}} f(\xi_i, t)\,dt + l_j \int\limits_{x_i}^{x_{i+1}} f(s, \eta_j)\,ds - h_i l_j f(\xi_i, \eta_j) \right] \right|$$

$$\leq \left[\frac{1}{4} h_i^2 + \left(\xi_i - \frac{x_i + x_{i+1}}{2} \right)^2 \right] \left[\frac{1}{4} l_j^2 + \left(\eta_j - \frac{y_j + y_{j+1}}{2} \right)^2 \right] \|f''_{s,t}\|_\infty$$

for all $i = 0, ..., n-1; \quad j = 0, ..., m-1.$

Summing over i from 0 to $n-1$ and over j from 0 to $m-1$ and using the generalized triangle inequality we deduce the first inequality in (3.3).

For the second part we observe that

$$\left| \xi_i - \frac{x_i + x_{i+1}}{2} \right| \leq \frac{1}{2} h_i \quad \text{and} \quad \left| \eta_j - \frac{y_j + y_{j+1}}{2} \right| \leq \frac{1}{2} l_j$$

for all i, j as above. ∎

Remark 3.1. *As*

$$\sum_{i=0}^{n-1} h_i^2 \leq \nu(h) \sum_{i=0}^{n-1} h_i = (b-a)\nu(h)$$

and

$$\sum_{j=0}^{m-1} l_j^2 \leq \mu(l) \sum_{j=0}^{m-1} l_j = (d-c)\mu(l)$$

where

$$\nu\left(h\right) = \max\left\{h_i : i = 0, ..., n-1\right\},$$

$$\mu\left(l\right) = \max\left\{l_j : j = 0, ..., m-1\right\},$$

the right membership of (3.3) *can be bounded by*

$$\frac{1}{4}\left\|f_{s,t}''\right\|_\infty (b-a)(d-c)\nu\left(h\right)\mu\left(l\right)$$

which is of order 2 *precision.*

Now, define the sum

$$C_M\left(f, I_n, J_m\right) := \sum_{i=0}^{n-1}\sum_{j=0}^{m-1} h_i \int_{y_j}^{y_{j+1}} f\left(\frac{x_i + x_{i+1}}{2}, t\right) dt$$

$$+ \sum_{i=0}^{n-1}\sum_{j=0}^{m-1} l_j \int_{x_i}^{x_{i+1}} f\left(s, \frac{y_j + y_{j+1}}{2}\right) ds - \sum_{i=0}^{n-1}\sum_{j=0}^{m-1} h_i l_j f\left(\frac{x_i + x_{i+1}}{2}, \frac{y_j + y_{j+1}}{2}\right).$$

Then we have the best cubature formula we can get from (3.2).

Corollary 3.2. *Under the above assumptions we have*

$$(3.4) \qquad \int_a^b \int_c^d f\left(s, t\right) ds dt = C_M\left(f, I_n, J_m\right) + R\left(f, I_n, J_m\right)$$

when the remainder $R\left(f, I_n, J_m\right)$ *satisfies the estimation:*

$$\left|R\left(f, I_n, J_m\right)\right| \leq \frac{\left\|f_{s,t}''\right\|_\infty}{16} \sum_{i=0}^{n-1} h_i^2 \sum_{j=0}^{m-1} l_j^2.$$

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