A Discrete Grüss Type Inequality and Applications for the Moments of Random Variables and Guessing Mappings

A DISCRETE GRÜSS TYPE INEQUALITY AND APPLICATIONS FOR THE MOMENTS OF RANDOM VARIABLES AND GUESSING MAPPINGS

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Abstract. A new discrete Grüss type inequality and applications for the moments of random variables and guessing mappings are given.

1. Introduction

In 1935, G. Grüss proved the following integral inequality which gives an approximation of the integral of a product in terms of the product of integrals as follows

$$\left|\frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx-\frac{1}{b-a}\int_a^b f(x)\,dx\cdot\frac{1}{b-a}\int_a^b g(x)\,dx\right| \le \frac{1}{4}(\Phi-\varphi)(\Gamma-\gamma), \tag{1.1}$$

where $f,g:[a,b]\to\mathbb{R}$ are integrable on $[a,b]$ and satisfying the assumption

$$\varphi\le f(x)\le\Phi,\ \gamma\le g(x)\le\Gamma \tag{1.2}$$

for each $x\in[a,b]$ where $\varphi,\Phi,\gamma,\Gamma$ are given real constants.

Moreover, the constant $\frac{1}{4}$ is sharp in the sense that it can not be replaced by a smaller one.

For a simple proof of (1.1) as well as for some other integral inequalities of Grüss' type see Chapter X of the recent book [4] by Mitrinović, Pečarić and Fink.

In 1950, M. Biernacki, H. Pidek and C. Ryll-Nardzewski established the following discrete version of Grüss' inequality [4, Chap. X]:

Theorem 1. *Let $a=(a_1,...,a_n)$, $b=(b_1,...,b_n)$ be two n-tuples of real numbers such that $r\le a_i\le R$ and $s\le b_i\le S$ for $i=1,...,n$. Then one has*

$$\left|\frac{1}{n}\sum_{i=1}^n a_ib_i-\frac{1}{n}\sum_{i=1}^n a_i\cdot\frac{1}{n}\sum_{i=1}^n b_i\right| \le \frac{1}{n}\left\lfloor\frac{n}{2}\right\rfloor\left(1-\frac{1}{n}\left\lfloor\frac{n}{2}\right\rfloor\right)(R-r)(S-s), \tag{1.3}$$

where $\lfloor x\rfloor$ is the integer part of x, $x\in\mathbb{R}$.

A weighted version of Grüss' discrete inequality was proved by J.E. Pečarić in 1979, [4, Chap. X]:

Date: June, 1999.
1991 *Mathematics Subject Classification.* Primary 26D15, 26D20; Secondary 62Xxx.
Key words and phrases. Grüss Inequality, Random Variables, Guessing Mappings.

Theorem 2. *Let a, b and p be three monotonic n-tuples with all elements of p positive. Then*

$$\left|\frac{1}{P_n}\sum_{i=1}^{n}p_ia_ib_i-\frac{1}{P_n}\sum_{i=1}^{n}p_ia_i\cdot\frac{1}{P_n}\sum_{i=1}^{n}p_ib_i\right| \leq |a_n-a_1|\,|b_n-b_1|\max_{1\leq k\leq n-1}\left(\frac{P_k\bar{P}_{k+1}}{P_n^2}\right), \tag{1.4}$$

where $P_n=\sum_{i=1}^{n}p_i$, $\bar{P}_{k+1}=P_n-P_{k+1}$.

In 1981 , A. Lupaş [4, Chap. X] proved some similar results for the first difference of a as follows :

Theorem 3. *Let a, b two monotonic n-tuples in the same sense and p a positive n-tuple. Then*

$$\begin{aligned}
&\min_{1\leq i\leq n-1}|a_{i+1}-a_i|\min_{1\leq i\leq n-1}|b_{i+1}-b_i|\left[\frac{1}{P_n}\sum_{i=1}^{n}i^2p_i-\left(\frac{1}{P_n}\sum_{i=1}^{n}ip_i\right)^2\right] \\
&\leq \frac{1}{P_n}\sum_{i=1}^{n}p_ia_ib_i-\frac{1}{P_n}\sum_{i=1}^{n}p_ia_i\cdot\frac{1}{P_n}\sum_{i=1}^{n}p_ib_i \\
&\leq \max_{1\leq i\leq n-1}|a_{i+1}-a_i|\max_{1\leq i\leq n-1}|b_{i+1}-b_i|\left[\frac{1}{P_n}\sum_{i=1}^{n}i^2p_i-\left(\frac{1}{P_n}\sum_{i=1}^{n}ip_i\right)^2\right].
\end{aligned} \tag{1.5}$$

If there exists the numbers $\bar{a}, \bar{a}_1, r, r_1, (rr_1>0)$ such that $a_k=\bar{a}+kr$ and $b_k=\bar{a}_1+kr_1$, then in (1.5) *the equality holds.*

In this paper we point out some other Grüss type inequalities and apply them for the moments of discrete random variables and for the moments of guessing mappings.

