An Inequality of Ostrowski Type and its Applications for Simpson's Rule and Special Means

AN INEQUALITY OF OSTROWSKI TYPE FOR MAPPINGS WHOSE SECOND DERIVATIVES BELONG TO $L_1(A,B)$ AND APPLICATIONS

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Abstract. An inequality of Ostrowski type for twice differentiable mappings whose derivatives belong to $L_1(a,b)$ and applications in Numerical Integration and for special means (logarithmic mean, identric mean, p-logarithmic mean etc...) are given.

1 Introduction

In 1938, Ostrowski (see for example [3, p. 468]) proved the following integral inequality:

Theorem 1.1. *Let $f : I \subseteq \mathbf{R} \to \mathbf{R}$ be a differentiable mapping on I° (I° is the interior of I), and let $a, b \in I^\circ$ with $a < b$. If $f' : (a,b) \to \mathbf{R}$ is bounded on (a,b), i.e., $\|f'\|_\infty := \sup_{t\in(a,b)} |f'(t)| < \infty$, then we have the inequality:*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \le \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right](b-a)\|f'\|_\infty \tag{1.1}$$

for all $x \in [a,b]$.
The constant $\frac{1}{4}$ is sharp in the sense that it can not be replaced by a smaller one.

For some applications of Ostrowski's inequality to some special means and some numerical quadrature rules, we refer to the recent paper [1] by S.S. Dragomir and S. Wang.

In paper [2], the same authors considered another inequality of Ostrowski type for $\|\cdot\|_1$ norm as follows:

Theorem 1.2. *Let $f : I \subseteq \mathbf{R} \to \mathbf{R}$ be a differentiable mapping in I° and $a, b \in I^\circ$ with $a < b$. If $f' \in L_1[a,b]$, then we have the inequality:*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \le \left[\frac{1}{2} + \frac{\left| x - \frac{a+b}{2} \right|}{(b-a)} \right]\|f'\|_1 \tag{1.2}$$

for all $x \in [a,b]$.

They also pointed out some applications of (1.2) in Numerical Integration as well as for special means.

In 1976, G.V. Milovanović and J.E. Pečarić proved a generalization of Ostrowski's inequality for n-time differentiable mappings (see for example [3, p. 468]) from which we would like to mention only the case of twice differentiable mappings [3, p. 470]:

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Theorem 1.3. *Let $f : [a,b] \to \mathbf{R}$ be a twice differentiable mapping such that $f'' : (a,b) \to \mathbf{R}$ is bounded on (a,b), i.e., $\|f''\|_\infty = \sup_{t\in(a,b)} |f''(t)| < \infty$. Then we have the inequality:*

$$\left|\frac{1}{2}\left[f(x) + \frac{(x-a)f(a) + (b-x)f(b)}{b-a}\right] - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \tag{1.3}$$

$$\leq \frac{\|f''\|_\infty}{4}(b-a)^2\left[\frac{1}{12} + \frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right]$$

for all $x \in (a,b)$.

In this paper we point out an inequality of Ostrowski type for twice differentiable mappings which is in terms of the $\|\cdot\|_1$ -norm of the second derivative f'' and apply it in numerical integration and for some special means such as : logarithmic mean, identric mean, p-logarithmic mean etc.

2 Some Integral Inequalities

The following inequality of Ostrowski's type for mappings which are twice differentiable, holds.

Theorem 2.1. *Let $f : [a,b] \to \mathbf{R}$ be continuous on $[a,b]$, twice differentiable on (a,b) and $f'' \in L_1(a,b)$. Then we have the inequality:*

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt - \left(x - \frac{a+b}{2}\right)f'(x)\right| \tag{2.1}$$

$$\leq \frac{1}{2(b-a)}\left(\left|x - \frac{a+b}{2}\right| + \frac{1}{2}(b-a)\right)^2 \|f''\|_1 \leq \frac{b-a}{2}\|f''\|_1$$

for all $x \in [a,b]$.

