# Generalized Abstracted Mean Values



Qi, Feng (1998) Generalized Abstracted Mean Values. RGMIA research report collection, 2 (5).

# GENERALIZED ABSTRACTED MEAN VALUES

FENG QI

Abstract. In this article, the author introduces the generalized abstracted mean values which extend the concepts of most means with two variables, and researches their basic properties and monotonicities.

## 1. Introduction

The simplest and classical means are the arithmetic mean, the geometric mean, and the harmonic mean. For a positive sequence $a = (a_1, \dots, a_n)$, they are defined respectively by

$$A_n(a) = \frac{1}{n}\sum_{i=1}^{n} a_i, \quad G_n(a) = \sqrt[n]{\prod_{i=1}^{n} a_i}, \quad H_n(a) = \frac{n}{\sum_{i=1}^{n} \frac{1}{a_i}}. \tag{1.1}$$

For a positive function $f$ defined on $[x, y]$, the integral analogues of (1.1) are given by

$$A(f) = \frac{1}{y-x}\int_x^y f(t)\,dt, \tag{1.2}$$

$$G(f) = \exp\left(\frac{1}{y-x}\int_x^y \ln f(t)\,dt\right), \tag{1.3}$$

$$H(f) = \frac{y-x}{\int_x^y \frac{dt}{f(t)}}. \tag{1.4}$$

It is well-known that

$$A_n(a) \ge G_n(a) \ge H_n(a), \quad A(f) \ge G(f) \ge H(f) \tag{1.5}$$

are called the arithmetic mean–geometric mean–harmonic mean inequalities.

These classical means have been generalized, extended and refined in many different directions. The study of various means has a rich literature, for details, please refer to [1, 2], [4]–[8] and [19], especially to [9], and so on.

Some mean values also have applications in medicine [3, 18].

---

1991 *Mathematics Subject Classification.* Primary 26A48; Secondary 26D15.

*Key words and phrases.* Generalized abstracted mean values, basic property, monotonicity.

The author was partially supported by NSF of Henan Province, The People's Republic of China.

This paper is typeset using $\mathcal{A}\mathcal{M}\mathcal{S}$-LaTeX.

Recently, the author [9] introduced the generalized weighted mean values $M_{p,f}(r,s;x,y)$ with two parameters $r$ and $s$, which are defined by

$$M_{p,f}(r,s;x,y)=\left(\frac{\int_x^y p(u)f^s(u)du}{\int_x^y p(u)f^r(u)du}\right)^{1/(s-r)}, \qquad (r-s)(x-y)\neq 0; \tag{1.6}$$

$$M_{p,f}(r,r;x,y)=\exp\left(\frac{\int_x^y p(u)f^r(u)\ln f(u)du}{\int_x^y p(u)f^r(u)du}\right), \qquad x-y\neq 0; \tag{1.7}$$

$$M_{p,f}(r,s;x,x)=f(x),$$

where $x,y,r,s\in\mathbb{R}$, $p(u)\not\equiv 0$ is a nonnegative and integrable function and $f(u)$ a positive and integrable function on the interval between $x$ and $y$.

It was shown in [9, 17] that $M_{p,f}(r,s;x,y)$ increases with both $r$ and $s$ and has the same monotonicities as $f$ in both $x$ and $y$. Sufficient conditions in order that

$$M_{p_1,f}(r,s;x,y)\geq M_{p_2,f}(r,s;x,y), \tag{1.8}$$

$$M_{p,f_1}(r,s;x,y)\geq M_{p,f_2}(r,s;x,y) \tag{1.9}$$

were also given in [9].

It is clear that $M_{p,f}(r,0;x,y) = M^{[r]}(f;p;x,y)$. For the definition of $M^{[r]}(f;p;x,y)$, please see [6].

*Remark* 1. As concrete applications of the monotonicities and properties of the generalized weighted mean values $M_{p,f}(r,s;x,y)$, some monotonicity results and inequalities of the gamma and incomplete gamma functions are presented in [10].

Moreover, an inequality between the extended mean values $E(r,s;x,y)$ and the generalized weighted mean values $M_{p,f}(r,s;x,y)$ for a convex function $f$ is given in [14], which generalizes the well-known Hermite-Hadamard inequality.

