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AN OSTROWSKI TYPE INEQUALITY FOR MAPPINGS WHOSE SECOND DERIVATIVES BELONG TO $\mathbf{L}_P(A,B)$ AND APPLICATIONS

P. CERONE, S.S. DRAGOMIR AND J. ROUMELIOTIS

Abstract. An inequality of the Ostrowski type for twice differentiable mappings whose derivatives belong to $L_p(a,b)$ $(p>1)$ and applications in Numerical Integration are investigated.

1 Introduction

The following inequality is well known in the literature as Ostrowski's integral inequality (see for example [1, p. 468])

Theorem 1.1. *Let $f: I \subseteq \mathbf{R} \to \mathbf{R}$ be a differentiable mapping on I° (I° is the interior of I) and let $a,b \in I^\circ$ with $a<b$. If $f':(a,b) \to \mathbf{R}$ is bounded, i.e., $\|f'\|_\infty := \sup_{t\in(a,b)} |f'(t)| < \infty$, then we have the inequality:*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \le \left[\frac{1}{4} + \frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right](b-a)\|f'\|_\infty \tag{1.1}$$

for all $x \in (a,b)$.

The constant $\frac{1}{4}$ is the best possible.

For a simple proof and some applications of Ostrowski's inequality to some special means and some numerical quadrature rules, we refer the reader to the recent paper [2] by S.S. Dragomir and A. Wang.

In [3], the same authors considered another inequality of Ostrowski type for $\|\cdot\|_p$ −norm $(p>1)$ as follows:

Theorem 1.2. *Let $f: I \subseteq \mathbf{R} \to \mathbf{R}$ be a differentiable mapping on I° and $a,b \in I^\circ$ with $a<b$. If $f' \in L_p(a,b)$ $\left(p>1, \frac{1}{p}+\frac{1}{q}=1\right)$ then we have the inequality:*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \le \frac{1}{b-a}\left[\frac{(x-a)^{q+1}+(b-x)^{q+1}}{q+1}\right]^{\frac{1}{q}}\|f'\|_p \tag{1.2}$$

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for all $x \in [a,b]$, *where*

$$\|f'\|_p := \left(\int_a^b |f(t)|^p \, dt \right)^{\frac{1}{p}},$$

is the $L_p(a,b)$ $-$*norm.*

They also pointed out some applications of (1.2) in Numerical Integration as well as for special means.

In 1976, G.V. Milovanović and J.E. Pečarić proved a generalization of Ostrowski inequality for $n-$times differentiable mappings (see for example [1, p. 468]). The case of twice differentiable mappings [1, p. 470] is as follows:

Theorem 1.3. *Let* $f : [a,b] \to \mathbf{R}$ *be a twice differentiable mapping such that* $f'' : (a,b) \to \mathbf{R}$ *is bounded on* (a,b), *i.e.,* $\|f''\|_\infty := \sup_{t \in (a,b)} |f''(t)| < \infty$. *Then we have the inequality:*

$$\left| \frac{1}{2} \left[f(x) + \frac{(x-a) f(a) + (b-x) f(b)}{b-a} \right] - \frac{1}{b-a} \int_a^b f(t) \, dt \right| \tag{1.3}$$

$$\leq \frac{\|f''\|_\infty}{4} (b-a)^2 \left[\frac{1}{12} + \frac{\left(x - \frac{a+b}{2} \right)^2}{(b-a)^2} \right]$$

for all $x \in [a,b]$.

In this paper, we point out an inequality of Ostrowski type for twice differentiable mappings which is in terms of the $\|\cdot\|_p$-norm of the second derivative f'' and apply it in Numerical Integration.

