An Inequality of the Ostrowski Type For Double Integrals and Applications for Cubature Formulae

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AN INEQUALITY OF THE OSTROWSKI TYPE FOR DOUBLE INTEGRALS AND APPLICATIONS FOR CUBATURE FORMULAE

S. S. DRAGOMIR, P. CERONE, N.S. BARNETT, AND J. ROUMELIOTIS

ABSTRACT. We point out a new inequality of the Ostrowski type for mappings of two independent variables, which complements, in a sense, some recent results and apply it to the approximation problem of double integrals by their Riemann sums.

1. INTRODUCTION

In the recent papers [1] and [2], the authors proved the following inequality of the Ostrowski type for double integrals.

Theorem 1. *Let $f : [a, b] \times [c, d] \to \mathbb{R}$ be continuous on $[a, b] \times [c, d]$. Then we have the inequality:*

$$
(1.1) \quad \left| \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(t, s) \, ds \, dt \right.
$$

$$
\left. - \frac{1}{d-c} \int_c^d f(x, s) \, ds - \frac{1}{b-a} \int_a^b f(t, y) \, dt + f(x, y) \right|
$$

$$
\leq
\begin{cases}
\left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] \left[\frac{1}{4} + \left(\frac{y - \frac{c+d}{2}}{d-c} \right)^2 \right] (b-a)(d-c) \left\| \frac{\partial^2 f}{\partial t \partial s} \right\|_\infty \\[2mm]
\text{if } \; \frac{\partial^2 f}{\partial t \partial s} \in L_\infty \left([a, b] \times [c, d] \right); \\[4mm]
\frac{1}{(q+1)^{\frac{2}{q}}} \left[\left(\frac{x-a}{b-a} \right)^{q+1} + \left(\frac{b-x}{b-a} \right)^{q+1} \right]^{\frac{1}{q}} \left[\left(\frac{y-c}{d-c} \right)^{q+1} + \left(\frac{d-y}{d-c} \right)^{q+1} \right]^{\frac{1}{q}} \\
\times \left[(b-a)(d-c) \right]^{\frac{1}{q}} \left\| \frac{\partial^2 f}{\partial t \partial s} \right\|_p \\[2mm]
\text{if } \; \frac{\partial^2 f}{\partial t \partial s} \in L_p \left([a, b] \times [c, d] \right), \; p > 1, \; \frac{1}{p} + \frac{1}{q} = 1; \\[4mm]
\left[\frac{1}{2} + \left| \frac{x - \frac{a+b}{2}}{b-a} \right| \right] \left[\frac{1}{2} + \left| \frac{y - \frac{c+d}{2}}{d-c} \right| \right] \left\| \frac{\partial^2 f}{\partial t \partial s} \right\|_1 \\[2mm]
\text{if } \; \frac{\partial^2 f}{\partial t \partial s} \in L_1 \left([a, b] \times [c, d] \right)
\end{cases}
$$

for all $(x, y) \in [a, b] \times [c, d]$, *where*

$$\left\| \frac{\partial^2 f}{\partial t \partial s} \right\|_\infty \quad : \quad = \quad \sup_{(t,s) \in [a,b] \times [c,d]} \left| \frac{\partial^2 f(t,s)}{\partial t \partial s} \right|,$$

$$\left\| \frac{\partial^2 f}{\partial t \partial s} \right\|_p \quad : \quad = \quad \left(\int_a^b \int_c^d \left| \frac{\partial^2 f(t,s)}{\partial t \partial s} \right|^p ds dt \right)^{\frac{1}{p}}$$

if $p \in [1, \infty)$.

The best inequality we can get from (1.1) is the one for which $x = \frac{a+b}{2}$ and $y = \frac{c+d}{2}$, obtaining:

Corollary 1. *With the assumptions in Theorem 1, we have the following mid-point type inequality:*

$$(1.2) \qquad \left| \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(t,s)\, ds dt - \frac{1}{d-c} \int_c^d f\left(\frac{a+b}{2}, s \right) ds \right.$$

$$\left. - \frac{1}{b-a} \int_a^b f\left(t, \frac{c+d}{2} \right) dt + f\left(\frac{a+b}{2}, \frac{c+d}{2} \right) \right|$$

$$\leq \begin{cases} \dfrac{1}{16} \left\| \dfrac{\partial^2 f}{\partial t \partial s} \right\|_\infty (b-a)(d-c) \quad \text{if} \ \dfrac{\partial^2 f}{\partial t \partial s} \in L_\infty\left([a,b] \times [c,d]\right); \\[3mm] \dfrac{1}{4(q+1)^{\frac{2}{q}}} \left[(b-a)(d-c)\right]^{\frac{1}{q}} \left\| \dfrac{\partial^2 f}{\partial t \partial s} \right\|_p \\[3mm] \quad \text{if} \ \dfrac{\partial^2 f}{\partial t \partial s} \in L_p\left([a,b] \times [c,d]\right), \quad \text{for } p > 1, \ \frac{1}{p} + \frac{1}{q} = 1; \\[3mm] \dfrac{1}{4} \left\| \dfrac{\partial^2 f}{\partial t \partial s} \right\|_1 \quad \text{if} \ \dfrac{\partial^2 f}{\partial t \partial s} \in L_1\left([a,b] \times [c,d]\right). \end{cases}$$

For some applications of the above results in Numerical Integration for cubature formulae see [1] and [2].

In this paper we point out some bounds for the quantity

$$\left| \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(t,s)\, ds dt - f(x,y) \right|,$$

where $(x, y) \in [a, b] \times [c, d]$ and use the results for the approximation of a double integral by its Riemann sums.

2. AN INTEGRAL IDENTITY

The following theorem holds.

