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AN OSTROWSKI TYPE INEQUALITY FOR DOUBLE INTEGRALS IN TERMS OF L_P- NORMS AND APPLICATIONS IN NUMERICAL INTEGRATION

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Abstract. An inequality of the Ostrowski type for double integrals and applications in Numerical Analysis in connection with cubature formulae are given.

1 Introduction

In 1938, A. Ostrowski proved the following integral inequality [5, p. 468]

Theorem 1.1. *Let $f:[a,b]\to\mathbf{R}$ be continuous on $[a,b]$ and differentiable on (a,b) whose derivative $f':(a,b)\to\mathbf{R}$ is bounded on (a,b), i.e., $\|f'\|_\infty:=\sup_{t\in(a,b)}|f'(t)|<\infty$. Then we have the inequality*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\leq\left[\frac{1}{4}+\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right](b-a)\|f'\|_\infty$$

for all $x\in[a,b]$.
The constant $\frac{1}{4}$ is the best possible.

For some generalizations see the book [5, p. 468-484] by Mitrinović, Pečarić and Fink.

Some applications of the above results in Numerical Integration and for special means have been given in [3] by S.S. Dragomir and S. Wang.

In [4] Dragomir and Wang established the following Ostrowski type inequality for differentiable mappings whose derivatives belong to L_p-spaces.

Theorem 1.2. *Let $f:I\subseteq\mathbf{R}\to\mathbf{R}$ be a differentiable mapping on I° and $a,b\in I^\circ$ with $a<b$. If $f'\in L_p(a,b)$ $\left(p>1,\frac{1}{p}+\frac{1}{q}=1\right)$, then we have the inequality:*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\leq\frac{1}{b-a}\left[\frac{(x-a)^{q+1}+(b-x)^{q+1}}{q+1}\right]^{\frac{1}{q}}\|f'\|_p$$

for all $x\in[a,b]$, where $\|f'\|_p:=\left(\int_a^b|f'(t)|^p\,dt\right)^{\frac{1}{p}}$, is the $L_p(a,b)-$norm.

Note that the above inequality can also be obtained from Theorem 1.1 [5, p. 471] due to A.M. Fink.

For other Ostrowski type inequalities, see the papers [1, 2 and 4].

In 1975, G.N. Milovanović generalized Theorem 1.1 where f is a function of several variables [5, p. 468].

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Theorem 1.3. *Let $f:\mathbf{R}^m\to\mathbf{R}$ be a differentiable function defined on $D=\{(x_1,...,x_m)\,|a_i\le x_i\le b_i\,(i=1,...,m)\}$ and let $\left|\frac{\partial f}{\partial x_i}\right|\le M_i\,(M_i>0,i=1,...,m)$ in D. Furthermore, let function $x\longmapsto p(x)$ be integrable and $p(x)>0$ for every $x\in D$. Then for every $x\in D$, we have the inequality:*

$$\left|f(x)-\frac{\int_D p(y)\,f(y)\,dy}{\int_D p(y)\,dy}\right|\le\frac{\sum_{i=1}^{m}M_i\int_D p(y)\,|x_i-y_i|\,dy}{\int_D p(y)\,dy}.$$

In the present paper we point out an Ostrowski type inequality for double integrals in terms of L_p-norms and apply it in Numerical Integration obtaining a general cubature formula.

2 The Results

The following inequality of Ostrowski's type for mappings of two variables holds:

Theorem 2.1. *Let $f:[a,b]\times[c,d]\to\mathbf{R}$ be a continuous mapping on $[a,b]\times[c,d]$, $f''_{x,y}=\frac{\partial^2 f}{\partial x\partial y}$ exists on $(a,b)\times(c,d)$ and is in $L_p((a,b)\times(c,d))$, i.e.,*

$$\left\|f''_{s,t}\right\|_p:=\left(\int_a^b\int_c^d\left|\frac{\partial^2 f(x,y)}{\partial x\partial y}\right|^p dxdy\right)^{\frac{1}{p}}<\infty,\qquad p>1$$

then we have the inequality:

$$\text{(2.1)}\quad \left|\int_a^b\int_c^d f(s,t)\,dsdt-\left[(b-a)\int_c^d f(x,t)\,dt+(d-c)\int_a^b f(s,y)\,ds\right.\right.$$
$$\left.\left.-(d-c)(b-a)\,f(x,y)\right]\right|$$
$$\le\left[\frac{(x-a)^{q+1}+(b-x)^{q+1}}{q+1}\right]^{\frac{1}{q}}\left[\frac{(y-c)^{q+1}+(d-y)^{q+1}}{q+1}\right]^{\frac{1}{q}}\left\|f''_{s,t}\right\|_p$$

for all $(x,y)\in[a,b]\times[c,d]$, where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Integrating by parts successively, we have the equality:

$$\text{(2.2)}\quad \int_a^x\int_c^y(s-a)(t-c)\,f''_{s,t}(s,t)\,dtds$$
$$=(y-c)(x-a)\,f(x,y)-(y-c)\int_a^x f(s,y)\,ds$$
$$-(x-a)\int_c^y f(x,t)\,dt+\int_a^x\int_c^y f(s,t)\,dsdt.$$