2 Some Integral Inequalities

The following inequality of Ostrowski type for mappings which are twice differentiable, holds:

Theorem 2.1. *Let* $f : [a,b] \to \mathbf{R}$ *be a twice differentiable mapping on* (a,b) *and* $f'' \in L_p(a,b)$ $(p > 1)$. *Then we have the inequality:*

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t) \, dt - \left(x - \frac{a+b}{2} \right) f'(x) \right| \tag{2.1}$$

$$\leq \frac{1}{2(b-a)(2q+1)^{\frac{1}{q}}} \left[(x-a)^{2q+1} + (b-x)^{2q+1} \right]^{\frac{1}{q}} \|f''\|_p$$

$$\leq \frac{(b-a)^{1+\frac{1}{q}} \|f''\|_p}{2(2q+1)^{\frac{1}{q}}}$$

for all $x \in [a,b]$, *where* $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Let us define the mapping $K(\cdot,\cdot):[a,b]^2\to\mathbf{R}$ given by

$$K(x,t):=\begin{cases}\dfrac{(t-a)^2}{2} & \text{if } t\in[a,x]\\[2ex] \dfrac{(t-b)^2}{2} & \text{if } t\in(x,b]\end{cases}.$$

Integrating by parts, we have successively,

$$\int_a^b K(x,t)f''(t)\,dt=\int_a^x\frac{(t-a)^2}{2}f''(t)\,dt+\int_x^b\frac{(t-b)^2}{2}f''(t)\,dt$$

$$=\left.\frac{(t-a)^2}{2}f'(t)\right|_a^x-\int_a^x(t-a)f'(t)\,dt+\left.\frac{(t-b)^2}{2}f'(t)\right|_x^b-\int_x^b(t-b)f'(t)\,dt$$

$$=\frac{(x-a)^2}{2}f'(x)-\left[(t-a)f(t)|_a^x-\int_a^x f(t)\,dt\right]$$

$$-\frac{(b-x)^2}{2}f'(x)-\left[(t-b)f(t)|_x^b-\int_x^b f(t)\,dt\right]$$

$$=\frac{1}{2}\left[(x-a)^2-(b-x)^2\right]f'(x)$$

$$-(x-a)f(x)+\int_a^x f(t)\,dt+(x-b)f(x)+\int_x^b f(t)\,dt$$

$$=(b-a)\left(x-\frac{a+b}{2}\right)f'(x)-(b-a)f(x)+\int_a^b f(t)\,dt$$

from which we get the integral identity

(2.2)
$$\int_a^b f(t)\,dt=(b-a)f(x)-(b-a)\left(x-\frac{a+b}{2}\right)f'(x)+\int_a^b K(x,t)f''(t)\,dt$$

for all $x\in[a,b]$.

Using (2.2), we have, by Hölder's integral inequality, that

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt-\left(x-\frac{a+b}{2}\right)f'(x)\right| \tag{2.3}$$

$$=\frac{1}{b-a}\left|\int_a^b K(x,t)\,f''(t)\,dt\right|\leq\frac{1}{b-a}\left(\int_a^b K^q(x,t)\,dt\right)^{\frac{1}{q}}\|f''\|_p$$

$$=\frac{1}{b-a}\left[\int_a^x\frac{(t-a)^{2q}}{2^q}dt+\int_x^b\frac{(t-b)^{2q}}{2^q}dt\right]^{\frac{1}{q}}\|f''\|_p$$

$$=\frac{1}{b-a}\left[\frac{(x-a)^{2q+1}}{2^q(2q+1)}+\frac{(b-x)^{2q+1}}{2^q(2q+1)}\right]^{\frac{1}{q}}\|f''\|_p$$

$$=\frac{1}{2(b-a)}\frac{1}{(2q+1)^{\frac{1}{q}}}\left[(x-a)^{2q+1}+(b-x)^{2q+1}\right]^{\frac{1}{q}}\|f''\|_p$$

and the first inequality in (2.1) is proved. The second inequality is obvious taking into account that

$$(x-a)^{2q+1}+(b-x)^{2q+1}\leq(b-a)^{2q+1}$$

for all $x\in[a,b]$. ∎

The following particular case for euclidean norms is interesting

Corollary 2.2. *Let $f:[a,b]\to\mathbf{R}$ be as above and $f''\in L_2(a,b)$. Then we have the inequality:*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt-\left(x-\frac{a+b}{2}\right)f'(x)\right| \tag{2.4}$$

$$\leq\frac{(b-a)^{\frac{3}{2}}}{2}\left[\frac{1}{80}+\frac{1}{2}\cdot\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}+\frac{\left(x-\frac{a+b}{2}\right)^4}{(b-a)^4}\right]^{\frac{1}{2}}\|f''\|_2\,.$$