2. The Results

The following inequality of Grüss type holds for sequences of complex numbers:

Theorem 4. *Let a_i, b_i $(i=1,...,n)$ be complex numbers and p_i $(i=1,...,n)$ be a probability distribution, i.e., $p_i\geq 0$ $(i=1,...,n)$ and $\sum_{i=1}^{n}p_i=1$. Then we have the inequality*

$$\left|\sum_{i=1}^{n}p_ia_ib_i-\sum_{i=1}^{n}p_ia_i\sum_{i=1}^{n}p_ib_i\right| \leq \begin{cases}
\max_{k=\overline{1,n-1}}|a_{k+1}-a_k|\max_{i=\overline{1,n}}|b_i|\sum_{i,j=1}^{n}p_ip_j|i-j|, \\[2mm]
n^{\frac{1}{p}}\max_{k=\overline{1,n-1}}|a_{k+1}-a_k|\left(\sum_{i=1}^{n}|b_i|^p\right)^{\frac{1}{p}}\left(\sum_{i,j=1}^{n}p_i^qp_j^q|i-j|^q\right)^{\frac{1}{q}}, \\[2mm]
\text{if } \frac{1}{p}+\frac{1}{q}=1,\ p>1, \\[2mm]
n\max_{k=\overline{1,n-1}}|a_{k+1}-a_k|\sum_{i=1}^{n}|b_i|\max_{i,j=\overline{1,n}}\{p_ip_j|i-j|\}.
\end{cases} \tag{2.1}$$

Proof. First of all, we observe that we have the identity

$$\begin{aligned}
I \quad : \quad &= \sum_{i=1}^{n} p_i a_i b_i - \sum_{i=1}^{n} p_i a_i \sum_{i=1}^{n} p_i b_i \\
&= \sum_{i=1}^{n} p_i a_i b_i - \sum_{j=1}^{n} p_j a_j \sum_{i=1}^{n} p_i b_i \\
&= \sum_{i=1}^{n} p_i b_i \left(a_i - \sum_{j=1}^{n} p_j a_j \right) \\
&= \sum_{i=1}^{n} p_i b_i \sum_{j=1}^{n} (a_i - a_j) \, p_j \\
&= \sum_{i=1}^{n} p_i b_i \left[\sum_{j=1}^{i-1} p_j (a_i - a_j) + \sum_{t=i+1}^{n} p_t (a_i - a_t) \right] \\
&= \sum_{i=1}^{n} p_i b_i \left[\sum_{j=1}^{i-1} p_j (a_i - a_j) - \sum_{t=i+1}^{n} p_t (a_t - a_i) \right] \\
&= \sum_{i=1}^{n} p_i b_i \left[\sum_{j=1}^{i-1} p_j \sum_{k=j}^{i-1} (a_{k+1} - a_k) - \sum_{t=i+1}^{n} p_t \sum_{l=i}^{t-1} (a_{l+1} - a_l) \right],
\end{aligned}$$

as it is easy to see that

$$a_i - a_j = \sum_{k=j}^{i-1} (a_{k+1} - a_k) \;\; (i > j)$$

and

$$a_t - a_i = \sum_{l=i}^{t-1} (a_{l+1} - a_l) \;\; (t > i) \, .$$

Taking the modulus and applying successively the triangle inequality, we can write:

$$\begin{aligned}
|I| \quad &= \quad \left| \sum_{i=1}^{n} p_i b_i \left[\sum_{j=1}^{i-1} p_j \sum_{k=j}^{i-1} (a_{k+1} - a_k) - \sum_{t=i+1}^{n} p_t \sum_{l=i}^{t-1} (a_{l+1} - a_l) \right] \right| \\
&\leq \sum_{i=1}^{n} p_i |b_i| \left| \sum_{j=1}^{i-1} p_j \sum_{k=j}^{i-1} (a_{k+1} - a_k) - \sum_{t=i+1}^{n} p_t \sum_{l=i}^{t-1} (a_{l+1} - a_l) \right| \\
&\leq \sum_{i=1}^{n} p_i |b_i| \left[\left| \sum_{j=1}^{i-1} p_j \sum_{k=j}^{i-1} (a_{k+1} - a_k) \right| + \left| \sum_{t=i+1}^{n} p_t \sum_{l=i}^{t-1} (a_{l+1} - a_l) \right| \right] \\
&\leq \sum_{i=1}^{n} p_i |b_i| \left[\sum_{j=1}^{i-1} p_j \left| \sum_{k=j}^{i-1} (a_{k+1} - a_k) \right| + \sum_{t=i+1}^{n} p_t \left| \sum_{p=i}^{t-1} (a_{l+1} - a_l) \right| \right] \leq
\end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{i=1}^{n} p_i |b_i| \left[\sum_{j=1}^{i-1} p_j \sum_{k=j}^{i-1} |a_{k+1} - a_k| + \sum_{t=i+1}^{n} p_t \sum_{l=i}^{t-1} |a_{l+1} - a_l| \right] \\
&\leq \sum_{i=1}^{n} p_i |b_i| \left[\max_{k=\overline{1,n-1}} |a_{k+1} - a_k| \sum_{j=1}^{i-1} p_j (i - 1 + 1 - j) \right. \\
&\quad \left. + \max_{k=\overline{1,n-1}} |a_{k+1} - a_k| \sum_{t=i+1}^{n} p_t (t + 1 - 1 - i) \right] \\
&= \max_{k=\overline{1,n-1}} |a_{k+1} - a_k| \sum_{i=1}^{n} p_i |b_i| \left[\sum_{j=1}^{i-1} p_j (i - j) + \sum_{t=i+1}^{n} p_t (t - i) \right] \\
&= \max_{k=\overline{1,n-1}} |a_{k+1} - a_k| \sum_{i=1}^{n} p_i |b_i| \sum_{j=1}^{n} p_j |i - j| \\
&= \max_{k=\overline{1,n-1}} |a_{k+1} - a_k| \sum_{i,i=1}^{n} p_i p_j |i - j| |b_i| .
\end{aligned}
$$

