VICTORIA UNIVERSITY

MELBOURNE AUSTRALIA

Further Bounds for the Estimation Error Variance of a Continuous Stream with Stationary Variogram

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FURTHER BOUNDS FOR THE ESTIMATION ERROR VARIANCE OF A CONTINUOUS STREAM WITH STATIONARY VARIOGRAM

N. S. BARNETT, S. S. DRAGOMIR, AND I. S. GOMM

ABSTRACT. In this paper we establish an upper bound for the estimation error variance of a continuous stream with a stationary variogram V which is assumed to be of r-Hölder type (Lipschitzian) on $[-d, d]$.

1. INTRODUCTION

In [1], the authors considered $X(t)$ as defining the quality of a product at time t where $X(t)$ is a continuous time stochastic process which may be non-stationary. Typically, $X(t)$ represents a continuous stream industrial process such as is common in many areas of the chemical industry.

The paper [1] was concerned with issues related to sampling the stream with a view to estimating the mean quality characteristic of the flow, $\bar{X}$, over the interval $[0, d]$. Specifically, focus was on obtaining the sampling location, said to be optimal, which minimizes the estimation error variance, $E\left[\left(\bar{X} - X(t)\right)^2\right]$, $0 \le t \le d$.

Given that t is as specified, the problem is to find the value of t (the sampling location) that minimizes $E\left[\left(\bar{X} - X(t)\right)^2\right]$. It is shown that for constant stream flows, the optimal sampling point is the midpoint of $[0, d]$ for situations where the process variogram,

$$V(u) = \frac{1}{2} E\left[\left(\bar{X} - X(t)\right)^2\right],$$

where

$$V(0) = 0, \ V(-u) = V(u)$$

is stationary (note that variogram stationarity is not equivalent to process stationarity).

The paper [1] continues to consider optimal sampling locations for situations where the stream flow rate varies. The optimal sampling location is seen to depend on both the flow rate function and the form of the process variogram - some examples are given.

In [2], rather than focussing on the optimal sampling point, the authors have focussed on the actual value of the estimation error variance itself. They obtained the following result by employing an inequality of the Ostrowski type for double integrals.

Theorem 1. *Let $V : (-d, d) \to \mathbb{R}$ be a twice differentiable variogram having the second derivative $V'' : (-d, d) \to \mathbb{R}$ which is bounded. If $\|V''\|_\infty := \sup_{t \in (-d,d)} |V''(t)| < \infty$, then*

$$(1.1) \qquad E\left[\left(\bar{X} - X(t)\right)^2\right] \leq \left[\frac{1}{4} + \frac{\left(t - \frac{d}{2}\right)^2}{d^2}\right]^2 \|V''\|_\infty$$

for all $t \in [0, d]$.

Note that the best inequality we can get from (1.1) is that one for which $t = t_0 = \frac{d}{2}$ giving the bound

$$E\left[\left(\bar{X} - X(t_0)\right)^2\right] \leq \frac{d^2}{16} \|V''\|_\infty.$$

It should be noted that the above result requires double differentiability of V in $(-d, d)$ and that this condition does not hold for the case of a linear variogram. That is,

$$V(u) = a|u|, \ u \in \mathbb{R}.$$

For other results on Ostrowski's inequality we refer to the recent papers [3]- [7] and the book [8].

In this note we point out another bound for the estimation error variance which does not require the differentiability of V. Some functional properties are also given.

2. The Results

Firstly, let us recall the concept of r-Hölder type mappings.

Definition 1. *The mapping $f : [a, b] \subset \mathbb{R} \to \mathbb{R}$ is called of r-Hölder type with $r \in (0, 1]$ if*

$$(2.1) \qquad |f(x) - f(y)| \leq H|x - y|^r$$

for all $x, y \in [a, b]$ with a certain $H > 0$.
If $r = 1$, we get the classical concept of Lipschitzian mappings.

Example 1. *If $r \in (0, 1]$, then the mapping $f(x) = x^r$ satisfies the condition*

$$(2.2) \qquad |f(x) - f(y)| = |x^r - y^r| \leq |x - y|^r \text{ for all } x, y \in [0, \infty),$$

which shows that f is of r-Hölder type with the constant $H = 1$, on every closed interval $[a, b]$.

Example 2. *Any differentiable mapping $f : [a, b] \to \mathbb{R}$ having the derivative bounded in (a, b) is Lipschitzian on (a, b).*

The following result holds.

Theorem 2. *Assume that the variogram $V : [-d, d] \to \mathbb{R}$ is of r-Hölder type on $[-d, d]$ with the constant $H > 0$. Then we have the inequality*

$$(2.3) \qquad E\left[\left(\bar{X} - X(t)\right)^2\right] \leq \frac{2H}{d}\left[\frac{t^{r+1} + (d - t)^{r+1}}{r + 1}\right] \leq \frac{2Hd}{r + 1}$$

for all $t \in [0, d]$.

