Error Estimates on Approximating the Finite Fourier Transform of Complex-Valued Functions via a Pre-Grüss Inequality

ERROR ESTIMATES ON APPROXIMATING THE FINITE FOURIER TRANSFORM OF COMPLEX-VALUED FUNCTIONS VIA A PRE-GRÜSS INEQUALITY

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Abstract. Some estimates for the error in approximating the Finite Fourier Transform in terms of exponential means via a pre-Grüss inequality for complex-valued functions are given.

1. Introduction

Let (Ω, Σ, μ) be a measure space consisting of a set Ω, Σ and $\sigma-$algebra of subsets of Ω and μ a countable additive and positive measure with values in $\mathbb{R} \cup \{\infty\}$. Denote by $L_\rho^2(\Omega, \mathbb{K})$ the Hilbert space of all measurable functions $f : \Omega \to \mathbb{K}$ that are $2-\rho-$integrable on Ω, i.e., $\int_\Omega \rho(s) |f(s)|^2 d\mu(s) < \infty$, where $\rho : \Omega \to [0, \infty)$ is a given measurable function on Ω.

Using the fact that $(\cdot, \cdot)_\rho : L_\rho^2(\Omega, \mathbb{K}) \times L_\rho^2(\Omega, \mathbb{K}) \to \mathbb{K}$,

$$(f, g)_\rho := \int_\Omega \rho(s) f(s) \overline{g(s)} d\mu(s)$$

is an inner product on $L_\rho^2(\Omega, \mathbb{K})$ that generates the norm $\|\cdot\|_\rho$,

$$\|f\|_\rho := \int_\Omega \rho(s) |f(s)|^2 d\mu(s),$$

and an earlier result due to Dragomir from [4], Dragomir and Gomm pointed out in [6] the following weighted Grüss type inequality for complex-valued functions

$$\begin{aligned}(1.1)\quad & \left| \frac{1}{\int_\Omega \rho(s) d\mu(s)} \int_\Omega \rho(s) f(s) \overline{g(s)} d\mu(s) \right. \\ & \left. - \frac{1}{\int_\Omega \rho(s) d\mu(s)} \int_\Omega \rho(s) f(s) d\mu(s) \cdot \frac{1}{\int_\Omega \rho(s) d\mu(s)} \int_\Omega \rho(s) \overline{g(s)} d\mu(s) \right| \\ & \leq \frac{1}{4} |Z - z| |T - t|,\end{aligned}$$

provided $\int_\Omega \rho(s) d\mu(s) > 0$, $f, g \in L_\rho^2(\Omega, \mathbb{K})$, and

$$\begin{aligned}(1.2)\quad \operatorname{Re}\left[(Z - f(s)) \left(\overline{f(s)} - \overline{z}\right)\right] &\geq 0, \\ \operatorname{Re}\left[(T - g(s)) \left(\overline{g(s)} - \overline{t}\right)\right] &\geq 0,\end{aligned}$$

for $\mu-$a.e. $s \in \Omega$. The constant $\frac{1}{4}$ is best possible in (1.1).

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Since for any $a, b, c \in \mathbb{C}$, the following two statements are clearly equivalent

(i) $\left|b - \frac{a+c}{2}\right| \leq \frac{1}{2}\left|c-a\right|$;
(ii) $\operatorname{Re}\left[(a-b)\left(\bar{b}-\bar{c}\right)\right] \geq 0$,

it follows that (1.2) is equivalent with the more intuitive formula

$$\begin{aligned} \left|f(s) - \frac{Z+z}{2}\right| &\leq \frac{1}{2}\left|Z-z\right|, \\ \left|g(s) - \frac{T+t}{2}\right| &\leq \frac{1}{2}\left|T-t\right|, \end{aligned} \tag{1.3}$$

for $\mu-$a.e. $s \in \Omega$.