Denote

$$J := \sum_{i,i=1}^{n} p_i p_j |i - j| |b_i| .$$

Then we have

$$J \leq \max_{j=\overline{1,n}} |b_j| \sum_{i,i=1}^{n} p_i p_j |i - j| ,$$

and the first inequality in (2.1) is proved.
Using Hölder's discrete inequality for double sums, we have

$$
\begin{aligned}
J &\leq \left(\sum_{i,i=1}^{n} |b_i|^p \right)^{\frac{1}{p}} \left(\sum_{i,i=1}^{n} p_i^q p_j^q |i - j|^q \right)^{\frac{1}{q}} \\
&= n^{\frac{1}{p}} \left(\sum_{i=1}^{n} |b_i|^p \right)^{\frac{1}{p}} \left(\sum_{i,j=1}^{n} p_i^q p_j^q |i - j|^q \right)^{\frac{1}{q}}
\end{aligned}
$$

and the second inequality in (2.1) is proved.
Finally, we observe that

$$
\begin{aligned}
J &\leq \max_{i,j=\overline{1,n}} \{p_i p_j |i - j|\} \sum_{i,j=1}^{n} |b_i| \\
&= n \max_{i,j=\overline{1,n}} \{p_i p_j |i - j|\} \sum_{i=1}^{n} |b_i| ,
\end{aligned}
$$

and the last part of (2.1) is proved. ∎

Corollary 1. *With above assumptions, we have:*

$$\left|\sum_{i=1}^{n} p_i a_i b_i - \sum_{i=1}^{n} p_i a_i \sum_{i=1}^{n} p_i b_i\right| \tag{2.2}$$
$$\leq \begin{cases} \frac{(n-1)n(n+1)}{3} P_M^2 \max_{k=\overline{1,n-1}} \left|a_{k+1}-a_k\right| \max_{i=\overline{1,n}} \left|b_i\right|, \\ \\ n^{\frac{1}{p}} P_M^2 \max_{k=\overline{1,n-1}} \left|a_{k+1}-a_k\right| \left(\sum_{i=1}^{n} \left|b_i\right|^p\right)^{\frac{1}{p}} \left(\sum_{i,j=1}^{n} \left|i-j\right|^q\right)^{\frac{1}{q}}, \\ \\ \textit{if } \frac{1}{p}+\frac{1}{q}=1,\ p>1, \\ \\ n\left(n-1\right) P_M^2 \max_{k=\overline{1,n-1}} \left|a_{k+1}-a_k\right| \sum_{i=1}^{n} \left|b_i\right|, \end{cases}$$

where $P_M = \max\left\{p_i | i=\overline{1,n}\right\}.$

Proof. The second and third inequalities are obvious by the corresponding inequalities in (2.1), taking into account that $p_i \leq P_M$ for all $i \in \{1, ..., n\}$.
To complete the proof, we have to compute

$$T := \sum_{i,j=1}^{n} \left|i-j\right|.$$

We observe that

$$\begin{aligned} \sum_{j=1}^{n} \left|i-j\right| &= \sum_{j=1}^{i} \left|i-j\right| + \sum_{j=i+1}^{n} \left|i-j\right| \\ &= \sum_{j=1}^{i} (i-j) + \sum_{j=i+1}^{n} (j-i) \\ &= i^2 - \frac{i\left(i+1\right)}{2} + \sum_{j=1}^{n} j - \sum_{j=1}^{i} j - i\left(n-i\right) \\ &= i^2 - \left(n+1\right) i + \frac{n\left(n+1\right)}{2} \\ &= \frac{n^2-1}{4} + \left(i - \frac{n+1}{2}\right)^2. \end{aligned}$$

Then

$$\begin{aligned} T &= \sum_{i=1}^{n} \left(\sum_{j=1}^{n} \left|i-j\right|\right) \\ &= \sum_{i=1}^{n} \left(i^2 - \left(n+1\right) i + \frac{n\left(n+1\right)}{2}\right) \\ &= \frac{n\left(n+1\right)\left(2n+1\right)}{6} - \frac{\left(n+1\right) n \left(n+1\right)}{2} + \frac{n\left(n+1\right)}{2} n \\ &= \frac{\left(n-1\right) n \left(n+1\right)}{3}, \end{aligned}$$

and the corollary is proved. ∎

If we choose in the above theorem $p_i = \frac{1}{n},\ i = 1, ..., n$, then we get the following unweighted version of (2.1) :