Proof. Let us define the mapping $K(\cdot,\cdot) : [a,b]^2 \to \mathbf{R}$ given by

$$K(x,t) := \begin{cases} \dfrac{(t-a)^2}{2} & \text{if } t \in [a,x] \\ \dfrac{(t-b)^2}{2} & \text{if } t \in (x,b] \end{cases}.$$

Integrating by parts, we have successively

$$\int_a^b K(x,t)\,f''(t)\,dt$$

$$= \int_a^x \frac{(t-a)^2}{2}f''(t)\,dt + \int_x^b \frac{(t-b)^2}{2}f''(t)\,dt$$

$$= \left.\frac{(t-a)^2}{2}f'(t)\right|_a^x - \int_a^x (t-a)\,f'(t)\,dt + \left.\frac{(t-b)^2}{2}f'(t)\right|_x^b - \int_x^b (t-b)\,f'(t)\,dt$$

$$
\begin{aligned}
&= \frac{(x-a)^2}{2} f'(x) - \left[(t-a) f(t)\Big|_a^x - \int_a^x f(t)\,dt \right] \\
&- \frac{(b-x)^2}{2} f'(x) - \left[(t-b) f(t)\Big|_x^b - \int_x^b f(t)\,dt \right] \\
&= \frac{1}{2}\left[(x-a)^2 - (b-x)^2 \right] f'(x) - (x-a) f(x) \\
&+ \int_a^x f(t)\,dt + (x-b) f(x) + \int_x^b f(t)\,dt \\
&= (b-a)\left(x - \frac{a+b}{2} \right) f'(x) - (b-a) f(x) + \int_a^b f(t)\,dt
\end{aligned}
$$

from where we get the integral identity:

$$
\int_a^b f(t)\,dt = (b-a) f(x) - (b-a)\left(x - \frac{a+b}{2} \right) f'(x) + \int_a^b K(x,t) f''(t)\,dt \tag{2.2}
$$

for all $x \in [a,b]$, which is interesting in itself, too.

Using the identity (2.2) we have

$$
\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt - \left(x - \frac{a+b}{2} \right) f'(x) \right| \tag{2.3}
$$

$$
\begin{aligned}
&= \frac{1}{b-a}\left| \int_a^b K(x,t) f''(t)\,dt \right| \\
&= \frac{1}{b-a}\left| \int_a^x \frac{(t-a)^2}{2} f''(t)\,dt + \int_x^b \frac{(t-b)^2}{2} f''(t)\,dt \right|
\end{aligned}
$$

$$
\begin{aligned}
&\le \frac{1}{b-a}\left[\int_a^x \frac{(t-a)^2}{2} |f''(t)|\,dt + \int_x^b \frac{(t-b)^2}{2} |f''(t)|\,dt \right] \\
&\le \frac{1}{b-a}\left[\frac{(x-a)^2}{2}\int_a^x |f''(t)|\,dt + \frac{(b-x)^2}{2}\int_x^b |f''(t)|\,dt \right] \\
&\le \frac{1}{b-a}\max\left\{ \frac{(x-a)^2}{2}, \frac{(b-x)^2}{2} \right\} \left[\int_a^x |f''(t)|\,dt + \int_x^b |f''(t)|\,dt \right].
\end{aligned}
$$

Now, let observe that

$$
\max\left\{ \frac{(x-a)^2}{2}, \frac{(b-x)^2}{2} \right\}
$$

$$
\begin{aligned}
&= \frac{1}{2}\frac{(x-a)^2+(b-x)^2+\left|(b-x)^2-(x-a)^2\right|}{2} \\
&= \frac{1}{2}\left[\frac{(x-a)^2+(b-x)^2}{2}+(b-a)\left|x-\frac{a+b}{2}\right|\right] \\
&= \frac{1}{2}\left[\frac{1}{4}(b-a)^2+\left(x-\frac{a+b}{2}\right)^2+(b-a)\left|x-\frac{a+b}{2}\right|\right] \\
&= \frac{1}{2}\left(\left|x-\frac{a+b}{2}\right|+\frac{1}{2}(b-a)\right)^2.
\end{aligned}
$$

Using (2.3) we deduce the desired inequality (2.1). ∎

Corollary 2.2. *Let f be as above. Then we have the mid-point inequality:*

$$
\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \le \frac{1}{8}(b-a)\|f''\|_1. \tag{2.4}
$$

The following trapezoid inequality also holds:

Corollary 2.3. *Under the above assumptions we have:*

$$
\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt-\frac{b-a}{4}(f'(b)-f'(a))\right| \le \frac{1}{2}(b-a)\|f''\|_1. \tag{2.5}
$$

Proof. Choose in (2.1) $x=a$ and $x=b$ to get:

$$
\left|f(a)-\frac{1}{b-a}\int_a^b f(t)\,dt+\frac{b-a}{2}f'(a)\right| \le \frac{b-a}{2}\|f''\|_1
$$

and

$$
\left|f(b)-\frac{1}{b-a}\int_a^b f(t)\,dt-\frac{b-a}{2}f'(b)\right| \le \frac{b-a}{2}\|f''\|_1.
$$