Theorem 2. *Let* $f : [a, b] \times [c, d] \rightarrow \mathbb{R}$ *be such that the partial derivatives* $\frac{\partial f(t,s)}{\partial t}$, $\frac{\partial f(t,s)}{\partial s}$, $\frac{\partial^2 f(t,s)}{\partial t \partial s}$ *exist and are continuous on* $[a, b] \times [c, d]$. *Then for all* $(x, y) \in$

$[a, b] \times [c, d]$, *we have the representation*

$$(2.1) \quad f(x, y) = \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(t, s)\, ds\, dt$$

$$+ \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d p(x, t) \frac{\partial f(t, s)}{\partial t}\, ds\, dt$$

$$+ \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d q(y, s) \frac{\partial f(t, s)}{\partial s}\, ds\, dt$$

$$+ \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d p(x, t) q(y, s) \frac{\partial^2 f(t, s)}{\partial t \partial s}\, ds\, dt,$$

where $p : [a, b]^2 \to \mathbb{R}$, $q : [c, d]^2 \to \mathbb{R}$ *and are given by*

$$(2.2) \quad p(x, t) := \begin{cases} t - a & \text{if} \quad t \in [a, x] \\ t - b & \text{if} \quad t \in (x, b] \end{cases},$$

and

$$(2.3) \quad q(y, s) := \begin{cases} s - c & \text{if} \quad s \in [c, y] \\ s - d & \text{if} \quad s \in (y, d] \end{cases}.$$

Proof. We use the following identity, which can be easily proved by integration by parts,

$$(2.4) \quad g(u) = \frac{1}{\beta - \alpha} \int_\alpha^\beta g(z)\, dz + \frac{1}{\beta - \alpha} \int_\alpha^\beta k(u, z) g'(z)\, dz,$$

where $k : [\alpha, \beta]^2 \to \mathbb{R}$ is given by

$$k(u, z) := \begin{cases} z - \alpha & \text{if} \quad z \in [\alpha, u] \\ z - \beta & \text{if} \quad z \in (u, \beta] \end{cases}$$

and g is absolutely continuous on $[\alpha, \beta]$.

Indeed, we have

$$\int_\alpha^u (z - \alpha) g'(z)\, dz = (u - \alpha) g(u) - \int_\alpha^u g(z)\, dz$$

and

$$\int_u^\beta (z - \beta) g'(z)\, dz = (\beta - u) g(u) - \int_u^\beta g(z)\, dz$$

which produces, by summation, the desired identity (2.4).

Now, write the identity (2.4) for the partial map $f(\cdot, y)$, $y \in [c, d]$ to obtain

$$(2.5) \quad f(x, y) = \frac{1}{b-a} \int_a^b f(t, y)\, dt + \frac{1}{b-a} \int_a^b p(x, t) \frac{\partial f(t, y)}{\partial t}\, dt$$

for all $(x, y) \in [a, b] \times [c, d]$.

Also, if we write (2.4) for the map $f(t, \cdot)$, we get

$$(2.6) \quad f(t, y) = \frac{1}{d-c} \int_c^d f(t, s)\, ds + \frac{1}{d-c} \int_c^d q(y, s) \frac{\partial f(t, s)}{\partial s}\, ds,$$

for all $(t, y) \in [a, b] \times [c, d]$.

The same formula (2.4) applied for the partial derivative $\frac{\partial f(\cdot,y)}{\partial t}$ will produce

$$(2.7) \qquad \frac{\partial f(t,y)}{\partial t} = \frac{1}{d-c}\int_c^d \frac{\partial f(t,s)}{\partial t}ds + \frac{1}{d-c}\int_c^d q(y,s)\frac{\partial^2 f(t,s)}{\partial t\partial s}ds$$

for all $(t,y) \in [a,b] \times [c,d]$.

Substituting (2.6) and (2.7) in (2.5), and using Fubini's theorem, we have

$$
\begin{aligned}
f(x,y) &= \frac{1}{b-a}\int_a^b \left[\frac{1}{d-c}\int_c^d f(t,s)\,ds + \frac{1}{d-c}\int_c^d q(y,s)\frac{\partial f(t,s)}{\partial s}ds\right]dt \\
&\quad + \frac{1}{b-a}\int_a^b p(x,t)\left[\frac{1}{d-c}\int_c^d \frac{\partial f(t,s)}{\partial t}ds \right. \\
&\qquad \left. + \frac{1}{d-c}\int_c^d q(y,s)\frac{\partial^2 f(t,s)}{\partial t\partial s}ds\right]dt \\
&= \frac{1}{(b-a)(d-c)}\left[\int_a^b\int_c^d f(t,s)\,dsdt + \int_a^b\int_c^d q(y,s)\frac{\partial f(t,s)}{\partial s}dsdt\right. \\
&\qquad \left. + \int_a^b\int_c^d p(x,t)\frac{\partial f(t,s)}{\partial t}dsdt + \int_a^b\int_c^d p(x,t)q(y,s)\frac{\partial^2 f(t,s)}{\partial t\partial s}dsdt\right],
\end{aligned}
$$

and the identity (2.1) is completely proved. ∎

A particular case which is of interest is embodied in the following corollary.