The main purposes of this paper are to establish the definitions of the generalized abstracted mean values, to research their basic properties, and to prove their monotonicities. In Section 2, we introduce some definitions of mean values and study their basic properties. In Section 3, the monotonicities of the generalized abstracted mean values, and the like, are proved.

## 2. Definitions and Basic Properties

**Definition 1.** Let $p$ be a defined, positive and integrable function on $[x,y]$ for $x,y\in\mathbb{R}$, $f$ a real-valued and monotonic function on $[\alpha,\beta]$. If $g$ is a function valued on $[\alpha,\beta]$ and $f\circ g$ integrable on $[x,y]$, the quasi-arithmetic non-symmetrical mean of $g$ is defined by

$$M_f(g;p;x,y)=f^{-1}\left(\frac{\int_x^y p(t)f(g(t))\ dt}{\int_x^y p(t)\ dt}\right), \tag{2.1}$$

where $f^{-1}$ is the inverse function of $f$.

For $g(t)=t$, $f(t)=t^{r-1}$, $p(t)=1$, the mean $M_f(g;p;x,y)$ reduces to the extended logarithmic means $S_r(x,y)$; for $p(t)=t^{r-1}$, $g(t)=f(t)=t$, to the one-parameter mean $J_r(x,y)$; for $p(t)=f'(t)$, $g(t)=t$, to the abstracted mean $M_f(x,y)$; for $g(t)=t$, $p(t)=t^{r-1}$, $f(t)=t^{s-r}$, to the extended mean values $E(r,s;x,y)$; for $f(t)=t^r$, to the weighted mean of order $r$ of the function $g$ with weight $p$ on $[x,y]$. If we replace $p(t)$ by $p(t)f^r(t)$, $f(t)$ by $t^{s-r}$, $g(t)$ by $f(t)$ in

(2.1), then we get the generalized weighted mean values $M_{p,f}(r,s;x,y)$. Hence, from $M_f(g;p;x,y)$ we can deduce most of the two variable means.

**Lemma 1** ([13]). *Suppose that $f$ and $g$ are integrable, and $g$ is non-negative, on $[a,b]$, and that the ratio $f(t)/g(t)$ has finitely many removable discontinuity points. Then there exists at least one point $\theta \in (a,b)$ such that*

$$\frac{\int_a^b f(t)dt}{\int_a^b g(t)dt} = \lim_{t\to\theta} \frac{f(t)}{g(t)}. \tag{2.2}$$

We call Lemma 1 the revised Cauchy's mean value theorem in integral form.

*Proof.* Since $f(t)/g(t)$ has finitely many removable discontinuity points, without loss of generality, suppose it is continuous on $[a,b]$. Furthermore, using $g(t) \geq 0$, from the mean value theorem for integrals, there exists at least one point $\theta \in (a,b)$ satisfying

$$\int_a^b f(t)dt = \int_a^b \left(\frac{f(t)}{g(t)}\right) g(t)dt = \frac{f(\theta)}{g(\theta)} \int_a^b g(t)dt. \tag{2.3}$$

Lemma 1 follows. ∎

**Theorem 1.** *The mean $M_f(g;p;x,y)$ has the following properties:*

$$\begin{gathered} \alpha \leq M_f(g;p;x,y) \leq \beta, \\ M_f(g;p;x,y) = M_f(g;p;y,x), \end{gathered} \tag{2.4}$$

*where $\alpha = \inf\limits_{t\in[x,y]} g(t)$ and $\beta = \sup\limits_{t\in[x,y]} g(t)$.*

*Proof.* This follows from Lemma 1 and standard arguments. ∎

**Definition 2.** For a sequence of positive numbers $a = (a_1, \dots, a_n)$ and positive weights $p = (p_1, \dots, p_n)$, the generalized weighted mean values of numbers $a$ with two parameters $r$ and $s$ is defined as

$$M_n(p;a;r,s) = \left(\frac{\sum\limits_{i=1}^n p_i a_i^r}{\sum\limits_{i=1}^n p_i a_i^s}\right)^{1/(r-s)}, \quad r-s \neq 0; \tag{2.5}$$

$$M_n(p;a;r,r) = \exp\left(\frac{\sum\limits_{i=1}^n p_i a_i^r \ln a_i}{\sum\limits_{i=1}^n p_i a_i^r}\right). \tag{2.6}$$

For $s=0$ we obtain the weighted mean $M_n^{[r]}(a;p)$ of order $r$ which is defined in [2, 5, 6, 7] and introduced above; for $s=0$, $r=-1$, the weighted harmonic mean; for $s=0$, $r=0$, the weighted geometric mean; and for $s=0$, $r=1$, the weighted arithmetic mean.

