Some Inequalities for Random Variables whose Probability Density Functions are Bounded using a Pre-Grüss Inequality

SOME INEQUALITIES FOR RANDOM VARIABLES WHOSE PROBABILITY DENSITY FUNCTIONS ARE BOUNDED USING A PRE-GRÜSS INEQUALITY

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Abstract. Using the pre-Grüss inequality considered by Matić-Pečarić-Ujević in a recent paper [1] and some related results, we point out some inequalities for random variables whose p.d.f.'s are bounded above and below by the assumed known constants γ and ϕ.

1. Introduction

In a recent paper [1], Matić, Pečarić and Ujević proved the following inequality, which has been called, in [2], the *pre-Grüss inequality*

$$\left|\frac{1}{b-a}\int_a^b f(t)\,g(t)\,dt-\frac{1}{b-a}\int_a^b f(t)\,dt\cdot\frac{1}{b-a}\int_a^b g(t)\,dt\right| \le \frac{1}{2}(\phi-\gamma)\left[\frac{1}{b-a}\int_a^b g^2(t)\,dt-\left(\frac{1}{b-a}\int_a^b g(t)\,dt\right)^2\right]^{\frac{1}{2}}, \tag{1.1}$$

provided that $\gamma \le f(t) \le \phi$ a.e. on $[a,b]$ and the integrals exist and are finite.

In [1], the authors used (1.1) to obtain some bounds for the remainder in certain Taylor like formulae whilst in [2], the authors applied (1.1) to estimation of the remainder in three point quadrature formulae.

Basically, (1.1) is a pre-Grüss inequality since, if we assume that $\alpha \le g(t) \le \beta$ a.e. in $[a,b]$, then, by the well known fact that (see for example [8])

$$\frac{1}{b-a}\int_a^b g^2(t)\,dt-\left(\frac{1}{b-a}\int_a^b g(t)\,dt\right)^2 \le \frac{1}{4}(\beta-\alpha)^2, \tag{1.2}$$

and, by (1.1) and (1.2), we can deduce the original *Grüss inequality:*

$$\left|\frac{1}{b-a}\int_a^b f(t)\,g(t)\,dt-\frac{1}{b-a}\int_a^b f(t)\,dt\cdot\frac{1}{b-a}\int_a^b g(t)\,dt\right| \le \frac{1}{4}(\phi-\gamma)(\beta-\alpha). \tag{1.3}$$

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In [1], Matić, Pečarić and Ujević observed that if a factor is known, for example $g(t)$, $t \in [a,b]$, then instead of using (1.3) in estimating the difference

$$\frac{1}{b-a}\int_a^b f(t)\,g(t)\,dt - \frac{1}{b-a}\int_a^b f(t)\,dt \cdot \frac{1}{b-a}\int_a^b g(t)\,dt,$$

it is better to use (1.1), as they have shown in their paper that it improves some recent results by the second author [4].

In this paper, by using the same approach, we obtain some inequalities for the expectation $E(X)$ and cumulative distribution function $F(\cdot)$ of a random variable having the probability distribution function $f:[a,b]\to\mathbb{R}$. It is assumed that we know the lower and the upper bound for f, i.e., the real numbers γ,ϕ such that $0\le\gamma\le f(t)\le\phi\le 1$ a.e. t on $[a,b]$. Some related results are also established.

2. Some Inequalities for Expectation and Dispersion

We start with the following result for expectation.

Theorem 1. *Let X be a random variable having the probability density function $f:[a,b]\to\mathbb{R}$. Assume that there exists the constants γ,ϕ such that $0\le\gamma\le f(t)\le\phi\le 1$ a.e. t on $[a,b]$. Then we have the inequality*

$$\left|E(X)-\frac{a+b}{2}\right|\le\frac{1}{4\sqrt{3}}(\phi-\gamma)(b-a)^2, \tag{2.1}$$

where $E(X)$ is the expectation of the random variable X.