Corollary 2. *Let f be as in Theorem 2, then we have the identity*

$$
\begin{aligned}
(2.8) \qquad & f\left(\frac{a+b}{2},\frac{c+d}{2}\right) \\
&= \frac{1}{(b-a)(d-c)}\left[\int_a^b\int_c^d f(t,s)\,dsdt + \int_a^b\int_c^d p_0(t)\frac{\partial f(t,s)}{\partial t}dsdt\right. \\
&\qquad \left. + \int_a^b\int_c^d q_0(s)\frac{\partial f(t,s)}{\partial s}dsdt + \int_a^b\int_c^d p_0(t)q_0(s)\frac{\partial^2 f(t,s)}{\partial t\partial s}dsdt\right],
\end{aligned}
$$

where $p_0 : [a,b] \to \mathbb{R}$, $q_0 : [c,d] \to \mathbb{R}$ are given by

$$
p_0(t) := \begin{cases} t-a & \text{if } t \in \left[a,\frac{a+b}{2}\right] \\ t-b & \text{if } t \in \left(\frac{a+b}{2},b\right] \end{cases},
$$

and

$$
q_0(s) := \begin{cases} s-c & \text{if } s \in \left[c,\frac{c+d}{2}\right] \\ s-d & \text{if } s \in \left(\frac{c+d}{2},d\right] \end{cases}.
$$

The following corollary which provides a trapezoid type identity is also of interest.

Corollary 3. *Let f be as in Theorem 2. Then we have the identity*

$$(2.9) \qquad \frac{f(a,c) + f(a,d) + f(b,c) + f(b,d)}{4}$$

$$= \frac{1}{(b-a)(d-c)} \left[\int_a^b \int_c^d f(t,s)\,dsdt + \int_a^b \int_c^d \left(t - \frac{a+b}{2}\right) \frac{\partial f(t,s)}{\partial t}\,dsdt \right.$$

$$+ \int_a^b \int_c^d \left(s - \frac{c+d}{2}\right) \frac{\partial f(t,s)}{\partial s}\,dsdt$$

$$\left. + \int_a^b \int_c^d \left(t - \frac{a+b}{2}\right)\left(s - \frac{c+d}{2}\right) \frac{\partial^2 f(t,s)}{\partial t \partial s}\,dsdt \right].$$

Proof. Letting $(x,y) = (a,c),\,(a,d),\,(b,c)$ and (b,d) in (2.1), we obtain successively

$$f(a,c) = \frac{1}{(b-a)(d-c)} \left[\int_a^b \int_c^d f(t,s)\,dsdt + \int_a^b \int_c^d (t-b) \frac{\partial f(t,s)}{\partial t}\,dsdt \right.$$

$$\left. + \int_a^b \int_c^d (s-d) \frac{\partial f(t,s)}{\partial s}\,dsdt + \int_a^b \int_c^d (t-b)(s-d) \frac{\partial^2 f(t,s)}{\partial s \partial t}\,dsdt \right],$$

$$f(a,d) = \frac{1}{(b-a)(d-c)} \left[\int_a^b \int_c^d f(t,s)\,dsdt + \int_a^b \int_c^d (t-b) \frac{\partial f(t,s)}{\partial t}\,dsdt \right.$$

$$\left. + \int_a^b \int_c^d (s-c) \frac{\partial f(t,s)}{\partial s}\,dsdt + \int_a^b \int_c^d (t-b)(s-c) \frac{\partial^2 f(t,s)}{\partial s \partial t}\,dsdt \right],$$

$$f(b,c) = \frac{1}{(b-a)(d-c)} \left[\int_a^b \int_c^d f(t,s)\,dsdt + \int_a^b \int_c^d (t-a) \frac{\partial f(t,s)}{\partial t}\,dsdt \right.$$

$$\left. + \int_a^b \int_c^d (s-d) \frac{\partial f(t,s)}{\partial s}\,dsdt + \int_a^b \int_c^d (t-a)(s-d) \frac{\partial^2 f(t,s)}{\partial s \partial t}\,dsdt \right],$$

and

$$f(b,d) = \frac{1}{(b-a)(d-c)} \left[\int_a^b \int_c^d f(t,s)\,dsdt + \int_a^b \int_c^d (t-a) \frac{\partial f(t,s)}{\partial t}\,dsdt \right.$$

$$\left. + \int_a^b \int_c^d (s-c) \frac{\partial f(t,s)}{\partial s}\,dsdt + \int_a^b \int_c^d (t-a)(s-c) \frac{\partial^2 f(t,s)}{\partial s \partial t}\,dsdt \right].$$

After summing over the above equalities, dividing by 4 and some simple computation, we arrive at the desired identity (2.9). ∎

3. Some Inequalities

We can state the following inequality of the Ostrowski type which holds for mappings of two independent variables.

Theorem 3. *Let* $f : [a, b] \times [c, d] \to \mathbb{R}$ *be a mapping as in Theorem 2. Then we have the inequality:*

$$(3.1) \qquad \left| f(x, y) - \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(t, s) \, ds dt \right|$$

$$\leq \quad M_1(x) + M_2(y) + M_3(x, y),$$

where

$$M_1(x) = \begin{cases} \dfrac{\left[\frac{1}{4}(b-a)^2 + \left(x - \frac{a+b}{2} \right)^2 \right]}{b-a} \left\| \dfrac{\partial f}{\partial t} \right\|_\infty, & if \quad \dfrac{\partial f(t,s)}{\partial t} \in L_\infty([a, b] \times [c, d]); \\[3ex] \dfrac{\left[\frac{(b-x)^{q_1+1} + (x-a)^{q_1+1}}{q_1+1} \right]^{\frac{1}{q_1}}}{(b-a)[(d-c)]^{\frac{1}{p_1}}} \left\| \dfrac{\partial f}{\partial t} \right\|_{p_1}, & if \quad \dfrac{\partial f(t,s)}{\partial t} \in L_{p_1}([a, b] \times [c, d]), \\[1ex] & \qquad p_1 > 1, \frac{1}{p_1} + \frac{1}{q_1} = 1; \\[2ex] \dfrac{\left[\frac{1}{2}(b-a) + \left| x - \frac{a+b}{2} \right| \right]}{(b-a)(d-c)} \left\| \dfrac{\partial f}{\partial t} \right\|_1, & if \quad \dfrac{\partial f(t,s)}{\partial t} \in L_1([a, b] \times [c, d]). \end{cases}$$

