Inequalities of Hadamard's Type for Lipschitzian Mappings and their Applications

INEQUALITIES OF HADAMARD'S TYPE FOR LIPSCHITZIAN MAPPINGS AND THEIR APPLICATIONS

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Abstract. In this paper, we give some inequalities of Hadamard's type for M-Lipschitzian functions. Some applications which are connected with arithmetic mean, geometric mean, harmonic mean, logarithmical mean, identric mean, etc., for two positive numbers are also given.

I. Introduction

The inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}, \tag{1.1}$$

where $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is a convex function defined on the interval I of $\mathbb{R}$, the set of real numbers, and $a, b \in I$ with $a < b$, is well known as Hadamard's inequality.

For some recent results which generalize, improve and extend this classic inequality, see the papers ([1]-[16]) where further references are given. Here we will list only some results we need for our further considerations.

If $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is a function, then we can define

$$H : [0,1] \to \mathbb{R},\ H(t) = \frac{1}{b-a}\int_a^b f\left(tx + (1-t)\frac{a+b}{2}\right)dx$$

and

$$F : [0,1] \to \mathbb{R},\ F(t) = \frac{1}{(b-a)^2}\int_a^b \int_a^b f(tx + (1-t)y)dxdy,$$

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respectively ([4], [5], [8]).

For these mappings and if f is convex on $[a, b]$, then we have the following main properties ([5], [8]):

(1) H and F are convex on $[0, 1]$.
(2) H is monotonically nondecreasing on $[0, 1]$ and F is monotonically nonincreasing on $[0, \frac{1}{2}]$ and nondecreasing on $[\frac{1}{2}, 1]$.
(3) We have the bounds:

$$\inf_{t\in[0,1]} H(t) = H(0) = f(\frac{a+b}{2}),$$

$$\sup_{t\in[0,1]} H(t) = H(1) = \frac{1}{b-a}\int_a^b f(x)dx,$$

$$\inf_{t\in[0,1]} F(t) = F(\frac{1}{2}) = \frac{1}{(b-a)^2}\int_a^b\int_a^b f(\frac{x+y}{2})dxdy,$$

and

$$\sup_{t\in[0,1]} F(t) = F(0) = F(1) = \frac{1}{b-a}\int_a^b f(x)dx.$$

(4) We have the inequality

$$F(t) \geq \max\{H(t), H(1-t)\}$$

for all $t \in [0, 1]$.

II. Hadamard's Type Inequality

We will start with the following theorem containing two inequalities of Hadamard's type.

Theorem 2.1. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be an M-Lipschitzian mapping on I and $a, b \in I$ with $a < b$. Then we have the inequalities:*

$$\left|f(\frac{a+b}{2}) - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{M}{4}(b-a), \tag{2.1}$$

and

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{M}{3}(b-a). \tag{2.2}$$

Proof. Let $t \in [0,1]$. Then we have, for all $a, b \in I$

$$\begin{aligned} &|tf(a) + (1-t)f(b) - f(ta + (1-t)b)| \\ &\quad = |t(f(a) - f(ta + (1-t)b) + (1-t)(f(b) - f(ta + (1-t)b)| \\ &\quad \le t|f(a) - f(ta + (1-t)b)| + (1-t)|f(b) - f(ta + (1-t)b)| \qquad (2.3) \\ &\quad \le tM|a - (ta + (1-t)b)| + (1-t)M|b - (ta + (1-t)b)| \\ &\quad = 2t(1-t)M|b-a|. \end{aligned}$$

If we choose $t = \frac{1}{2}$, we have also

$$\left| \frac{f(a) + f(b)}{2} - f(\frac{a+b}{2}) \right| \le \frac{M}{2}|b-a|. \qquad (2.4)$$