Proof. Apply inequality (2.1) for $p=q=2$, to get

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt-\left(x-\frac{a+b}{2}\right)f'(x)\right| \tag{2.5}$$

$$\leq \frac{1}{2(b-a)\sqrt{5}}\left[(x-a)^5+(b-x)^5\right]^{\frac{1}{2}}\|f''\|_2 .$$

Denote $t := x - \frac{a+b}{2}$. Then

$$x-a = t+\frac{b-a}{2}, b-x = \frac{b-a}{2}-t.$$

Let us compute

$$I := (x-a)^5+(b-x)^5 = \left(t+\frac{b-a}{2}\right)^5+\left(\frac{b-a}{2}-t\right)^5 .$$

We know that, for numbers $A, B \in \mathbf{R}$, we have

$$\begin{aligned} A^5+B^5 &= (A+B)\left(A^4-A^3B+A^2B^2-AB^3+B^4\right) \\ &= (A+B)\left[A^4+B^4-AB\left(A^2+B^2\right)+A^2B^2\right] \\ &= (A+B)\left[\left(A^2+B^2\right)^2-A^2B^2-AB\left(A^2+B^2\right)\right]. \end{aligned}$$

Now, if we put $A := t+\frac{b-a}{2}$, $\quad B := \frac{b-a}{2}-t$, then we get

$$A^2+B^2 = 2t^2+\frac{(b-a)^2}{2}, AB = \frac{(b-a)^2}{4}-t^2$$

and then

$$\begin{aligned} J &:= \left(A^2+B^2\right)^2-A^2B^2-AB\left(A^2+B^2\right) \\ &= \left[2t^2+\frac{(b-a)^2}{2}\right]^2-\left[t^2-\frac{(b-a)^2}{4}\right]^2-\left[\frac{(b-a)^2}{4}-t^2\right]\left[2t^2+\frac{(b-a)^2}{2}\right] \\ &= 5t^4+\frac{5}{2}(b-a)^2t^2+\frac{(b-a)^4}{16} = 5\left[t^4+2\left(\frac{b-a}{2}\right)^2t^2+\frac{1}{5}\left(\frac{b-a}{2}\right)^4\right]. \end{aligned}$$

Consequently,

$$\begin{aligned} I &= (b-a)\left[5\left(x-\frac{a+b}{2}\right)^4+\frac{5}{2}(b-a)^2\left(x-\frac{a+b}{2}\right)^2+\frac{(b-a)^4}{16}\right] \\ &= 5(b-a)^5\left[\frac{1}{80}+\frac{1}{2}\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}+\frac{\left(x-\frac{a+b}{2}\right)^4}{(b-a)^4}\right]. \end{aligned}$$

Finally, using the inequality (2.5), we get the desired result (2.4) ∎

Remark 2.1. *Let $f : [a,b] \to \mathbf{R}$ be as above . Then we have the midpoint inequality:*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \frac{1}{8(2q+1)^{\frac{1}{q}}}(b-a)^{1+\frac{1}{q}}\|f''\|_p . \tag{2.6}$$

Taking into account the fact that the mapping

$$h : [a, b] \to \mathbf{R}, \quad h(x) = (x-a)^{2q+1} + (b-x)^{2q+1}$$

has the property that

$$\inf_{x \in [a,b]} h(x) = h\left(\frac{a+b}{2}\right) = \frac{(b-a)^{2q+1}}{2^{2q}}$$

and

$$\sup_{x \in [a,b]} h(x) = h(a) = h(b) = (b-a)^{2q+1};$$

then, the best estimation we can get from (2.1) is that one for which $x = \frac{a+b}{2}$, obtaining the inequality (2.6).

Remark 2.2. *If in* (2.1) *we choose* $x = a$ *we get*

$$\left| f(a) - \frac{1}{b-a} \int_a^b f(t)\, dt + \frac{b-a}{2} f'(a) \right| \leq \frac{(b-a)^{1+\frac{1}{q}}}{2(2q+1)^{\frac{1}{q}}} \|f''\|_p$$

and putting $x = b$, *we also get*

$$\left| f(b) - \frac{1}{b-a} \int_a^b f(t)\, dt - \frac{b-a}{2} f'(b) \right| \leq \frac{(b-a)^{1+\frac{1}{q}}}{2(2q+1)^{\frac{1}{q}}} \|f''\|_p .$$