$$M_2(y) = \begin{cases} \dfrac{\left[\frac{1}{4}(d-c)^2 + \left(y - \frac{c+d}{2} \right)^2 \right]}{d-c} \left\| \dfrac{\partial f}{\partial s} \right\|_\infty, & if \quad \dfrac{\partial f(t,s)}{\partial s} \in L_\infty([a, b] \times [c, d]); \\[3ex] \dfrac{\left[\frac{(d-y)^{q_2+1} + (y-c)^{q_2+1}}{q_2+1} \right]^{\frac{1}{q_2}}}{[(b-a)]^{\frac{1}{p_2}}(d-c)} \left\| \dfrac{\partial f}{\partial s} \right\|_{p_2}, & if \quad \dfrac{\partial f(t,s)}{\partial s} \in L_{p_2}([a, b] \times [c, d]); \\[1ex] & \qquad p_2 > 1, \frac{1}{p_2} + \frac{1}{q_2} = 1; \\[2ex] \dfrac{\left[\frac{1}{2}(d-c) + \left| y - \frac{c+d}{2} \right| \right]}{(b-a)(d-c)} \left\| \dfrac{\partial f}{\partial s} \right\|_1, & if \quad \dfrac{\partial f(t,s)}{\partial s} \in L_1([a, b] \times [c, d]); \end{cases}$$

and

$$M_3(x, y)$$
$$= \begin{cases} \dfrac{\left[\frac{1}{4}(b-a)^2 + \left(x - \frac{a+b}{2} \right)^2 \right]\left[\frac{1}{4}(d-c)^2 + \left(y - \frac{c+d}{2} \right)^2 \right]}{(b-a)(d-c)} \left\| \dfrac{\partial^2 f}{\partial t \partial s} \right\|_\infty, \\[2ex] \quad if \quad \dfrac{\partial^2 f(t,s)}{\partial s \partial t} \in L_\infty([a, b] \times [c, d]); \\[3ex] \dfrac{\left[\frac{(b-x)^{q_3+1} + (x-a)^{q_3+1}}{q_3+1} \right]^{\frac{1}{q_3}}\left[\frac{(d-y)^{q_3+1} + (y-c)^{q_3+1}}{q_3+1} \right]^{\frac{1}{q_3}}}{(b-a)(d-c)} \left\| \dfrac{\partial^2 f}{\partial t \partial s} \right\|_{p_3}, \\[2ex] \quad if \quad \dfrac{\partial^2 f(t,s)}{\partial s \partial t} \in L_{p_3}([a, b] \times [c, d]), \ p_3 > 1, \frac{1}{p_3} + \frac{1}{q_3} = 1; \\[3ex] \dfrac{\left[\frac{1}{2}(b-a) + \left| x - \frac{a+b}{2} \right| \right]\left[\frac{1}{2}(d-c) + \left| y - \frac{c+d}{2} \right| \right]}{(b-a)(d-c)} \left\| \dfrac{\partial^2 f}{\partial t \partial s} \right\|_1, \\[2ex] \quad if \quad \dfrac{\partial^2 f(t,s)}{\partial s \partial t} \in L_1([a, b] \times [c, d]); \end{cases}$$

for all $(x, y) \in [a, b] \times [c, d]$, *where* $\|\cdot\|_p$ $(1 \leq p \leq \infty)$ *are the usual* $p-$*norms on* $[a, b] \times [c, d]$.

Proof. Using the identity (2.1), we can state that

$$(3.2) \quad \left| f(x,y) - \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(t,s)\,dsdt \right|$$

$$\leq \left| \frac{1}{(b-a)(d-c)} \left[\int_a^b \int_c^d p(x,t) \frac{\partial f(t,s)}{\partial t} dsdt \right. \right.$$

$$\left. \left. + \int_a^b \int_c^d q(y,s) \frac{\partial f(t,s)}{\partial s} dsdt + \int_a^b \int_c^d p(x,t) q(y,s) \frac{\partial^2 f(t,s)}{\partial t \partial s} dsdt \right] \right|$$

$$\leq \frac{1}{(b-a)(d-c)} \left[\int_a^b \int_c^d |p(x,t)| \left| \frac{\partial f(t,s)}{\partial t} \right| dsdt \right.$$

$$\left. + \int_a^b \int_c^d |q(y,s)| \left| \frac{\partial f(t,s)}{\partial s} \right| dsdt + \int_a^b \int_c^d |p(x,t)| |q(y,s)| \left| \frac{\partial^2 f(t,s)}{\partial t \partial s} \right| dsdt \right].$$

We have that

$$(3.3) \quad \int_a^b \int_c^d |p(x,t)| \left| \frac{\partial f(t,s)}{\partial t} \right| dsdt$$

$$\leq \begin{cases} \left\| \frac{\partial f}{\partial t} \right\|_\infty \int_a^b \int_c^d |p(x,t)|\,dsdt, & \text{if} \quad \frac{\partial f(t,s)}{\partial t} \in L_\infty([a,b] \times [c,d]); \\[2ex] \left\| \frac{\partial f}{\partial t} \right\|_{p_1} \left(\int_a^b \int_c^d |p(x,t)|^{q_1}\,dsdt \right)^{\frac{1}{q_1}}, & \text{if} \quad \frac{\partial f(t,s)}{\partial t} \in L_{p_1}([a,b] \times [c,d]), \\ & \qquad p_1 > 1, \; \frac{1}{p_1} + \frac{1}{q_1} = 1; \\[2ex] \left\| \frac{\partial f}{\partial t} \right\|_1 \sup_{t \in [a,b]} |p(x,t)|, & \text{if} \quad \frac{\partial f(t,s)}{\partial t} \in L_1([a,b] \times [c,d]). \end{cases}$$