As a particular case of (1.1) we should note the following reverse of the Cauchy-Bunyakovsky-Schwarz inequality

$$\begin{aligned} 0 &\leq \frac{1}{\int_\Omega \rho(s)\, d\mu(s)} \int_\Omega \rho(s) \left|f(s)\right|^2 d\mu(s) - \left|\frac{1}{\int_\Omega \rho(s)\, d\mu(s)} \int_\Omega \rho(s) f(s)\, d\mu(s)\right|^2 \\ &\leq \frac{1}{4}\left|Z-z\right|^2, \end{aligned} \tag{1.4}$$

provided $f \in L^2_\rho(\Omega, \mathbb{K})$ and f satisfies either (1.2) or, equivalently, (1.3) and $\int_\Omega \rho(s)\, d\mu(s) > 0$.

The main aim of this paper is to point out a pre-Grüss inequality for complex-valued functions and apply it in approximating the finite Fourier transform of complex-valued mappings.

2. A Pre-Grüss Type Inequality for Complex Valued Functions

The following result provides an inequality of Grüss type that may be useful in applications where one of the factors is known and some bounds for the second factor are provided.

Theorem 1. *Let $\rho : \Omega \to [0, \infty)$ be a $\mu-$measurable function on Ω with $\int_\Omega \rho(s)\, d\mu(s) = 1$. If $f, g \in L^2_\rho(\Omega, \mathbb{K})$ and there exists the constants $\gamma, \Gamma \in \mathbb{K}$ with the property that either*

$$\operatorname{Re}\left[(\Gamma - f(s))\left(\overline{f(s)} - \overline{\gamma}\right)\right] \geq 0 \quad \text{for} \quad \mu - a.e.\ s \in \Omega \tag{2.1}$$

or, equivalently,

$$\left|f(s) - \frac{\gamma+\Gamma}{2}\right| \leq \frac{1}{2}\left|\Gamma - \gamma\right| \quad \text{for} \quad \mu - a.e.\ s \in \Omega \tag{2.2}$$

holds, then

$$\begin{aligned} &\left|\int_\Omega \rho(s) f(s) \overline{g(s)} d\mu(s) - \int_\Omega \rho(s) f(s)\, d\mu(s) \int_\Omega \rho(s) \overline{g(s)} d\mu(s)\right| \\ &\qquad \leq \frac{1}{2}\left|\Gamma - \gamma\right| \left[\int_\Omega \rho(s) \left|g(s)\right|^2 d\mu(s) - \left|\int_\Omega \rho(s) g(s)\, d\mu(s)\right|^2\right]^{\frac{1}{2}}. \end{aligned} \tag{2.3}$$

Proof. We know, by Korkine's identity, that

$$\int_{\Omega}\rho(s) f(s)\overline{g(s)}d\mu(s)-\int_{\Omega}\rho(s) f(s)\,d\mu(s)\int_{\Omega}\rho(s)\overline{g(s)}d\mu(s)$$
$$=\frac{1}{2}\int_{\Omega}\int_{\Omega}\rho(s)\rho(t)[f(s)-f(t)]\left[\overline{g(s)}-\overline{g(t)}\right]d\mu(s)\,d\mu(t).$$

Applying Schwarz's integral inequality for double integrals, we have

$$\left|\int_{\Omega}\int_{\Omega}\rho(s)\rho(t)[f(s)-f(t)]\left[\overline{g(s)}-\overline{g(t)}\right]d\mu(s)\,d\mu(t)\right| \tag{2.4}$$
$$\le \left[\int_{\Omega}\int_{\Omega}\rho(s)\rho(t)|f(s)-f(t)|^2\,d\mu(s)\,d\mu(t)\right.$$
$$\left.\times\int_{\Omega}\int_{\Omega}\rho(s)\rho(t)\left|\overline{g(s)}-\overline{g(t)}\right|^2 d\mu(s)\,d\mu(t)\right]^{\frac{1}{2}}$$
$$=\left(\int_{\Omega}\rho(s)|f(s)|^2\,d\mu(s)-\left|\int_{\Omega}\rho(s) f(s)\,d\mu(s)\right|^2\right)^{\frac{1}{2}}$$
$$\times\left(\int_{\Omega}\rho(s)|g(s)|^2\,d\mu(s)-\left|\int_{\Omega}\rho(s)\overline{g(s)}d\mu(s)\right|^2\right)^{\frac{1}{2}},$$

and, for the last identity, we have also used Korkine's identity for one function $(f=g)$.