By similar computations we have,

$$\int_a^x \int_y^d (s-a)(t-d) f''_{s,t}(s,t)\, ds dt \tag{2.3}$$

$$= (x-a)(d-y) f(x,y) - (d-y) \int_a^x f(s,y)\, ds$$

$$-(x-a) \int_y^d f(x,t)\, dt + \int_a^x \int_c^y f(s,t)\, ds dt.$$

Now,

$$\int_x^b \int_y^d (s-b)(t-d) f''_{s,t}(s,t)\, ds dt \tag{2.4}$$

$$= (d-y)(b-x) f(x,y) - (d-y) \int_x^b f(s,y)\, ds$$

$$-(b-x) \int_y^d f(x,t)\, dt + \int_x^b \int_y^d f(s,t)\, ds dt$$

and finally

$$\int_x^b \int_c^y (s-b)(t-c) f''_{s,t}(s,t)\, ds dt \tag{2.5}$$

$$= (y-c)(b-x) f(x,y) - (y-c) \int_x^b f(s,y)\, ds$$

$$-(b-x) \int_c^y f(x,t)\, dt + \int_x^b \int_c^y f(s,t)\, ds dt.$$

If we add the equalities $(2.2)-(2.5)$ we get, in the right hand side:

$$[(y-c)(x-a) + (x-a)(d-y)$$

$$+(d-y)(b-x) + (y-c)(b-x)] f(x,y)$$

$$-(d-c)\int_a^x f(s,y)\,ds-(d-c)\int_x^b f(s,y)\,ds-(b-a)\int_c^y f(x,t)\,dt$$

$$-(b-a)\int_y^d f(x,t)\,dt+\int_a^x\int_c^y f(s,t)\,dsdt+\int_a^x\int_y^d f(s,t)\,dsdt$$

$$+\int_x^b\int_y^d f(s,t)\,dsdt+\int_x^b\int_c^y f(s,t)\,dsdt$$

$$=(d-c)(b-a)f(x,y)-(d-c)\int_a^b f(s,y)\,ds$$

$$-(b-a)\int_c^b f(x,t)\,dt+\int_a^b\int_c^d f(s,t)\,dsdt.$$

For the first part, let us define the kernels: $p:[a,b]^2\to\mathbf{R}$, $q:[c,d]^2\to\mathbf{R}$ given by:

$$p(x,s):=\begin{cases}s-a & \text{if } s\in[a,x]\\ s-b & \text{if } s\in(x,b]\end{cases}$$

and

$$q(y,t):=\begin{cases}t-c & \text{if } t\in[c,y]\\ t-d & \text{if } t\in(y,d].\end{cases}$$

Now, we deduce that the left part can be represented as :

$$\int_a^b\int_c^d p(x,s)\,q(y,t)\,f''_{s,t}(s,t)\,dsdt.$$

Consequently, we get the identity

$$\int_a^b\int_c^d p(x,s)\,q(y,t)\,f''_{s,t}(s,t)\,dsdt \tag{2.6}$$

$$=(d-c)(b-a)f(x,y)-(d-c)\int_a^b f(s,y)\,ds$$

$$-(b-a)\int_c^d f(x,t)\,dt+\int_a^b\int_c^d f(s,t)\,dsdt$$

for all $(x,y) \in [a,b] \times [c,d]$.