and as

$$\int_a^b \int_c^d |p(x,t)|\,dsdt = \int_c^d \left(\int_a^b |p(x,t)|\,dt \right) ds$$

$$= (d-c) \left[\int_a^x |p(x,t)|\,dt + \int_x^b |p(x,t)|\,dt \right]$$

$$= (d-c) \left[\int_a^x (t-a)\,dt + \int_x^b (b-t)\,dt \right]$$

$$= (d-c) \left[\frac{(x-a)^2 + (b-x)^2}{2} \right]$$

$$= (d-c) \left[\frac{1}{4}(b-a)^2 + \left(x - \frac{a+b}{2} \right)^2 \right],$$

$$\left[\int_a^b \int_c^d |p(x,t)|^{q_1} \, ds dt\right]^{\frac{1}{q_1}} = \left[\int_c^d \left(\int_a^b |p(x,t)|^{q_1} \, dt\right) ds\right]^{\frac{1}{q_1}}$$

$$= (d-c)^{\frac{1}{q_1}} \left[\int_a^x |p(x,t)|^{q_1} \, dt + \int_x^b |p(x,t)|^{q_1} \, dt\right]^{\frac{1}{q_1}}$$

$$= (d-c)^{\frac{1}{q_1}} \left[\int_a^x (t-a)^{q_1} \, dt + \int_x^b (b-t)^{q_1} \, dt\right]^{\frac{1}{q_1}}$$

$$= (d-c)^{\frac{1}{q_1}} \left[\frac{(b-x)^{q_1+1} + (x-a)^{q_1+1}}{q_1+1}\right]^{\frac{1}{q_1}}$$

and

$$\sup_{t\in[a,b]} |p(x,t)| = \max\{x-a, b-x\} = \frac{b-a}{2} + \left|x - \frac{a+b}{2}\right|,$$

then, by (3.3), we obtain

$$(3.4) \qquad \int_a^b \int_c^d |p(x,t)| \left|\frac{\partial f(t,s)}{\partial t}\right| ds dt$$

$$\leq \begin{cases} (d-c)\left[\frac{1}{4}(b-a)^2 + \left(x - \frac{a+b}{2}\right)^2\right] \left\|\frac{\partial f}{\partial t}\right\|_\infty, & \text{if } \frac{\partial f(t,s)}{\partial t} \in L_\infty\left([a,b]\times[c,d]\right); \\ \\ (d-c)^{\frac{1}{q_1}} \left[\frac{(b-x)^{q_1+1}+(x-a)^{q_1+1}}{q_1+1}\right]^{\frac{1}{q_1}} \left\|\frac{\partial f}{\partial t}\right\|_{p_1}, & \text{if } \frac{\partial f(t,s)}{\partial t} \in L_{p_1}\left([a,b]\times[c,d]\right), \\ & \quad p_1 > 1, \frac{1}{p_1} + \frac{1}{q_1} = 1; \\ \left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right] \left\|\frac{\partial f}{\partial t}\right\|_1. & \text{if } \frac{\partial f(t,s)}{\partial t} \in L_1\left([a,b]\times[c,d]\right). \end{cases}$$

In a similar fashion, we can state that the following inequality holds

$$(3.5) \qquad \int_a^b \int_c^d |q(y,s)| \left|\frac{\partial f(t,s)}{\partial s}\right| ds dt$$

$$\leq \begin{cases} (b-a)\left[\frac{1}{4}(d-c)^2 + \left(y - \frac{c+d}{2}\right)^2\right] \left\|\frac{\partial f}{\partial s}\right\|_\infty, & \text{if } \frac{\partial f(t,s)}{\partial s} \in L_\infty\left([a,b]\times[c,d]\right); \\ \\ (b-a)^{\frac{1}{q_2}} \left[\frac{(d-y)^{q_2+1}+(y-c)^{q_2+1}}{q_2+1}\right]^{\frac{1}{q_2}} \left\|\frac{\partial f}{\partial s}\right\|_{p_2}, & \text{if } \frac{\partial f(t,s)}{\partial s} \in L_{p_2}\left([a,b]\times[c,d]\right), \\ & \quad p_2 > 1, \frac{1}{p_2} + \frac{1}{q_2} = 1; \\ \left[\frac{1}{2}(d-c) + \left|y - \frac{c+d}{2}\right|\right] \left\|\frac{\partial f}{\partial s}\right\|_1. & \text{if } \frac{\partial f(t,s)}{\partial s} \in L_1\left([a,b]\times[c,d]\right). \end{cases}$$

In addition, we have

$$(3.6) \qquad \int_a^b \int_c^d |p(x,t)| \, |q(y,s)| \left|\frac{\partial^2 f(t,s)}{\partial t \partial s}\right| ds dt$$

$$\leq \begin{cases} \left\| \frac{\partial^2 f(t,s)}{\partial t \partial s} \right\|_\infty \int_a^b |p(x,t)| \, dt \int_c^d |q(y,s)| \, ds, \\[2ex] \quad \text{if} \quad \frac{\partial^2 f(t,s)}{\partial s \partial t} \in L_\infty([a,b] \times [c,d]); \\[3ex] \left\| \frac{\partial^2 f(t,s)}{\partial t \partial s} \right\|_{p_3} \left(\int_a^b |p(x,t)|^{q_3} \, dt \right)^{\frac{1}{q_3}} \left(\int_c^d |q(y,s)|^{q_3} \, ds \right)^{\frac{1}{q_3}}, \\[2ex] \quad \text{if} \quad \frac{\partial^2 f(t,s)}{\partial s \partial t} \in L_{p_3}([a,b] \times [c,d]), \; p_3 > 1, \; \frac{1}{p_3} + \frac{1}{q_3} = 1; \\[3ex] \left\| \frac{\partial^2 f(t,s)}{\partial t \partial s} \right\|_1 \sup_{t \in [a,b]} |p(x,t)| \sup_{s \in [c,d]} |q(y,s)| \\[2ex] \quad \text{if} \quad \frac{\partial^2 f(t,s)}{\partial s \partial t} \in L_1([a,b] \times [c,d]). \end{cases}$$