The mean $M_n(p;a;r,s)$ has some basic properties similar to those of $M_{p,f}(r,s;x,y)$, for instance

**Theorem 2.** *The mean $M_n(p;a;r,s)$ is a continuous function with respect to $(r,s)\in\mathbb{R}^2$ and has the following properties:*

$$
\begin{gathered}
m \le M_n(p;a;r,s) \le M,\\
M_n(p;a;r,s) = M_n(p;a;s,r),\\
M_n^{s-r}(p;a;r,s) = M_n^{s-t}(p;a;t,s)\cdot M_n^{t-r}(p;a;r,t),
\end{gathered} \tag{2.7}
$$

*where $m=\min_{1\le i\le n}\{a_i\}$, $M=\max_{1\le i\le n}\{a_i\}$.*

*Proof.* For an arbitrary sequence $b=(b_1,\dots,b_n)$ and a positive sequence $c=(c_1,\dots,c_n)$, the following elementary inequalities [6, p. 204] are well-known

$$
\min_{1\le i\le n}\left\{\frac{b_i}{c_i}\right\} \le \frac{\sum_{i=1}^n b_i}{\sum_{i=1}^n c_i} \le \max_{1\le i\le n}\left\{\frac{b_i}{c_i}\right\}. \tag{2.8}
$$

This implies the inequality property.

The other properties follow from standard arguments. ∎

**Definition 3.** Let $f_1$ and $f_2$ be real-valued functions such that the ratio $f_1/f_2$ is monotone on the closed interval $[\alpha,\beta]$. If $a=(a_1,\dots,a_n)$ is a sequence of real numbers from $[\alpha,\beta]$ and $p=(p_1,\dots,p_n)$ a sequence of positive numbers, the generalized abstracted mean values of numbers $a$ with respect to functions $f_1$ and $f_2$, with weights $p$, is defined by

$$
M_n(p;a;f_1,f_2) = \left(\frac{f_1}{f_2}\right)^{-1}\left(\frac{\sum_{i=1}^n p_i f_1(a_i)}{\sum_{i=1}^n p_i f_2(a_i)}\right), \tag{2.9}
$$

where $(f_1/f_2)^{-1}$ is the inverse function of $f_1/f_2$.

The integral analogue of Definition 3 is given by

**Definition 4.** Let $p$ be a positive integrable function defined on $[x,y]$, $x,y\in\mathbb{R}$, $f_1$ and $f_2$ real-valued functions and the ratio $f_1/f_2$ monotone on the interval $[\alpha,\beta]$. In addition, let $g$ be defined on $[x,y]$ and valued on $[\alpha,\beta]$, and $f_i\circ g$ integrable on $[x,y]$ for $i=1,2$. The generalized abstracted mean values of function $g$ with respect to functions $f_1$ and $f_2$ and with weight $p$ is defined as

$$
M(p;g;f_1,f_2;x,y) = \left(\frac{f_1}{f_2}\right)^{-1}\left(\frac{\int_x^y p(t)f_1(g(t))\,dt}{\int_x^y p(t)f_2(g(t))\,dt}\right), \tag{2.10}
$$

where $(f_1/f_2)^{-1}$ is the inverse function of $f_1/f_2$.

*Remark* 2. Set $f_2\equiv 1$ in Definition 4, then we can obtain Definition 1 easily. Replacing $f$ by $f_1/f_2$, $p(t)$ by $p(t)f_2(g(t))$ in Definition 1, we arrive at Definition 4 directly. Analogously, formula (2.9) is equivalent to $M_f(a;p)$, see [6, p. 77]. Definition 1 and Definition 4 are equivalent to each other. Similarly, so are Definition 3 and the quasi-arithmetic non-symmetrical mean $M_f(a;p)$ of numbers $a=(a_1,\dots,a_n)$ with weights $p=(p_1,\dots,p_n)$.

