New (Probabilistic) Derivation of Diaz-Metcalf and Pólya-Szegő Inequalities and Consequences

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NEW (PROBABILISTIC) DERIVATION OF DIAZ - METCALF AND PÓLYA - SZEGŐ INEQUALITIES AND CONSEQUENCES

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Dedicated to Professor M.I.Yadrenko on the occasion of his 70th birthday

Abstract. Classical inequalities of Diaz - Metcalf and Pólya - Szegő are generalized to probabilistic setting which covers the initial deterministic (both discrete and integral) variants. From these two inequalities, by the probabilistic derivation method further well - known inequalities are obtained (that ones by Kantorovich, Rennie and Schweitzer).

1. Introduction

Probably the most frequently mentioned inequality is the celebrated Cauchy - Buniakowski - Schwarz inequality. In probabilistic description it is

$$|\mathsf{E}\xi\eta|^2 \le \mathsf{E}\xi^2\mathsf{E}\eta^2, \tag{1}$$

where ξ, η are random variables defined on certain probability space $(\Omega, \mathfrak{F}, \mathsf{P})$. An inequality reverse to (1) is known but just in few situations. These are both deterministic - discrete and integral variants of Diaz - Metcalf and Pólya - Szegő inequalities, such that includes many other famous ones, e.g. inequalities by Kantorovich, Rennie and Schweitzer and some of its generalizations, see [6, §2.11.], [7].

In this paper new reverse evaluations are given to (1) for almost surely P bounded random variables ξ, involving just its first two moments $\mathsf{E}\xi, \mathsf{E}\xi^2$ and the limiting parameters m, M in $\mathsf{P}\{m \le \xi \le M\} = 1$ and precise necessary and sufficient conditions are given to hold equalities in all considered cases. These results generate the classical Diaz - Metcalf and Pólya - Szegő inequalities such that include all other above mentioned deterministic inequalities together with few new ones both in plane and weighted versions. The derived results cover the discrete and the integral cases. The used derivation methods are quite simple and elementary; we use mainly the monotonicity and linearity of the expectation operator E.

In the article we will write $\xi \sim \psi$ for r.v. ξ possessing distribution/density ψ, the symbol $\mathcal{I}_A$ denotes the indicator of random event A and $\chi_S(t)$ stands for the characteristic function of the set S, $\mathbb{N}_0$ denotes the set of nonnegative integers and $\delta_{\lambda\mu}$ is the Kronecker symbol. Finally, under $L^2_\varphi[A]$ we mean the function space $\{h \mid \int_A |h(t)|^2\varphi(t)dt < \infty\}$ and $\mathrm{supp}(h) = \overline{\{t|h(t) \ne 0\}}$.

Key words and phrases. almost surely bounded random variable, Diaz - Metcalf inequality, discrete inequality, integral inequality, Kantorovich inequality, mathematical expectation, Pólya - Szegő inequality, Rennie inequality, Schweitzer inequality.

2. On Diaz - Metcalf Inequality

In the sequel we will deal with almost surely P bounded real random variable ξ and we will assume the existence of real constants $m, M,\ m \le M$ such that

$$\mathsf{P}\{m \le \xi \le M\} = 1.$$

Of course, there is the possibility to estimate all moments $\mathsf{E}\xi^r,\ r > 0;\ m \ge 0$, by the obvious relation $m^r \le \mathsf{E}\xi^r \le M^r$, while the variance satisfies $\mathsf{D}\xi \le (M-m)^2/4$. In the continuation we list certain more sophisticated results.

First of all we give a result we will need in the equality discussion concerning the Diaz - Metcalf inequality. This theorem was communicated to the author by O.I.Klesov, [5].

Equality Theorem. *Let $\mathfrak{X}$ be a random variable for which there exist two real numbers $\mathfrak{m}, \mathfrak{M},\ \mathfrak{m} < \mathfrak{M}$ such that*

$$\mathsf{P}(\mathfrak{m} \le \mathfrak{X} \le \mathfrak{M}) = 1.$$

Then two conditions are equivalent:

$$\mathsf{E}(\mathfrak{X} - \mathfrak{m})(\mathfrak{M} - \mathfrak{X}) = 0 \tag{2}$$

and

there exist two events A, B such that

$$\mathfrak{X} = \mathfrak{m}\mathcal{I}_A + \mathfrak{M}\mathcal{I}_B, \qquad a.s. \tag{3}$$

$$\mathsf{P}(A \cup B) = 1, \tag{4}$$

$$\mathsf{P}(A \cap B) = 0. \tag{5}$$

Proof. Put $\delta = (\mathfrak{X} - \mathfrak{m})(\mathfrak{M} - \mathfrak{X})$. Conditions (3-5) imply condition (2), since $\delta = 0$ almost surely under (3-5).