Applying the inequality (1.4) for the function f, and taking into account the hypothesis of the theorem, we may state that

$$\int_{\Omega}\rho(s)|f(s)|^2\,d\mu(s)-\left|\int_{\Omega}\rho(s) f(s)\,d\mu(s)\right|^2\le\frac{1}{4}|\Gamma-\gamma|^2. \tag{2.5}$$

Utilising (2.4) and (2.5), we deduce the desired result (2.3). The fact that $c=\frac{1}{2}$ is the best possible constant follows by the fact that $\frac{1}{4}$ is best possible in Grüss' inequality (1.1) and we omit the details. ∎

Remark 1. *If the function f is a real-valued function, then the condition (2.2) may be replaced with the equivalent condition (say $\Gamma>\gamma$)*

$$\gamma\le f(s)\le\Gamma \quad \text{for} \quad \mu-a.e. \; s\in\Omega, \tag{2.6}$$

and thus, (2.3) becomes the corresponding real-valued pre-Grüss inequality used by several authors in the last five years to obtain perturbed quadrature rules (see for instance [7]*).*

3. Applications for Uni-Dimensional Fourier Transform

The *Fourier Transform* has applications in a wide variety of fields in science and engineering [3, p. xi].

Let $g:[a,b]\to\mathbb{K}$ $(\mathbb{K}=\mathbb{C},\mathbb{R})$ be a Lebesgue integrable mapping defined on the finite interval $[a,b]$ and $\mathcal{F}(g)$ its finite Fourier transform, i.e.,

$$\mathcal{F}(g)(t):=\int_a^b g(s)\,e^{-2\pi its}ds.$$

The following inequality in approximating the finite Fourier trasnform in terms of the exponential mean was obtained in [1].

Theorem 2. *Let g be an absolutely continuous mapping on $[a,b]$. Then we have the inequality*

$$\left|\mathcal{F}(g)(x)-E(-2\pi ixa,-2\pi ixb)\int_a^b g(t)\,dt\right| \tag{3.1}$$
$$\leq\begin{cases}\frac{1}{3}\|g'\|_\infty (b-a)^2 & \text{if } g'\in L_\infty[a,b];\\[2ex] \frac{2^{\frac{1}{q}}}{[(q+1)(q+2)]^{\frac{1}{q}}}(b-a)^{1+\frac{1}{q}}\|g'\|_p & \text{if } g'\in L_p[a,b];\\[2ex] (b-a)\|g'\|_1, & \end{cases}$$

for each $x\in[a,b]$, $x\neq 0$, when E is the exponential mean of two complex numbers defined by

$$E(z,w):=\begin{cases}\frac{e^z-e^w}{z-w}, & \text{if } z\neq w\\[2ex] \exp(w) & \text{if } z=w\end{cases},\quad z,w\in\mathbb{C}. \tag{3.2}$$

For functions of bounded variation, the following result holds as well (see [2]):

Theorem 3. *Let $g:[a,b]\to\mathbb{K}$ be a mapping of bounded variation on $[a,b]$. Then we have the inequality*

$$\left|\mathcal{F}(g)(x)-E(-2\pi ixa,-2\pi ixb)\int_a^b g(s)\,ds\right|\leq\frac{3}{4}(b-a)\bigvee_a^b(g), \tag{3.3}$$

for each $x\in[a,b]$, $x\neq 0$, where $\bigvee_a^b(g)$ is the total variation of g on $[a,b]$.