Proof. From [1], using an identity given in [9], it can be shown that

$$
\begin{aligned}
0 \;\le\; & E\left[\left(\bar{X} - X\left(t\right)\right)^2\right] \\
= \;& \frac{2}{d}\left\{\int_0^t V\left(u\right) du + \int_0^{d-t} V\left(u\right) du\right\} - \frac{1}{d^2}\int_0^b\int_0^d V\left(v - u\right) dudv.
\end{aligned}
$$

Also, observe that (see [1])

$$
\int_0^d V\left(v - t\right) dv = \int_0^t V\left(u\right) du + \int_0^{d-t} V\left(u\right) du
$$

and

$$
\int_0^d V\left(t - u\right) du = \int_0^t V\left(u\right) du + \int_0^{d-t} V\left(u\right) du
$$

and then we get the identity

$$
\begin{aligned}
(2.4) \qquad 0 \;\le\; & E\left[\left(\bar{X} - X\left(t\right)\right)^2\right] \\
= \;& \frac{1}{d}\int_0^d V\left(v - t\right) dv + \frac{1}{d}\int_0^d V\left(t - u\right) du \\
= \;& \frac{1}{d^2}\left[d\int_0^d V\left(v - t\right) dv + d\int_0^d V\left(t - u\right) du\right. \\
& \left. - \int_0^b\int_0^d V\left(v - u\right) dvdu\right] \\
= \;& \frac{1}{d^2}\int_0^b\int_0^d \left[V\left(v - t\right) + V\left(t - u\right) - V\left(v - u\right)\right] dvdu.
\end{aligned}
$$

Using the fact that V is of $r-$Hölder type, we can write that

$$
\begin{aligned}
(2.5) \qquad \left|V\left(v - t\right) - V\left(v - u\right)\right| \;\le\;& H\left|v - t - v + u\right|^r \\
= \;& H\left|u - t\right|^r
\end{aligned}
$$

for all $u, v, t \in [0, d]$ and

$$
(2.6) \qquad \left|V\left(t - u\right)\right| = \left|V\left(t - u\right) - V\left(0\right)\right| \le H\left|t - u\right|^r
$$

for all $t, x \in [0, d]$.

Now, using $(2.4) - (2.6)$, we get

$$
\begin{aligned}
E\left[\left(\bar{X} - X\left(t\right)\right)^2\right] &= \left| \frac{1}{d^2} \int_0^d \int_0^d \left[V\left(v - t\right) + V\left(t - u\right) - V\left(v - u\right)\right] dv du \right| \\
&\leq \frac{1}{d^2} \int_0^d \int_0^d \left| V\left(v - t\right) - V\left(v - u\right) + V\left(t - u\right) \right| dv du \\
&\leq \frac{1}{d^2} \int_0^d \int_0^d \left| V\left(v - t\right) - V\left(v - u\right) \right| + \left| V\left(t - u\right) \right| dv du \\
&\leq \frac{1}{d^2} \int_0^d \int_0^d \left[H \left|t - u\right|^r + H \left|t - u\right|^r\right] dv du \\
&= \frac{2H}{d} \int_0^d \left|t - u\right|^r du \\
&= \frac{2H}{d} \left[\int_0^t \left(t - u\right)^r du + \int_t^d \left(u - t\right)^r du \right] \\
&= \frac{2H}{d} \left[\frac{t^{r+1} + \left(d - t\right)^{r+1}}{r + 1} \right]
\end{aligned}
$$

and the first inequality in (2.3) is proved. The second part is obvious. ∎

Corollary 1. *If V is Lipschitzian with the constant $L > 0$, then we have the inequality:*

$$
(2.7) \qquad E\left[\left(\bar{X} - X\left(t\right)\right)^2\right] \leq \left[\frac{1}{2} + \frac{\left(t - \frac{d}{2}\right)^2}{d^2} \right] Ld.
$$

Proof. Choose $r = 1$ to get in the right hand side of the inequality

$$
\frac{t^2 + \left(d - t\right)^2}{2} = \left[\frac{1}{4} + \frac{\left(t - \frac{d}{2}\right)^2}{d^2} \right] d^2.
$$

Then, by (2.3), we deduce (2.7). ∎

Remark 1. *It is easy to see that the mapping $g : [0, d] \to \mathbb{R}$, $g\left(t\right) := t^{r+1} + \left(d - t\right)^{r+1}$ has the properties*

$$
\inf_{t \in [0, d]} g\left(t\right) = g\left(\frac{d}{2}\right) = \frac{d^{r+1}}{2^r}
$$

and

$$
\sup_{t \in [0, d]} g\left(t\right) = g\left(0\right) = g\left(d\right) = d^{r+1}
$$

which shows that the best inequality we can get from (2.3) is that one for which $t = t_0 = \frac{d}{2}$, getting

$$
(2.8) \qquad E\left[\left(\bar{X} - X\left(t_0\right)\right)^2\right] \leq \frac{2^{1-r} H d^r}{r + 1}.
$$