If we put $ta + (1-t)b$ instead of a and $(1-t)a + tb$ instead of b in (2.4), respectively, then we have

$$\left| \frac{f(ta + (1-t)b) + f((1-t)a + tb)}{2} - f(\frac{a+b}{2}) \right| \le \frac{M|2t-1|}{2}|b-a| \qquad (2.5)$$

for all $t \in [0,1]$. If we integrate the inequality (2.5) on $[0,1]$, we have

$$\begin{aligned} &\left| \frac{1}{2}\left[\int_0^1 f(ta + (1-t)b)dt + \int_0^1 f((1-t)a + tb)dt \right] - f(\frac{a+b}{2}) \right| \\ &\qquad \le \frac{M|b-a|}{2} \int_0^1 |2t-1|dt. \end{aligned}$$

Thus, from

$$\begin{aligned} \int_0^1 f(ta + (1-t)b)dt &= \int_0^1 f((1-t)a + tb)dt \\ &= \frac{1}{b-a} \int_a^b f(x)dx \end{aligned}$$

and

$$\int_0^1 |2t-1|dt = \frac{1}{2},$$

we obtain the inequality (2.1).

Note that, by the inequality (2.3), we have

$$|tf(a) + (1-t)f(b) - f(ta + (1-t)b)| \le 2t(1-t)M(b-a)$$

for all $t \in [0,1]$ and $a, b \in I$ with $a < b$. Integrating on $[0,1]$, we have

$$\left| f(a)\int_0^1 tdt + f(b)\int_0^1 (1-t)dt - \int_0^1 f(ta+(1-t)b)dt \right| \\ \leq 2M(b-a)\int_0^1 t(1-t)dt.$$

Hence, from

$$\int_0^1 tdt = \int_0^1 (1-t)dt = \frac{1}{2}, \int_0^1 t(1-t)dt = \frac{1}{6},$$

we have

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx \right| \leq \frac{M}{3}(b-a)$$

and so we have the inequality (2.2). This completes the proof.

The following corollary is important in applications:

Corollary 2.2. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable convex mapping on I, $a, b \in I$ with $a < b$ and $M := \sup_{t\in[a,b]} |f'(t)| < \infty$. Then we have the following complements of Hadamard's inequalities:*

$$0 \leq \frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right) \leq \frac{M}{4}(b-a) \tag{2.6}$$

and

$$0 \leq \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{M}{3}(b-a). \tag{2.7}$$

Proof. The proof is obvious by Lagrange's theorem, i.e., we recall that for any $x, y \in (a, b)$ there exists a c between them so that

$$|f(x) - f(y)| = |x-y|\,|f'(c)| \leq M\,|x-y|,$$

and Theorem 2.1. We shall omit the details.

The following corollaries for elementary inequalities holds:

Corollary 2.3. (1) *Let $p \geq 1$ and $a, b \in \mathbb{R}$ with $0 \leq a < b$. Then we have the inequalities:*

$$0 \leq \frac{b^{p+1} - a^{p+1}}{(p+1)(b-a)} - \Big(\frac{a+b}{2}\Big)^p \leq \frac{pb^{p-1}}{4}(b-a)$$

and

$$0 \leq \frac{a^p + b^p}{2} - \frac{b^{p+1} - a^{p+1}}{(p+1)(b-a)} \leq \frac{pb^{p-1}}{3}(b-a).$$

(2) *Let $a, b \in \mathbb{R}$ with $0 < a < b$. Then we have the inequalities:*

$$0 \leq \frac{\ln b - \ln a}{b-a} - \frac{2}{a+b} \leq \frac{1}{4a^2}(b-a)$$

and

$$0 \leq \frac{a+b}{2ab} - \frac{\ln b - \ln a}{b-a} \leq \frac{1}{3a^2}(b-a).$$

(3) *Let $a, b \in \mathbb{R}$ with $a < b$. Then we have the inequalities*

$$0 \leq \frac{\exp(b) - \exp(a)}{b-a} - \exp\Big(\frac{a+b}{2}\Big) \leq \frac{\exp(b)}{4}(b-a)$$

and

$$0 \leq \frac{\exp(a) + \exp(b)}{2} - \frac{\exp(b) - \exp(a)}{b-a} \leq \frac{\exp(b)}{3}(b-a).$$

(4) *Let $a, b \in \mathbb{R}$ with $0 < a < b$. Then we have the inequalities*

$$1 \leq e\Big(\frac{a^a}{b^b}\Big)^{\frac{1}{b-a}} \frac{a+b}{2} \leq \exp\Big(\frac{1}{4a}(b-a)\Big)$$

and

$$1 \leq \frac{\big(\frac{b^b}{a^a}\big)^{\frac{1}{b-a}}}{e\sqrt{ab}} \leq \exp\Big(\frac{1}{3a}(b-a)\Big).$$

Proof. (1) The proof follows by Corollary 2.2 applied for the convex mapping $f(x) = x^p$ on $[a, b]$.

