# *Some New Bounds for the Čebyšev Functional in Terms of the First Derivative and Applications*

# SOME NEW BOUNDS FOR THE ČEBYŠEV FUNCTIONAL IN TERMS OF THE FIRST DERIVATIVE AND APPLICATIONS

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ABSTRACT. Some new inequalities for the Čebyšev functional in terms of the first derivative and applications for Taylor's expansion and generalised trapezoid formula are pointed out.

## 1. Introduction

For two Lebesgue integrable functions $f,g:[a,b]\to\mathbb{R}$, consider the Čebyšev functional:

$$T(f,g):=\frac{1}{b-a}\int_a^b f(t)\,g(t)\,dt-\frac{1}{b-a}\int_a^b f(t)\,dt\cdot\frac{1}{b-a}\int_a^b g(t)\,dt. \tag{1.1}$$

In 1934, G. Grüss [7] showed that

$$|T(f,g)|\le\frac{1}{4}(M-m)(N-n), \tag{1.2}$$

provided $m,M,n,N$ are real numbers with the property

$$-\infty<m\le f\le M<\infty,\ -\infty<n\le g\le N<\infty \ \text{ a.e. on } [a,b]. \tag{1.3}$$

The constant $\frac{1}{4}$ is best possible in (1.2) in the sense that it cannot be replaced by a smaller one. Less known appears to be another inequality for $T(f,g)$ derived in 1882 by Čebyšev [3] under the assumptions that $f'$, $g'$ exist and are continuous in $[a,b]$,

$$|T(f,g)|\le\frac{1}{12}\|f'\|_\infty\|g'\|_\infty(b-a)^2, \tag{1.4}$$

where $\|f'\|_\infty:=\sup_{t\in[a,b]}|f'(t)|$.

The constant $\frac{1}{12}$ cannot be improved in the general case.

Čebyšev's inequality (1.4) also holds if $f,g:[a,b]\to\mathbb{R}$ are assumed to be absolutely continuous and $f',g'\in L_\infty[a,b]$.

In 1970, A.M. Ostrowski [8] proved amongst others the following result that is somehow a mixture of the Čebyšev and Grüss results

$$|T(f,g)|\le\frac{1}{8}(b-a)(M-m)\|g'\|_\infty, \tag{1.5}$$

provided $f$ is Lebesgue integrable on $[a,b]$ and satisfying (1.3) while $g:[a,b]\to\mathbb{R}$ is absolutely continuous and $g'\in L_\infty[a,b]$. The constant $\frac{1}{8}$ in (1.5) is also sharp.

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In 1973, [4], A. Lupaş pointed out another inequality in terms of the Euclidean norms of $f', g'$

$$|T(f,g)| \leq \frac{1}{\pi^2}(b-a)\|f'\|_2 \|g'\|_2, \tag{1.6}$$

where $\|f'\|_2 := \left(\int_a^b |f'(t)|^2 dt\right)^{\frac{1}{2}}$, provided $f, g$ are absolutely continuous and $f', g' \in L_2[a,b]$. The quantity $\frac{1}{\pi^2}$ is best possible in the sense that it cannot be replaced by a smaller constant.

In this paper, some other inequalities in terms of the derivatives of $f, g$ are pointed out. Applications for Taylor's expansion and the generalised trapezoid formula are also provided.

## 2. Some Bounds for the Čebyšev Functional

The following lemma holds.

**Lemma 1.** *If $\varphi : [a,b] \to \mathbb{R}$ is an absolutely continuous function with*

$$(\cdot - a)(b - \cdot)(\varphi')^2 \in L[a,b],$$

*then we have the inequality*

$$T(\varphi,\varphi) \leq \frac{1}{2(b-a)} \int_a^b (x-a)(b-x)[\varphi'(x)]^2 dx. \tag{2.1}$$

*The constant $\frac{1}{2}$ is best possible.*

*Proof.* By Korkine's identity represented by (see [6, p. 242]),

$$T(f,g) = \frac{1}{2(b-a)^2} \int_a^b \int_a^b (f(x) - f(y))(g(x) - g(y))\, dxdy \tag{2.2}$$

we have

$$T(\varphi,\varphi) = \frac{1}{2(b-a)^2} \int_a^b \int_a^b (\varphi(t) - \varphi(s))^2 dtds.$$