Proof. If we put $g(t)=t$ in (1.1), we obtain

$$\begin{aligned}(2.2)\quad &\left|\frac{1}{b-a}\int_a^b tf(t)\,dt-\frac{1}{b-a}\int_a^b f(t)\,dt\cdot\frac{1}{b-a}\int_a^b tdt\right|\\ &\le \frac{1}{2}(\phi-\gamma)\left[\frac{1}{b-a}\int_a^b t^2dt-\left(\frac{1}{b-a}\int_a^b tdt\right)^2\right]^{\frac{1}{2}}\end{aligned}$$

and as

$$\begin{aligned}\int_a^b tf(t)\,dt &= E(X),\\ \int_a^b f(t)\,dt &= 1,\quad \frac{1}{b-a}\int_a^b tdt=\frac{a+b}{2}\end{aligned}$$

and

$$\frac{1}{b-a}\int_a^b t^2dt-\left(\frac{1}{b-a}\int_a^b tdt\right)^2=\frac{(b-a)^2}{12},$$

then by (2.2) we deduce (2.1). ∎

To point out a result for the $p-$moments of the random variable X, $p\in\mathbb{R}\backslash\{-1,0\}$, we need the following $p-$*Logarithmic mean*

$$M_p(a,b):=\left[\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right]^{\frac{1}{p}},$$

where $0<a<b$.

Theorem 2. *Let X and f be as in Theorem 1 and $E_p(X)$ be the p-moment of X, i.e.,*

$$E_p(X) := \int_a^b t^p f(t)\,dt,$$

which is assumed to be finite.
Then we have the inequality

$$\left|E_p(X) - M_p^p(a,b)\right| \le \frac{1}{2}(\phi-\gamma)\left[M_{2p}^{2p}(a,b) - M_p^{2p}(a,b)\right]^{\frac{1}{2}}. \tag{2.3}$$

The proof is obvious by the inequality (1.1) in which we choose $g(t) = t^p,\ p \in \mathbb{R}\backslash\{-1,0\}$ and use the definition of $p-$logarithmic means.

If we consider the Logarithmic mean

$$M_{-1}(a,b) := L(a,b) = \frac{b-a}{\ln b - \ln a},\ 0 < a < b$$

and define the $(-1)-$moment of the random variable X by

$$E_{-1}(X) := \int_a^b \frac{f(t)}{t}dt,$$

then we can also state the following theorem.

Theorem 3. *Let X and f be as in Theorem 1. Then we have the inequality:*

$$\left|E_{-1}(X) - M_{-1}^{-1}(a,b)\right| \le \frac{1}{2}(\phi-\gamma)\left[M_{-2}^{-2}(a,b) - M_{-1}^{-2}(a,b)\right]^{\frac{1}{2}}, \tag{2.4}$$

provided the $(-1)-$moment of X is finite.

The proof is obvious by the inequality (1.1) and we omit the details.

The following theorem also holds.

Theorem 4. *Let X and f be as above. If*

$$\sigma_\mu(X) := \left[\int_a^b (t-\mu)^2 f(t)\,dt\right]^{\frac{1}{2}},\ \mu \in [a,b],$$

then we have the inequality

$$\begin{aligned} &\left|\sigma_\mu^2(X) - \left(\mu - \frac{a+b}{2}\right)^2 - \frac{(b-a)^2}{12}\right| \\ \le\ & \frac{1}{2}(\phi-\gamma)(b-a)^2\left[\frac{1}{3}\left(\mu - \frac{a+b}{2}\right)^2 + \frac{1}{180}(b-a)^2\right]^{\frac{1}{2}} \\ \le\ & \frac{1}{3\sqrt{5}}(\phi-\gamma)(b-a)^3. \end{aligned} \tag{2.5}$$

