Jensen-Type Inequalities for Invex Functions

JENSEN-TYPE INEQUALITIES FOR INVEX FUNCTIONS

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Abstract. Jensen's inequality for a real convex function f on a convex domain is generalised in several ways to vector functions with a cone inequality, and to invex functions generalizing convex functions.

1. Introduction

Jensen's inequality for a real convex function f on a convex domain C states that, whenever $x_1, x_2, ... \in C$ and $\alpha_1, \alpha_2, ... \geq 0$ with $\sum \alpha_i = 1$,

$$f\left(\sum \alpha_i x_i\right) \leq \sum \alpha_i f(x_i).$$

This paper presents several generalisations of this inequality, to vector functions F with a cone inequality, and to invex functions generalizing convex functions. This introduces additional gradient terms into various inequalities.

2. Basic Concepts and Definitions

Definition 1. *Let X and Z be normed spaces, $Q \subset Z$ a closed convex cone, and $E \subset X$ a convex open set. A function $F : E \to Z$ is $Q-$convex on E if*

$$(\forall x, y \in E)(\forall \alpha \in (0,1)) \quad \alpha F(x) + (1-\alpha) F(y) - F(\alpha x + (1-\alpha) y) \in Q.$$

A (Fréchet or linear Gateaux) differentiable function $F : E \to Z$ is $Q-$invex (see [1], [2], [3]*) if, for some scale function $\omega : E \times E \to X$,*

$$(\forall x, y \in E) \quad F(x) - F(y) - F'(y)\,\omega(x-y, y) \in Q.$$

Remark 1. *$F'(y)$ denotes the derivative. Invex properties have been extensively used with optimization problems. It is well-known that $Q-$invex implies $Q-$convex if $(\forall x, y \in E)$ $\omega(x-y, y) = x-y$. If F is real-valued and $Q = \mathbb{R}_+$, then the usual convexity follows. From $Q-$convex there follows readily*

$$(\forall x_i \in E)\left(\forall \alpha_i \geq 0,\ \sum \alpha_i = 1\right) \sum \alpha_i F(x_i) - F\left(\sum \alpha_i x_i\right) \in Q.$$

In this paper, invex shall require the function to be differentiable. (Extensions to nondifferentiable functions will be considered elsewhere.)

Denote by $\geq_Q$ the ordering defined by Q thus $b \geq_Q a \Leftrightarrow b - a \in Q$.

If $\omega(\cdot, y)$ is linear, then $(\forall z = x - y)\, F(y+z) - F(z) \geq_Q F'(y)\, Cz$, where C is a linear mapping (that may depend on y), hence $(\forall z)$ $F'(y)\, z \geq_Q F'(y)\, Cz$;

if Q is pointed (thus if $Q \cap (-Q) = \{0\}$) then F is convex, since

$$(\forall x) \ F(x) - F(y) \geq_Q F'(y)(x-y) + F'(y)(C-I)(x-y) = F'(y)(x-y).$$

Thus functions where $\omega(\cdot, y)$ is linear for each y are equivalent to convex functions.

Date: 2nd August, 1999.

Notation 1. *Various results require a suitable notation, in order that the concept is not obscured by algebraic details. Let* $x = (x_1, x_2, ..., x_n)$, *where each* x_i *is a vector in* E. *Let* $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$ *where each* $\alpha_i \geq 0$ *and* $\sum \alpha_i = 1$. *Denote by* $\alpha \cdot x$ *the inner product* $\sum \alpha_i x_i$, *and similarly for other inner products. Denote by* S *a sequence of indices* $(i_1, i_2, ..., i_r)$ *taken from* $\{1, 2, ..., n\}$ *with repetitions allowed; similarly denote by* S' *a sequence of indices* $\{i_1, i_2, ..., i_{r+1}\}$. *Denote by* x_S *the vector of* x_i *when* i *runs through* S; *and similarly define* α_S. *Similarly* $F'(x_S)$ *denotes the sequence of* $F'(x_i)$ *as* i *runs through* S. *Denote by* M_{x_S} *the mean of the sequence* x_S, *and by* Π_{α_S} *the product of the items in* α_S. *If* $\Phi(S)$ *denotes a function of* S, *denote by* $A\Phi(S)$ *the sum of the items in* $\Phi(S)$, *when* S *runs over all sequences of length* r, *and denote by* $A\Phi(S')$ *the sum of the items in* $\Phi(S')$, *where* S' *runs over the sequences of length* $r+1$. *Thus* $A(\Pi_{\alpha_S})\Phi(S)$ *represents the sum of all terms*

$$\alpha_{i_1}\alpha_{i_2}...\alpha_{i_r}\Phi(i_1, i_2, ..., i_r)$$

as $i_1, i_2, ..., i_r$ *run through* $1, 2, ..., n$.