$$= \begin{cases} \left[\frac{1}{4}(b-a)^2 + \left(x - \frac{a+b}{2}\right)^2 \right] \left[\frac{1}{4}(d-c)^2 + \left(y - \frac{c+d}{2}\right)^2 \right] \left\| \frac{\partial^2 f}{\partial t \partial s} \right\|_\infty, \\[2ex] \quad \text{if} \quad \frac{\partial^2 f(t,s)}{\partial s \partial t} \in L_\infty([a,b] \times [c,d]); \\[3ex] \left[\frac{(b-x)^{q_3+1} + (x-a)^{q_3+1}}{q_3+1} \right]^{\frac{1}{q_3}} \left[\frac{(d-y)^{q_3+1} + (y-c)^{q_3+1}}{q_3+1} \right]^{\frac{1}{q_3}} \left\| \frac{\partial^2 f}{\partial t \partial s} \right\|_{p_3}, \\[2ex] \quad \text{if} \quad \frac{\partial^2 f(t,s)}{\partial s \partial t} \in L_{p_3}([a,b] \times [c,d]); \; p_3 > 1, \; \frac{1}{p_3} + \frac{1}{q_3} = 1; \\[3ex] \left[\frac{1}{2}(b-a) + \left| x - \frac{a+b}{2} \right| \right] \left[\frac{1}{2}(d-c) + \left| y - \frac{c+d}{2} \right| \right] \left\| \frac{\partial^2 f}{\partial t \partial s} \right\|_1, \\[2ex] \quad \text{if} \quad \frac{\partial^2 f(t,s)}{\partial s \partial t} \in L_1([a,b] \times [c,d]). \end{cases}$$

and the theorem is proved. ∎

The following corollary holds by taking $x = \frac{a+b}{2}$, $y = \frac{c+d}{2}$.

Corollary 4. *With the assumptions in Theorem 2, we have the inequality*

$$(3.7) \qquad \left| f\left(\frac{a+b}{2}, \frac{c+d}{2} \right) - \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(t,s) \, ds \, dt \right|$$
$$\leq \quad \tilde{M}_1 + \tilde{M}_2 + \tilde{M}_3,$$

where

$$\tilde{M}_1 := \begin{cases} \frac{1}{4}(b-a) \left\| \frac{\partial f}{\partial t} \right\|_\infty, & \text{if} \quad \frac{\partial f}{\partial t} \in L_\infty([a,b] \times [c,d]) \\[2ex] \frac{1}{2} \left[\frac{(b-a)^{\frac{1}{q_1}}}{(q_1+1)^{\frac{1}{q_1}}(d-c)^{\frac{1}{p_1}}} \right] \left\| \frac{\partial f}{\partial t} \right\|_{p_1}, & \text{if} \quad \frac{\partial f}{\partial t} \in L_{p_1}([a,b] \times [c,d]) \\ & \qquad p_1 > 1, \; \frac{1}{p_1} + \frac{1}{q_1} = 1; \\[2ex] \frac{1}{2(d-c)} \left\| \frac{\partial f}{\partial t} \right\|_1, & \text{if} \quad \frac{\partial f}{\partial t} \in L_1([a,b] \times [c,d]) \end{cases}$$

$$\tilde{M}_2 := \begin{cases} \frac{1}{4}\left(d-c\right)\left\|\frac{\partial f}{\partial s}\right\|_\infty, & \text{if } \frac{\partial f}{\partial t} \in L_\infty\left([a,b]\times[c,d]\right) \\[3mm] \frac{1}{2}\left[\frac{(d-c)^{\frac{1}{q_2}}}{(q_2+1)^{\frac{1}{q_2}}(b-a)^{\frac{1}{p_2}}}\right]\left\|\frac{\partial f}{\partial s}\right\|_{p_2} & \text{if } \frac{\partial f}{\partial t}\in L_{p_2}\left([a,b]\times[c,d]\right) \\[1mm] & \quad p_2>1,\ \frac{1}{p_2}+\frac{1}{q_2}=1; \\[3mm] \frac{1}{2(b-a)}\left\|\frac{\partial f}{\partial s}\right\|_1, & \text{if } \frac{\partial f}{\partial t}\in L_1\left([a,b]\times[c,d]\right) \end{cases}$$

and

$$\tilde{M}_3 := \begin{cases} \frac{1}{16}\left(b-a\right)\left(d-c\right)\left\|\frac{\partial^2 f}{\partial t\partial s}\right\|_\infty, & \text{if } \frac{\partial^2 f}{\partial t\partial s}\in L_\infty\left([a,b]\times[c,d]\right); \\[3mm] \frac{1}{4}\cdot\frac{(b-a)^{\frac{1}{q_3}}(d-c)^{\frac{1}{q_3}}}{(q_3+1)^{\frac{2}{q_3}}}\left\|\frac{\partial^2 f}{\partial t\partial s}\right\|_{p_3}, & \text{if } \frac{\partial^2 f}{\partial t\partial s}\in L_{p_3}\left([a,b]\times[c,d]\right); \\[1mm] & \quad p_3>1,\ \frac{1}{p_3}+\frac{1}{q_3}=1, \\[3mm] \frac{1}{4}\left\|\frac{\partial^2 f}{\partial t\partial s}\right\|_1, & \text{if } \frac{\partial^2 f}{\partial t\partial s}\in L_1\left([a,b]\times[c,d]\right). \end{cases}$$