**Lemma 2.** *Suppose the ratio $f_1/f_2$ is monotonic on a given interval. Then*

$$\left(\frac{f_1}{f_2}\right)^{-1}(x) = \left(\frac{f_2}{f_1}\right)^{-1}\left(\frac{1}{x}\right), \tag{2.11}$$

*where $(f_1/f_2)^{-1}$ is the inverse function of $f_1/f_2$.*

*Proof.* This is a direct consequence of the definition of an inverse function. ∎

**Theorem 3.** *The means $M_n(p;a;f_1,f_2)$ and $M(p;g;f_1,f_2;x,y)$ have the following properties:*

(i) *Under the conditions of Definition 3, we have*

$$\begin{gathered} m \le M_n(p;a;f_1,f_2) \le M, \\ M_n(p;a;f_1,f_2) = M_n(p;a;f_2,f_1), \end{gathered} \tag{2.12}$$

*where $m = \min_{1\le i\le n}\{a_i\}$, $M = \max_{1\le i\le n}\{a_i\}$;*

(ii) *Under the conditions of Definition 4, we have*

$$\begin{gathered} \alpha \le M(p;g;f_1,f_2;x,y) \le \beta, \\ M(p;g;f_1,f_2;x,y) = M(p;g;f_1,f_2;y,x), \\ M(p;g;f_1,f_2;x,y) = M(p;g;f_2,f_1;x,y), \end{gathered} \tag{2.13}$$

*where $\alpha = \inf_{t\in[x,y]} g(t)$ and $\beta = \sup_{t\in[x,y]} g(t)$.*

*Proof.* These follow from inequality (2.8), Lemma 1, Lemma 2, and standard arguments. ∎

## 3. Monotonicities

**Lemma 3** ([16])**.** *Assume that the derivative of second order $f''(t)$ exists on $\mathbb{R}$. If $f(t)$ is an increasing (or convex) function on $\mathbb{R}$, then the arithmetic mean of function $f(t)$,*

$$\phi(r,s) = \begin{cases} \dfrac{1}{s-r}\displaystyle\int_r^s f(t)\,dt, & r \ne s, \\ f(r), & r = s, \end{cases} \tag{3.1}$$

*is also increasing (or convex, respectively) with both $r$ and $s$ on $\mathbb{R}$.*

*Proof.* Direct calculation yields

$$\frac{\partial\phi(r,s)}{\partial s} = \frac{1}{(s-r)^2}\Big[(s-r)f(s) - \int_r^s f(t)dt\Big], \tag{3.2}$$

$$\frac{\partial^2\phi(r,s)}{\partial s^2} = \frac{(s-r)^2 f'(s) - 2(s-r)f(s) + 2\int_r^s f(t)dt}{(s-r)^3} \equiv \frac{\varphi(r,s)}{(s-r)^3}, \tag{3.3}$$

$$\frac{\partial\varphi(r,s)}{\partial s} = (s-r)^2 f''(s). \tag{3.4}$$

In the case of $f'(t) \ge 0$, we have $\partial\phi(r,s)/\partial s \ge 0$, thus $\phi(r,s)$ increases in both $r$ and $s$, since $\phi(r,s) = \phi(s,r)$.

In the case of $f''(t) \ge 0$, $\varphi(r,s)$ increases with $s$. Since $\varphi(r,r) = 0$, we have $\partial^2\phi(r,s)/\partial s^2 \ge 0$. Therefore $\phi(r,s)$ is convex with respect to either $r$ or $s$, since $\phi(r,s) = \phi(s,r)$. This completes the proof. ∎

**Theorem 4.** *The mean $M_n(p;a;r,s)$ of numbers $a=(a_1,\dots,a_n)$ with weights $p=(p_1,\dots,p_n)$ and two parameters $r$ and $s$ is increasing in both $r$ and $s$.*

*Proof.* Set $N_n=\ln M_n$, then we have

$$N_n(p;a;r,s)=\frac{1}{r-s}\int_s^r\frac{\sum_{i=1}^n p_i a_i^t\ln a_i}{\sum_{i=1}^n p_i a_i^t}dt,\quad r-s\neq 0; \tag{3.5}$$

$$N_n(p;a;r,r)=\frac{\sum_{i=1}^n p_i a_i^r\ln a_i}{\sum_{i=1}^n p_i a_i^r}. \tag{3.6}$$