Now, using the identity (2.6) we get

$$\left| \int_a^b \int_c^d f(s,t)\, dsdt - \left[(b-a) \int_c^d f(x,t)\, dt + (d-c) \int_a^b f(s,y)\, ds \right.\right.$$

$$\left.\left. - (d-c)(b-a) f(x,y) \right] \right|$$

$$\leq \int_a^b \int_c^d |p(x,s)\, q(y,t)| \left| f''_{s,t}(s,t) \right| dsdt.$$

Using Hölder's integral inequality for double integrals, we get

$$\int_a^b \int_c^d |p(x,s)\, q(y,t)| \left| f''_{s,t}(s,t) \right| dsdt$$

$$\leq \left(\int_a^b \int_c^d |p(x,s)\, q(y,t)|^q\, dsdt \right)^{\frac{1}{q}} \left(\int_a^b \int_c^d \left| f''_{s,t}(s,t) \right|^p dsdt \right)^{\frac{1}{p}}$$

$$= \left(\int_a^b |p(x,s)|^q\, ds \right)^{\frac{1}{q}} \left(\int_c^d |q(y,t)|^q\, dt \right)^{\frac{1}{q}} \left\| f''_{s,t} \right\|_p$$

$$= \left[\frac{(x-a)^{q+1} + (b-x)^{q+1}}{q+1} \right]^{\frac{1}{q}} \left[\frac{(y-c)^{q+1} + (d-y)^{q+1}}{q+1} \right]^{\frac{1}{q}} \left\| f''_{s,t} \right\|_p$$

and the theorem is proved. ∎

Corollary 2.2. *Under the above assumptions, we have the inequality:*

$$\left| \int_a^b \int_c^d f(s,t)\, dsdt - \left[(b-a) \int_c^d f\left(\frac{a+b}{2}, t \right) dt \right.\right. \tag{2.7}$$

$$\left.\left. + (d-c) \int_a^b f\left(s, \frac{c+d}{2} \right) ds - (d-c)(b-a) f\left(\frac{a+b}{2}, \frac{c+d}{2} \right) \right] \right|$$

$$\leq \frac{(b-a)^{1+\frac{1}{q}} (d-c)^{1+\frac{1}{q}}}{4(q+1)^{\frac{2}{q}}} \left\| f''_{s,t} \right\|_p.$$

Remark 2.1. *Consider the mapping* $g : [\alpha, \beta] \to \mathbf{R}, g(t) = (t-\alpha)^m + (\beta - t)^m$, $(m \geq 1)$. *Taking into account the fact that one has the properties*

$$\inf_{t\in[\alpha,\beta]} g(t) = g\left(\frac{\alpha+\beta}{2}\right) = \frac{(\beta-\alpha)^m}{2^{m-1}}$$

and

$$\sup_{t\in[\alpha,\beta]} g(t) = g(\alpha) = g(\beta) = (\beta-\alpha)^m$$

then, the above inequality (2.7) *is the best that can be obtained from* (2.1).

Remark 2.2. *Now, if we assume that* $f(s,t) = h(s)\,h(t)$, $h : [a,b] \to \mathbf{R}$ *is continuous on* $[a,b]$ *and suppose that* $\|h'\|_p < \infty$, *then from* (2.1) *we get (for* $x = y$*)*

$$\left|\int_a^b h(s)\,ds \int_a^b h(s)\,ds - h(x)(b-a)\int_a^b h(s)\,ds\right.$$

$$\left. -h(x)(b-a)\int_a^b h(s)\,ds + (b-a)^2 h^2(x)\right|$$

$$\leq \left[\frac{(x-a)^{q+1} + (b-x)^{q+1}}{q+1}\right]^{\frac{2}{q}} \|h'\|_p^2$$

i.e.

$$\left[\int_a^b h(s)\,ds - h(x)(b-a)\right]^2 \leq \left[\frac{(x-a)^{q+1} + (b-x)^{q+1}}{q+1}\right]^{\frac{2}{q}} \|h'\|_p^2$$

which is clearly equivalent to Ostrowski's inequality. Consequently (2.1) *can be also regarded as a generalization for double integrals of the result embodied in Theorem 1.2.*

3 Applications For Cubature Formulae

Let us consider the arbitrary division $I_n : a = x_0 < x_1 < ... < x_{n-1} < x_n = b$ and $J_m : c = y_0 < y_1 < ... < y_{m-1} < y_m = b$ and $\xi_i \in [x_i, x_{i+1}]$ $(i = 0, ..., n-1), \eta_j \in [y_j, y_{j+1}]$ $(j = 0, ..., m-1)$ be intermediate points. Consider the sum

$$C(f, I_n, J_m, \boldsymbol{\xi}, \boldsymbol{\eta}) := \sum_{i=0}^{n-1}\sum_{j=0}^{m-1} h_i \int_{y_j}^{y_{j+1}} f(\xi_i, t)\,dt$$

$$+\sum_{i=0}^{n-1}\sum_{j=0}^{m-1} l_j \int_{x_i}^{x_{i+1}} f(s, \eta_j)\,ds - \sum_{i=0}^{n-1}\sum_{j=0}^{m-1} h_i l_j f(\xi_i, \eta_j)$$

for which we assume that the involved integrals can more easily be computed than the original double integral

$$D := \int_a^b \int_c^d f(s,t)\,dsdt,$$

and

$$h_i := x_{i+1} - x_i \,(i = 0, ..., n-1)\,, \quad l_j := y_{j+1} - y_j \,(j = 0, ..., m-1)\,.$$