Proposition 1. *Let* $F : E \to Z$ *be* $Q-$*invex; let* $x_1, x_2, ... \in E$ *(allowing possible repetitions), and let* $\alpha_1, \alpha_2, ...$ *be nonnegative with sum* 1; *let* $w = \sum \alpha_i x_i$. *Then*

$$\sum \alpha_i F(x_i) - F(w) \geq_Q F'(w) \sum \omega(x_i - w, w).$$

Proof. $Q-$invex requires, for each i, that

$$F(x_i) - F(w) \geq_Q F'(w)\,\omega(x_i - w, w).$$

Multiplication by α_i and summing gives the result. ∎

Proposition 2. *Let* $F : E \to Z$ *be differentiable* $Q-$*invex. Let* $x_1, x_2, ... \in E$ *(allowing possible repetitions), and let* $\alpha_1, \alpha_2, ...$ *be nonnegative with sum* 1. *Then*

$$\begin{aligned} &-\sum_j \alpha_j F'(x_j)\,\omega(\alpha \cdot (x - x_j), x_j) \\ &\geq_Q \alpha \cdot F(x_S) - F(\alpha \cdot x) \\ &\geq_Q F'(\alpha \cdot x) \sum \alpha_j \omega(\alpha \cdot (x_j - x), \alpha \cdot x). \end{aligned}$$

Proof. Let $w = \alpha \cdot x$. From invex,

$$F'(x_j)\,\omega(x - x_j, x_j) \geq_Q F(x_j) - F(w) \geq_Q F'(w)\,\omega(x_i - w, w).$$

Multiplication by α_j and summing gives the result. ∎

Corollary 1. *Assume also that* E *is a linear subspace, and* $\omega(\cdot, w)$ *is linear. Then* $\alpha \cdot F(x_S) - F(\alpha \cdot x) \geq_Q 0$.

Proof. $\sum_j \alpha_j \omega(\alpha \cdot (x - x_j), \alpha \cdot x) = \omega\left(\sum_{j,i} \alpha_j \alpha_i (x_i - x_j), \alpha \cdot x\right) = 0.$ ∎

Corollary 2. *Under hypothesis of Corollary 1,*

$$\sum_j \alpha_j F'(x_j)\,\omega(x_j, x_j) - \sum_{i,j} \alpha_i \alpha_j F'(x_j)\,\omega(x_j, x_j)$$

$$\geq_Q \sum_j \alpha_j F(x_j) - F\left(\sum_j \alpha_j x_j\right) \geq_Q 0.$$

Proof. Since $\omega(\cdot, w)$ is linear,

$$
\begin{aligned}
& -\sum_j \alpha_j F'(x_j)\, \omega(\alpha \cdot (x - x_j), x_j) \\
= & -\sum_j F'(x_j)\, \omega\left(\sum_i \alpha_j \alpha_i \cdot (x_i - x_j), x_j\right) \\
= & \sum_j \alpha_j F'(x_j)\, \omega(x_j, x_j) - \sum_{i,j} \alpha_i \alpha_j F'(x_j)\, \omega(x_j, x_j).
\end{aligned}
$$

Then the result follows from Corollary 1. ∎

Corollary 3. *If F is Q-convex, then*

$$
\begin{aligned}
& \sum_j \alpha_j F'(x_j)\left(\sum_i \alpha_i x_i\right) - \sum_j \alpha_j F'(x_j)\left(\sum_i \alpha_i x_i\right) \\
\geq_Q & \sum_j \alpha_j F(x_j) - F\left(\sum_j \alpha_j x_j\right) \geq_Q 0.
\end{aligned}
$$

Proof. By substituting $\omega(\alpha \cdot (x - x_j), x_j) = \sum_j \alpha_j (x_i - x_j)$ into the left inequality of Proposition 2, then rearranging the terms. ∎

3. Generalized Jensen Inequalities for Invex Functions

The following generalizations of Jensen's inequalities, involving multiple summations, hold for $Q-$invex functions.