By Cauchy's inequality, direct calculation arrives at

$$\left(\frac{\sum_{i=1}^n p_i a_i^t\ln a_i}{\sum_{i=1}^n p_i a_i^t}\right)_t=\frac{\sum_{i=1}^n p_i a_i^t(\ln a_i)^2\sum_{i=1}^n p_i a_i^t-\left(\sum_{i=1}^n p_i a_i^t\ln a_i\right)^2}{\left(\sum_{i=1}^n p_i a_i^t\right)^2}\geq 0. \tag{3.7}$$

Combination of (3.7) with Lemma 3 yields the statement of Theorem 4. ∎

**Theorem 5.** *For a monotonic sequence of positive numbers $0<a_1\leq a_2\leq\cdots$ and positive weights $p=(p_1,p_2,\dots)$, if $m<n$, then*

$$M_m(p;a;r,s)\leq M_n(p;a;r,s). \tag{3.8}$$

*Equality holds if $a_1=a_2=\cdots$.*

*Proof.* For $r\geq s$, inequality (3.8) reduces to

$$\frac{\sum_{i=1}^m p_i a_i^r}{\sum_{i=1}^m p_i a_i^s}\leq\frac{\sum_{i=1}^n p_i a_i^r}{\sum_{i=1}^n p_i a_i^s}. \tag{3.9}$$

Since $0<a_1\leq a_2\leq\cdots$, $p_i>0$, $i\geq 1$, the sequences $\{p_i a_i^r\}_{i=1}^\infty$ and $\{p_i a_i^s\}_{i=1}^\infty$ are positive and monotonic.

By mathematical induction and the elementary inequalities (2.8), we can easily obtain the inequality (3.9). The proof of Theorem 5 is completed. ∎

**Lemma 4.** *If $A=(A_1,\dots,A_n)$ and $B=(B_1,\dots,B_n)$ are two nondecreasing (or nonincreasing) sequences and $P=(P_1,\dots,P_n)$ is a nonnegative sequence, then*

$$\sum_{i=1}^n P_i\sum_{i=1}^n P_iA_iB_i\geq\sum_{i=1}^n P_iA_i\sum_{i=1}^n P_iB_i, \tag{3.10}$$

*with equality if and only if at least one of the sequences $A$ or $B$ is constant.*

*If one of the sequences $A$ or $B$ is nonincreasing and the other nondecreasing, then the inequality in* (3.10) *is reversed.*

The inequality (3.10) is known in the literature as Tchebycheff's (or Čebyšev's) inequality in discrete form [7, p. 240].

**Theorem 6.** *Let $p = (p_1, \dots, p_n)$ and $q = (q_1, \dots, q_n)$ be positive weights, $a = (a_1, \dots, a_n)$ a sequence of positive numbers. If the sequences $(p_1/q_1, \dots, p_n/q_n)$ and $a$ are both nonincreasing or both nondecreasing, then*

$$M_n(p; a; r, s) \geq M_n(q; a; r, s). \tag{3.11}$$

*If one of the sequences of $(p_1/q_1, \dots, p_n/q_n)$ or $a$ is nonincreasing and the other nondecreasing, the inequality* (3.11) *is reversed.*

*Proof.* Substitution of $P = (q_1 a_1^s, \dots, q_n a_n^s)$, $A = (a_1^{r-s}, \dots, a_n^{r-s})$ and $B = (p_1/q_1, \dots, p_n/q_n)$ into inequality (3.10) and the standard arguments produce inequality (3.11). This completes the proof of Theorem 6. ∎

**Theorem 7.** *Let $p = (p_1, \dots, p_n)$ be positive weights, $a = (a_1, \dots, a_n)$ and $b = (b_1, \dots, b_n)$ two sequences of positive numbers. If the sequences $(a_1/b_1, \dots, a_n/b_n)$ and $b$ are both increasing or both decreasing, then*

$$M_n(p; a; r, s) \geq M_n(p; b; r, s) \tag{3.12}$$

*holds for $a_i/b_i \geq 1$, $n \geq i \geq 1$, and $r, s \geq 0$ or $r \geq 0 \geq s$. The inequality* (3.12) *is reversed for $a_i/b_i \leq 1$, $n \geq i \geq 1$, and $r, s \leq 0$ or $s \geq 0 \geq r$.*