The same identity applied for $\ell(x) = x$, $g$ an integrable function, produces

$$T(\ell,g) = \frac{1}{2(b-a)^2} \int_a^b \int_a^b (x-y)(g(x) - g(y))\, dxdy. \tag{2.3}$$

Sonin's identity given by (see [6, p. 246]),

$$T(f,g) = \frac{1}{b-a} \int_a^b \left(f(x) - \frac{1}{b-a}\int_a^b f(y)\, dy\right)(g(x) - \gamma)\, dx, \gamma \in \mathbb{R},$$

produces

$$T(\ell,g) = \frac{1}{b-a} \int_a^b \left(x - \frac{a+b}{2}\right) g(x)\, dx.$$

Integrating by parts, we have

$$\int_a^b \left(x - \frac{a+b}{2}\right) g(x)\, dx = \frac{1}{2}\int_a^b (x-a)(b-x)\, g'(x)\, dx$$

to give (see also [8, p. 366])

$$T(\ell,g) = \frac{1}{2(b-a)} \int_a^b (x-a)(b-x)\, g'(x)\, dx. \tag{2.4}$$

Since $\varphi$ is absolutely continuous, $\varphi(t)-\varphi(s)=\int_s^t \varphi'(u)\,du$, and by the Cauchy-Schwarz inequality, we have from (2.2)

$$\begin{aligned}
T(\varphi,\varphi) &= \frac{1}{2(b-a)^2}\int_a^b\int_a^b (t-s)^2\left(\frac{\varphi(t)-\varphi(s)}{t-s}\right)^2 dtds \\
&= \frac{1}{2(b-a)^2}\int_a^b\int_a^b (t-s)^2\left(\frac{\int_s^t \varphi'(u)\,du}{t-s}\right)^2 dtds \\
&\leq \frac{1}{2(b-a)^2}\int_a^b\int_a^b (t-s)^2\left(\frac{1}{t-s}\int_s^t [\varphi'(u)]^2\,du\right) dtds \\
&= \frac{1}{2(b-a)^2}\int_a^b\int_a^b (t-s)\left(\int_s^t [\varphi'(u)]^2\,du\right) dtds \\
&\left(\text{by (2.3) and (2.4) for } g(x)=\int_a^x [\varphi'(u)]^2\,du\right) \\
&= \frac{1}{2(b-a)}\int_a^b (u-a)(b-u)[\varphi'(u)]^2\,du,
\end{aligned}$$

and the inequality (2.1) is proved.

To prove the sharpness of the constant $\frac{1}{2}$, assume that (2.1) holds with a constant $C>0$, namely,

$$T(\varphi,\varphi)\leq \frac{C}{b-a}\int_a^b (x-a)(b-x)[\varphi'(x)]^2\,dx. \tag{2.5}$$

If we choose $\varphi(x)=x$, then we observe that

$$T(\varphi,\varphi)=\frac{(b-a)^2}{12},$$

$$\frac{1}{b-a}\int_a^b (x-a)(b-x)[\varphi'(x)]^2\,dx=\frac{(b-a)^2}{6}$$

and by (2.5) we deduce $C\geq\frac{1}{2}$. ∎

**Remark 1.** *The inequality (2.1) in an equivalent form, with $a=0$ and $b=1$ was obtained by Ostrowski in* [8, p. 372]*. However, he did not consider the sharpness of the constant $\frac{1}{2}$.*

The following Grüss type inequality holds.