Proof. If we put $g(t) = (t-\mu)^2$ in (1.1) we get

$$\begin{aligned} &\left|\frac{1}{b-a}\int_a^b f(t)(t-\mu)^2\,dt - \frac{1}{b-a}\int_a^b f(t)\,dt \cdot \frac{1}{b-a}\int_a^b (t-\mu)^2\,dt\right| \\ \le\ & \frac{1}{2}(\phi-\gamma)\left[\frac{1}{b-a}\int_a^b (t-\mu)^4\,dt - \left(\frac{1}{b-a}\int_a^b (t-\mu)^2\,dt\right)^2\right]^{\frac{1}{2}}, \end{aligned} \tag{2.6}$$

and as

$$\begin{aligned}
\int_a^b f(t)\,dt &= 1, \\
\frac{1}{b-a}\int_a^b (t-\mu)^2\,dt &= \frac{(b-\mu)^3+(\mu-a)^3}{3(b-a)} \\
&= \frac{(b-\mu)^2-(b-\mu)(\mu-a)+(\mu-a)^2}{3} \\
&= \left(\mu-\frac{a+b}{2}\right)^2+\frac{(b-a)^2}{12},
\end{aligned}$$

$$\begin{aligned}
&\frac{1}{b-a}\int_a^b (t-\mu)^4\,dt-\left(\frac{1}{b-a}\int_a^b (t-\mu)^2\,dt\right)^2 \\
&= \frac{(b-\mu)^5+(\mu-a)^5}{5(b-a)}-\left[\frac{(b-\mu)^3+(\mu-a)^3}{3(b-a)}\right]^2 \\
&= \frac{(b-\mu)^4-(b-\mu)^3(\mu-a)+(b-\mu)^2(\mu-a)^2-(b-\mu)(\mu-a)^3+(\mu-a)^4}{5} \\
&\quad -\left[\frac{(b-\mu)^2-(b-\mu)(\mu-a)+(\mu-a)^2}{3}\right]^2 \\
&= \frac{1}{45}\Big[9(b-\mu)^4-(b-\mu)^3(\mu-a)+(b-\mu)^2(\mu-a)^2 \\
&\quad -(b-\mu)(\mu-a)^3+(\mu-a)^4-5(b-\mu)^4-5(b-\mu)^2(\mu-a)^2 \\
&\quad -5(\mu-a)^4+10(b-\mu)^3(\mu-a)+10(b-\mu)(\mu-a)^3-10(b-\mu)^2(\mu-a)^2\Big] \\
&= \frac{1}{45}\Big[4(b-\mu)^4+4(\mu-a)^4-8(b-\mu)^2(\mu-a)^2+2(b-\mu)^3(\mu-a)^2 \\
&\quad +(\mu-a)(b-\mu)\left[(b-\mu)^2+(\mu-a)^2\right]\Big]
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{45}\Big[4\left[(b-\mu)^2-(\mu-a)^2\right]^2 \\
&\quad +2(b-\mu)^2(\mu-a)^2+(\mu-a)(b-\mu)\left[(b-\mu)^2+(\mu-a)^2\right]\Big] \\
:&= A.
\end{aligned}$$

However,

$$\begin{aligned}
(b-\mu)^2-(\mu-a)^2 &= (b-a)(b+a-2\mu)=2(b-a)\left(\frac{b+a}{2}-\mu\right), \\
(b-\mu)(\mu-a) &= \frac{1}{4}(b-a)^2-\left(\mu-\frac{a+b}{2}\right)^2, \\
(b-\mu)^2+(\mu-a)^2 &= \frac{1}{2}(b-a)^2+2\left(\mu-\frac{a+b}{2}\right)^2.
\end{aligned}$$