*If one of the sequences of $(a_1/b_1, \dots, a_n/b_n)$ or $b$ is nonincreasing and the other nondecreasing, then inequality* (3.12) *is valid for $a_i/b_i \geq 1$, $n \geq i \geq 1$ and $r, s \geq 0$ or $s \geq 0 \geq r$; the inequality* (3.12) *reverses for $a_i/b_i \leq 1$, $n \geq i \geq 1$, and $r, s \geq 0$ or $r \geq 0 \geq s$,.*

*Proof.* The inequality (3.10) applied to

$$P_i = p_i b_i^r, \quad A_i = \left(\frac{a_i}{b_i}\right)^r, \quad B_i = b_i^{s-r}, \quad 1 \leq i \leq n \tag{3.13}$$

and the standard arguments yield Theorem 7. ∎

**Theorem 8.** *Suppose $p$ and $g$ are defined on $\mathbb{R}$. If $f_1 \circ g$ has constant sign and if $(f_1/f_2) \circ g$ is increasing (or decreasing, respectively), then $M(p; g; f_1, f_2; x, y)$ have the inverse (or same) monotonicities as $f_1/f_2$ with both $x$ and $y$.*

*Proof.* Without loss of generality, suppose $(f_1/f_2) \circ g$ increases. By straightforward computation and using Lemma 1, we obtain

$$\begin{aligned} &\frac{d}{dy}\left(\frac{\int_x^y p(t) f_1(g(t))dt}{\int_x^y p(t) f_2(g(t))dt}\right) \\ &\quad = \frac{p(y) f_1(g(y)) \int_x^y p(t) f_1(g(t))dt}{\left(\int_x^y p(t) f_2(g(t))dt\right)^2} \left(\frac{\int_x^y p(t) f_2(g(t))dt}{\int_x^y p(t) f_1(g(t))dt} - \frac{f_2(g(y))}{f_1(g(y))}\right) \leq 0. \end{aligned} \tag{3.14}$$

From Definition 4 and its suitable basic properties, Theorem 8 follows. ∎

**Lemma 5.** *Let $G, H : [a, b] \to \mathbb{R}$ be integrable functions, both increasing or both decreasing. Furthermore, let $Q : [a, b] \to [0, +\infty)$ be an integrable function. Then*

$$\int_a^b Q(u)G(u)\, du \int_a^b Q(u)H(u)\, du \leq \int_a^b Q(u)\, du \int_a^b Q(u)G(u)H(u)\, du. \tag{3.15}$$

*If one of the functions of $G$ or $H$ is nonincreasing and the other nondecreasing, then the inequality* (3.15) *reverses.*

Inequality (3.15) is called Tchebycheff's integral inequality, please refer to [1] and [4]–[7].

*Remark* 3. Using Tchebycheff's integral inequality, some inequalities of the complete elliptic integrals are established in [15], many inequalities concerning the probability function, the error function, and so on, are improved in [12].

**Theorem 9.** *Suppose $f_2 \circ g$ has constant sign on $[x, y]$. When $g(t)$ increases on $[x, y]$, if $p_1/p_2$ is increasing, we have*

$$M(p_1; g; f_1, f_2; x, y) \geq M(p_2; g; f_1, f_2; x, y); \tag{3.16}$$

*if $p_1/p_2$ is decreasing, inequality* (3.16) *reverses.*

*When $g(t)$ decreases on $[x, y]$, if $p_1/p_2$ is increasing, then inequality* (3.16) *is reversed; if $p_1/p_2$ is decreasing, inequality* (3.16) *holds.*

*Proof.* Substitution of $Q(t) = f_2(g(t))p_2(t)$, $G(t) = (f_1/f_2) \circ g(t)$ and $H(t) = p_1(t)/p_2(t)$ into Lemma 5 and the standard arguments produce inequality (3.16). The proof of Theorem 9 is completed. ∎

**Theorem 10.** *Suppose $f_2 \circ g_2$ does not change its sign on $[x, y]$.*

(i) *When $f_2 \circ (g_1/g_2)$ and $(f_1/f_2) \circ g_2$ are both increasing or both decreasing, inequality*

$$M(p; g_1; f_1, f_2; x, y) \geq M(p; g_2; f_1, f_2; x, y) \tag{3.17}$$

*holds for $f_1/f_2$ being increasing, or reverses for $f_1/f_2$ being decreasing.*