**Theorem 1.** *Let $f,g:[a,b]\to\mathbb{R}$ be two absolutely continuous functions on $[a,b]$ with $(\cdot-a)(b-\cdot)[f']^2,\ (\cdot-a)(b-\cdot)[g']^2\in L[a,b]$. Then we have the inequality*

$$|T(f,g)|\le\frac{1}{\sqrt{2}}[T(f,f)]^{\frac{1}{2}}\frac{1}{\sqrt{b-a}}\left(\int_a^b(x-a)(b-x)[g'(x)]^2dx\right)^{\frac{1}{2}} \tag{2.6}$$
$$\le\frac{1}{2(b-a)}\left(\int_a^b(x-a)(b-x)[f'(x)]^2dx\right)^{\frac{1}{2}}\times\left(\int_a^b(x-a)(b-x)[g'(x)]^2dx\right)^{\frac{1}{2}}.$$

*The constant $\frac{1}{\sqrt{2}}$ and $\frac{1}{2}$ are best possible in (2.6).*

The proof follows by (2.1) and by the fact that, using Korkine's identity and Cauchy-Schwartz's inequality for double integrals,

$$(T(f,g))^2\le T(f,f)\,T(g,g).$$

We omit the details.

The following inequality of Grüss type holds.

**Theorem 2.** *Assume that $g:[a,b]\to\mathbb{R}$ is monotonic nondecreasing on $[a,b]$ and $f:[a,b]\to\mathbb{R}$ is absolutely continuous with $f'\in L_\infty[a,b]$. Then we have the inequality*

$$|T(f,g)|\le\frac{1}{2(b-a)}\|f'\|_\infty\int_a^b(x-a)(b-x)\,dg(x). \tag{2.7}$$

*The constant $\frac{1}{2}$ is best possible.*

*Proof.* We have, by Korkine's identity, that

$$\begin{aligned}|T(f,g)|&=\frac{1}{2(b-a)^2}\left|\int_a^b\int_a^b(f(x)-f(y))(g(x)-g(y))\,dxdy\right|\\&\le\frac{1}{2(b-a)^2}\int_a^b\int_a^b\left|\frac{f(x)-f(y)}{x-y}\right||(x-y)(g(x)-g(y))|\,dxdy\\&\le\frac{\|f'\|_\infty}{2(b-a)^2}\int_a^b\int_a^b|(x-y)(g(x)-g(y))|\,dxdy\\&=\frac{\|f'\|_\infty}{2(b-a)^2}\int_a^b\int_a^b(x-y)(g(x)-g(y))\,dxdy\\&=\|f'\|_\infty T(\ell,g),\end{aligned}$$

where $\ell(x)=x$, $x\in[a,b]$.

Using the following identity obtained by Ostrowski in [8, p. 366] for the monotonic function $y:[a,b]\to\mathbb{R}$

$$T(\ell,y)=\frac{1}{2(b-a)}\int_a^b(b-x)(x-a)\,dy(x),$$

that may easily be proved on applying the integration by parts formula for Stieltjes integrals, we deduce (2.7).

Now for the sharpness. Assume that (2.7) holds with a constant $D > 0$, that is

$$|T(f,g)| \leq \frac{D}{b-a} \|f'\|_{\infty} \int_a^b (x-a)(b-x)\,dg(x). \tag{2.8}$$

If we choose $f(x) = g(x) = x$, $x \in [a,b]$, then obviously

$$T(f,f) = \frac{1}{12}(b-a)^2, \quad \|f'\|_{\infty} = 1,$$
$$\frac{1}{b-a}\int_a^b (x-a)(b-x)\,dx = \frac{1}{6}(b-a)^2$$

and so, by (2.8), we deduce $D \geq \frac{1}{2}$. ∎

**Remark 2.** *If, in addition to the hypotheses of Theorem 2, we assume that the function $g$ is absolutely continuous on $[a,b]$ and $g' \in L_{\infty}[a,b]$, then*

$$\begin{aligned}\int_a^b (x-a)(b-x)\,dg(x) &= \int_a^b (x-a)(b-x)\,g'(x)\,dx \\ &\leq \|g'\|_{\infty}\int_a^b (x-a)(b-x)\,dx \\ &= \frac{1}{6}(b-a)^3\|g'\|_{\infty}\end{aligned}$$

*providing the following refinement of the Čebyšev result (1.4)*

$$\begin{aligned}|T(f,g)| &\leq \frac{1}{2(b-a)}\|f'\|_{\infty}\int_a^b (x-a)(b-x)\,dg(x) \\ &\leq \frac{1}{12}\|f'\|_{\infty}\|g'\|_{\infty}(b-a)^2.\end{aligned} \tag{2.9}$$

*In (2.9) the constants $\frac{1}{2}$ and $\frac{1}{12}$ are best possible.*

Another result of this type is incorporated in the following.