Corollary 2. *Under the above assumptions, we have*

$$\left|\frac{1}{n}\sum_{i=1}^{n} a_i b_i - \frac{1}{n}\sum_{i=1}^{n} a_i \cdot \frac{1}{n}\sum_{i=1}^{n} b_i\right| \tag{2.3}$$

$$\leq \begin{cases} \frac{n^2-1}{3n} \max_{k=\overline{1,n-1}} |a_{k+1} - a_k| \max_{i=\overline{1,n}} |b_i|, \\ \\ \frac{1}{n \cdot n^{\frac{1}{q}}} \max_{k=\overline{1,n-1}} |a_{k+1} - a_k| \left(\sum_{i=1}^{n} |b_i|^p\right)^{\frac{1}{p}} \left(\sum_{i,j=1}^{n} |i-j|^q\right)^{\frac{1}{q}}, \\ \\ \frac{n-1}{n} \max_{k=\overline{1,n-1}} |a_{k+1} - a_k| \sum_{i=1}^{n} |b_i|. \end{cases}$$

Remark 1. *Suppose that* $p = q = 2$ *in* (2.3) *. Then we get*

$$\begin{aligned} \sum_{i,j=1}^{n} |i-j|^2 &= \sum_{i,j=1}^{n} \left(i^2 - 2ij + j^2\right) = 2\left[n\sum_{i=1}^{n} i^2 - \left(\sum_{i=1}^{n} i\right)^2\right] \\ &= 2\left[n \cdot \frac{n(n+1)(2n+1)}{6} - \left[\frac{n(n+1)}{2}\right]^2\right] \\ &= \frac{n^2(n+1)(n-1)}{6}. \end{aligned}$$

In addition,

$$\frac{1}{n \cdot n^{\frac{1}{2}}} \cdot \left[\frac{n^2(n+1)(n-1)}{6}\right]^{\frac{1}{2}} = \left[\frac{(n+1)(n-1)}{6n}\right]^{\frac{1}{2}}$$

and then we get the inequality:

$$\begin{aligned} &\left|\frac{1}{n}\sum_{i=1}^{n} a_i b_i - \frac{1}{n}\sum_{i=1}^{n} a_i \cdot \frac{1}{n}\sum_{i=1}^{n} b_i\right| \\ &\leq \left[\frac{(n+1)(n-1)}{6n}\right]^{\frac{1}{2}} \max_{k=\overline{1,n-1}} |a_{k+1} - a_k| \left(\sum_{i=1}^{n} |b_i|^2\right)^{\frac{1}{2}}. \end{aligned} \tag{2.4}$$

The following theorem also holds:

Theorem 5. *Under the assumptions of Theorem 4, we have the inequality:*

$$\left|\sum_{i=1}^{n} p_i a_i b_i - \sum_{i=1}^{n} p_i a_i \sum_{i=1}^{n} p_i b_i\right| \tag{2.5}$$

$$\leq \begin{cases} \max_{k=\overline{1,n-1}} |a_{k+1} - a_k| \max_{i=\overline{1,n}} \{p_i |b_i|\} \sum_{i,j=1}^{n} p_i |i-j|, \\ \\ \max_{k=\overline{1,n-1}} |a_{k+1} - a_k| \left(\sum_{i=1}^{n} p_i |b_i|^p\right)^{\frac{1}{p}} \left(\sum_{i,j=1}^{n} p_i p_j |i-j|^q\right)^{\frac{1}{q}}, \\ \\ \text{if } \frac{1}{p} + \frac{1}{q} = 1,\ p > 1, \\ \\ (n-1) \max_{k=\overline{1,n-1}} |a_{k+1} - a_k| \sum_{i=1}^{n} p_i |b_i|. \end{cases}$$

Proof. As in the proof of Theorem 4, we have

$$\begin{aligned} & \left|\sum_{i=1}^{n} p_i a_i b_i - \sum_{i=1}^{n} p_i a_i \sum_{i=1}^{n} p_i b_i\right| \\ \leq & \max_{k=\overline{1,n-1}} |a_{k+1} - a_k| \sum_{i,j=1}^{n} p_i p_j |b_i| |i-j| . \end{aligned}$$

Using the above assumptions, we have:

$$J \leq \max_{i=\overline{1,n}} \{p_i |b_i|\} \sum_{i,j=1}^{n} p_j |i-j|$$

and the first in equality in (2.5) is obtained.
Using Hölder's discrete inequality for double sums and weighted means, we have:

$$\begin{aligned} J & \leq \left(\sum_{i,j=1}^{n} p_i p_j |b_i|^p\right)^{\frac{1}{p}} \left(\sum_{i,j=1}^{n} p_i p_j |i-j|^q\right)^{\frac{1}{q}} \\ & = \left(\sum_{j=1}^{n} p_j \sum_{i=1}^{n} p_i |b_i|^p\right)^{\frac{1}{p}} \left(\sum_{i,i=1}^{n} p_i p_j |i-j|^q\right)^{\frac{1}{q}} \\ & = \left(\sum_{i=1}^{n} p_i |b_i|^p\right)^{\frac{1}{p}} \left(\sum_{i,i=1}^{n} p_i p_j |i-j|^q\right)^{\frac{1}{q}} . \end{aligned}$$

Finally, we have

$$\begin{aligned} J & \leq \max_{i,j=\overline{1,n-1}} |i-j| \sum_{i,j=1}^{n} p_i p_j |b_i| \\ & = (n-1) \sum_{j=1}^{n} p_j \sum_{i=1}^{n} p_i |b_i| \\ & = (n-1) \sum_{i=1}^{n} p_i |b_i| , \end{aligned}$$

and the proof is completed. ∎

The following corollary is useful.