Conversely, put $A = \{\omega |\ \mathfrak{X}(\omega) = \mathfrak{m}\}$ and $B = \{\omega |\ (\omega) = \mathfrak{M}\}$. Obviously by (2) we have

$$\mathsf{P}(\mathfrak{X} \in \mathbb{R} \setminus \{\mathfrak{m}, \mathfrak{M}\}) = 0,$$

accordingly (3),(4) hold. Since $\mathfrak{m} < \mathfrak{M}$ (5) holds too. $\square$

Now, we are ready to formulate the Diaz - Metcalf inequality for bounded r.v. case.

Theorem 1. *Let ξ, η be real random variables defined on the same probability space $(\Omega, \mathfrak{F}, \mathsf{P})$. When $\mathsf{P}\{m_1 \le \xi \le M_1\} = 1,\ \mathsf{P}\{m_2 \le \eta \le M_2\} = 1$ with $m_j \le M_j,\ m_2 > 0$, it holds true*

$$\mathsf{E}\xi^2 + \frac{m_1 M_1}{m_2 M_2}\mathsf{E}\eta^2 \le \left(\frac{m_1}{M_2} + \frac{M_1}{m_2}\right)\mathsf{E}\xi\eta, \tag{6}$$

and the equality holds **iff** *either* **(i)** $m_1/M_2 = M_1/m_2$ *or* **(ii)** $m_1/M_2 < M_1/m_2$ *and*

$$\mathsf{P}\left(\frac{\xi}{\eta} \in \left\{\frac{m_1}{M_2}, \frac{M_1}{m_2}\right\}\right) = 1. \tag{7}$$

Proof. It is easy to see that by the assumptions of the theorem it is

$$\frac{m_1}{M_2} \leq \frac{\xi}{\eta} \leq \frac{M_1}{m_2}. \tag{8}$$

Now by monotonicity of E, we have

$$\mathsf{E}\left(\frac{M_1\eta}{m_2} - \xi\right)\left(\xi - \frac{m_1\eta}{M_2}\right) \geq 0 \tag{9}$$

and since E is a linear operator we deduce the desired relation (6) from (9).

Now, it remains the equality discussion in (6), i.e. in (9). Since the case **(i)** is obvious, consider (**ii**), i.e. $m_1/M_2 < M_1/m_2$ and (7). But we can write

$$\mathsf{P}(m_2\xi = M_1\eta \ \ \vee \ \ M_2\xi = m_1\eta) = 1 \quad \textit{and} \quad \mathsf{P}(m_2\xi = M_1\eta \ \ \wedge \ \ M_2\xi = m_1\eta) = 0$$

instead of **(ii)**. Then in (9) the equality appears for the case (8). Conversely, when (9) becomes equality, that means

$$\mathsf{P}\{M_2\xi = m_1\eta \ \ \vee \ \ m_2\xi = M_1\eta\} = 1. \tag{10}$$

Denote $A = \{\omega \in \Omega | M_2\xi = m_1\eta\}$, $B = \{\omega \in \Omega | m_2\xi = M_1\eta\}$ and apply the Equality Theorem with

$$\mathfrak{X} = \xi/\eta, \quad \mathfrak{m} = m_1/M_2, \quad \mathfrak{M} = M_1/m_2.$$