Finally, we mention the following result obtained in [5] providing an approximation of the Fourier transform of Lebesgue integrable functions:

Theorem 4. *Let $g:[a,b]\to\mathbb{R}$ be a measurable function on $[a,b]$. Then we have the estimates:*

$$\left|\mathcal{F}(g)(x)-E(-2\pi ixa,-2\pi ixb)\int_a^b g(s)\,ds\right| \tag{3.4}$$
$$\leq\begin{cases}\frac{2\pi}{3}|x|(b-a)^2\|g\|_\infty & \text{if } g\in L_\infty[a,b];\\[2ex] \frac{2^{1+\frac{1}{q}}(b-a)^{1+\frac{1}{q}}}{[(q+1)(q+2)]^{\frac{1}{q}}}|x|\|g\|_p & \text{if } g\in L_p[a,b],\ p>1,\\ & \frac{1}{p}+\frac{1}{q}=1;\\ 2\pi|x|(b-a)\|g\|_1 & \text{if } g\in L_1[a,b]\end{cases}$$

for each $x\in[a,b]$, $x\neq 0$.

In the following we apply the pre-Grüss inequality obtained in Theorem 1 to approximate the finite Fourier transform $\mathcal{F}(\cdot)$.

Theorem 5. *Let $g : [a,b] \to \mathbb{K}$ be a real or complex-valued function with $g \in L^2_\rho(\Omega, \mathbb{K})$, and there exists the constants $\varphi, \phi \in \mathbb{K}$ with the property that, either*

$$\left| g(s) - \frac{\phi + \varphi}{2} \right| \leq \frac{1}{2} |\phi - \varphi| \quad \text{for } \mu - a.e.\ s \in [a,b] \tag{3.5}$$

or, equivalently

$$\operatorname{Re}\left[(\phi - g(s)) \left(\overline{g(s)} - \overline{\varphi} \right) \right] \geq 0 \quad \text{for } \ a.e.\ s \in [a,b] \tag{3.6}$$

hold. Then we have the inequality

$$\begin{aligned} &\left| \mathcal{F}(g)(x) - E(-2\pi i x a, -2\pi i x b) \int_a^b g(s)\, ds \right| \\ &\qquad \leq \frac{1}{2} |\phi - \varphi| (b-a) \left[1 - \frac{\sin^2[\pi x (b-a)]}{\pi^2 |x|^2 (b-a)^2} \right]^{\frac{1}{2}}, \end{aligned} \tag{3.7}$$

for each $x \in [a,b]$ $(x \neq 0)$, where $E(\cdot,\cdot)$ is the exponential mean defined above.

Proof. We apply the pre-Grüss inequality (2.3) to get

$$\begin{aligned} &\left| \frac{1}{b-a} \int_a^b g(s) e^{-2\pi i x s} ds - \frac{1}{b-a} \int_a^b e^{-2\pi i x s} ds \cdot \frac{1}{b-a} \int_a^b g(s)\, ds \right| \\ &\qquad \leq \frac{1}{2} |\phi - \varphi| \left[\frac{1}{b-a} \int_a^b \left| e^{2\pi i x s} \right|^2 ds - \left| \frac{1}{b-a} \int_a^b e^{2\pi i x s} ds \right|^2 \right]^{\frac{1}{2}}. \end{aligned} \tag{3.8}$$

However,

$$\int_a^b e^{-2\pi i x s} ds = (b-a) E(-2\pi i x a, -2\pi i x b),$$

$$\left| e^{2\pi i x s} \right|^2 = 1,$$

$$\int_a^b e^{2\pi i x s} ds = \frac{1}{2\pi i x} \left[e^{2\pi i x b} - e^{2\pi i x a} \right],$$