Corollary 3. *Under the above assumptions, we have the inequality*

$$\begin{aligned} & \left|\sum_{i=1}^{n} p_i a_i b_i - \sum_{i=1}^{n} p_i a_i \sum_{i=1}^{n} p_i b_i\right| \\ \leq & \sqrt{2} \max_{k=\overline{1,n-1}} |a_{k+1} - a_k| \left(\sum_{i=1}^{n} p_i |b_i|^2\right)^{\frac{1}{2}} \left[n \sum_{i=1}^{n} i^2 p_i - \left(\sum_{i=1}^{n} i p_i\right)^2\right]^{\frac{1}{2}} . \end{aligned} \tag{2.6}$$

Proof. Putting in (2.5) $p=q=2$, we obtain

$$\begin{aligned}\sum_{i,i=1}^{n} p_i p_j (i-j)^2 &= \sum_{i,j=1}^{n} p_i p_j \left(i^2-2ij+j^2\right)\\ &= 2\left[\sum_{i=1}^{n} i^2 p_i p_j - \sum_{i,j} ij p_i p_j\right]\\ &= 2\left[\sum_{i=1}^{n} i^2 p_i - \left(\sum_{i=1}^{n} i p_i\right)^2\right],\end{aligned}$$

and the corollary is proved. ∎

3. Applications for the Moments of Discrete Random Variables

Consider the discrete random variable

$$X:\begin{pmatrix} x_1,...,x_n \\ p_1,...,p_n \end{pmatrix}$$

which is taking the real positive values $x_1,x_2,...,x_n$ with the probabilities $p_1,...,p_n$. Define the $a-$moment $(a>0)$ as follows:

$$M_a(X) := \sum_{i=1}^{n} p_i x_i^a.$$

Using Theorem 4 and the Corollary 1 we can state the following approximation result which allows us to compare the $(a+b)-$Moment of X with the product of $a-$Moment and $b-$Moment of X :

Proposition 1. *Under the above assumptions, we have:*

$$(3.1)\qquad \begin{aligned} &|M_{a+b}(X) - M_a(X) M_b(X)| \\ &\le \begin{cases} \max_{k=\overline{1,n-1}} \left|x_{k+1}^a - x_k^a\right| \max_{k=\overline{1,n}}\left\{x_k^b\right\} \sum_{i,j=1}^{n} p_i p_j |i-j|, \\ n^{\frac{1}{p}} \max_{k=\overline{1,n-1}} \left|x_{k+1}^a - x_k^a\right| \left(\sum_{i=1}^{n} x_i^{pb}\right)^{\frac{1}{p}} \left(\sum_{i,j=1}^{n} p_i^q p_j^q |i-j|^q\right)^{\frac{1}{q}}, \\ \text{if } \frac{1}{p}+\frac{1}{q}=1,\ p>1, \\ n \max_{k=\overline{1,n-1}} \left|x_{k+1}^a - x_k^a\right| \left(\sum_{i=1}^{n} x_i^b\right) \max_{i,j=\overline{1,n}} \{p_i p_j |i-j|\} \end{cases} \\ &\le \begin{cases} \frac{(n-1)n(n+1)}{3} P_M^2 \max_{k=\overline{1,n-1}} \left|x_{k+1}^a - x_k^a\right| \max_{k=\overline{1,n}}\left\{x_k^b\right\}; \\ n^{\frac{1}{p}} P_M^2 \max_{k=\overline{1,n-1}} \left|x_{k+1}^a - x_k^a\right| \left(\sum_{i=1}^{n} x_i^{pb}\right)^{\frac{1}{p}} \left(\sum_{i,j=1}^{n} |i-j|^q\right)^{\frac{1}{q}}; \\ \text{if } \frac{1}{p}+\frac{1}{q}=1,\ p>1, \\ n(n-1) P_M^2 \max_{k=\overline{1,n-1}} \left|x_{k+1}^a - x_k^a\right| \sum_{i=1}^{n} x_i^b, \end{cases} \end{aligned}$$

where $P_M := \max\left\{p_i | i=\overline{1,n}\right\}$ *and* $a,b>0$.

Using Theorem 5 we can state the similar result:

Proposition 2. *Under the above assumptions, we have*

$$(3.2)\qquad \left|M_{a+b}(X)-M_a(X)M_b(X)\right| \leq \begin{cases} \max_{k=\overline{1,n-1}}\left|x_{k+1}^a-x_k^a\right|\max_{k=\overline{1,n}}\left\{p_ix_k^b\right\}\sum_{i,j=1}^n p_j\left|i-j\right|; \\ \max_{k=\overline{1,n-1}}\left|x_{k+1}^a-x_k^a\right|\left(\sum_{i=1}^n p_ix_i^{pb}\right)^{\frac{1}{p}}\left(\sum_{i,j=1}^n p_ip_j\left|i-j\right|^q\right)^{\frac{1}{q}}; \\ \text{if } \frac{1}{p}+\frac{1}{q}=1,\ p>1, \\ n(n-1)\max_{k=\overline{1,n-1}}\left|x_{k+1}^a-x_k^a\right|M_b(X), \end{cases}$$

for all $a,b>0$.