So, (7) is proved. □

Corollary 1.1 (Diaz - Metcalf Type Weighted Discrete Inequality)**.** *Suppose* $m_1 \leq x_k \leq M_1$, $k = \overline{1,n}$ *and let* $0 < m_2 \leq y_l \leq M_2$, $l = \overline{1,m}$. *Assume* $p_{kl} \geq 0$, *and* $\sum_{k=1}^n \sum_{l=1}^n p_{kl} = 1$. *Then it holds*

$$\sum_{k=1}^n x_k^2 q_k + \frac{m_1 M_1}{m_2 M_2} \sum_{l=1}^m y_l^2 r_l \leq \left(\frac{m_1}{M_2} + \frac{M_1}{m_2}\right) \sum_{k=1}^n \sum_{l=1}^m x_k y_l p_{kl}, \tag{11}$$

where

$$\sum_{l=1}^m p_{kl} = q_k, \quad k = \overline{1,n}, \qquad \sum_{k=1}^n p_{kl} = r_l, \quad l = \overline{1,m}.$$

The equality in (11) *is valid* **iff** *either* **(i)** $m_1/M_2 = M_1/m_2$ *or* **(ii)** $m_1/M_2 < M_1/m_2$ *and*

$$\sum_{k,l \in \mathbf{I}_{x,y}} p_{kl} = 1, \tag{12}$$

where $\mathbf{I}_{x,y} := \{(k,l) : x_k/y_l \in \{m_1/M_2, M_1/m_2\}\}$.

Proof. Let (ξ, η) be a twodimensional discrete random variable which coordinates are defined on probability space $(\Omega, \mathcal{P}(\Omega), \mathsf{P})$, η being positive, and $\mathcal{P}(\Omega) = \{S | S \subseteq \Omega\}$. Set $(\xi, \eta) \sim \mathsf{P}\{\xi = x_k, \eta = y_l\} = p_{kl}$, $k = \overline{1,n}$, $l = \overline{1,m}$ and $\sum_{k=1}^n \sum_{l=1}^m p_{kl} = 1$. Now, Theorem 1 by the inequality (6) gives (11). The equality discussion in Theorem 1 clearifies the equality in (11). □

Remark 1. Putting $m = n$ and $p_{kl} = \frac{1}{n}\delta_{kl},\ k,l = \overline{1,n}$, into (11) (such that implies $q_k = r_l = 1/n$), we get the original equal - weight Diaz - Metcalf inequality:

$$\sum_{k=1}^{n} x_k^2 + \frac{m_1 M_1}{m_2 M_2}\sum_{l=1}^{n} y_l^2 \le \left(\frac{m_1}{M_2} + \frac{M_1}{m_2}\right)\sum_{k=1}^{n} x_k y_k,$$

published firstly in [2]. Here the equality is valid **iff** $x_k/y_l \in \{m_1/M_2, M_1/m_2\}$ for all $k = \overline{1,n}$. This follows from (12), since $p_{kl} = 0$ for all $k \ne l$ and $p_{kk} = 1/n$.

Extensions to complex n - tuples were proved by the author couple Diaz - Metcalf in [2], compare the list of references in [6, p.66-67] as well.

Corollary 1.2 (Diaz - Metcalf Weighted Integral Inequality)**.** *Let f, g be Borel - functions, such that*

$$m_1 \le f(x) \le M_1, \qquad 0 < m_2 \le g(y) \le M_2, \qquad a.e.\ (x,y) \in [a,b]\times[c,d]. \tag{13}$$

Let $w(x,y)$ be a nonnegative normalized weight function such that $\mathrm{supp}(w) = [a,b]\times[c,d]$, *i.e.* $\int_a^b\int_c^d w(x,y)dxdy = 1$ *and let*

$$\int_c^d w(x,y)dy = w_1(x), \qquad \int_a^b w(x,y)dx = w_2(y).$$

Then it is valid

$$\int_a^b f^2(x)w_1(x)dx + \frac{m_1M_1}{m_2M_2}\int_c^d g^2(y)w_2(y)dy \le \left(\frac{m_1}{M_2} + \frac{M_1}{m_2}\right)\int_a^b\int_c^d f(x)g(y)w(x,y)dxdy. \tag{14}$$

In (14) *the equality appears* **iff** *either* **(i)** $m_1/M_2 = M_1/m_2$ *or* **(ii)** $m_1/M_2 < M_1/m_2$ *and*

$$\int_{\mathbb{I}_{x,y}} w(x,y) = 1,$$

where $\mathbb{I}_{x,y} := \{(x,y):\ f(x)/g(y) \in \{m_1/M_2, M_1/m_2\}\}$.