Summing the above two inequalities, using the triangle inequality and dividing by 2, *we get the perturbed trapezoid formula*

(2.7)
$$\left| \frac{f(a)+f(b)}{2} - \frac{b-a}{4} \left(f'(b) - f'(a)\right) - \frac{1}{b-a} \int_a^b f(t)\, dt \right| \leq \frac{(b-a)^{1+\frac{1}{q}}}{2(2q+1)^{\frac{1}{q}}} \|f''\|_p .$$

Remark 2.3. *If* $p = q = 2$, *then we get for the euclidean norm, from* (2.6),

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(t)\, dt \right| \leq \frac{\sqrt{5}(b-a)^{\frac{3}{2}}}{40} \|f''\|_2 \tag{2.8}$$

and, from(2.7),

$$\left| \frac{f(a)+f(b)}{2} - \frac{b-a}{4} \left(f'(b) - f'(a)\right) - \frac{1}{b-a} \int_a^b f(t)\, dt \right| \leq \frac{\sqrt{5}(b-a)^{\frac{3}{2}}}{10} \|f''\|_2 . \tag{2.9}$$

3 Applications in Numerical Integration

Let $I_n : a = x_0 < x_1 < ... < x_{n-1} < x_n = b$ be a division of the interval, $\xi_i \in [x_i, x_{i+1}] \quad (i = 0, ..., n-1)$. We have the following quadrature formula:

Theorem 3.1. *Let $f : [a,b] \to \mathbf{R}$ be a twice differentiable mapping on (a,b) whose second derivative $f'' : (a,b) \to \mathbf{R}$ belongs to $L_p(a,b)$ $(p > 1)$, i.e.,*

$$\|f''\|_p := \left(\int_a^b |f''(t)^p| \, dt \right)^{\frac{1}{p}} < \infty.$$

Then the following perturbed Riemann type quadrature formula holds:

$$\int_a^b f(x)\,dx = A(f, f', \boldsymbol{\xi}, I_n) + R(f, f', \boldsymbol{\xi}, I_n) \tag{3.1}$$

where $A(f, f', \boldsymbol{\xi}, I_n)$ is given by

$$A(f, f', \boldsymbol{\xi}, I_n) := \sum_{i=0}^{n-1} h_i f(\xi_i) - \sum_{i=0}^{n-1} f'(\xi_i)\left(\xi_i - \frac{x_i + x_{i+1}}{2}\right) h_i$$

and the remainder satisfies the estimation:

$$| R(f, f', \boldsymbol{\xi}, I_n) | \tag{3.2}$$

$$\leq \frac{1}{2(2q+1)^{\frac{1}{q}}} \left(\sum_{i=0}^{n-1} (\xi_i - x_i)^{2q+1} + (x_{i+1} - \xi_i)^{2q+1} \right)^{\frac{1}{q}} \|f''\|_p$$

$$\leq \frac{1}{2(2q+1)^{1/q}} \left(\sum_{i=0}^{n-1} h_i^{2q+1} \right)^{\frac{1}{q}} \|f''\|_p ,$$

for all $\xi_i \in [x_i, x_{i+1}] \quad (i = 0, ..., n-1)$.

Proof. Apply inequality (2.1) on the interval $[x_i, x_{i+1}]$ $(i = 0, ..., n-1)$ to get

$$\left| f(\xi_i) h_i - \int_{x_i}^{x_{i+1}} f(t)\,dt - \left(\xi_i - \frac{x_i + x_{i+1}}{2}\right) f'(\xi_i) h_i \right|$$

$$\leq \frac{1}{2(2q+1)^{1/q}} \left[(\xi_i - x_i)^{2q+1} + (x_{i+1} - \xi_i)^{2q+1} \right]^{\frac{1}{q}} \left(\int_{x_i}^{x_{i+1}} |f''(t)|^p \, dt \right)^{\frac{1}{p}}$$

for all $i \in \{0, ..., n-1\}$.