(2) The proof follows by Corollary 2.2 applied for the convex mapping $f(x) = \frac{1}{x}$ on $[a, b]$.

(3) The proof is obvious by Corollary 2.2 applied for the convex mapping $f(x) = \exp(x)$ on $\mathbb{R}$.

(4) The proof follows by Corollary 2.2 applied for the convex mapping $f(x) = -\ln x$ on $[a, b]$. This completes the proof.

Now, we shall point out some other inequalities of the types in Corollary 2.3, but these hold for the mappings which are not convex on $[a, b]$.

Corollary 2.4. (1) *Let $a, b \in \mathbb{R}$ with $a < b$ and $k \in \mathbb{N}$. Then we have the inequalities:*

$$\left|\left(\frac{a+b}{2}\right)^{2k+1} - \frac{b^{2k+2}-a^{2k+2}}{(2k+2)(b-a)}\right| \le \frac{(2k+1)\max\{a^{2k}, b^{2k}\}}{4}(b-a)$$

and

$$\left|\frac{a^{2k+1}+b^{2k+1}}{2} - \frac{b^{2k+2}-a^{2k+2}}{(2k+2)(b-a)}\right| \le \frac{(2k+1)\max\{a^{2k}, b^{2k}\}}{3}(b-a).$$

(2) *Let $a, b \in \mathbb{R}$ with $a < b$. Then we have the inequalities:*

$$\left|\cos\left(\frac{a+b}{2}\right) - \frac{\sin b - \sin a}{b-a}\right| \le \frac{b-a}{4}$$

and

$$\left|\frac{\cos a + \cos b}{2} - \frac{\sin b - \sin a}{b-a}\right| \le \frac{b-a}{3}.$$

Proof. (1) The proof follows by Theorem 2.1 applied for the mapping $f(x) = x^{2k+1}$ on $[a, b]$.

(2) The proof is obvious by Theorem 2.1 applied for the mapping $f(x) = \cos x$ on $[a, b]$. This completes the proof.

III. The Mapping H

For an M-Lipschitzian function $f : I \subseteq \mathbb{R} \to \mathbb{R}$, we can define a mapping $H : [0, 1] \to \mathbb{R}$ by

$$H(t) = \frac{1}{b-a}\int_a^b f\left(tx + (1-t)\frac{a+b}{2}\right)dx$$

for all $t \in [0, 1]$ and we shall give some properties of the mapping H:

Theorem 3.1. *Let a mapping $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be M-Lipschitzian on I and $a, b \in I$ with $a < b$. Then*

(1) *The mapping H is $\frac{M}{4}(b-a)$-Lipschitzian on $[0, 1]$.*
(2) *We have the inequalities:*

$$\left|H(t) - \frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{M(1-t)}{4}(b-a), \tag{3.1}$$

$$\left|f\big(\frac{a+b}{2}\big) - H(t)\right| \le \frac{Mt}{4}(b-a), \tag{3.2}$$

and

$$\left|H(t) - t\frac{1}{b-a}\int_a^b f(x)dx - (1-t)f\big(\frac{a+b}{2}\big)\right| \le \frac{t(1-t)M}{2}(b-a) \tag{3.3}$$

for all $t \in [0,1]$.