(ii) *When one of the functions $f_2 \circ (g_1/g_2)$ or $(f_1/f_2) \circ g_2$ is decreasing and the other increasing, inequality* (3.17) *holds for $f_1/f_2$ being decreasing, or reverses for $f_1/f_2$ being increasing.*

*Proof.* The inequality (3.15) applied to $Q(t) = p(t)(f_2 \circ g_2)(t)$, $G(t) = f_2 \circ \left(\frac{g_1}{g_2}\right)(t)$ and $H(t) = \left(\frac{f_1}{f_2}\right) \circ g_2(t)$, and standard arguments yield Theorem 10. ∎

## References

[1] E. F. Beckenbach and R. Bellman, *Inequalities*, Springer, Berlin, 1983.

[2] P. S. Bullen, D. S. Mitrinović and P. M. Vasić, *Means and Their Inequalities*, D. Reidel Publ. Company, Dordrecht/Boston/Lancaster/Tokyo, 1988.

[3] Yong Ding, *Two classes of means and their applications*, Mathematics in Practice and Theory **25** (1995), no. 2, 16–20. (Chinese)

[4] G. H. Hardy, J. E. Littlewood and G. Pólya, *Inequalities*, 2nd edition, Cambridge University Press, Cambridge, 1952.

[5] Ji-Chang Kuang, *Applied Inequalities*, 2nd edition, Hunan Education Press, Changsha, China, 1993. (Chinese)

[6] D. S. Mitrinović, *Analytic Inequalities*, Springer-Verlag, Berlin, 1970.

[7] D. S. Mitrinović, J. E. Pečarić and A. M. Fink, *Classical and New Inequalities in Analysis*, Kluwer Academic Publishers, Dordrecht/Boston/London, 1993.

[8] J. Pečarić, Feng Qi, V. Šimić and Sen-Lin Xu, *Refinements and extensions of an inequality, III*, J. Math. Anal. Appl. **227** (1998), no. 2, 439–448.

[9] Feng Qi, *Generalized weighted mean values with two parameters*, Proc. Roy. Soc. London Ser. A **454** (1998), no. 1978, 2723–2732.

[10] Feng Qi, *Monotonicity results and inequalities for the gamma and incomplete gamma functions*, submitted for publication.

[11] Feng Qi, *Studies on Problems in Topology and Geometry and on Generalized Weighted Abstracted Mean Values*, Thesis submitted for the degree of Doctor of Philosophy at University of Science and Technology of China, Hefei City, Anhui Province, China, Winter 1998. (Chinese)

[12] Feng Qi, Li-Hong Cui and Sen-Lin Xu, *Some inequalities constructed by Tchebycheff's integral inequality*, submitted for publication.

[13] Feng Qi and Lokenath Debnath, *Inequalities of power-exponential functions*, Intern. J. Math. Math. Sci. **23** (2000), in the press.

[14] Feng Qi and Bai-Ni Guo, *Inequalities for generalized weighted mean values of convex function*, submitted for publication.

[15] Feng Qi and Zheng Huang, *Inequalities of the complete elliptic integrals*, Tamkang Journal of Mathematics **29** (1998), no. 3, 165–169.

[16] Feng Qi, Sen-Lin Xu and Lokenath Debnath, *A new proof of monotonicity for extended mean values*, Intern. J. Math. Math. Sci. **22** (1999), no. 2, 415–420.

[17] Feng Qi and Shi-Qin Zhang, *Note on monotonicity of generalized weighted mean values*, Proc. Roy. Soc. London Ser. A **455** (1999), no. 1989, 3259–3260.

[18] Ming-Bao Sun, *Inequalities for two-parameter mean of convex function*, Mathematics in Practice and Theory **27** (1997), no. 3, 193–197. (Chinese)

[19] Da-Feng Xia, Sen-Lin Xu, and Feng Qi, *A proof of the arithmetic mean-geometric mean-harmonic mean inequalities,* RGMIA Research Report Collection 2 (1999), no. 1, 85–88.

Department of Mathematics, Jiaozuo Institute of Technology, Jiaozuo City, Henan 454000, The People's Republic of China

*E-mail address*: qifeng@jzit.edu.cn