Using the inequality (2.9) in Corollary 3 and a similar argument to the one used in Theorem 3, we can point out the following trapezoid type inequality:

Corollary 5. *With the assumption in Theorem 2, we have the inequality*

$$(3.8) \qquad \left|\frac{f\left(a,c\right)+f\left(a,d\right)+f\left(b,c\right)+f\left(b,d\right)}{4}-\frac{1}{\left(b-a\right)\left(d-c\right)}\int_a^b\int_c^d f\left(t,s\right)dsdt\right|$$

$$\leq \quad \tilde{M}_1+\tilde{M}_2+\tilde{M}_3,$$

where the $\tilde{M}_i\ \ (i=1,2,3)$ *are as given above.*

4. Applications for Cubature Formulae

Let us consider the arbitrary divisions $I_n := a = x_0 < x_1 < \dots < x_{n-1} < x_n = b$ and $J_m : c = y_0 < y_1 < \dots < y_{m-1} < y_m = d$, where $\xi_i \in [x_i, x_{i+1}]$ $(i=0,\dots,n-1)$, $\eta_j \in [y_j, y_{j+1}]$ $(j=0,\dots,m-1)$ are intermediate points. Consider the Riemann sum:

$$(4.1) \qquad R\left(f,I_n,J_m,\xi,\eta\right)=\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}h_i\,l_j\,f\left(\xi_i,\eta_j\right),$$

where $h_i := x_{i+1}-x_i$, $l_j := y_{j+1}-y_j$, $i=0,\dots,n-1$, $j=0,\dots,m-1$.

Using Theorem 3, we can state twenty-seven different inequalities in bounding the quantity

$$(4.2) \qquad \left|f\left(x,y\right)-\frac{1}{\left(b-a\right)\left(d-c\right)}\int_a^b\int_c^d f\left(t,s\right)dsdt\right|,\quad \left(x,y\right)\in[a,b]\times[c,d].$$

Let us consider only one case, namely, when all the partial derivatives $\frac{\partial f}{\partial t}$, $\frac{\partial f}{\partial s}$, $\frac{\partial^2 f}{\partial t \partial s}$ are bounded. That is,

(4.3)
$$\left| f(x,y) - \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(t,s)\,dsdt \right|$$

$$\leq \frac{1}{b-a}\left[\frac{1}{4}(b-a)^2 + \left(x - \frac{a+b}{2} \right)^2 \right] \left\| \frac{\partial f}{\partial t} \right\|_\infty$$

$$+ \frac{1}{d-c}\left[\frac{1}{4}(d-c)^2 + \left(y - \frac{c+d}{2} \right)^2 \right] \left\| \frac{\partial f}{\partial s} \right\|_\infty$$

$$+ \frac{1}{(b-a)(d-c)}\left[\frac{1}{4}(b-a)^2 + \left(x - \frac{a+b}{2} \right)^2 \right]$$

$$\times \left[\frac{1}{4}(d-c)^2 + \left(y - \frac{c+d}{2} \right)^2 \right] \left\| \frac{\partial^2 f}{\partial t \partial s} \right\|_\infty$$

for all $[x,y] \in [a,b] \times [c,d]$.

Using this inequality, we can state the following theorem.

Theorem 4. *Let $f:[a,b] \times [c,d] \to \mathbb{R}$ be as in Theorem 2. Then we have*

(4.4)
$$\int_a^b \int_c^d f(t,s)\,dsdt = R(f, I_n, J_m, \xi, \eta) + W(f, I_n, J_m, \xi, \eta),$$

where $R(f, I_n, J_m, \xi, \eta)$ is the Riemann sum defined by (4.1) and the remainder through the approximation $W(f, I_n, J_m, \xi, \eta)$ satisfies the bound

(4.5) $|W(f, I_n, J_m, \xi, \eta)|$

$$\leq (d-c)\left\| \frac{\partial f}{\partial t} \right\|_\infty \sum_{i=0}^{n-1}\left[\frac{1}{4}h_i^2 + \left(\xi_i - \frac{x_i + x_{i+1}}{2} \right)^2 \right]$$

$$+ (b-a)\left\| \frac{\partial f}{\partial s} \right\|_\infty \sum_{j=0}^{m-1}\left[\frac{1}{4}l_j^2 + \left(\eta_j - \frac{y_j + y_{j+1}}{2} \right)^2 \right]$$

$$+ \left\| \frac{\partial^2 f}{\partial t \partial s} \right\|_\infty \sum_{i=0}^{n-1}\left[\frac{1}{4}h_i^2 + \left(\xi_i - \frac{x_i + x_{i+1}}{2} \right)^2 \right]$$

$$\times \sum_{j=0}^{m-1}\left[\frac{1}{4}l_j^2 + \left(\eta_j - \frac{y_j + y_{j+1}}{2} \right)^2 \right]$$

$$\leq \frac{1}{2}(d-c)\left\| \frac{\partial f}{\partial t} \right\|_\infty \sum_{i=0}^{n-1}h_i^2 + \frac{1}{2}(b-a)\left\| \frac{\partial f}{\partial s} \right\|_\infty \sum_{j=0}^{m-1}l_j^2$$

$$+ \frac{1}{4}\left\| \frac{\partial^2 f}{\partial t \partial s} \right\|_\infty \sum_{i=0}^{n-1}h_i^2 \sum_{j=0}^{m-1}l_j^2$$

$$\leq \frac{1}{2}(d-c)(b-a)\left[\nu(h)\left\| \frac{\partial f}{\partial t} \right\|_\infty + \nu(l)\left\| \frac{\partial f}{\partial s} \right\|_\infty + \frac{1}{2}\left\| \frac{\partial^2 f}{\partial t \partial s} \right\|_\infty \nu(h)\nu(l) \right]$$

for all ξ, η intermediate points, where $\nu(h) := \max\{h_i, i = 0, ..., n-1\}$ and $\nu(l) := \max\{l_j, j = 0, ..., m-1\}$.