Proof. Let $(\xi, \eta > 0)$ be twodimensional continuous r.v. with density function $w(x,y)$ having $\mathrm{supp}(w) = [a,b]\times[c,d]$. We conclude $w_1(x),\ w_2(y)$ are the marginal density functions of ξ,η respectively. Since f,g are Borel - functions, they are from $L^2_{w_1}[a,b],\ L^2_{w_2}[c,d]$ respectively. Choosing m_j, M_j in way that $\mathsf{P}\{m_1 \le f(\xi) \le M_1\} = \mathsf{P}\{m_2 \le g(\eta) \le M_2\} = 1$, with the help of Theorem 1, we deduce

$$\mathsf{E}f^2(\xi) + \frac{m_1M_1}{m_2M_2}\mathsf{E}g^2(\eta) \le \left(\frac{m_1}{M_2} + \frac{M_1}{m_2}\right)\mathsf{E}f(\xi)g(\eta), \tag{15}$$

or in equivalent notation (14).

The equality is obvious in case **(i)**. Considering

$$\mathsf{E}f^2(\xi) + \frac{m_1M_1}{m_2M_2}\mathsf{E}g^2(\eta) - \left(\frac{m_1}{M_2} + \frac{M_1}{m_2}\right)\mathsf{E}f(\xi)g(\eta) = \mathsf{E}\left(f(\xi) - \frac{M_1}{m_2}g(\eta)\right)\left(f(\xi) - \frac{m_1}{M_2}g(\eta)\right)$$

by the Equality Theorem we deduce the assertion **(ii)** of the Corollary. □

Remark 2. Take $(\xi,\eta)\sim w(x,y)=(b-a)^{-1}\chi_{[a,b]^2}(x,y)\delta_{xy}$. This specification results with the classical Diaz - Metcalf equal - weight integral inequality:

$$\int_a^b f^2(x)dx+\frac{m_1M_1}{m_2M_2}\int_a^b g^2(x)dx\le\left(\frac{m_1}{M_2}+\frac{M_1}{m_2}\right)\int_a^b f(x)g(x)dx,$$

because of the marginal density functions become $w_1(x)=(b-a)^{-1}\chi_{[a,b]}(x)=w_2(x)$. The equality holds **iff** either **(i)** $m_1/M_2=M_1/m_2$ or **(ii)** $m_1/M_2<M_1/m_2$ and

$$\int_{\mathbb{I}_x} w_1(x)dx=1,$$

where $\mathbb{I}_x\equiv\mathbb{I}_{x,x},\ x\in[a,b]$, compare Corollary 1.2 and the article [2](where the inequality appears firstly) as well. See [6, p.64] for further informations on the subject.

3. On Pólya - Szegő Inequality

The classical Pólya - Szegő inequality in both cases (discrete and integral) was published in [8, p.57 & pp.213-214]. Here we give the basic probabilistic moment inequality which generalize the classical results to weighted variants.

Theorem 2. *Let ξ,η be real random variables defined on fixed probability space $(\Omega,\mathfrak{F},\mathsf{P})$. When $\mathsf{P}\{m_1\le\xi\le M_1\}=1,\ \mathsf{P}\{m_2\le\eta\le M_2\}=1$ with $0<m_j\le M_j,\ j=1,2$, it is valid*

$$\frac{\mathsf{E}\xi^2\mathsf{E}\eta^2}{(\mathsf{E}\xi\eta)^2}\le\frac{1}{4}\left(\sqrt{\frac{m_1m_2}{M_1M_2}}+\sqrt{\frac{M_1M_2}{m_1m_2}}\right)^2. \tag{16}$$

In (16) *the equality holds* **iff**

$$\xi=m_1\mathcal{I}_A(\omega)+M_1\mathcal{I}_{\Omega\setminus A}(\omega)\quad a.s.,\qquad \eta=m_2\mathcal{I}_{\Omega\setminus A}(\omega)+M_2\mathcal{I}_A(\omega)\quad a.s., \tag{17}$$

where $A\in\mathfrak{F}$ is chosen so that

$$\mathsf{P}(A)=\frac{M_1m_2}{m_1M_2+M_1m_2}. \tag{18}$$

Proof. Suppose that ξ,η are positive a.s. P bounded random variables. Then the left - hand expression in the Diaz - Metcalf inequality (6), with the aid of AG - means inequality, becomes

$$\mathsf{E}\xi^2+\frac{m_1M_1}{m_2M_2}\mathsf{E}\eta^2\ge 2\sqrt{\frac{m_1M_1}{m_2M_2}\mathsf{E}\xi^2\mathsf{E}\eta^2}. \tag{19}$$

Now, straightforward calculation gives us the asserted result.