Denote $\delta := b - a$ and $x = \mu - \frac{a+b}{2}$. Then we get

$$\begin{aligned} 45A &= 4(2\delta x)^2 + 2\left(\frac{1}{4}\delta^2 - x^2\right)^2 + \left(\frac{1}{4}\delta^2 - x^2\right)\left(\frac{1}{2}\delta + 2x^2\right) \\ &= 16\delta^2 x^2 + \left(\frac{1}{4}\delta^2 - x^2\right)\left[2\left(\frac{1}{4}\delta^2 - x^2\right) + \frac{1}{2}\delta^2 + 2x^2\right] \\ &= \delta^2\left(15x^2 + \frac{1}{4}\delta^2\right). \end{aligned}$$

Then

$$\begin{aligned} A &= \frac{(b-a)^2}{45}\left[15\left(\mu - \frac{a+b}{2}\right)^2 + \frac{1}{4}(b-a)^2\right] \\ &= (b-a)^2\left[\frac{1}{3}\left(\mu - \frac{a+b}{2}\right)^2 + \frac{1}{180}(b-a)^2\right]. \end{aligned}$$

Using the inequality (2.6), we deduce the desired inequality (2.5). ∎

The best inequality we can obtain from (2.5) is that one for which $\mu = \frac{a+b}{2}$ and therefore, we can state the following corollary.

Corollary 1. *With the above assumptions and denoting $\sigma_0(X) := \sigma_{\frac{a+b}{2}}(X)$, we have the inequality:*

$$\left|\sigma_0^2(X) - \frac{(b-a)^2}{12}\right| \leq \frac{1}{12\sqrt{5}}(\phi - \gamma)(b-a)^3. \tag{2.7}$$

The following theorem also holds.

Theorem 5. *Let X and f be as above. If*

$$A_\mu(X) := \int_a^b |t - \mu| f(t)\,dt,\ \mu \in [a,b],$$

then we have the inequality

$$\begin{aligned} &\left|A_\mu(X) - \frac{1}{b-a}\left[\frac{(b-a)^2}{4} + \left(\mu - \frac{a+b}{2}\right)^2\right]\right| \qquad (2.8) \\ &\leq \frac{1}{2}(\phi-\gamma)(b-a)\left[\frac{(b-a)^2}{48} + \left(\frac{\mu - \frac{a+b}{2}}{b-a}\right)^2\left[\frac{1}{2}(b-a)^2 + \left(\mu - \frac{a+b}{2}\right)^2\right]\right]^{\frac{1}{2}}. \end{aligned}$$

for all $\mu \in [a,b]$.

Proof. If we put $g(t) = |t - \mu|$ in (1.1), we have

$$\begin{aligned} &\left|\frac{1}{b-a}\int_a^b |t-\mu| f(t)\,dt - \frac{1}{b-a}\int_a^b f(t)\,dt \cdot \frac{1}{b-a}\int_a^b |t-\mu|\,dt\right| \qquad (2.9) \\ &\leq \frac{1}{2}(\phi-\gamma)\left[\frac{1}{b-a}\int_a^b |t-\mu|^2\,dt - \left(\frac{1}{b-a}\int_a^b |t-\mu|\,dt\right)^2\right]^{\frac{1}{2}} \end{aligned}$$

and as

$$\int_a^b f(t)\,dt = 1,$$

$$\begin{aligned}
\frac{1}{b-a}\int_a^b |t-\mu|\,dt &= \frac{1}{b-a}\left[\int_a^\mu (\mu-t)\,dt + \int_\mu^b (t-\mu)\,dt\right] \\
&= \frac{1}{b-a}\left[\frac{(b-\mu)^2+(\mu-a)^2}{2}\right] \\
&= \frac{1}{b-a}\left[\frac{(b-a)^2}{4} + \left(\mu-\frac{a+b}{2}\right)^2\right],
\end{aligned}$$

$$\begin{aligned}
\frac{1}{b-a}\int_a^b (t-\mu)^2\,dt &= \frac{(b-\mu)^3+(\mu-a)^3}{3(b-a)} \\
&= \frac{(b-a)^2}{12} + \left(\mu-\frac{a+b}{2}\right)^2,
\end{aligned}$$