Proof. (1) Let $t_1, t_2 \in [0,1]$. Then we have

$$\begin{aligned}
|H(t_2) - H(t_1)| &= \frac{1}{b-a}\bigg|\int_a^b f\big(t_2 + (1-t_2)\frac{a+b}{2}\big)dx \\
&\qquad - \int_a^b f\big(t_1 x + (1-t_1)\frac{a+b}{2}\big)dx\bigg| \\
&\le \frac{1}{b-a}\int_a^b \Big|f\big(t_2 x + (1-t_2)\frac{a+b}{2}\big) - f\big(t_1 x + (1-t_1)\frac{a+b}{2}\big)\Big|dx \\
&\le \frac{M}{b-a}\int_a^b \Big|t_2 x + (1-t_2)\frac{a+b}{2} - t_1 x - (1-t_1)\frac{a+b}{2}\Big|dx \\
&= \frac{M|t_2 - t_1|}{b-a}\int_a^b \Big|x - \frac{a+b}{2}\Big|dx \\
&= \frac{M(b-a)}{4}|t_2 - t_1|,
\end{aligned}$$

i.e., for all $t_1, t_2 \in [0,1]$,

$$|H(t_2) - H(t_1)| \le \frac{M(b-a)}{4}|t_2 - t_1|, \tag{3.4}$$

which yields that the mapping H is $\frac{M(b-a)}{4}$-Lipschitzian on $[0,1]$.

(2) The inequalities (3.1) and (3.2) follow from (3.4) by choosing $t_1 = 0$, $t_2 = t$ and $t_1 = 1$, $t_2 = t$, respectively.

Inequality (3.3) follows by adding t times (3.1) and $(1-t)$ times (3.2). This completes the proof.

Another result which is connected in a sense with the inequality (2.2) is also given in the following:

Theorem 3.2. *With the above assumptions, we have the inequality:*

$$\left|\frac{f\left(tb+(1-t)\frac{a+b}{2}\right)+f\left(ta+(1-t)\frac{a+b}{2}\right)}{2}-H(t)\right|\le\frac{Mt}{3}(b-a) \tag{3.5}$$

for all $t\in[0,1]$.

Proof. If we denote $u=tb+(1-t)\frac{a+b}{2}$ and $v=ta+(1-t)\frac{a+b}{2}$, then we have

$$H(t)=\frac{1}{u-v}\int_v^u f(z)dz.$$

Now, using the inequality (2.2) applied for u and v, we have

$$\left|\frac{f(u)+f(v)}{2}-\frac{1}{u-v}\int_v^u f(z)dz\right|\le\frac{M}{3}(u-v),$$

from which we have the inequality (3.5). This completes the proof.

Theorems 3.1 and 3.2 imply the following theorem which is important in applications for convex functions:

Theorem 3.3. *Let* $f:I\subseteq\mathbb{R}\to\mathbb{R}$ *be a differentiable convex mapping on* I, $a,b\in I$ *with* $a<b$ *and* $M=\sup_{x\in[a,b]}|f'(x)|<\infty$. *Then we have the inequalities:*

$$0\le\frac{1}{b-a}\int_a^b f(x)dx-H(t)\le\frac{M(1-t)}{4}(b-a), \tag{3,6}$$

$$0\le H(t)-f\Big(\frac{a+b}{2}\Big)\le\frac{Mt}{4}(b-a), \tag{3.7}$$

$$0\le\frac{f\left(tb+(1-t)\frac{a+b}{2}\right)+f\left(ta+(1-t)\frac{a+b}{2}\right)}{2}-H(t)\le\frac{Mt}{3}(b-a) \tag{3.8}$$

for all $t\in[0,1]$.

IV. The Mapping F

For an M-Lipschitzian function $f:I\subseteq\mathbb{R}\to\mathbb{R}$ we can define a mapping $F:[0,1]\to\mathbb{R}$ by

$$F(t)=\frac{1}{(b-a)^2}\int_a^b\int_a^b f(tx+(1-t)y)dxdy$$

and give some properties of the mapping F as follows:

Theorem 4.1. *Let a mapping $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be M-Lipschitzian on I and $a, b \in I$ with $a < b$. Then*