Proposition 3. *Let $F : E \to Z$ be differentiable $Q-$invex; let $r \geq 1$. Then*

$$
\begin{aligned}
& -A(\Pi\alpha_S)\, F'(M_{x_S})\, \omega((\alpha \cdot (x - M_{x_S}), M_{x_S})) \\
\geq_Q & \; A(\Pi\alpha_S)\, F(M_{x_S}) - F(\alpha \cdot x) \\
\geq_Q & \; F'(\alpha \cdot x)\, A(\Pi\alpha_S)\, \omega(\alpha \cdot (M_{x_S} - x), M_{x_S}).
\end{aligned}
$$

Proof. By the $Q-$invex property,

$$
\begin{aligned}
& -F'(M_{x_S})\, \omega(\alpha \cdot (x - M_{x_S}), M_{x_S}) \\
\geq_Q & \; F(M_{x_S}) - F(\alpha \cdot x) \\
\geq_Q & \; F'(\alpha \cdot x)\, \omega((M_{x_S} - \alpha \cdot x), \alpha \cdot x).
\end{aligned}
$$

The result follows by applying $J \equiv A^r \Pi\alpha_S$ on the left, noting that cone inequalities are thus unchanged, and $J\xi = \xi$ for any argument ξ independent of S. ∎

Corollary 4. *Assume the hypothesis of Proposition 3. If E is a linear subspace and $\omega(\cdot, M_{x_S})$ is linear, then*

$$
A(\Pi\alpha_S 0)\, F(M_{x_S}) - F(\alpha \cdot x) \geq_Q 0.
$$

Proof. ¿From the linear assumption,

$$
A(\Pi\alpha_S)\, \omega(\alpha \cdot (M_{x_S} - x), M_{x_S}) = \omega(u, M_{x_S}),
$$

where, after simplification,

$$
u = (A(\Pi\alpha_S))\, \alpha \cdot (M_{x_S} - x) = \alpha \cdot (M_{x_S} - x) = 0.
$$

∎

In the following Corollary, which generalizes Corollary 2, note that $\xi = x_{i_1}$ in the first summation, and S has r terms; in the second summation, M_{x_S} is unchanged, but S' has $r+1$ terms, and $\eta = x_{i_{r+1}}$.

Corollary 5. *Under the hypothesis of Corollary 4,*

$$\begin{aligned} &-A\left(\Pi\alpha_S\right)F'\left(M_{x_S}\right)\omega\left(\xi, M_{x_S}\right) - A\left(\Pi\alpha_{S'}\right)F'\left(M_{x_S}\right)\omega\left(\eta, M_{x_S}\right) \\ \geq_Q\ & A\left(\Pi\alpha_S\right)F\left(M_{x_S}\right) - F\left(\alpha\cdot x\right). \end{aligned}$$

Proof. Since $\omega\left(\cdot, M_{x_S}\right)$ is linear,

$$\begin{aligned} & -A\left(\Pi\alpha_S\right)F'\left(M_{x_S}\right)\omega\left(\alpha\cdot\left(\xi - M_{x_S}\right), M_{x_S}\right) \\ =\ & A\left(\Pi\alpha_S\right)F'\left(M_{x_S}\right)\omega\left(M_{x_S}, M_{x_S}\right) - A\left(\Pi\alpha_S\right)F'\left(M_{x_S}\right)\alpha\cdot\omega\left(\xi, M_{x_S}\right) \\ =\ & A\left(\Pi\alpha_S\right)F'\left(M_{x_S}\right)\omega\left(\xi, M_{x_S}\right) - A\left(\Pi\alpha_{S'}\right)F'\left(M_{x_S}\right)\omega\left(\eta, M_{x_S}\right). \end{aligned}$$