**Theorem 3.** *Assume that $f, g : [a,b] \to \mathbb{R}$ are continuous on $[a,b]$ and differentiable on $[a,b]$ with $g'(t) \neq 0$ for each $t \in (a,b)$. Then we have the inequality*

$$\begin{aligned}|T(f,g)| &\leq \left\|\frac{f'}{g'}\right\|_{\infty} \cdot T(g,g) \\ &\leq \frac{1}{2(b-a)}\left\|\frac{f'}{g'}\right\|_{\infty}\int_a^b (x-a)(b-x)\left[g'(x)\right]^2 dx.\end{aligned} \tag{2.10}$$

*The first inequality in (2.10) and the constant $\frac{1}{2}$ in the second inequality are sharp.*

*Proof.* Applying Cauchy's mean value theorem, for any $t, s \in [a,b]$, with $t \neq s$, there is an $\eta$ between $t$ and $s$ such that

$$[f(t) - f(s)]\,g'(\eta) = [g(t) - g(s)]\,f'(\eta),$$

and thus

$$\left|\frac{f(t)-f(s)}{g(t)-g(s)}\right| \leq \left\|\frac{f'}{g'}\right\|_{\infty}$$

for any $t, s \in [a,b]$ with $t \neq s$.

Using Korkine's identity (2.2), we deduce

$$\begin{aligned}|T(f,g)| &\leq \frac{1}{2(b-a)^2}\int_a^b\int_a^b |(f(t)-f(s))(g(t)-g(s))|\,dtds \\ &= \frac{1}{2(b-a)^2}\int_a^b\int_a^b \left|\frac{f(t)-f(s)}{g(t)-g(s)}\right|(g(t)-g(s))^2\,dtds \\ &\leq \frac{1}{2(b-a)^2}\left\|\frac{f'}{g'}\right\|_\infty \int_a^b\int_a^b (g(t)-g(s))^2\,dtds \\ &= \left\|\frac{f'}{g'}\right\|_\infty T(g,g),\end{aligned}$$

and the first inequality in (2.10) follows.

The second inequality is obvious by Lemma 1.

The sharpness of the inequalities may be proved in a similar way as in Theorem 2 and we omit the details. ∎

## 3. Applications for Taylor's Expansion

Let $I \subset \mathbb{R}$ be a closed interval, let $a \in I$ and let $n$ be a positive integer. If $f : I \to \mathbb{R}$ is such that $f^{(n)}$ is absolutely continuous, then for each $x \in I$

$$f(x) = T_n(f;a,x) + R_n(f;a,x), \tag{3.1}$$

where $T_n(f;a,x)$ is Taylor's polynomial, i.e.,

$$T_n(f;a,x) = \sum_{k=0}^{n} \frac{(x-a)^k}{k!} f^{(k)}(a) \tag{3.2}$$

(note that $f^{(0)} = f$ and $0! = 1$), and the remainder is given by

$$R_n(f;a,x) = \frac{1}{n!}\int_a^x (x-t)^n f^{(n+1)}(t)\,dt. \tag{3.3}$$

We note that, many authors have considered recently different perturbations for Taylor's formula and pointed out bounds for the remainders that are, some times, better than the ones provided in the classical case. The reader may consult for example [5], [1] and the references therein. This motivates our interest to apply the Grüss type inequalities obtained before in pointing out different bounds for the remainder in the perturbed Taylor's formula below.

Using Theorem 1, we may point out the following perturbation of the Taylor's expansion.