and

$$\begin{aligned} \left| \int_a^b e^{2\pi i x s} ds \right|^2 &= \left(\frac{1}{2\pi |x|} \right)^2 \left[\left| e^{2\pi i x b} \right|^2 - 2 \operatorname{Re}\left[e^{2\pi i x b} e^{-2\pi i x a} \right] + \left| e^{2\pi i x a} \right|^2 \right] \\ &= \frac{1}{4\pi^2 |x|^2} \left[1 - 2 \operatorname{Re}\left[e^{2\pi i x (b-a)} \right] + 1 \right] \\ &= \frac{1}{2\pi^2 |x|^2} \left[1 - \operatorname{Re}\left[\cos(2\pi x (b-a)) + i \sin(2\pi x (b-a)) \right] \right] \\ &= \frac{1}{2\pi^2 |x|^2} \left[1 - \cos(2\pi x (b-a)) \right] \\ &= \frac{1}{2\pi^2 |x|^2} \left[1 - \left(1 - 2\sin^2(\pi x (b-a)) \right) \right] \\ &= \frac{\sin^2[\pi x (b-a)]}{\pi^2 |x|^2}. \end{aligned}$$

Using (3.8) multiplied with $b-a>0$, we deduce

$$\left|\mathcal{F}(g)(x)-E(-2\pi ixa,-2\pi ixb)\int_a^b g(s)\,ds\right| \leq \frac{1}{2}|\phi-\varphi|(b-a)\left[1-\frac{\sin^2[\pi x(b-a)]}{\pi^2|x|^2(b-a)^2}\right]^{\frac{1}{2}},$$

giving the desired inequality (3.7). ∎

Remark 2. *If g takes real values, then the condition (3.5) may be replaced by the equivalent condition (for $\Gamma>\gamma$)*

$$\gamma\leq g(s)\leq\Gamma \text{ for a.e. } s\in[a,b]. \tag{3.9}$$

4. Errors in a Quadrature Formula

Let $I_n := a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$ be a division of the interval $[a,b]$, put $h_k := x_{k+1}-x_k$ $(k=0,\ldots,n-1)$ and $\nu(h) := \max\{h_k|k=0,\ldots,n-1\}$. Define the sum (see also [1] and [2])

$$\mathcal{E}(g,I_n,x) := \sum_{k=0}^{n-1} E(-2\pi ix_k,-2\pi ix_{k+1})\int_{x_k}^{x_{k+1}} g(t)\,dt, \tag{4.1}$$

where $x\in[a,b]$, $x\neq 0$.

The following approximation result holds.

Theorem 6. *Let $g:[a,b]\to\mathbb{K}$ be such that $g\in L_\rho^2(\Omega,\mathbb{K})$ and there exists the constants $\phi,\varphi\in\mathbb{K}$ such that either (3.5) or, equivalently, (3.6) holds true. Then we have*

$$\mathcal{F}(g)(x)=\mathcal{E}(g,I_n,x)+\mathcal{R}(g,I_n,x), \tag{4.2}$$

for each $x\in[a,b]$ $(x\neq 0)$, where $\mathcal{E}(g,I_n,x)$ is as defined in (4.1) and the remainder $\mathcal{R}(g,I_n,x)$ satisfies the estimate:

$$\begin{aligned}|\mathcal{R}(g,I_n,x)| &\leq \frac{1}{2}|\phi-\varphi|\left(\sum_{k=0}^{n-1}h_k^2\right)^{\frac{1}{2}}\left(\sum_{k=0}^{n-1}\left(1-\frac{\sin^2(\pi xh_k)}{\pi^2|x|^2h_k^2}\right)\right)^{\frac{1}{2}} \\ &\leq \frac{1}{2}|\phi-\varphi|(b-a)^{\frac{1}{2}}[\nu(h)]^{\frac{1}{2}}\left(\sum_{k=0}^{n-1}\left(1-\frac{\sin^2(\pi xh_k)}{\pi^2|x|^2h_k^2}\right)\right)^{\frac{1}{2}}.\end{aligned} \tag{4.3}$$

Proof. If we apply Theorem 5 on every subinterval $[x_n,x_{n+1}]$, $k=0,\ldots,n-1$, then we can state that

$$\left|\int_{x_k}^{x_{k+1}} g(t)e^{-2\pi ixt}dt - E(-2\pi ix_k,-2\pi ix_{k+1})\cdot\int_{x_k}^{x_{k+1}} g(t)\,dt\right| \leq \frac{1}{2}|\phi-\varphi|h_k\left[1-\frac{\sin^2(\pi xh_k)}{\pi^2|x|^2h_k^2}\right]^{\frac{1}{2}},$$

for all $k\in\{0,\ldots,n-1\}$.