To prove the **if** part of the equality assertion, assume that ξ, η are given by (17) and (18). Consequently, easy calculation gives us

$$\begin{aligned} \mathsf{E}\xi^2 &= \frac{m_1M_1(m_1m_2+M_1M_2)}{m_1M_2+M_1m_2}\,, \\ \mathsf{E}\eta^2 &= \frac{m_2M_2(m_1m_2+M_1M_2)}{m_1M_2+M_1m_2}\,, \\ \mathsf{E}\xi\eta &= \frac{2m_1m_2M_1M_2}{m_1M_2+M_1m_2}\,, \end{aligned}$$

which results with equality in (16).

The **only if** part of the equality in (16) we get when equality holds in Diaz - Metcalf inequality (6) and in the AG - mean inequality (19). Assume at first that $m_1m_2 < M_1M_2$ otherwise the theorem is obvious. Then it have to be satisfied

$$\mathsf{P}\left(\frac{\xi}{\eta} \in \left\{\frac{m_1}{M_2}, \frac{M_1}{m_2}\right\}\right) = 1, \tag{a}$$

$$\mathsf{E}\xi^2 = m_1M_1(m_2M_2)^{-1}\mathsf{E}\eta^2. \tag{b}$$

The requirement (a) is satisfied **iff** there are $A, B \in \mathfrak{F},\ A \cup B = \Omega,\ A \cap B = \emptyset$, so

$$\xi = m_1\mathcal{I}_A(\omega) + M_1\mathcal{I}_B(\omega), \qquad \eta = m_2\mathcal{I}_B(\omega) + M_2\mathcal{I}_A(\omega). \tag{20}$$

Replacing (20) into (b) we get

$$\frac{\mathsf{P}(A)}{\frac{M_1}{m_1}} = \frac{\mathsf{P}(B)}{\frac{M_2}{m_2}},$$

i.e.

$$\mathsf{P}(A) = \frac{M_1m_2}{m_1M_2+M_1m_2}, \quad \mathsf{P}(B) = \frac{M_2m_1}{m_1M_1+M_1m_2}.$$

Therefore (17) and (18) are proved. □

Corollary 2.1 (Pólya - Szegő Type Weighted Discrete Inequality)**.** *Suppose* $0 < m_1 \le x_k \le M_1,\ k = \overline{1,n}$ *and* $0 < m_2 \le y_l \le M_2,\ l = \overline{1,m}$. *Then let* $p_{kl} \ge 0$, *and* $\sum_{k=1}^n \sum_{l=1}^n p_{kl} = 1$. *It follows*

$$\sum_{k=1}^n x_k^2 q_k \sum_{l=1}^m y_l^2 r_l \le \frac{1}{4}\left(\sqrt{\frac{m_1m_2}{M_1M_2}} + \sqrt{\frac{M_1M_2}{m_1m_2}}\right)^2 \left(\sum_{k=1}^n \sum_{l=1}^m x_k y_l p_{kl}\right)^2. \tag{21}$$

Here is

$$\sum_{l=1}^m p_{kl} = q_k,\ k = \overline{1,n}; \quad \sum_{k=1}^n p_{kl} = r_l,\ l = \overline{1,m}.$$

The equality in (21) *holds* **iff** $m = n$, *and*

(1)

$$\kappa = \frac{M_1m_2n}{m_1M_2+M_1m_2} \in \mathbb{N}_0, \tag{22}$$

(2)

$$x_{i_1} = \cdots = x_{i_\kappa} = m_1, \quad x_{i_{\kappa+1}} = \cdots = x_{i_n} = M_1, \tag{23}$$

$$y_{i_1} = \cdots = y_{i_\kappa} = M_2, \quad y_{i_{\kappa+1}} = \cdots = y_{i_n} = m_2,\ i_j \in \{1, \cdots, n\}, \tag{24}$$

(3)

$$p_{kl} = 0,\ \ (k,l) \in \{i_1, \cdots, i_\kappa\} \times \{i_{\kappa+1}, \cdots, i_n\} \cup \{i_{\kappa+1}, \cdots, i_n\} \times \{i_1, \cdots, i_\kappa\}. \tag{25}$$

Proof. Specify

$$0 < (\xi, \eta) \sim \mathsf{P}\{\xi = x_k, \eta = y_l\} = p_{kl}$$

with $\sum_{k=1}^n \sum_{l=1}^m p_{kl} = 1$ on fixed probability space $(\Omega, \mathcal{P}(\Omega), \mathsf{P})$. Easy calculation gives us (21) by means of (16).