**Theorem 4.** *Let $f : I \to \mathbb{R}$ be such that $f^{(n+1)}$ is absolutely continuous and $a \in I$. Then we have the perturbed Taylor's formula:*

$$f(x) = T_n(f;a,x) + \frac{(x-a)^{n+1}}{(n+1)!}\left[f^{(n)};a,x\right] + G_n(f;a,x) \tag{3.4}$$

*and the remainder $G_n(f;a,x)$ satisfies the estimation*

$$\begin{aligned}&|G_n(f;a,x)| \\ &\leq \frac{1}{\sqrt{2}}\cdot\frac{n}{(n+1)!}|x-a|^{n+\frac{1}{2}}\left|\int_a^x (t-a)(x-t)\left[f^{(n+2)}(t)\right]^2 dt\right|^{\frac{1}{2}}\end{aligned} \tag{3.5}$$

*for any* $x \in I$, *where*

$$\left[f^{(n)};a,x\right] = \frac{f^{(n)}(x) - f^{(n)}(a)}{x-a},$$

*is the divided difference.*

*Proof.* If we apply Theorem 1 for $f \to (x-\cdot)^n$, $g \to f^{(n+1)}$, we deduce

$$\begin{aligned}(3.6)\qquad & \left|\frac{1}{x-a}\int_a^x (x-t)^n f^{(n+1)}(t)\,dt\right.\\ &\qquad \left. - \frac{1}{x-a}\int_a^x (x-t)^n\,dt \cdot \frac{1}{x-a}\int_a^x f^{(n+1)}(t)\,dt\right|\\ &\leq \frac{1}{\sqrt{2}}\left[T\left((x-\cdot)^n,(x-\cdot)^n\right)\right]^{\frac{1}{2}}\\ &\qquad \times \frac{1}{\sqrt{x-a}}\left|\int_a^x (t-a)(x-t)\left[f^{(n+2)}(t)\right]^2 dt\right|^{\frac{1}{2}}\\ &= \frac{1}{\sqrt{2}}\cdot\frac{n}{n+1}\left|x-a\right|^{n-\frac{1}{2}}\left|\int_a^x (t-a)(x-t)\left[f^{(n+2)}(t)\right]^2 dt\right|^{\frac{1}{2}},\end{aligned}$$

since

$$\begin{aligned}T\left((x-\cdot)^n,(x-\cdot)^n\right) &= \frac{1}{x-a}\int_a^x (x-t)^{2n}\,dt - \left[\frac{1}{x-a}\int_a^x (x-t)^n\,dt\right]^n\\ &= \frac{n^2}{(n+1)^2}(x-a)^{2n}.\end{aligned}$$

Using (3.1) and (3.6), we deduce the representation (3.4) and the bound (3.5). ∎

The following result also holds.

**Theorem 5.** *Let* $f : I \to \mathbb{R}$ *be such that* $f^{(n+1)}$ *is absolutely continuous and* $f^{(n+1)} \geq 0$ *on* $I$. *If* $a \in I$, *then we have the representation (3.4) and the remainder* $G_n(f;a,x)$ *satisfies the bound*

$$(3.7)\qquad \left|G_n(f;a,x)\right| \leq \frac{|x-a|^n}{(n-1)!}\left\{\frac{f^{(n)}(x)+f^{(n)}(a)}{2} - \left[f^{(n-1)};a,x\right]\right\},$$

*for any* $x \in I$.

*Proof.* We apply Theorem 2 for $f \to (x-\cdot)^n$ and $g \to f^{(n+1)}$, to get

$$\begin{aligned}(3.8)\qquad & \left|\frac{1}{x-a}\int_a^x (x-t)^n f^{(n+1)}(t)\,dt\right.\\ &\qquad \left. - \frac{1}{x-a}\int_a^x (x-t)^n\,dt \cdot \frac{1}{x-a}\int_a^x f^{(n+1)}(t)\,dt\right|\\ &\qquad \leq \frac{1}{2|x-a|}n\left|x-a\right|^{n-1}\left|\int_a^x (t-a)(x-t)f^{(n+2)}(t)\,dt\right| := K.\end{aligned}$$

Since

$$\begin{aligned}\int_a^x (t-a)(x-t)f^{(n+2)}(t)\,dt &= \int_a^x f^{(n+1)}(t)\left[2t-(a+x)\right]dt\\ &= (x-a)\left[f^{(n)}(x)+f^{(n)}(a)\right] - 2\left(f^{(n-1)}(x)-f^{(n-1)}(a)\right),\end{aligned}$$

then

$$K = n|x-a|^{n-1}\left\{\frac{f^{(n)}(x)+f^{(n)}(a)}{2} - \left[f^{(n-1)};a,x\right]\right\}.$$

Using the representation (3.1) and the inequality (3.8), we deduce (3.7). ∎

Finally, we may point out the following result as well.