Summing over i from 0 to $n-1$ and utilising the generalised triangle inequality and the Cauchy-Bunyakovsky-Schwarz inequality

$$\left|\sum_{i=1}^{n}\alpha_i\beta_i\right|^2 \le \sum_{i=1}^{n}|\alpha_i|^2\sum_{i=1}^{n}|\beta_i|^2,$$

where $\alpha_i, \beta_i \in \mathbb{K}$, we deduce

$$\begin{aligned}
|\mathcal{R}(g, I_n, x)| &= |\mathcal{F}(g)(x) - \mathcal{E}(g, I_n, x)| \\
&\le \frac{1}{2}|\phi-\varphi|\sum_{k=0}^{n-1}h_k\left[1-\frac{\sin^2(\pi x h_k)}{\pi^2|x|^2h_k^2}\right]^{\frac{1}{2}} \\
&\le \frac{1}{2}|\phi-\varphi|\left\{\sum_{k=0}^{n-1}h_k^2\sum_{k=0}^{n-1}\left[\left(1-\frac{\sin^2(\pi x h_k)}{\pi^2|x|^2h_k^2}\right)^{\frac{1}{2}}\right]^2\right\}^{\frac{1}{2}} \\
&= \frac{1}{2}|\phi-\varphi|\left(\sum_{k=0}^{n-1}h_k^2\right)^{\frac{1}{2}}\left[\sum_{k=0}^{n-1}\left(1-\frac{\sin^2(\pi x h_k)}{\pi^2|x|^2h_k^2}\right)\right]^{\frac{1}{2}}
\end{aligned}$$

and the first part of (4.3) is proved.

For the second part, let us observe that

$$\sum_{k=0}^{n-1}h_k^2 \le \nu(h)\sum_{k=0}^{n-1}h_k = (b-a)\,\nu(h).$$

The proof is thus completed. ∎

In practical applications, it is more convenient to consider the equidistant partitioning on the interval $[a,b]$. Thus, let

$$I_n : x_j = a + j\cdot\frac{b-a}{n}, \quad j = 0,\dots,n;$$

be an equidistant partition of $[a,b]$, and define the sum (see [1] and [2])

$$\mathcal{E}_n(g,x) := \sum_{k=0}^{n-1}E\left[-2\pi i x\left(a+k\cdot\frac{b-a}{n}\right), -2\pi i x\left(a+(k+1)\cdot\frac{b-a}{n}\right)\right] \times\int_{a+k\cdot\frac{b-a}{n}}^{a+(k+1)\cdot\frac{b-a}{n}}g(t)\,dt. \tag{4.4}$$

We may state the following corollary as well.

Corollary 1. *Let g be as defined in Theorem 6. Then we have*

$$\mathcal{F}(g)(x) = \mathcal{E}_n(g,x) + \mathcal{R}_n(g,x), \tag{4.5}$$

where $\mathcal{E}_n(g,x)$ approximates the Fourier transform at any point $x\in[a,b]$ $(x\neq 0)$. The error of approximation $\mathcal{R}_n(g,x)$ satisfies the bound

$$|\mathcal{R}_n(g,x)| \le \frac{1}{2n}|\phi-\varphi|(b-a)\left(\sum_{k=0}^{n-1}\left(1-\frac{\sin^2\left[\pi x\left(a+k\cdot\frac{b-a}{n}\right)\right]}{\pi^2x^2\left(a+k\cdot\frac{b-a}{n}\right)^2}\right)\right)^{\frac{1}{2}} \tag{4.6}$$

for any $x\in[a,b]$, $(x\neq 0)$.

Remark 3. *For the computer numerical implementation of the quadrature formula outlined above we refer the reader to either* [1] *or* [2].

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