$$\begin{aligned}
&\frac{1}{b-a}\int_a^b |t-\mu|^2\,dt - \left(\frac{1}{b-a}\int_a^b |t-\mu|\,dt\right)^2 \\
&= \frac{(b-a)^2}{12} + \left(\mu-\frac{a+b}{2}\right)^2 - \left[\frac{(b-a)}{4} + \frac{1}{b-a}\left(\mu-\frac{a+b}{2}\right)^2\right]^2 \\
&= \frac{(b-a)^2}{48} + \frac{1}{2}\left(\mu-\frac{a+b}{2}\right)^2 - \frac{1}{(b-a)^2}\left(\mu-\frac{a+b}{2}\right)^4 \\
&= \frac{(b-a)^2}{48} + \left(\frac{\mu-\frac{a+b}{2}}{b-a}\right)^2\left[\frac{1}{2}(b-a)^2 - \left(\mu-\frac{a+b}{2}\right)^2\right].
\end{aligned}$$

Finally, using (2.9) we deduce the desired inequality. ∎

The best inequality we can get from (2.8) is embodied in the following corollary.

Corollary 2. *The best inequality we can get from (2.8) is for* $\mu = \mu_0 := \frac{a+b}{2}$, *obtaining*

$$\left|A_{\mu_0}(X) - \frac{b-a}{4}\right| \le \frac{1}{8\sqrt{3}}(\phi-\gamma)(b-a)^2. \tag{2.10}$$

Proof. Consider the mapping $g(\mu) := \frac{(b-a)^2}{48} + \frac{1}{2}\left(\mu-\frac{a+b}{2}\right)^2 - \frac{1}{(b-a)^2}\left(\mu-\frac{a+b}{2}\right)^4$.

We have

$$\begin{aligned}
\frac{dg(\mu)}{d\mu} &= \left(\mu-\frac{a+b}{2}\right) - \frac{4}{(b-a)^2}\left(\mu-\frac{a+b}{2}\right)^3 \\
&= \left(\mu-\frac{a+b}{2}\right)\left[1-\frac{4}{(b-a)^2}\left(\mu-\frac{a+b}{2}\right)^2\right].
\end{aligned}$$

We observe that $\frac{dg(\mu)}{d\mu} = 0$ if $\mu = a$ or $\mu = \frac{a+b}{2}$ or $\mu = b$ and as

$$\frac{dg(\mu)}{d\mu} < 0 \text{ for } \mu \in \left(a, \frac{a+b}{2}\right) \text{ and } \frac{dg(\mu)}{d\mu} > 0 \text{ for } \mu \in \left(\frac{a+b}{2}, \mu\right),$$

we deduce that $\mu = \frac{a+b}{2}$ is the point realizing the global minimum on (a,b) and as $g(\mu_0) = \frac{(b-a)^2}{48}$, the inequality (2.10) is indeed the best inequality we can get from (2.8). ∎

Another inequality which can be useful for obtaining different inequalities for dispersion is the following weighted Grüss type result (see for example [8] or [6]).

Lemma 1. *Let $g, p : [a,b] \to \mathbb{R}$ be measurable functions and such that $\alpha \le g \le \beta$ a.e., $p \ge 0$ a.e. on $[a,b]$ and $\int_a^b p(x)\,dx > 0$.*
Then

$$0 \le \frac{\int_a^b p(x)\, g^2(x)\,dx}{\int_a^b p(x)\,dx} - \left(\frac{\int_a^b p(x)\,g(x)\,dx}{\int_a^b p(x)\,dx}\right)^2 \le \frac{1}{4}(\beta - \alpha)^2, \tag{2.11}$$

provided that all the integrals in (2.11) exist and are finite.

Using the above lemma we shall be able to prove the following result for dispersion.