Summing over i from 0 to $n-1$, using the generalized triangle inequality and Hölder's discrete inequality, we get:

$$
\begin{aligned}
&\left|R\left(f,f',\boldsymbol{\xi},I_n\right)\right| \\
&\leq \sum_{i=0}^{n-1}\left|f\left(\xi_i\right)h_i-\left(\xi_i-\frac{x_i+x_{i+1}}{2}\right)f'\left(\xi_i\right)h_i-\int_{x_i}^{x_{i+1}}f\left(t\right)dt\right| \\
&\leq \frac{1}{2\left(2q+1\right)^{1/q}}\sum_{i=0}^{n-1}\left[\left(\xi_i-x_i\right)^{2q+1}+\left(x_{i+1}-\xi_i\right)^{2q+1}\right]^{\frac{1}{q}}\left(\int_{x_i}^{x_{i+1}}\left|f''\left(t\right)\right|^p dt\right)^{\frac{1}{p}} \\
&\leq \frac{1}{2\left(2q+1\right)^{1/q}}\left(\sum_{i=0}^{n-1}\left(\left[\left(\xi_i-x_i\right)^{2q+1}+\left(x_{i+1}-\xi_i\right)^{2q+1}\right]^{\frac{1}{q}}\right)^q\right)^{\frac{1}{q}} \\
&\quad\times\left(\sum_{i=0}^{n-1}\left(\left(\int_{x_i}^{x_{i+1}}\left|f''\left(t\right)\right|^p dt\right)^{\frac{1}{p}}\right)^p\right)^{\frac{1}{p}} \\
&= \frac{1}{2\left(2q+1\right)^{1/q}}\left(\sum_{i=0}^{n-1}\left[\left(\xi_i-x_i\right)^{2q+1}+\left(x_{i+1}-\xi_i\right)^{2q+1}\right]\right)^{\frac{1}{q}}\left\|f''\right\|_p
\end{aligned}
$$

and the first inequality in (3.2) is proved.

The last part is obvious from the fact that

$$\left(\xi_i-x_i\right)^{2q+1}+\left(x_{i+1}-\xi_i\right)^{2q+1}\leq h_i^{2q+1}$$

for all $i\in\{0,...,n-1\}$. ∎

Now, if we consider the midpoint formula

$$M\left(f,I_n\right):=\sum_{i=0}^{n-1}f\left(\frac{x_i+x_{i+1}}{2}\right)h_i$$

then we have

$$\int_a^b f\left(t\right)dt=M\left(f,I_n\right)+R\left(f,I_n\right) \tag{3.3}$$

and the remainder $R\left(f,I_n\right)$ can be estimated in terms of the $p-$norm of f'' as follows:

$$\left|R\left(f,I_n\right)\right|\leq\frac{1}{8\left(2q+1\right)^{1/q}}\left(\sum_{i=0}^{n-1}h_i^{2q+1}\right)^{\frac{1}{q}}\left\|f''\right\|_p \tag{3.4}$$

which is, in a certain sense, the best estimation we can obtain from (3.2).

Also, we can construct the following perturbed trapezoid formula

$$T_p(f,f',I_n) := \frac{1}{2}\sum_{i=0}^{n-1}\frac{f(x_i)+f(x_{i+1})}{2}h_i + \frac{1}{4}\sum_{i=0}^{n-1}h_i^2\left(f'(x_i)-f'(x_{i+1})\right).$$

Then we have

$$\int_a^b f(t)\,dt = T_p(f,f',I_n) + R_p(f,f',I_n) \tag{3.5}$$

and the remainder can be estimated (see the inequality (2.7)) as follows:

$$|R_p(f,f',I_n)| \le \frac{1}{2(2q+1)^{1/q}}\left(\sum_{i=0}^{n-1}h_i^{2q+1}\right)^{\frac{1}{q}}\|f''\|_p. \tag{3.6}$$

Remark 3.1. *To derive the corresponding results for the euclidean norm $\|f''\|_2$, we put in the above $p=q=2$.*

We omit the details.

Remark 3.2. *The reader can obtain the corresponding quadrature formulae for equidistant partitioning by choosing $x_i = a + i\cdot\frac{b-a}{n}$ $(i=0,...,n-1)$.*

Remark 3.3. *If we consider equidistant partitioning of $[a,b]$ then the perturbed trapezoid formula we considered above will involve the calculation for f' only at the endpoints a and b , which is a good advantage for practical applications.*

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School of Communications and Informatics, Victoria University of Technology, PO Box 14428, Melbourne City MC, Victoria 8001, Australia.

E-mail address: {pc, sever, johnr}@matilda.vut.edu.au