For the Lipschitzian case, we get

$$
(2.9) \qquad E\left[\left(\bar{X} - X\left(t_0\right)\right)^2\right] \leq \frac{1}{2} Ld.
$$

Define the mapping $\xi : [0, d] \to \mathbb{R}$ given by

$$\xi(t) = E\left[\left(\bar{X} - X(t)\right)^2\right].$$

The following property of continuity for ξ holds.

Theorem 3. *If V is of r-Hölder type with the constant $H > 0$ on the interval $[0, d]$, then ξ is of r-Hölder type with the constant $2H$.*

Proof. Let $t_1, t_2 \in [0, d]$. Then we have

$$
\begin{aligned}
&|\xi(t_2) - \xi(t_1)| \\
=\ & \left| \frac{1}{d^2} \int_0^d \int_0^d [V(v - t_2) + V(t_2 - u) - V(v - u)]\, du\, dv \right. \\
& \left. - \frac{1}{d^2} \int_0^d \int_0^d [V(v - t_1) + V(t_1 - u) - V(v - u)]\, du\, dv \right| \\
=\ & \left| \frac{1}{d^2} \int_0^d \int_0^d [(V(v - t_2) - V(v - t_1)) + (V(t_2 - u) - V(t_1 - u))]\, du\, dv \right| \\
\leq\ & \frac{1}{d^2} \int_0^d \int_0^d [|V(v - t_2) - V(v - t_1)| + |V(t_2 - u) - V(t_1 - u)|]\, du\, dv \\
\leq\ & \frac{1}{d^2} \int_0^d \int_0^d [H\, |t_2 - t_1|^r + H\, |t_2 - t_1|^r]\, du\, dv \\
=\ & \frac{2H\, |t_2 - t_1|^r\, d^2}{d^2} \\
=\ & 2H\, |t_2 - t_1|^r
\end{aligned}
$$

and the theorem thus proved. ∎

Corollary 2. *If V is L-Lipschitzian on $[0, d]$, then ξ is $2L$-Lipschitzian on $[0, d]$.*

The following result concerning the convexity property of the mapping ξ defined above on $[0, d]$ holds.

Theorem 4. *If the variogram $V : [-d, d]$ is monotonic nondecreasing on the interval $[0, d]$, then $\xi(\cdot)$ is convex on $[0, d]$.*

Proof. We know that for all $t \in [0, d]$

$$\xi(t) = \frac{2}{d} \left\{ \int_0^t V(u)\, du + \int_0^{d-t} V(u)\, du \right\} - \frac{1}{d^2} \int_0^d \int_0^d V(v - u)\, du\, dv.$$

Then

$$\xi'(t) = \frac{2}{d} [V(t) - V(d - t)].$$

Now, let $t_1, t_2 \in [0, d]$ with $t_2 > t_1$. Then

$$\xi(t_2) - \xi(t_1) - (t_2 - t_1)\xi'(t_1)$$

$$= \frac{2}{d}\left\{\int_0^{t_2} V(u)\,du + \int_0^{d-t_2} V(u)\,du\right\} - \frac{2}{d}\left\{\int_0^{t_1} V(u)\,du + \int_0^{d-t_1} V(u)\,du\right\}$$

$$-\frac{2}{d}[V(t_1) - V(d-t_1)](t_2 - t_1)$$

$$= \frac{2}{d}\left\{\int_{t_1}^{t_2} V(u)\,du - \int_{d-t_2}^{d-t_1} V(u)\,du - (t_2 - t_1)V(t_1) + (t_2 - t_1)V(d-t_1)\right\}.$$

As V is nondecreasing on $[0, d]$, then

$$\int_{t_1}^{t_2} V(u)\,du \ge (t_2 - t_1)V(t_1)$$

and

$$\int_{d-t_2}^{d-t_1} V(u)\,du \le (t_2 - t_1)V(d-t_1)$$

which implies that

$$\xi(t_2) - \xi(t_1) \ge (t_2 - t_1)\xi'(t_1)$$

for all $t_2 > t_1 \in [0, d]$, which shows that the mapping $\xi(\cdot)$ is convex on $[0, d]$. ∎

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School of Communications and Informatics, Victoria University of Technology, PO Box 14428, MC melbourne City, 8001 Victoria, Australia.

E-mail address: {neil, sever, isg}@matilda.vu.edu.au

URL: http://matilda.vu.edu.au/~rgmia

E-mail address: sever@matilda.vu.edu.au