Now, in connection with the random variable X, let us consider the *uniformly distributed random variable*

$$U:\begin{pmatrix} x_1,\ldots..,x_n \\ \frac{1}{n},\ldots..,\frac{1}{n}\end{pmatrix}$$

and its $a-$Moment

$$M_n(U)=\frac{1}{n}\sum_{i=1}^n x_i^a.$$

Now, let us consider the Corollary 2 and put in it $a_i=p_i,\ p_i\geq 0$, but not necessarily a probability distribution, and $b_i:=x_i^a$ to get the following inequality:

$$(3.3)\qquad \left|\frac{1}{n}\sum_{i=1}^n p_ix_i^a-\frac{1}{n}\sum_{i=1}^n p_i\cdot\frac{1}{n}\sum_{i=1}^n x_i^a\right| \leq \begin{cases} \frac{n^2-1}{3n}\max_{k=\overline{1,n-1}}\left|p_{k+1}-p_k\right|\max_{i=\overline{1,n}}\left\{x_i^a\right\}; \\ \frac{1}{n^{1+\frac{1}{q}}}\max_{k=\overline{1,n-1}}\left|p_{k+1}-p_k\right|\left(\sum_{i=1}^n x_i^{pa}\right)^{\frac{1}{p}}\left(\sum_{i,j=1}^n\left|i-j\right|^q\right)^{\frac{1}{q}}, \\ \frac{n-1}{n}\max_{k=\overline{1,n-1}}\left|p_{k+1}-p_k\right|\sum_{i=1}^n x_i^a. \end{cases}$$

If in the same corollary we choose $a_i=x_i^a,\ b_i=p_i$, we get the inequality:

$$(3.4)\qquad \left|\frac{1}{n}\sum_{i=1}^n p_ix_i^a-\frac{1}{n}\sum_{i=1}^n p_i\cdot\frac{1}{n}\sum_{i=1}^n x_i^a\right| \leq \begin{cases} \frac{n^2-1}{3n}\max_{k=\overline{1,n-1}}\left|x_{k+1}^a-x_k^a\right|P_M^2; \\ \frac{1}{n\cdot n^{\frac{1}{q}}}\max_{k=\overline{1,n-1}}\left|x_{k+1}^a-x_k^a\right|\left(\sum_{i=1}^n x_i^p\right)^{\frac{1}{p}}\left(\sum_{i,j=1}^n\left|i-j\right|^q\right)^{\frac{1}{q}}P_M^{1+\frac{1}{q}}; \\ \text{if } \frac{1}{p}+\frac{1}{q}=1,\ p>1, \\ \frac{n-1}{n}\max_{k=\overline{1,n-1}}\left|x_{k+1}^a-x_k^a\right|P_M. \end{cases}$$

In terms of moments of random variable, the previous inequalities can be stated as follows.

Proposition 3. *Under the above assumptions, we have the estimations*

$$
\begin{aligned}
(3.5)\qquad & |M_a(X)-M_a(U)| \\
& \leq \begin{cases} \frac{n^2-1}{3n^2}\max_{k=\overline{1,n-1}}|p_{k+1}-p_k|\max_{i=\overline{1,n}}\{x_i^a\}; \\ \\ \frac{1}{n^{\frac{1}{q}}}\max_{k=\overline{1,n-1}}|p_{k+1}-p_k|\left(\sum_{i=1}^n x_i^{pa}\right)^{\frac{1}{p}}\left(\sum_{i,j=1}^n|i-j|^q\right)^{\frac{1}{q}}; \\ \\ (n-1)\max_{k=\overline{1,n-1}}|p_{k+1}-p_k|\sum_{i=1}^n x_i^a \end{cases}
\end{aligned}
$$

and

$$
\begin{aligned}
(3.6)\qquad & |M_a(X)-M_a(U)| \\
& \leq \begin{cases} \frac{n^2-1}{3n^2}\max_{k=\overline{1,n-1}}\left|x_{k+1}^a-x_k^a\right|P_M^2; \\ \\ \frac{1}{n^{\frac{1}{q}}}\max_{k=\overline{1,n-1}}\left|x_{k+1}^a-x_k^a\right|\left(\sum_{i=1}^n x_i^{p}\right)^{\frac{1}{p}}\left(\sum_{i,j=1}^n|i-j|^q\right)^{\frac{1}{q}}P_M^{1+\frac{1}{q}}, \\ \text{if } \frac{1}{p}+\frac{1}{q}=1,\ p>1, \\ \\ (n-1)\max_{k=\overline{1,n-1}}\left|x_{k+1}^a-x_k^a\right|P_M, \end{cases}
\end{aligned}
$$

respectively, where $a>0$.

4. Applications for the Guessing Mapping

J.L. Massey in [1] considered the problem of guessing the value of realization of random variable X by asking questions of the form: "Is X equal to x ? " until the answer is "Yes" .

Let $G(X)$ denote the number of guesses required by a particular guessing strategy when $X=x$.

Massey observed that $E(G(x))$, the average number of guesses, is minimized by a guessing strategy that guesses the possible values of X in decreasing order of probability.

