Characterization of Best Approximants from Convex Subsets and Level Sets in Normed Linear Spaces

CHARACTERIZATION OF BEST APPROXIMANTS FROM CONVEX SUBSETS AND LEVEL SETS IN NORMED LINEAR SPACES

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Abstract. Some new characterization of best approximants from convex subsets and level sets of convex mappings in normed linear spaces in terms of norm derivatives and sub-orthogonality in Birkhoff's sense are given.

1. Introduction.

Let $(X, \|\cdot\|)$ be a real normed space and consider the norm derivatives

$$(x,y)_{i(s)} = \lim_{t\to-(+)0}(\frac{\|y+tx\|^2-\|y\|^2)}{2t}.$$

Note that these mappings are well defined on $X \times X$ and the following properties are valid (see also [1], [3]):

(i) $(x,y)_i = -(-x,y)_s$ if x, y are in X;
(ii) $(x,x)_p = \|x\|^2$ for all x in X;
(iii) $(\alpha x, \beta y)_p = \alpha\beta\,(x,y)_p$ for all x, y in X and $\alpha\beta \geq 0$;
(iv) $(\alpha x + y, x)_p = \alpha\,\|x\|^2 + (y,x)_p$ for all x, y in X and α a real number;
(v) $(x+y,z)_p \leq \|x\| \cdot \|z\| + (y,z)_p$ for all x, y, z in X;
(vi) The element x in X is Birkhoff orthogonal over y in X (we denote $x\perp y(B)$),i.e., $\|x+ty\| \geq \|x\|$ for all t a real number iff $(y,x)_i \leq 0 \leq (y,x)_s$;
(vii) The space X is smooth iff $(y,x)_i = (y,x)_s$ for all x, y in X iff $(\cdot,\cdot)_p$ is linear in the first variable;
(viii) We have the representation:

$$(y,x)_i = \inf\{f(y) : f \in J(x)\} \text{ and } (y,x)_s = \sup\{f(y) : f \in J(x)\}$$

where J is the *normalized duality mapping*, i.e.,

$$J(x) = \{f \in X^* : f(x) = \|f\| \cdot \|x\|, \|f\| = \|x\|\},$$

where $p = s$ or $p = i$.

Now, let $(X, \|\cdot\|)$ be a normed linear space and G a nondense subset in X. Suppose $x_0 \in X \setminus Cl(G)$ and $g_0 \in G$.

Definition 1. *The element g_0 will be called the best approximation element of x_0 in G if*

$$\|x_0 - g_0\| = \inf_{g\in G}\|x_0 - g\| \tag{1.1}$$

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and we shall denote by $\mathcal{P}_G(x_0)$ the set of all elements which satisfy (1.1).

The main aim of this paper is to prove some characterization of best approximants from convex subsets in normed linear spaces.

For the classical results in domain, see the monograph [4] due to Ivan Singer.

2. The Results

We shall consider the concept of sub-orthogonality in the sense of Birkhoff introduced by the author in the paper [2]:

Definition 2. *Let $(X, \|\cdot\|)$ be a normed linear space and $x, y \in X$. The element x will be called sub-orthogonal in the sense of Birkhoff over y if $(y, x)_i \le 0$. We shall denote this by $x \perp_S y(B)$.*

The following elementary properties of sub-orthogonality hold:

(i) $0 \perp_S y(B)$ and $x \perp_S 0(B)$ for all $x, y \in X$,
(ii) $x \perp_S y(B)$ implies $(\alpha x) \perp_S (\beta y)(B)$ for $\alpha\beta \ge 0$,
(iii) $x \perp_S x(B)$ implies $x = 0$.

The following characterization of best approximants from convex sets in normed linear spaces which completes the classical results from the book [4] holds.

Theorem 1. *Let C be a nondense convex set in the normed linear spaces X. If $x_0 \in X \backslash Cl(C)$ and $g_0 \in C$, then the following statements are equivalent:*

(i) $g_0 \in \mathcal{P}_G(x_0)$;
(ii) *We have the relation:*

$$x_0 - g_0 \perp_S (C - g_0)(B); \tag{2.1}$$

(iii) *The following inclusion holds*

$$C - g_0 \subset \cup_{f \in J(x_0 - g_0)} K_-(f); \tag{2.2}$$

where J is the normalized duality mapping and $K_-(f)$ is the half space $\{x \in X : f(x) \le 0\}$;

(vi) *We have the bound*

$$\inf_{g \in C} (g - x_0, g_0 - x_0)_s = \|g_0 - x_0\|^2. \tag{2.3}$$

Proof. "*(i)* $\Rightarrow$ *(ii)*". If $g_0 \in \mathcal{P}_G(x_0)$, then $\|x_0 - g_0\| = \inf_{g \in G} \|x_0 - g\|$, which implies that

$$\|x_0 - g_0\|^2 \le \|x_0 - ((1-t)g_0 + tg)\|^2$$

for each $g \in C$ and $t \in [0, 1]$.
Denoting $w_0 := x_0 - g_0$ and $u_0 := g_0 - g$ we get $\|w_0\|^2 \le \|w_0 + tu_0\|^2$ for all $t \in [0, 1]$, which implies

$$\frac{\left(\|w_0 + tu_0\|^2 - \|w_0\|^2\right)}{2t} \ge 0 \text{ for all } t \in (0, 1].$$