Theorem 6. *Let X be a random variable whose probability density function f is defined on the finite interval $[a,b]$ and $\sigma(X) < \infty$. Then we have the inequality*

$$0 \le \sigma_\mu^2(X) - (E(X) - \mu)^2 \le \frac{1}{4}(b-a)^2 \tag{2.12}$$

for all $\mu \in [a,b]$, or, equivalently,

$$0 \le \sigma(X) \le \frac{1}{2}(b-a). \tag{2.13}$$

Proof. Let us choose in (2.11), $g(x) = x - \mu$, $p(x) = f(x)$. Then, obviously, $\sup_{x\in[a,b]} g(x) = b - \mu$, $\inf_{x\in[a,b]} g(x) = a - \mu$, $\int_a^b f(x)\,dx = 1$, and then by (2.11), we get

$$0 \le \int_a^b (x-\mu)^2 f(x)\,dx - \left(\int_a^b (x-\mu) f(x)\,dx\right)^2 \le \frac{1}{4}(b-a)^2$$

and the inequality (2.12) is proved. ∎

The following inequality connecting $\sigma_\mu(X)$ and $A_\mu(X)$ also holds.

Theorem 7. *Let X be as in Theorem 6 and assume that $\sigma_\mu(X)$, $A_\mu(X) < \infty$ for all $\mu \in [a,b]$. Then we have the inequality*

$$0 \le \sigma_\mu^2(X) - A_\mu^2(X) \le \frac{1}{2}\left|\mu - \frac{a+b}{2}\right| \tag{2.14}$$

for all $\mu \in [a,b]$.

Proof. Choose in Lemma 1, $p(x) = f(x)$, $g(x) = |x-\mu|$, $\mu \in [a,b]$. Then

$$\begin{aligned}\beta &= \sup_{x\in[a,b]} g(x) = \max\{\mu - a, b-\mu\} = \frac{b-a+|\mu-a-b+\mu|}{2},\\ \alpha &= \inf_{x\in[a,b]} g(x) = \min\{\mu - a, b-\mu\} = \frac{b-a-|\mu-a-b+\mu|}{2},\end{aligned}$$

which gives us

$$\beta - \alpha = 2\left|\mu - \frac{a+b}{2}\right|.$$

Applying (2.11), we deduce (2.14). ∎

3. Some Inequalities for the Cumulative Distribution Function

The following theorem contains an inequality which connects the expectation $E(X)$, the cumulative distributive function $F(X) := \int_a^x f(t)\,dt$ and the bounds γ and ϕ of the probability density function $f : [a,b] \to \mathbb{R}$.

Theorem 8. *Let X, f, $E(X)$, $F(\cdot)$ and γ, ϕ be as above. Then we have the inequality:*

$$\left|E(X) + (b-a)F(x) - x - \frac{b-a}{2}\right| \le \frac{1}{4\sqrt{3}}(\phi-\gamma)(b-a)^2, \tag{3.1}$$

for all $x \in [a,b]$.

Proof. We have the following equality established by Barnett and Dragomir in [3]

$$\begin{aligned}(b-a)F(x) + E(X) - b &= \int_a^b p(x,t)\,dF(t)\\ &= \int_a^b p(x,t) f(t)\,dt,\end{aligned} \tag{3.2}$$

where

$$p(x,t) := \begin{cases} t-a & \text{if} \quad a \le t \le x \le b \\ \\ t-b & \text{if} \quad a \le x < t \le b \end{cases}.$$

Now, if we apply the inequality (1.1) for $g(t) = p(x,t)$, we get

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b p(x,t) f(t)\,dt - \frac{1}{b-a}\int_a^b p(x,t)\,dt \cdot \frac{1}{b-a}\int_a^b f(t)\,dt\right| \\ &\le \frac{1}{2}(\phi-\gamma)\left[\frac{1}{b-a}\int_a^b p^2(x,t)\,dt - \left(\frac{1}{b-a}\int_a^b p(x,t)\,dt\right)^2\right]^{\frac{1}{2}}.\end{aligned} \tag{3.3}$$