The conclusion follows by applying Proposition 3. ∎

Corollary 6. *If F is convex, then (denoting $\xi = x_{i_1}$)*

$$\begin{aligned} &-A\left(\Pi\alpha_S\right)F'\left(M_{x_S}\right)\xi - A\left(\Pi\alpha_S\right)F'\left(M_{x_S}\right)\left(M_{x_S}\right) \\ \geq_Q\ & A\left(\Pi\alpha_S\right)F\left(M_{x_S}\right) - F\left(\alpha\cdot x\right) \\ \geq_Q\ & 0. \end{aligned}$$

Proof. Substitute $\omega\left(\alpha\cdot\left(x - M_{x_S}\right), M_{x_S}\right) = \alpha\left(x - M_{x_S}\right)$. The details are omitted. ∎

4. Further Refinements

The notation of Corollary 5 is used to state Proposition 4, namely $\xi = x_i$, $\eta = x_{i_{r+1}}$. Note that S has r terms, S' has $r+1$ terms.

Proposition 4. *Let $F : E \to Z$ be differentiable $Q-$invex. Then, with $\varepsilon = (r+1)^{-1}$,*

$$\begin{aligned} &-A\left(\Pi\alpha_{S'}\right)F'\left(M_{x_S}\right)\omega\left(\varepsilon\left(\eta - M_{x_S}\right), M_{x_S}\right) \\ \geq_Q\ & A\left(\Pi\alpha_S\right)F'\left(M_{x_S}\right) - A\left(\Pi\alpha_{S'}\right)F\left(M_{x_{S'}}\right) \\ \geq_Q\ & A\left(\Pi\alpha_{S'}\right)F'\left(M_{x_{S'}}\right)\omega\left(\varepsilon\left(M_{x_S} - \eta\right), M_{x_{S'}}\right). \end{aligned}$$

Proof. ¿From $Q-$invex,

$$\begin{aligned} & F'\left(M_{x_S}\right)\omega\left(\varepsilon\left(\eta - M_{x_S}\right), M_{x_S}\right) \\ \geq_Q\ & F\left(M_{x_S}\right) - F\left(M_{x_{S'}}\right) \\ \geq_Q\ & F'\left(M_{x_{S'}}\right)\omega\left(\varepsilon\left(M_{x_S} - \eta\right), M_{x_{S'}}\right). \end{aligned}$$

The result follows on applying $\left(\Pi\alpha_{S'}\right)$ and averaging with A. ∎

Corollary 7. *If F is as in Proposition 4, and $\omega\left(\cdot, M_{x_{S'}}\right)$ is linear, then, for each $r = 1, 2, ...$,*

$$A\left(\Pi\alpha_S\right)F\left(M_{x_S}\right) \geq_Q A\left(\Pi\alpha_{S'}\right)F\left(M_{x_{S'}}\right).$$

Proof. This follows from the second inequality of Proposition 4, noting that, when $\omega\left(\cdot, M_{x_{S'}}\right)$ is linear,

$$\omega\left(\varepsilon\left(M_{x_S} - \eta\right), M_{x_{S'}}\right) = \left(\frac{\varepsilon}{r}\right)\sum_{1}^{n}\omega\left(x_{i_j}, M_{x_{S'}}\right) = \varepsilon\omega\left(x_{i_{r+1}}, M_{x_{S'}}\right),$$

and application of $A\left(\Pi\alpha_{S'}\right)$ gives an equal value for each summand, so the result is 0. ∎

Corollary 8. *Under the same hypotheses as Corollary 7,*

$$\begin{aligned} & A\left(\Pi\alpha_S\right) F'\left(M_{x_S}\right)\xi - A\left(\Pi\alpha_S\right) F'\left(M_{x_S}\left(M_{x_S}\right)\right) \\ \geq_Q & \; A\left(\Pi\alpha_S\right) F\left(M_{x_S}\right) - A\left(\Pi x_{S'}\right) F\left(M_{x_{s'}}\right) \\ \geq_Q & \; 0. \end{aligned}$$

Proof. ¿From the first inequality of Proposition 4, using the assumed linearity to expand $\omega\left(\cdot, M_{x_S}\right)$ in its first argument. ∎

For other recent results connected with Jensen's discrete inequality, see the papers [4]-[9], where further references are given

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