Letting $t \to 0+$ we deduce $(u_0, w_0)_s \ge 0$ which is equivalent to $(g - g_0, x_0 - x_0)_i \le 0$ for all $g \in C$ and then the relation (2.1) holds.
"*(ii)* $\Leftrightarrow$ *(iii)*". If $w_0 \perp_S (C - g_0)$, then $(g - g_0, w_0)_i \le 0$ for all $g \in C$ and then there exists (see the property (viii) from introduction) a continuous linear functional f so that $f \in J(w_0)$ and $f(g - g_0) = (g - g_0, w_0)_i$ and then $f(g - g_0) \le 0$, i.e., $g - g_0 \in K_-(f)$. Consequently the inclusion (2.2) holds.

Conversely, if the inclusion (2.2) holds, then for each $g \in C$ there exists a functional $f_0 \in J(x_0 - g_0)$ so that $g - g_0 \in K_-(f_0)$. But by property (viii) stated above, we have

$$(g - g_0, x_0 - g_0)_i = \inf\{f_0(g - g_0) : f \in J(x_0 - g_0)\}$$

and as $f_0 \in J(x_0 - g_0)$ and $f_0(g - g_0) \leq 0$ it follows that $(g - g_0, x_0 - g_0)_i \leq 0$. Consequently the relation (2.1) holds and the implication is proved.
"$(ii) \Rightarrow (iv)$". Relation (2.1) is equivalent to

$$(g_0 - g, x_0 - g_0)_s \geq 0 \text{ for all } g \in C.$$

A simple calculation shows that

$$\begin{aligned}(g_0 - g, x_0 - g_0)_s &= (x_0 - g - (x_0 - g_0), x_0 - g_0)_s \\ &= (x_0 - g, x_0 - g_0)_s - \|x_0 - g_0\|^2 \\ &= (g - x_0, g_0 - x_0)_s - \|x_0 - g_0\|^2\end{aligned}$$

and then by the above inequality we deduce

$$(g - x_0, g_0 - x_0)_s \geq \|g_0 - x_0\|^2 \text{ for all } g \in C$$

which is equivalent to (2.3).
"$(iv) \Rightarrow (i)$". Using the properties of semi-inner product $(\cdot, \cdot)_s$ we have

$$(g - x_0, g_0 - x_0)_s \leq \|g - x_0\| \cdot \|g_0 - x_0\| \text{ for each } g \in C.$$

From (2.3) we get

$$\|g_0 - x_0\|^2 \leq (g - x_0, g_0 - x_0)_s \text{ for each } g \in C,$$

consequently, by the previous two inequalities we deduce that $\|g_0 - x_0\| \leq \|g - x_0\|$ for all $g \in C$, *i.e.*, $g_0 \in \mathcal{P}_G(x_0)$. ∎

Remark 1. *The relation* (2.3) *is equivalent to the fact that the element* $g_0 \in C$ *minimizes the (nonlinear) functional* $F_{x_0,g_0} : C \to \mathbb{R}, F_{x_0,g_0}(u) := (u - x_0, g_0 - x_0)_s$.

The following corollary holds.

Corollary 1. *Let* G *be a nondense linear subspace in* X. *If* $x_0 \in X \backslash Cl(G)$ *and* $g_0 \in G$, *then the following statement are equivalent:*

(i) $g_0 \in \mathcal{P}_G(x_0)$,
(ii) $x_0 - g_0 \perp G(B)$,
(iii) $G \subset \cup_{f \in J(x_0 - g_0)} K_-(f)$.

The equivalence "$(i) \Leftrightarrow (ii)$" is a well known result due to Singer and follows from the fact that a vector is sub-orthogonal on a linear subspace iff it is orthogonal on that subspace.

Now, let us denote by

$$F^{\leq}(r) := \{x \in X : F(x) \leq r\}, r \in \mathbb{R}$$

the $r-$*level set* of F and assume that r is so that $F^{\leq}(r)$ is nonempty.

The following theorem characterizes best approximants by elements of the level set $F^{\leq}(r)$. This result can also be viewed as an estimation theorem for the continuous convex mappings defined on a normed space in terms of semi-inner product $(\cdot, \cdot)_i$.