Adding the above two inequalities, using the triangle inequality and dividing by 2, we get the desired inequality (2.5). ∎

3 Applications in Numerical Integration

Let $I_n : a = x_0 < x_1 < ... < x_{n-1} < x_n = b$ be a division of the interval $[a,b]$, $\xi_i \in [x_i, x_{i+1}]$ $(i = 0, ..., n-1)$. We have the following quadrature formula:

Theorem 3.1. *Let $f : [a,b] \to \mathbf{R}$ be continuous on $[a,b]$ and twice differentiable on (a,b), whose second derivative $f'' : (a,b) \to \mathbf{R}$ belongs to $L_1(a,b)$, i.e., $\|f''\|_1 := \int_a^b |f''(t)|\,dt < \infty$. Then the following perturbed Riemann's type quadrature formula holds:*

$$
\int_a^b f(x)\,dx = A(f, f', \boldsymbol{\xi}, I_n) + R(f, f', \boldsymbol{\xi}, I_n) \tag{3.1}
$$

where

$$A(f,f',\boldsymbol{\xi},I_n)=\sum_{i=0}^{n-1}h_if(\xi_i)-\sum_{i=0}^{n-1}f'(\xi_i)\left(\xi_i-\frac{x_i+x_{i+1}}{2}\right)h_i$$

and the remainder satisfies the estimation:

$$|R(f,f',\boldsymbol{\xi},I_n)|\le\frac{1}{2}\left[\frac{1}{2}\nu(h)+\sup_{i=0,\dots,n-1}\left|\xi_i-\frac{x_i+x_{i+1}}{2}\right|\right]^2\|f''\|_1 \tag{3.2}$$

$$\le\frac{\nu^2(h)}{2}\|f''\|_1$$

for all ξ_i *as above, where* $\nu(h)=\max\{x_{i+1}-x_i|i=0,\dots,n-1\}$.

Proof. Apply Theorem 2.1 on the interval $[x_i,x_{i+1}]$ $(i=0,\dots,n-1)$ to get

$$\left|\int_{x_i}^{x_{i+1}}f(t)\,dt-h_if(\xi_i)+\left(\xi_i-\frac{x_i+x_{i+1}}{2}\right)f'(\xi_i)\right|$$

$$\le\frac{1}{2}\left(\left|\xi_i-\frac{x_i+x_{i+1}}{2}\right|+\frac{1}{2}(x_{i+1}-x_i)\right)^2\int_{x_i}^{x_{i+1}}|f''(t)|\,dt.$$

Summing over i from 0 to $n-1$ and using the generalized triangle inequality we deduce :

$$|R(f,f',\boldsymbol{\xi},I_n)|\le\frac{1}{2}\sum_{i=0}^{n-1}\left[\frac{1}{2}(x_{i+1}-x_i)+\left|\xi_i-\frac{x_i+x_{i+1}}{2}\right|\right]^2\int_{x_i}^{x_{i+1}}|f''(t)|\,dt$$

$$\le\frac{1}{2}\sup_{i=0,\dots,n-1}\left[\frac{1}{2}(x_{i+1}-x_i)+\left|\xi_i-\frac{x_i+x_{i+1}}{2}\right|\right]^2\sum_{i=0}^{n-1}\int_{x_i}^{x_{i+1}}|f''(t)|\,dt$$

$$\le\frac{1}{2}\left[\frac{1}{2}\nu(h)+\sup_{i=0,\dots,n-1}\left|\xi_i-\frac{x_i+x_{i+1}}{2}\right|\right]^2\|f''\|_1$$

and the estimation (3.2) is obtained. ∎

Remark 3.1. *If we choose above* $\xi_i=\frac{x_i+x_{i+1}}{2}$, *we recapture the midpoint quadrature formula*

$$\int_a^b f(x)\,dx=A_M(f,I_n)+R_M(f,I_n)$$

where the remainder $R_M(f,I_n)$ *satisfies the estimation*

$$|R_M(f,I_n)|\le\frac{1}{8}\nu^2(h)\|f''\|_1.$$

4 Applications for Special Means

Let us recall the following means :

(a) The arithmetic mean

$$A = A(a,b) := \frac{a+b}{2}, \qquad a,b \geq 0;$$

(b) The geometric mean:

$$G = G(a,b) := \sqrt{ab}, \qquad a,b \geq 0;$$

(c) The harmonic mean:

$$H = H(a,b) := \frac{2}{\frac{1}{a}+\frac{1}{b}}, \qquad a,b \geq 0;$$

(d) The logarithmic mean:

$$L = L(a,b) := \begin{cases} a & \text{if } a = b \\ \dfrac{b-a}{\ln b - \ln a} & \text{if } a \neq b \end{cases} \qquad a,b > 0;$$