With this assumption, we can state the following cubature formula:

Theorem 3.1. *Let $f : [a,b] \times [c,d] \to \mathbf{R}$ be as in Theorem 2.1 and $I_n, J_m, \boldsymbol{\xi}$ and $\boldsymbol{\eta}$ be as above. Then we have the cubature formula:*

$$\int_a^b \int_c^d f(s,t)\, ds dt = C(f, I_n, J_m, \boldsymbol{\xi}, \boldsymbol{\eta}) + R(f, I_n, J_m, \boldsymbol{\xi}, \boldsymbol{\eta})$$

where the remainder term $R(f, I_n, J_m, \boldsymbol{\xi}, \boldsymbol{\eta})$ satisfies the estimation:

$$(3.1) \qquad |R(f, I_n, J_m, \boldsymbol{\xi}, \boldsymbol{\eta})|$$

$$\leq \left\| f''_{s,t} \right\|_p \left[\sum_{i=0}^{n-1} \left(\frac{(x_{i+1} - \xi_i)^{q+1} + (\xi_i - x_i)^{q+1}}{q+1} \right) \right]^{\frac{1}{q}}$$

$$\times \left[\sum_{j=0}^{m-1} \left(\frac{\left(y_{j+1} - \eta_j\right)^{q+1} + \left(\eta_j - y_j\right)^{q+1}}{q+1} \right) \right]^{\frac{1}{q}}$$

$$\leq \frac{\left\| f''_{s,t} \right\|_p}{(q+1)^{\frac{2}{q}}} \sum_{i=0}^{n-1} h_i^{1+\frac{1}{q}} \sum_{j=0}^{m-1} l_j^{1+\frac{1}{q}}.$$

for all $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ as above.

Proof. Apply Theorem 2.1 on the interval $[x_i, x_{i+1}] \times [y_j, y_{j+1}]$ $(i = 0, ..., n-1; j = 0, ..., m-1)$ to get:

$$\left| \int_{x_i}^{x_{i+1}} \int_{y_j}^{y_{j+1}} f(s,t)\, ds dt - \left[h_i \int_{y_j}^{y_{j+1}} f(\xi_i, t)\, dt + l_j \int_{x_i}^{x_{i+1}} f\left(s, \eta_j\right) ds - h_i l_j f\left(\xi_i, \eta_j\right) \right] \right|$$

$$\leq \left[\left(\frac{(x_{i+1} - \xi_i)^{q+1} + (\xi_i - x_i)^{q+1}}{q+1} \right) \left(\frac{\left(y_{j+1} - \eta_j\right)^{q+1} + \left(\eta_j - y_j\right)^{q+1}}{q+1} \right) \right]^{\frac{1}{q}}$$

$$\times \left(\int_{x_i}^{x_{i+1}} \int_{y_j}^{y_{j+1}} |\, f(s,t)\, |^p \, ds dt \right)^{\frac{1}{p}}$$

for all $i = 0, ..., n-1; j = 0, ..., m-1$.

Summing over i from 0 to $n-1$ and over j from 0 to $m-1$ and using the generalized triangle inequality and Hölder's discrete inequality for double sums, we deduce

$$|R(f, I_n, J_m, \boldsymbol{\xi}, \boldsymbol{\eta})|$$

$$\leq \sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\left[\left(\frac{(x_{i+1}-\xi_i)^{q+1}+(\xi_i-x_i)^{q+1}}{q+1}\right)\right.$$

$$\left.\times\left(\frac{(y_{j+1}-\eta_j)^{q+1}+(\eta_j-y_j)^{q+1}}{q+1}\right)\right]^{\frac{1}{q}}\times\left(\int_{x_i}^{x_{i+1}}\int_{y_j}^{y_{j+1}}|\,f(s,t)\,|^p\,dsdt\right)^{\frac{1}{p}}$$

$$\leq\left[\sum_{i=0}^{n-1}\left(\frac{(x_{i+1}-\xi_i)^{q+1}+(\xi_i-x_i)^{q+1}}{q+1}\right)\right]^{\frac{1}{q}}$$