Theorem 2. *Let $(X, \|\cdot\|)$ be a normed linear space, $F : X \to \mathbb{R}$ a continuous convex mapping on X, $r \in \mathbb{R}$ such that $F^{\leq}(r) \neq \emptyset, x_0 \in X \backslash F^{\leq}(r)$ and $g_0 \in F^{\leq}(r)$. The following statements are equivalent:*

(i) $g_0 \in \mathcal{P}_{F^{\leq}(r)}(x_0)$,
(ii) *We have the estimation:*

$$F(x) \geq r + \frac{F(x_0) - r}{\|x_0 - g_0\|^2}(x - g_0, x_0 - g_0)_i \text{ for all } x \in F^{\leq}(r), \tag{2.4}$$

or, equivalently, the estimation

$$F(x) \geq F(x_0) + \frac{F(x_0) - r}{\|x_0 - g_0\|^2}(x - x_0, x_0 - g_0)_i \text{ for all } x \in F^{\leq}(r). \tag{2.5}$$

Proof. "$(i) \Rightarrow (ii)$". Let us observe first as $x_0 \in X \backslash F^{\leq}(r)$ we have that $F(x_0) > r$. Now, let $x \in F^{\leq}(r)$. Then $F(x) \leq r$ and if we choose $\alpha := F(x_0) - r, \beta := r - F(x)$, then, obviously, $\alpha > 0, \beta \geq 0$ and $0 < \alpha + \beta = F(x_0) - F(x)$.
Let us consider the element

$$u := \frac{\alpha x + \beta x_0}{\alpha + \beta}.$$

Then, by the convexity of F we have:

$$F(u) \leq \frac{\alpha F(x) + \beta F(x_0)}{\alpha + \beta} = \frac{(F(x_0) - r)F(x) + (r - F(x))F(x_0)}{F(x_0) - F(x)}$$

which shows that $u \in F^{\leq}(r)$.
As $g_0 \in \mathcal{P}_{F^{\leq}(r)}(x_0)$ and as $F^{\leq}(r)$ is a convex set, we get that (see Theorem 1, "$(i) \Rightarrow (ii)$")

$$(g - g_0, x_0 - x_0)_i \leq 0 \text{ for all } g \in F^{\leq}(r).$$

Choose $g = u$, where u is defined as above. Then

$$\left(\frac{(F(x_0) - r)\, x + (r - F(x))x_0}{F(x_0) - F(x)} - g_0, x_0 - g_0 \right)_i \leq 0 \tag{2.6}$$

for all $x \in F^{\leq}(r)$. However,

$$\begin{aligned} & \left(\frac{(F(x_0) - r)x + (r - F(x))x_0}{F(x_0) - F(x)} - g_0, x_0 - g_0 \right)_i \\ = \; & \frac{1}{F(x_0) - F(x)} ((r - F(x))(x_0 - g_0) + (F(x_0) - r)(x - g_0), x_0 - g_0)_i \\ = \; & \frac{1}{F(x_0) - F(x)} ((r - F(x))\|x_0 - g_0\|^2 + (F(x_0) - r)(x - g_0, x_0 - g_0)_i) \end{aligned}$$

and then by (2.6) we get

$$(r - F(x))\|x_0 - g_0\|^2 + (F(x_0) - r)(x - g_0, x_0 - g_0)_i \geq 0$$

which is equivalent with the desired estimation (2.4).
Now, let us observe that

$$\begin{aligned} & r+\frac{F(x_0)\ -\ r}{\|x_0-g_0\|^2}(x-g_0,x_0-g_0)_i \\ = & r+\frac{F(x_0)\ -\ r}{\|x_0-g_0\|^2}(x-x_0+x_0-g_0,x_0-g_0)_i \\ = & r+\frac{F(x_0)\ -\ r}{\|x_0-g_0\|^2}[(x-x_0,x_0-g_0)_i+\|x_0-g_0\|^2] \\ = & r+F(x_0)-r+\frac{F(x_0)\ -\ r}{\|x_0-g_0\|^2}(x-x_0,x_0-g_0)_i \\ = & F(x_0)+\frac{F(x_0)\ -\ r}{\|x_0-g_0\|^2}(x-x_0,x_0-g_0)_i \end{aligned}$$

which shows that (2.4) and (2.5) are equivalent.
"$(ii)\Rightarrow(i)$". As $x\in F^{\leq}(r)$, then $0\geq F\,(x)-r$.
On the other hand, by (2.4) we have

$$F\,(x)-r\geq+\frac{F(x_0)-r}{\|x_0-g_0\|^2}(x-g_0,x_0-g_0)_i$$

for all $x\in F^{\leq}(r)$. Consequently,

$$0\geq\frac{F(x_0)-r}{\|x_0-g_0\|^2}(x-g_0,x_0-g_0)_i \text{ for all } x\in F^{\leq}(r).$$

As $F(x_0)-r>0$, we get

$$0\geq(x-g_0,x_0-g_0)_i \text{ for all } x\in F^{\leq}(r).$$

Now, using the implication "$(ii)\Rightarrow(i)$" of Theorem 1, we deduce that $g_0\in\mathcal{P}_{F^{\leq}(r)}\,(x_0)$, and the theorem is proved. ∎

Remark 2. *If $g_0\in\mathcal{P}_{F^{\leq}(r)}\,(x_0)$, then $F(g_0)=r$. Indeed, as $g_0\in F^{\leq}(r)$, then $F(g_0)\leq r$. On the other hand, choosing $x=g_0$ in (2.4) we get $F(g_0)\geq r$, and then the required equality holds.*

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