(e) The identric mean:

$$I = I(a,b) := \begin{cases} a & \text{if } a = b \\ \dfrac{1}{e}\left(\dfrac{b^b}{a^a}\right)^{\frac{1}{b-a}} & \text{if } a \neq b \end{cases} \qquad a,b > 0;$$

(f) The p-logarithmic mean:

$$L_p = L_p(a,b) := \begin{cases} \left[\dfrac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right]^{\frac{1}{p}} & \text{if } a \neq b; \\ a & \text{if } a = b \end{cases}$$

where, $p \in \mathbf{R} \backslash \{-1, 0\}, a, b > 0$.

The following simple relationships are known in the literature

$$H \leq G \leq L \leq I \leq A.$$

It is also known that L_p is monotonically increasing in $p \in \mathbf{R}$ with $L_0 = I$ and $L_{-1} = L$.

Consider the mapping $f : (0,\infty) \to \mathbf{R}$, $f(x) = x^r$, $r \in \mathbf{R}\backslash\{-1,0\}$.
Then we have for $0 < a < b$:

$$\frac{1}{b-a}\int_a^b f(x)\,dx = L_r^r(a,b)$$

and

$$\|f''\|_1 = |r(r-1)|(b-a)L_{r-1}^{r-1}(a,b).$$

Using the inequality (2.1) we get :

$$\left|x^r - L_r^r - r(x-A)x^{r-1}\right| \tag{4.1}$$

$$\leq \frac{1}{2}\left[|x-A| + \frac{1}{2}(b-a)\right]^2 |r(r-1)| L_{r-1}^{r-1}$$

for all $x \in [a,b]$.
If in (4.1) we choose $x = A$, we get

$$|A^r - L_r^r| \leq \frac{|r(r-1)|(b-a)^2}{8} L_{r-1}^{r-1}. \tag{4.2}$$

Consider the mapping $f : (0,\infty) \to \mathbf{R}$, $f(x) = \frac{1}{x}$.
Then we have for $0 < a < b$:

$$\frac{1}{b-a}\int_a^b f(x)\,dx = L^{-1}(a,b)$$

and

$$\|f''\|_1 = 2(b-a)L_{-3}^{-3}(a,b).$$

Using the inequality (2.1), we get :

$$\left|\frac{1}{x} - \frac{1}{L} + \frac{x-A}{x^2}\right| \leq L_{-3}^{-3}\left[|x-A| + \frac{1}{2}(b-a)\right]^2$$

which is equivalent to

$$|x(L-x) + L(x-A)| \leq x^2 L L_{-3}^{-3}\left[|x-A| + \frac{1}{2}(b-a)\right]^2 \tag{4.3}$$

for all $x \in [a,b]$.
Now, if we choose in (4.3), $x = A$, we get

$$0 \leq A - L \leq \frac{1}{4} A L L_{-3}^{-3}(b-a)^2. \tag{4.4}$$

If in (4.3) we choose $x = L$, we get

$$0 \leq A - L \leq L^2 L_{-3}^{-3}\left[L - A + \frac{1}{2}(b-a)\right]^2. \tag{4.5}$$

Let us consider the mapping $f(x) = \ln x$, $x \in [a,b] \subset (0,\infty)$.
Then we have :

$$\frac{1}{b-a}\int_a^b f(x)\,dx = \ln I(a,b),$$

and

$$\|f''\|_1 = (b-a)L_{-2}^{-2}(a,b).$$

Then the inequality (2.1) gives us

$$\left|\ln x-\ln I-\frac{x-A}{x}\right| \tag{4.6}$$

$$\leq \frac{1}{2}\left[|x-A|+\frac{1}{2}(b-a)\right]^{2} L_{-2}^{-2}$$

for all $x \in [a,b]$.
Now, if in (4.6) we choose $x=A$, we get

$$1 \leq \frac{A}{I} \leq \exp\left[\frac{1}{8}(b-a)^{2} L_{-2}^{-2}\right]. \tag{4.7}$$

If in (4.6) we choose $x=I$, we get

$$0 \leq A-I \leq \frac{I}{2}\left[A-I+\frac{1}{2}(b-a)\right]^{2} L_{-2}^{-2}. \tag{4.8}$$

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