Now, we observe that

$$\begin{aligned}\frac{1}{b-a}\int_a^b p(x,t)\,dt &= x - \frac{a+b}{2},\\ \int_a^b f(t)\,dt &= 1,\end{aligned}$$

and

$$\begin{aligned} D \quad : \quad &= \frac{1}{b-a}\int_a^b p^2(x,t)\,dt - \left(\frac{1}{b-a}\int_a^b p(x,t)\,dt\right)^2 \\ &= \frac{1}{b-a}\left[\frac{(b-x)^3+(x-a)^3}{3}\right] - \left(x-\frac{a+b}{2}\right)^2 \\ &= \frac{(b-x)^2-(b-x)(x-a)+(x-a)^2}{3} - \left(x-\frac{a+b}{2}\right)^2 . \end{aligned}$$

As a simple calculation shows that

$$\begin{aligned} &(b-x)^2-(b-x)(x-a)+(x-a)^2 \\ = \;& 3\left(x-\frac{a+b}{2}\right)^2 + \frac{1}{4}(b-a)^2 , \end{aligned}$$

then we get

$$D = \frac{1}{12}(b-a)^2 .$$

Using (3.3), we deduce (3.1). ∎

Remark 1. *If in (3.1) we choose either $x=a$ or $x=b$, we get*

$$\left|E(X) - \frac{a+b}{2}\right| \le \frac{1}{4\sqrt{3}}(\phi-\gamma)(b-a)^2 ,$$

which is the inequality (2.1).

Remark 2. *If in (3.1) we choose $x=\frac{a+b}{2}$, then we get the inequality*

$$\left|E(X) + (b-a)\Pr\left(X \le \frac{a+b}{2}\right) - b\right| \le \frac{1}{4\sqrt{3}}(\phi-\gamma)(b-a)^2 . \tag{3.4}$$

The following theorem also holds.

Theorem 9. *Let X, f, γ,ϕ and $F(\cdot)$ be as above. Then we have the inequality:*

$$\begin{aligned} &\left|E(X) + \frac{b-a}{2}F(x) - \frac{b+x}{2}\right| \\ \le\;& \frac{1}{2\sqrt{3}}(\phi-\gamma)\left[\frac{1}{4}(b-a)^2 + \left(x-\frac{a+b}{2}\right)^2\right] \\ \le\;& \frac{1}{4\sqrt{3}}(\phi-\gamma)(b-a)^2 , \end{aligned} \tag{3.5}$$

for all $x\in[a,b]$.

Proof. We use the following identity proved by Barnett and Dragomir in [3]

$$\begin{aligned} (b-a)F(x) + E(X) - b &= \int_a^x (t-a)\,dF(t) + \int_x^b (t-b)\,dF(t) \\ &= \int_a^x (t-a)f(t)\,dt + \int_x^b (t-b)f(t)\,dt \end{aligned} \tag{3.6}$$

for all $x\in[a,b]$.

Applying the pre-Grüss inequality (1.1), we get for $x \in [a, b]$

$$\begin{aligned} &\left| \frac{1}{x-a} \int_a^x (t-a) f(t)\, dt - \frac{1}{x-a} \int_a^x (t-a)\, dt \cdot \frac{1}{x-a} \int_a^x f(t)\, dt \right| \\ \leq\ & \frac{1}{2} (\phi - \gamma) \left[\frac{1}{x-a} \int_a^x (t-a)^2\, dt - \left[\frac{1}{x-a} \int_a^x (t-a)\, dt \right]^2 \right]^{\frac{1}{2}} \\ =\ & \frac{1}{4\sqrt{3}} (\phi - \gamma) (x-a) \end{aligned} \tag{3.7}$$

and, similarly, we have

$$\begin{aligned} &\left| \frac{1}{b-x} \int_x^b (t-b) f(t)\, dt - \frac{1}{b-x} \int_x^b (t-b)\, dt \cdot \frac{1}{b-x} \int_x^b f(t)\, dt \right| \\ \leq\ & \frac{1}{4\sqrt{3}} (\phi - \gamma) (b-x), \quad x \in (a, b). \end{aligned} \tag{3.8}$$