**Theorem 6.** *Let $f: I \to \mathbb{R}$ be such that $f^{(n+1)}$ is absolutely continuous. If $a, x \in I$ and there exists a constant $M(x)$ such that*

$$\left|f^{(n+2)}(t)\right| \leq M(x)|x-t|^{n-1} \quad \text{for} \quad t \in [a,x] \ \ ([x,a]), \tag{3.9}$$

*then we have the representation (3.4) and the remainder $G_n(f;a,x)$ satisfies the estimate*

$$|G_n(f;a,x)| \leq \frac{n}{n+1} \cdot \frac{1}{(n+1)!} M(x)|x-a|^{2n+1}. \tag{3.10}$$

*Proof.* We apply Theorem 3 for $g \to (x-\cdot)^n$ and $f \to f^{(n+1)}$, to get

$$\begin{aligned} &\left|\frac{1}{x-a}\int_a^x (x-t)^n f^{(n+1)}(t)\,dt \right.\\ &\qquad\left. - \frac{1}{x-a}\int_a^x (x-t)^n\,dt \cdot \frac{1}{x-a}\int_a^x f^{(n+1)}(t)\,dt\right| \\ &\leq \frac{M(x)}{n} \cdot T\left((x-\cdot)^n, (x-\cdot)^n\right) \\ &= \frac{M(x)}{n} \cdot \frac{n^2}{(n+1)^2}(x-a)^{2n}, \end{aligned} \tag{3.11}$$

giving the desired result (3.9). ∎

## 4. Applications for the Generalised Trapezoid Formula for $n-$Time Differentiable Functions

Let $f : [a,b] \to \mathbb{R}$ be a function such that the derivative $f^{(n-1)}$ $(n \geq 1)$ is absolutely continuous on $[a,b]$. In [2], the authors have obtained the following generalisation of the trapezoid formula:

$$\begin{aligned} \int_a^b f(t)\,dt = \sum_{k=0}^{n-1} \frac{1}{(k+1)!}\left[(x-a)^{k+1} f^{(k)}(a) + (-1)^k (b-x)^{k+1} f^{(k)}(b)\right] \\ + \frac{1}{n!}\int_a^b (x-t)^n f^{(n)}(t)\,dt. \end{aligned} \tag{4.1}$$

The following perturbed version of (4.1) holds.

**Theorem 7.** *Let $f : [a,b] \to \mathbb{R}$ be such that $f^{(n)}$ is absolutely continuous on $[a,b]$. Then we have the representation*

$$\begin{aligned} \int_a^b f(t)\,dt = \sum_{k=0}^{n-1} \frac{1}{(k+1)!}\left[(x-a)^{k+1} f^{(k)}(a) + (-1)^k (b-x)^{k+1} f^{(k)}(b)\right] \\ + \frac{(x-a)^{n+1} + (-1)^n (b-x)^{n+1}}{(n+1)!}\left[f^{(n-1)};a,b\right] + S_n(f,x), \end{aligned} \tag{4.2}$$

*where the remainder* $S_n(f,x)$ *satisfies the estimate*

$$|S_n(f,x)| \le \frac{1}{\sqrt{2}} \cdot \frac{1}{n!} \cdot \sqrt{b-a}\,[B_n(x)]^{\frac{1}{2}} \times \left(\int_a^b (t-a)(b-t)\left[f^{(n+1)}(t)\right]^2 dt\right)^{\frac{1}{2}}, \tag{4.3}$$

*and* $B_n(x)$ *is defined by*

$$B_n(x) = \frac{(b-x)^{2n+1} + (x-a)^{2n+1}}{(b-a)(2n+1)} - \left[\frac{(b-x)^{n+1} + (-1)^n (x-a)^{n+1}}{(n+1)(b-a)}\right]^2 \tag{4.4}$$

*for any* $x \in [a,b]$.