We omit the equality proof since the descriptionary character of the Corollary formulation in displays (22-25) with $m = n$. □

Special case of the discrete weighted Pólya - Szegő inequality is the Kantorovich inequality, see [4]. If we set $m = n;\ x_j^2 = \gamma_j,\ y_k^2 = \gamma_k^{-1}$ we deduce $0 < \sqrt{\mu} \le \gamma_k \le \sqrt{M}$ where $m_1 = \mu, M_1 = M$. Moreover if

$$p_{kl} = \frac{u_k^2}{\sum\limits_{j=1}^n u_j^2}\, \delta_{kl}, \qquad k, l = \overline{1, n}, \tag{26}$$

where at least one $u_k \neq 0$, we deduce from (21) the Kantorovich inequality, reads as follows

$$\sum_{j=1}^n \gamma_j u_j^2 \sum_{j=1}^n \frac{u_j^2}{\gamma_j} \le \frac{1}{4}\left(\sqrt{\frac{\mu}{M}} + \sqrt{\frac{M}{\mu}}\right)^2 \left(\sum_{j=1}^n u_j^2\right)^2,$$

see [6, p.61] too.

Putting $m = n$ and specifying p_{kl} in manner like (26) we get

$$\sum_{k=1}^n x_k^2 u_k^2 \sum_{k=1}^n y_k^2 u_k^2 \le \frac{1}{4}\left(\sqrt{\frac{m_1 m_2}{M_1 M_2}} + \sqrt{\frac{M_1 M_2}{m_1 m_2}}\right)^2 \left(\sum_{k=1}^n x_k y_k u_k^2\right)^2,$$

which is exactly the result by Greub and Rheinboldt, compare [3], [6, p.61].

Taking $m = n$ and $p_{kl} = n^{-1}\delta_{kl}$, the weighted discrete Pólya - Szegő inequality (21) becomes the classical one:

$$\sum_{k=1}^n x_k^2 \sum_{k=1}^n y_k^2 \le \frac{1}{4}\left(\sqrt{\frac{m_1 m_2}{M_1 M_2}} + \sqrt{\frac{M_1 M_2}{m_1 m_2}}\right)^2 \left(\sum_{k=1}^n x_k y_k\right)^2, \tag{27}$$

for all positive real n - tuples $\{x_k\},\ \{y_k\}, k = \overline{1, n}$, having bounds mentioned in Corollary 2.1, see e.g. [6, pp.60-61.], [8].

The equality condition is now somewhat simpler then in the Corollary 2.1., i.e. (27) becomes equality under (22-24).

Corollary 2.2 (Pólya - Szegő Weighted Integral Inequality). *Let f, g be positive Borel - functions, such that*

$$0 < m_1 \le f(x) \le M_1, \qquad 0 < m_2 \le g(y) \le M_2, \qquad a.s.\ (x,y) \in [a,b] \times [c,d]. \tag{28}$$

Let $w(x,y)$ be a nonnegative weight function such that $\mathrm{supp}(w) = [a,b] \times [c,d]$, *and let*

$$\int_c^d w(x,y)dy = w_1(x), \qquad \int_a^b w(x,y)dx = w_2(y).$$

Then it is valid

$$\frac{\int_a^b f^2(x)w_1(x)dx \int_c^d g^2(y)w_2(y)dy}{\left(\int_a^b \int_c^d f(x)g(y)w(x,y)dxdy\right)^2} \le \frac{1}{4}\left(\sqrt{\frac{m_1m_2}{M_1M_2}} + \sqrt{\frac{M_1M_2}{m_1m_2}}\right)^2. \tag{29}$$