1. *The mapping F is symmetrical, i.e., $F(t) = F(1-t)$ for all $t \in [0,1]$.*
2. *The mapping F is $\frac{M(b-a)}{3}$-Lipschitzian on $[0,1]$.*
3. *We have the inequalities:*

$$\left|F(t) - \frac{1}{(b-a)^2}\int_a^b\int_a^b f(\frac{x+y}{2})dxdy\right| \le \frac{M|2t-1|}{6}(b-a), \tag{4.1}$$

$$\left|F(t) - \frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{Mt}{3}(b-a), \tag{4.2}$$

and

$$|F(t) - H(t)| \le \frac{M(1-t)}{4}(b-a) \tag{4.3}$$

for all $t \in [0,1]$.

Proof. (1) It is obvious by the definition of the mapping F.

(2) Let $t_1, t_2 \in [0,1]$. Then we have

$$\begin{aligned}
&|F(t_2) - F(t_1)| \\
&= \frac{1}{(b-a)^2}\left|\int_a^b\int_a^b [f(t_2x + (1-t_2)y) - f(t_1x + (1-t_1)y)]dxdy\right| \\
&\le \frac{1}{(b-a)^2}\int_a^b\int_a^b |f(t_2x + (1-t_2)y) - f(t_1x + (1-t_1)y|dxdy \\
&\le \frac{M|t_2-t_1|}{(b-a)^2}\int_a^b\int_a^b |x-y|dxdy.
\end{aligned} \tag{4.4}$$

Now, note that

$$\int_a^b\int_a^b |x-y|\,dxdy = \frac{(b-a)^3}{3}. \tag{4.5}$$

Therefore, from (4.4) and (4.5), it follows that

$$|F(t_2) - F(t_1)| \le \frac{M|t_2-t_1|}{3}(b-a) \tag{4.6}$$

for all $t_1, t_2 \in [0,1]$ and so the mapping F is $\frac{M(b-a)}{3}$-Lipschitzian on $[0,1]$.

(3) The inequalities (4.1) and (4.2) follow from (4.6) if we choose $t_1 = \frac{1}{2}$, $t_2 = t$ and $t_1 = 0$, $t_2 = t$, respectively.

Now, we prove the inequality (4.3). Since f is M-Lipschitzian, we can write

$$\begin{aligned} &\Big| f(tx + (1-t)y) - f\big(tx + (1-t)\frac{a+b}{2}\big)\Big| \\ &\leq M\big|tx + (1-t)y - tx - (1-t)\frac{a+b}{2}\big| \\ &= (1-t)M\Big|y - \frac{a+b}{2}\Big| \end{aligned} \tag{4.7}$$

for all $t \in [0,1]$ and $x, y \in [a,b]$. Integrating the inequality (4.7) on $[a,b] \times [a,b]$, we have

$$\begin{aligned} &\left| \frac{1}{(b-a)^2} \int_a^b \int_a^b f(tx + (1-t)y)\,dxdy - \frac{1}{b-a}\int_a^b f\big(tx + (1-t)\frac{a+b}{2}\big)\,dx \right| \\ &\leq (1-t)M\frac{1}{b-a}\int_a^b \big|y - \frac{a+b}{2}\big|\,dy \\ &= \frac{M(1-t)(b-a)}{4} \end{aligned}$$

for all $t \in [0,1]$ and so the inequality (4.3) is proved. This completes the proof.

Theorem 4.1 implies the following converses of the known results holding for convex functions (see the results listed in section I).

Corollary 4.2. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable convex mapping and $M := \sup_{x \in [a,b]} |f'(x)|$ for $a, b \in I$ with $a < b$. Then we have the inequalities:*

$$0 \leq F(t) - \frac{1}{(b-a)^2}\int_a^b \int_a^b f\big(\frac{x+y}{2}\big)\,dxdy \leq \frac{M|2t-1|}{6}(b-a),$$

$$0 \leq \frac{1}{b-a}\int_a^b f(x)\,dx - F(t) \leq \frac{Mt}{2}(b-a),$$

and

$$0 \leq F(t) - H(t) \leq \frac{M(1-t)}{4}(b-a)$$

for all $t \in [0,1]$.

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