*Proof.* If we apply Theorem 1 for $f \to (x-\cdot)^n$ and $g \to f^{(n)}$, we obtain

$$\left|\frac{1}{b-a}\int_a^b (x-t)^n f^{(n)}(t)\,dt - \frac{1}{b-a}\int_a^b (x-t)^n dt \cdot \frac{1}{b-a}\int_a^b f^{(n)}(t)\,dt\right| \le \frac{1}{\sqrt{2}}\left[T\left((x-\cdot)^n,(x-\cdot)^n\right)\right]^{\frac{1}{2}} \times \frac{1}{\sqrt{b-a}}\left(\int_a^b (t-a)(b-t)\left[f^{(n+1)}(t)\right]^2 dt\right)^{\frac{1}{2}}. \tag{4.5}$$

Since

$$T\left((x-\cdot)^n,(x-\cdot)^n\right) = \frac{1}{b-a}\int_a^b (x-t)^{2n} dt - \left(\frac{1}{b-a}\int_a^b (x-t)^n dt\right)^2 = B_n(x),$$

then, by (4.1) and (4.5) we deduce the representation (4.2) and the bound (4.3). We omit the details. ∎

It is natural to consider the following particular case.

**Corollary 1.** *With the assumptions in Theorem 7, we have*

$$\int_a^b f(t)\,dt = \sum_{k=0}^{n-1} \frac{1}{(k+1)!}\left(\frac{b-a}{2}\right)^{k+1}\left[f^{(k)}(a) + (-1)^k f^{(k)}(b)\right] + \left(\frac{b-a}{2}\right)^n \frac{\left[1+(-1)^n\right]}{(n+1)!}\left[f^{(n-1)};a,b\right] + S_n(f) \tag{4.6}$$

*and the remainder* $S_n(f)$ *satisfies the bound:*

$$|S_n(f)| \le \frac{1}{n!} \cdot \frac{(b-a)^{n+\frac{1}{2}}}{2^{n+\frac{1}{2}}}\left[\frac{1}{2n+1} - \frac{\left[1+(-1)^n\right]^2}{(n+1)^2}\right]^{\frac{1}{2}} \times \left(\int_a^b (t-a)(b-t)\left[f^{(n+1)}(t)\right]^2 dt\right)^{\frac{1}{2}}. \tag{4.7}$$

The following result also holds.

**Theorem 8.** *Assume that* $f^{(n)}$ $(n \geq 2)$ *is absolutely continuous and* $f^{(n+1)} \geq 0$ *on* $[a,b]$. *Then we have the representation (4.2) and the remainder* $S_n(f,x)$ *satisfies the estimate*

$$|S_n(f,x)| \leq \frac{1}{(n-1)!}\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]^{n-1} \times (b-a)\left\{\frac{f^{(n-1)}(a)+f^{(n-1)}(b)}{2}-\left[f^{(n-2)};b,a\right]\right\} \tag{4.8}$$

*for any* $x \in [a,b]$.

*Proof.* If we use in Theorem 2 for $f \to (x-\cdot)^n$ and $g \to f^{(n)}$, we get

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b (x-t)^n f^{(n)}(t)\,dt - \frac{1}{b-a}\int_a^b (x-t)^n\,dt \cdot \frac{1}{b-a}\int_a^b f^{(n)}(t)\,dt\right| \\ &\leq \frac{1}{2(b-a)} n\left[\max(x-a,b-x)\right]^{n-1}\int_a^b (t-a)(b-t) f^{(n+1)}(t)\,dt \\ &= n\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]^{n-1} \times \left\{\frac{f^{(n-1)}(a)+f^{(n-1)}(b)}{2}-\left[f^{(n-2)};b,a\right]\right\}.\end{aligned} \tag{4.9}$$

Using (4.1) and (4.9) we deduce (4.8). ∎

**Corollary 2.** *With the assumptions in Theorem 8, we have the representation (4.6). The remainder* $S_n(f)$ *satisfies the bound*

$$|S_n(f)| \leq \frac{1}{2^{n-1}(n-1)!}(b-a)^n \times \left\{\frac{f^{(n-1)}(a)+f^{(n-1)}(b)}{2}-\left[f^{(n-2)};b,a\right]\right\}.$$

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