The equality in (29) *holds* **iff** *the following conditions are fulfilled:* $[a,b] \equiv [c,d]$ *and*

$$f(x) = m_1\chi_S(x) + M_1\chi_{[a,b]\setminus S}(x), \quad g(y) = M_2\chi_{[a,b]\setminus S}(y) + m_2\chi_S(y), \tag{30}$$

for a Borel set $S \subseteq [a,b]$ of Lebesgue measure

$$|S| = \frac{M_1m_2(b-a)}{m_1M_2 + M_1m_2}, \tag{31}$$

moreover

$$w(x,y) = 0, \qquad \forall (x,y) \in S^2 \cup ([a,b] \setminus S)^2. \tag{32}$$

Since the proof of the inequality (29) is very similar to the proof of the Corollary 1.2 and Corollary 2.1, it is omitted.

Remark 3. When $[a,b] = [c,d]$ and $(\xi, \eta) \sim w(x,y) = (b-a)^{-1}\chi_{[a,b]^2}(x,y)\delta_{xy}$, we get

$$\frac{\int_a^b f^2(x)dx \int_a^b g^2(x)dx}{\left(\int_a^b f(x)g(x)dx\right)^2} \le \frac{1}{4}\left(\sqrt{\frac{m_1m_2}{M_1M_2}} + \sqrt{\frac{M_1M_2}{m_1m_2}}\right)^2, \tag{33}$$

for all $f, g \in L^2[a,b]$ satisfying

$$0 < m_1 \le f(x) \le M_1, \ 0 < m_2 \le g(x) \le M_2.$$

When f, g satisfy (30) and (31), then (33) becomes equality and *vice versa*. The inequality (33) originates back to Pólya and Szegő, compare [8, p.81 & pp.251-252, Problem **93**] where no equality analysis was given.

4. On Rennie and Schweitzer Inequalities

In this section we give some generalizations of Rennie and Schweitzer inequalities to probabilistic setting. We can do this in two directions: **1.** applying the linearity and monotonicity of the expectation operator E; **2.** direct use of Diaz - Metcalf and Pólya - Szegő inequalities respectively.

Let ζ be a.s. P bounded, positive random variables, i.e. $\mathsf{P}\{0 < m \le \zeta \le M\} = 1$. Then it is clear that

$$\mathsf{E}(M\zeta^{-1} - 1)(\zeta - m) \ge 0.$$

Therefore we deduce

$$\mathsf{E}\zeta + mM\mathsf{E}\zeta^{-1} \le m + M.$$

This is the Rennie - type inequality for positive, bounded random variables. Now, the AG - mean inequality, applied to the left side term in Rennie inequality gives us $2\sqrt{mM\mathsf{E}\zeta\mathsf{E}(1/\zeta)}$ and we finish the derivation with

$$\mathsf{E}\zeta\mathsf{E}\zeta^{-1} \le \frac{(m+M)^2}{4mM},$$

which is the Schweitzer - type inequality.

Theorem 3. *Let ζ be positive, a.s. P bounded random variable defined on the probability space $(\Omega, \mathfrak{F}, \mathsf{P})$ and $\mathsf{P}\{0 < m \le \zeta \le M\} = 1$. Then*

$$\mathsf{E}\zeta + mM\mathsf{E}\zeta^{-1} \le m + M, \tag{34}$$

where the equality holds **iff** $\zeta = m\mathcal{I}_A(\omega) + M\mathcal{I}_{\Omega\setminus A}(\omega)$, *for certain $A \in \mathfrak{F}$ and*

$$\mathsf{E}\zeta\mathsf{E}\zeta^{-1} \le \tfrac{(m+M)^2}{4mM}, \tag{35}$$

where the equality is valid **iff** $\zeta = m\mathcal{I}_A(\omega) + M\mathcal{I}_{\Omega\setminus A}(\omega)$, *for $A \in \mathfrak{F}$, $\mathsf{P}(A) = 1/2$.*

Proof. Take $\xi^2 = \zeta = \eta^{-2}$ in (6) and (16) respectively. $\square$

The following result is given in [9], without any equality quotation, compare [6, p.63] as well.

Corollary 3.1 (Discrete Rennie Inequality)**.** *Let $\{x_j\}$ be a positive, real n - tuple, $\pi_j \ge 0$, $\sum_{j=1}^n \pi_j = 1$. Then*

$$\sum_{j=1}^n x_j\pi_j + mM\sum_{j=1}^m \frac{\pi_j}{x_j} \le m + M, \tag{36}$$

where m, M are the minimal, maximal element of $\{x_j\}$ respectively.