$$\times\left[\sum_{j=0}^{m-1}\left(\frac{(y_{j+1}-\eta_j)^{q+1}+(\eta_j-y_j)^{q+1}}{q+1}\right)\right]^{\frac{1}{q}}$$

$$\times\left[\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\int_{x_i}^{x_{i+1}}\int_{y_j}^{y_{j+1}}|\,f(s,t)\,|^p\,dsdt\right]^{\frac{1}{p}}$$

$$=\left[\sum_{i=0}^{n-1}\left(\frac{(x_{i+1}-\xi_i)^{q+1}+(\xi_i-x_i)^{q+1}}{q+1}\right)\right.$$

$$\left.\times\sum_{j=0}^{m-1}\left(\frac{(y_{j+1}-\eta_j)^{q+1}+(\eta_j-y_j)^{q+1}}{q+1}\right)\right]^{\frac{1}{q}}\times\left\|f''_{s,t}\right\|_p.$$

To prove the second part, we observe that

$$(x_{i+1}-\xi_i)^{q+1}+(\xi_i-x_i)^{q+1}\leq(x_{i+1}-x_i)^{q+1}$$

and

$$(y_{j+1}-\eta_j)^{q+1}+(\eta_j-y_j)^{q+1}\leq(y_{j+1}-y_j)^{q+1}$$

for all i,j as above and the intermediate points ξ_i and η_j.

We omit the details. ∎

Remark 3.1. *As*

$$\sum_{i=0}^{n-1}h_i^{1+\frac{1}{q}}\leq[\nu(h)]^{\frac{1}{q}}\sum_{i=0}^{n-1}h_i=(b-a)[\nu(h)]^{\frac{1}{q}}$$

and

$$\sum_{j=0}^{m-1}l_j^{1+\frac{1}{q}}\leq[\mu(l)]^{\frac{1}{q}}\sum_{j=0}^{m-1}l_j=(d-c)[\mu(l)]^{\frac{1}{q}}$$

where

$$\nu(h)=\max\{h_i:i=0,...,n-1\},$$

and

$$\mu(l) = \max\{l_j : j = 0, ..., m-1\},$$

the right hand side of (3.1) *can be bounded by*

$$\frac{1}{(q+1)^{\frac{2}{q}}} \left\| f''_{s,t} \right\|_p (b-a)(d-c)[\nu(h)\mu(l)]^{\frac{1}{q}}.$$

Now, define the sum

$$C_M(f, I_n, J_m) := \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} h_i \int_{y_j}^{y_{j+1}} f\left(\frac{x_i + x_{i+1}}{2}, t\right) dt$$

$$+ \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} l_j \int_{x_i}^{x_{i+1}} f\left(s, \frac{y_j + y_{j+1}}{2}\right) ds - \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} h_i l_j f\left(\frac{x_i + x_{i+1}}{2}, \frac{y_j + y_{j+1}}{2}\right).$$

Then we have the best cubature formula we can get from Theorem 3.1.

Corollary 3.2. *Under the above assumptions we have*

$$\int_a^b \int_c^d f(s,t)\, dsdt = C_M(f, I_n, J_m) + R(f, I_n, J_m),$$

where the remainder $R(f, I_n, J_m)$ *satisfies the estimation:*

$$|R(f, I_n, J_m)| \le \frac{1}{4(q+1)^{\frac{2}{q}}} \left\| f''_{s,t} \right\|_p \sum_{i=0}^{n-1} h_i^{1+\frac{1}{q}} \sum_{j=0}^{m-1} l_j^{1+\frac{1}{q}}.$$

References

[1] S.S. DRAGOMIR and S. WANG , A new inequality of Ostrowski's type in L_1-norm and applications to some special means and to some numerical quadrature rules, *Tamkang J. of Math.*, **28**(1997), 239-244.

[2] S.S. DRAGOMIR and S. WANG, An inequality of Ostrowski-Grüss' type and its applications to the estimation of error bounds for some special means and for some numerical quadrature rules, *Computers Math. Applic.,* **33**(1997), 15-20.

[3] S.S. DRAGOMIR and S. WANG, Applications of Ostrowski's inequality to the estimation of error bounds for some special means and some numerical quadrature rules, *Appl. Math. Lett.*, **11**(1998), 105-109.

[4] S.S. DRAGOMIR and S. WANG, A new inequality of Ostrowski's type in L_p-norm and applications to some special means and to some numerical quadrature rules, *submitted for publication*

[5] D.S. MITRINOVIĆ, J.E. PEČARIĆ and A.M. FINK, *Inequalities Involving Functions and Their Integrals and Derivatives*, Kluwer Academic Publishers, Dordrecht, 1994.

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