We begin by giving a formal and generalized statement of the above problem by following E. Arikan [2].

Let (X,Y) be a pair of random variables with X taking values in a finite set $\mathcal{X}$ of size n, Y taking values in a countable set $\mathcal{Y}$. Call a function $G(X)$ of the random variable X a *guessing function* for X if $G:\mathcal{X}\to\{1,...,n\}$ is one-to-one. Call a function $G(X\mid Y)$ a *guessing function for X given Y* if for any fixed value $Y=y, G(X\mid y)$ is a guessing function for X . $G(X\mid y)$ will be thought of as the number of guessing required to determine X when the value of Y is given.

The following inequalities on the moments of $G(X)$ and $G(X|Y)$ were proved by E. Arikan in the recent paper [2].

Theorem 6. *For an arbitrary guessing function $G(X)$ and $G(X\mid Y)$ and any $p>0$, we have:*

$$
E\left(G(X)^p\right)\geq(1+\ln n)^{-p}\left[\sum_{x\in\mathcal{X}}P_X(x)^{\frac{1}{1+p}}\right]^{1+p} \tag{4.1}
$$

and

$$E\left(G\left(X\mid Y\right)^{p}\right)\geq\left(1+\ln n\right)^{-p}\sum_{y\in\mathcal{Y}}\left[\sum_{x\in\mathcal{X}}P_{X,Y}\left(x,y\right)^{\frac{1}{1+p}}\right]^{1+p} \tag{4.2}$$

where $P_{X,Y}$ *and* P_X *are probability distributions of* (X,Y) *and* X, *respectively.*

Note that, for $p=1$, we get the following estimations on the average number of guesses:

$$E\left(G\left(X\right)\right)\geq\frac{\left[\sum_{x\in\mathcal{X}}P_{X}\left(x\right)^{\frac{1}{2}}\right]^{2}}{1+\ln n}$$

and

$$E\left(G\left(X\right)\right)\geq\frac{\sum_{y\in\mathcal{Y}}\left[\sum_{x\in\mathcal{X}}P_{X,Y}\left(x,y\right)^{\frac{1}{2}}\right]^{2}}{1+\ln n}.$$

In paper [3], Boztaş proved the following analytic inequality and applied it for the moments of guessing mappings:

Theorem 7. *The relation*

$$\left[\sum_{k=1}^{n}p_{k}^{\frac{1}{r}}\right]^{r}\geq\sum_{k=1}^{n}\left(k^{r}-\left(k-1\right)^{r}\right)p_{k} \tag{4.3}$$

where $r\geq1$ *holds for any positive integer* n, *provided that the weights* $p_1,...,p_n$ *are nonnegative real numbers satisfying the condition:*

$$p_{k+1}^{\frac{1}{r}}\leq\frac{1}{k}\left(p_{1}^{\frac{1}{r}}+...+p_{k}^{\frac{1}{r}}\right),k=1,2,...,n-1. \tag{4.4}$$

To simplify the notation further, we assume that the x_i are numbered such that x_k is always the k^{th} guess. This yields:

$$E\left(G^{p}\right)=\sum_{k=1}^{n}k^{p}p_{k},p\geq0.$$

If we now consider the guessing problem, we note that (4.1) can be written as [3]:

$$\left[\sum_{k=1}^{n}p_{k}^{\frac{1}{1+p}}\right]^{1+p}\geq E\left(G^{1+p}\right)-E\left(\left(G-1\right)^{1+p}\right)$$

for guessing sequences obeying (4.4).

In particular, using the binomial expansion of $(G-1)^{1+p}$ we have the following corollary [3]:

Corollary 4. *For guessing sequences obeying* (4.4) *with* $r=1+m$, *the* m^{th} *guessing moment, when* $m\geq1$ *is an integer satisfies:*

$$\begin{aligned}(4.5)\quad & E\left(G^{m}\right)\\ \leq\; & \frac{1}{1+m}\left[\sum_{k=1}^{n}p_{k}^{\frac{1}{1+m}}\right]^{1+m}\\ & +\frac{1}{1+m}\left\{\binom{m+1}{2}E\left(G^{m-1}\right)-\binom{m+1}{3}E\left(G^{m-2}\right)+...+\left(-1\right)^{m+1}\right\}.\end{aligned}$$

The following inequalities immediately follow from Corollary 4:

$$E(G) \leq \frac{1}{2}\left[\sum_{k=1}^{n} p_k^{\frac{1}{2}}\right]^2 + \frac{1}{2}$$

and

$$E\left(G^2\right) \leq \frac{1}{3}\left[\sum_{k=1}^{n} p_k^{\frac{1}{3}}\right]^3 + E(G) - \frac{1}{3}.$$

We are able now to point out some new results for the p-moment of guessing mapping as follows.