The equality holds in (36) for $x_j \in \{m, M\}$, $j = \overline{1, n}$.

Proof. Take $\zeta \sim \mathsf{P}\{\zeta = x_j\} = \pi_j$. By (34) we get the desired result (36) immediately.

Now, let $I \subseteq \{1, \cdots, n\}$. Put $x_j = m, j \in I$ and $x_j = M$ when $j \notin I$. As

$$\sum_{j=1}^n x_j\pi_j + mM\sum_{j=1}^m \frac{\pi_j}{x_j} = m\sum_{j\in I}\pi_j + M\sum_{j\notin I}\pi_j + mM\left(\frac{1}{m}\sum_{j\in I}\pi_j + \frac{1}{M}\sum_{j\notin I}\pi_j\right) = m + M,$$

the Corollary 3.1. is proved. $\square$

Corollary 3.2 (Integral Rennie Inequality). *Let $f, 1/f \in L_w^1[a,b]$, where $w(x) \geq 0$ and $\int_a^b w(x)dx = 1$. When $0 < m \leq f(x) \leq M$ for all $x \in [a,b]$, then*

$$\int_a^b f(x)w(x)dx + mM \int_a^b \frac{w(x)}{f(x)}dx \leq m + M; \tag{37}$$

here the equality is valid **iff** *$f(x) = m\chi_S(x) + M\chi_{[a,b]\setminus S}(x)$, for some Borel $S \subseteq [a,b]$.*

Corollary 3.3 (Weighted Discrete Schweitzer Inequality). *Let $0 < m \leq x_j \leq M,\ \pi_j \geq 0,\ j = \overline{1,n}$ where $\sum_{j=1}^n \pi_j = 1$. Then*

$$\sum_{j=1}^n x_j\pi_j \sum_{j=1}^m \frac{\pi_j}{x_j} \leq \frac{(m+M)^2}{4mM}. \tag{38}$$

The equality appears **iff** *n is even ($n = 2\lambda$), $\sum_{l=1}^\lambda \pi_l = 1/2$ and*

$$x_{j_1} = \cdots = x_{j_\lambda} = m, \quad x_{j_{\lambda+1}} = \cdots = x_{j_n} = M, \qquad j_l \in \{1, \cdots, n\}.$$

Corollary 3.4 (Weighted Integral Schweitzer Inequality). *Let $f, 1/f \in L_w^1[a,b]$, where $w(x) \geq 0$ and $\int_a^b w(x)dx = 1$. When $0 < m \leq f(x) \leq M$ for all $x \in [a,b]$, then*

$$\int_a^b f(x)w(x)dx \int_a^b \frac{w(x)}{f(x)}dx \leq \frac{(m+M)^2}{4mM} \tag{39}$$

and the equality is valid **iff** *$f(x) = m\chi_S(x) + M\chi_{[a,b]\setminus S}(x)$, for certain Borel $S \subseteq [a,b]$ which possesses Lebesgue measure equal $|S| = (b-a)/2$.*

The Schweitzer inequalities (in both cases, discrete and integral) we can find in [10]. More precisely, when we specify on certain probability space $(\Omega, \mathcal{P}(\Omega), \mathsf{P})$ the equal weight case by $\xi \sim \mathsf{P}\{\xi = x_j\} = \pi_j = 1/n,\ j = \overline{1,n}$ in (34), we get

$$\sum_{j=1}^n x_j \sum_{j=1}^n \frac{1}{x_j} \leq \frac{(m+M)^2 n^2}{4mM},$$

which is the original discrete Schweitzer result.

Additionally, taking $\xi \sim \mathcal{U}[a,b]$, i.e. $w(x) = (b-a)^{-1}\chi_{[a,b]}(x)$ in (39), we deduce

$$\int_a^b f(x)dx \int_a^b \frac{dx}{f(x)} \leq \frac{(m+M)^2}{4mM}\ (b-a)^2.$$

This inequality is originally proved by Schweitzer in [10] too. Further informations about the subject can be found in [6, pp.60-61], [8, pp.80-81 & pp.250-251].

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