Proof. Apply (4.3) in the intervals $[x_i, x_{i+1}] \times [y_j, y_{j+1}]$ to obtain

$$\left| \int_{x_i}^{x_{i+1}} \int_{y_j}^{y_{j+1}} f(t,s)\, ds dt - h_i l_j f(\xi_i, \eta_j) \right|$$

$$\leq \left[\frac{1}{4} h_i^2 + \left(\xi_i - \frac{x_i + x_{i+1}}{2} \right)^2 \right] l_j \left\| \frac{\partial f}{\partial t} \right\|_\infty$$

$$+ \left[\frac{1}{4} l_j^2 + \left(\eta_j - \frac{y_j + y_{j+1}}{2} \right)^2 \right] h_i \left\| \frac{\partial f}{\partial s} \right\|_\infty$$

$$+ \left[\frac{1}{4} h_i^2 + \left(\xi_i - \frac{x_i + x_{i+1}}{2} \right)^2 \right] \left[\frac{1}{4} l_j^2 + \left(\eta_j - \frac{y_j + y_{j+1}}{2} \right)^2 \right] \left\| \frac{\partial^2 f}{\partial t \partial s} \right\|_\infty$$

for all $i = 0, ..., n-1$, $j = 0, ..., m-1$.

Summing over i from 0 to $n-1$ and over j from 0 to $m-1$, we get the desired estimation (4.5). We omit the details. ∎

Consider the mid-point formula:

$$(4.6) \qquad M(f, I_n, J_m) := \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} h_i l_j f\left(\frac{x_i + x_{i+1}}{2}, \frac{y_j + y_{j+1}}{2} \right).$$

The following corollary contains the best quadrature formula we can obtain from (4.5).

Corollary 6. *Let f be as in Theorem 2, then we have:*

$$(4.7) \qquad \int_a^b \int_c^d f(t,s)\, ds dt = M(f, I_n, J_m) + L(f, I_n, J_m),$$

where $M(f, I_n, J_m)$ is the midpoint formula given by (4.6) and the remainder $L(f, I_n, J_m)$ satisfies the estimate

$$(4.8) \qquad |L(f, I_n, J_m)|$$

$$\leq \frac{1}{4}(d-c) \left\| \frac{\partial f}{\partial t} \right\|_\infty \sum_{i=0}^{n-1} h_i^2 + \frac{1}{4}(b-a) \left\| \frac{\partial f}{\partial s} \right\|_\infty \sum_{j=0}^{m-1} l_j^2$$

$$+ \frac{1}{16} \left\| \frac{\partial^2 f}{\partial t \partial s} \right\|_\infty \sum_{i=0}^{n-1} h_i^2 \sum_{j=0}^{m-1} l_j^2$$

$$: \quad = \mathcal{M}_1(f, I_n, J_m)$$

$$\leq \frac{1}{4}(d-c)(b-a) \left[\nu(h) \left\| \frac{\partial f}{\partial t} \right\|_\infty + \nu(l) \left\| \frac{\partial f}{\partial s} \right\|_\infty + \frac{1}{4}\nu(h)\nu(l) \left\| \frac{\partial^2 f}{\partial t \partial s} \right\|_\infty \right]$$

$$: \quad = \mathcal{M}_2(f, I_n, J_m).$$

We can also consider the trapezoid formula

$$(4.9) \qquad T(f, I_n, J_m)$$

$$: \quad = \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} h_i l_j \cdot \frac{f(x_i, y_j) + f(x_i, y_{j+1}) + f(x_{i+1}, y_j) + f(x_{i+1}, y_{j+1})}{4}.$$

Using Corollary 5 and a similar argument to the one used in the proof of Theorem 4, we can state the following corollary.

Corollary 7. *Let f be as in Theorem 2, then we have*

$$(4.10) \qquad \int_a^b \int_c^d f(t,s)\, ds dt = T(f, I_n, J_m) + P(f, I_n, J_m),$$

where $T(f, I_n, J_m)$ is the trapezoid formula obtained by (4.9) and the remainder $P(f, I_n, J_m)$ satisfies the estimate

$$(4.11) \qquad |P(f, I_n, J_m)| \leq \mathcal{M}_1(f, I_n, J_m) \leq \mathcal{M}_2(f, I_n, J_m),$$

where $\mathcal{M}_1$ and $\mathcal{M}_2$ are as defined above.

References

[1] N.S. BARNETT and S.S. DRAGOMIR, An Ostrowski type inequality for double integrals and applications for cubature formulae, *Soochow J. of Math.,* (in press).

[2] S.S. DRAGOMIR, N.S. BARNETT and P. CERONE, An Ostrowski type inequality for double integrals in terms of L_p-norms and applications in numerical integration, *Anal. Num. Theor. Approx.* (Romania), (in press).

[3] N.S. BARNETT and S.S. DRAGOMIR, A note on bounds for the estimation error variance of a continuous stream with stationary variogram, *J. KSIAM,* **2**(2) (1998), 49-56.

SCHOOL OF COMMUNICATIONS AND INFORMATICS, VICTORIA UNIVERSITY OF TECHNOLOGY, PO BOX 14428, MELBOURNE CITY MC 8001, AUSTRALIA
E-mail address: sever@matilda.vu.edu.au

E-mail address: pc@matilda.vu.edu.au

E-mail address: neil@matilda.vu.edu.au

E-mail address: John.Roumeliotis@vu.edu.au
URL: http://matilda.vu.edu.au/~rgmia