Let us observe that for $p \in (0,1)$, the sequence $x_k = k^p,\ k = 1, ..., n,$ is concave and for $p \in [1,\infty)$ it is convex, so

$$\delta_p(n) := \max_{k=1,\ldots,n-1}\left|x_{k+1}^p - x_k^p\right| = \begin{cases} n^p - (n-1)^p & \text{if } p \in [1,\infty), \\ 2^p - 1 & \text{if } p \in (0,1). \end{cases}$$

Using Proposition 1, we can state that:

Proposition 4. *If $a, b \in (0,\infty)$ and G is a guessing mapping as above, then we have the inequality*

$$\left|E\left(G^{a+b}\right) - E\left(G^a\right)E\left(G^b\right)\right| \tag{4.6}$$

$$\leq \begin{cases} \delta_a(n)\, n^b \sum_{i,j=1}^{n} p_i p_j |i-j|; \\ \delta_a(n)\, n^{\frac{1}{p}} \left[S_{pb}(n)\right]^{\frac{1}{p}} \left[\sum_{i,j=1}^{n} p_i^q p_j^q |i-j|^q\right]^{\frac{1}{q}} \\ \text{if } \frac{1}{p}+\frac{1}{q}=1,\ p>1; \\ n\delta_a(n)\, S_b(n) \max_{i=1,\ldots,n}\{p_i p_j |i-j|\}; \end{cases}$$

$$\leq \begin{cases} \frac{(n-1)n^{b+1}(n+1)}{3} P_M^2 \delta_a(n); \\ n^{\frac{1}{p}} \delta_a(n) P_M^2 \left[S_{pb}(n)\right]^{\frac{1}{p}} \left[\sum_{i,j=1}^{n} |i-j|^q\right]^{\frac{1}{q}} \\ \text{if } \frac{1}{p}+\frac{1}{q}=1,\ p>1; \\ n(n-1) P_M^2 \delta_a(n) S_b(n), \end{cases}$$

where $S_p(n) := \sum_{k=1}^{n} k^p,\ p > 0.$

A similar result can be stated if we use Proposition 2, but we omit the details.

Finally, by the use of Proposition 3, we have

Proposition 5. *Under the above assumtions, we have*

$$\left|E\left(G^{a}\right)-\frac{1}{n}S_{a}\left(n\right)\right| \tag{4.7}$$
$$\leq\begin{cases}\frac{\left(n^{2}-1\right)n^{a-2}}{3}\max_{k=1,\dots,n-1}\left|p_{k+1}-p_{k}\right|;\\ \\ \frac{1}{n^{\frac{1}{q}}}\left[S_{pa}\left(n\right)\right]^{\frac{1}{p}}\left(\sum_{i,j=1}^{n}\left|i-j\right|^{q}\right)^{\frac{1}{q}}\max_{k=1,\dots,n-1}\left|p_{k+1}-p_{k}\right|\\ \\ \text{if } \frac{1}{p}+\frac{1}{q}=1,\ p>1;\\ \\ n\left(n-1\right)S_{a}\left(n\right),\max_{k=1,\dots,n-1}\left|p_{k+1}-p_{k}\right|;\end{cases}$$

and

$$\left|E\left(G^{a}\right)-\frac{1}{n}S_{a}\left(n\right)\right| \tag{4.8}$$
$$\leq\begin{cases}\frac{n^{2}-1}{3n^{2}}\delta_{a}\left(n\right)P_{M}^{2};\\ \\ \frac{1}{n^{\frac{1}{q}}}\delta_{a}\left(n\right)S_{p}\left(n\right)\left(\sum_{i,j=1}^{n}\left|i-j\right|^{q}\right)^{\frac{1}{q}}P_{M}^{1+\frac{1}{q}}\\ \\ \text{if } \frac{1}{p}+\frac{1}{q}=1,\ p>1;\\ \\ \left(n-1\right)\delta_{a}\left(n\right)P_{M},\end{cases}$$

for all $a>0$.

Remark 2. *If we assume that* $a=1,2$ *or* 3 *in the above inequalities, we can obtain some bounds for* $E\left(G\right),\ E\left(G^{2}\right)$ *or* $E\left(G^{3}\right)$, *which will complement the results from Corollary 4 for* $m=1,2,$ *and* 3. *We omit the details.*

References

[1] J.L. MASSEY, Guessing and entropy, *Proc. 1994, IEEE Int. Symp. on Information Theory*, Trodheim, Norway, 1994, p. 204.

[2] E. ARIKAN, An inequality on guessing and its applications to sequential decoding, *IEEE Transactions on Information Theory*, **42**(1996), 99-105.

[3] S. BOZTAŞ, Comments on " An Inequality on Guessing and Its Applications to Sequential Decoding", to appear in *IEEE Transactions on Information Theory.*

[4] D.S. MITRINOVIĆ, J.E. PEČARIĆ and A.M. FINK, *Classical and New Inequalities in Analysis,* Kluwer Academic Publishers, Dordrecht, 1993.

[5] S.S. DRAGOMIR and J.VAN DER HOEK, Some new inequalities for the average number of guesses, *Kyungpook Math. Journal , accepted.*

[6] S.S. DRAGOMIR and J.VAN DER HOEK, New inequalities for the moments of guessing mapping, *East Asian Math. Journal, **14**(1)(1998), 1-14.*

[7] S.S. DRAGOMIR and J.VAN DER HOEK, Some new analytic inequalities and their applications in guessing theory, *J. Math. Anal. Appl.*, **225**(1998), 542-556.

[8] S.S. DRAGOMIR, J.E. PEČARIĆ, P. SARUNIC and J. VAN DER HOEK, On some inequalities for the moments of guessing mapping, *submitted.*

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