From (3.7) and (3.8) we can write

$$\left| \int_a^x (t-a) f(t)\, dt - \frac{x-a}{2} F(x) \right| \leq \frac{1}{4\sqrt{3}} (\phi - \gamma) (x-a)^2 \tag{3.9}$$

and

$$\left| \int_x^b (t-b) f(t)\, dt + \frac{b-x}{2} (1 - F(x)) \right| \leq \frac{1}{4\sqrt{3}} (\phi - \gamma) (b-x)^2, \tag{3.10}$$

for all $x \in [a, b]$.

Summing (3.9) and (3.10) and using the triangle inequality, we deduce that

$$\begin{aligned} &\left| \int_a^x (t-a) f(t)\, dt + \int_x^b (t-b) f(t)\, dt - \frac{b-a}{2} F(x) + \frac{b-x}{2} \right| \\ \leq\ & \frac{1}{4\sqrt{3}} (\phi - \gamma) \left[(x-a)^2 + (b-x)^2 \right] \\ =\ & \frac{1}{2\sqrt{3}} (\phi - \gamma) \left[\frac{1}{4} (b-a)^2 + \left(x - \frac{a+b}{2} \right)^2 \right]. \end{aligned} \tag{3.11}$$

Using the identity (3.6), we deduce the desired inequality (3.5). ∎

Remark 3. *If we choose in (3.5), either $x = a$ or $x = b$, we get the inequality*

$$\left| E(X) - \frac{a+b}{2} \right| \leq \frac{1}{4\sqrt{3}} (\phi - \gamma) (b-a)^2 \tag{3.12}$$

and thus recapture (2.1).

Remark 4. *If we choose in (3.5), $x = \frac{a+b}{2}$, then we get*

$$\left| E(X) + \left(\frac{b-a}{2} \right) \Pr\left(X \leq \frac{a+b}{2} \right) - \frac{a+3b}{4} \right| \leq \frac{1}{8\sqrt{3}} (\phi - \gamma) (b-a)^2, \tag{3.13}$$

which is the best inequality that can be obtained.

References

[1] M. MATIĆ, J.E. PEČARIĆ and N. UJEVIĆ, On new estimation of the remainder in generalized Taylor's formula, *Math. Ineq. & Appl.*, 3,**2**(1999), 343-361.

[2] P. CERONE and S.S. DRAGOMIR, Three Point Quadrature Rules Involving, at Most, a First Derivative, *RGMIA Res. Rep. Coll.*, 4,**2**(1999), Article 8. http : //melba.vu.edu.au/~rgmia/v2n4.html

[3] N.S. BARNETT and S.S. DRAGOMIR, An Inequality of Ostrowski's Type for Cumulative Distribution Functions, *RGMIA Re. Rep. Coll.*, 1,**1**(1998), Article 1., http://melba.vu.edu.au/~rgmia/v1n1.html

[4] S.S. DRAGOMIR, New Esttimation of the Remainder in Taylor's Formula using Grüss' Type Inequalities and Applications, *Math. Ineq. & Appl.*, 2,**2**(1999), 183-194.

[5] S.S. DRAGOMIR, A generalization of Grüss' inequality in inner product spaces and applications, *J. Math. Anal. Appl.*, *accepted*.

[6] S.S. DRAGOMIR, A Grüss type integral inequality for mappings of r-Hölder's type and applications for trapezoid formula, *Tamkang J. of Math.*, (*in press*)

[7] S.S. DRAGOMIR, Some discrete inequalities of Grüss type and applications in guessing theory, *submitted*.

[8] S.S. DRAGOMIR, Some integral inequalities of Grüss type, *Indian J. of Pure and Appl. Math.,* (*accepted).*

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