Trapezoidal Type Rules from an Inequalities Point of View

TRAPEZOIDAL TYPE RULES FROM AN INEQUALITIES POINT OF VIEW

P. CERONE AND S.S. DRAGOMIR

ABSTRACT. The article investigates trapezoid type rules and obtains explicit bounds through the use of a Peano kernel approach and the modern theory of inequalities. Both Riemann-Stieltjes and Riemann integrals are evaluated with a variety of assumptions about the integrand enabling the characterisation of the bound in terms of a variety of norms. Perturbed quadrature rules are obtained through the use of Grüss, Chebychev and Lupaş inequalities, producing a variety of tighter bounds. The implementation is demonstrated through the investigation of a variety of composite rules based on inequalities developed. The analysis allows the determination of the partition required that would assure that the accuracy the result would be within a prescribed error tolerance.

1. INTRODUCTION

The following inequality is well known in the literature as the *trapezoid inequality*:

$$\left|\int_a^b f(x)\,dx - \frac{b-a}{2}\left[f(a)+f(b)\right]\right| \le \frac{(b-a)^3}{12}\left\|f''\right\|_\infty, \tag{1.1}$$

where the mapping $f:[a,b]\subset\mathbb{R}\to\mathbb{R}$ is assumed to be twice differentiable on the interval (a,b), with the second derivative bounded on (a,b). That is, $\|f''\|_\infty := \sup_{x\in(a,b)}|f''(x)| < \infty$.

Now if we assume that $I_n : a = x_0 < x_1 < ... < x_{n-1} < x_n = b$ is a partition of the interval $[a,b]$ and f is as above, then we can approximate the integral $\int_a^b f(x)\,dx$ by the *trapezoidal quadrature formula* $A_T(f,I_n)$, having an error given by $R_T(f,I_n)$, where

$$A_T(f,I_n) := \frac{1}{2}\sum_{i=0}^{n-1}\left[f(x_i)+f(x_{i+1})\right]h_i, \tag{1.2}$$

and the remainder satisfies the estimation

$$|R_T(f,I_n)| \le \frac{1}{12}\|f''\|_\infty \sum_{i=0}^{n-1} h_i^3, \tag{1.3}$$

with $h_i := x_{i+1}-x_i$ for $i=0,...,n-1$.

Expression (1.2) is known as the trapezoidal rule, if $n=1$, and as the composite trapezoidal rule for $n>1$. The trapezoidal rule is the simplest closed Newton-Cotes quadrature rule in which function evaluation is restricted at the ends of equispaced intervals.

Date: October 21, 1999.
1991 *Mathematics Subject Classification.* Primary 41A55; Secondary 26D15.
Key words and phrases. Trapezoidal type rules, Analytic Inequalities, *A priori* bounds.

The trapezoidal rule is widely used in practice since it is easy to implement and in an efficient fashion, especially if the partitioning is done in a uniform manner. It is also very accurate for periodic functions. It forms the basic building block for intricate closed Newton-Cotes formulae (Press et al. [37])

The current work investigates trapezoidal type rules and obtains explicit bounds through the use of a Peano kernel approach and the modern theory of inequalities. This approach allows for the investigation of quadrature rules that place fewer restrictions on the behaviour of the integrand and thus allow us to cope with larger classes of functions. Expression (1.1) relies on the behaviour of the second derivative whereas bounds for the trapezoidal rules are obtained in terms of Riemann-Stieltjes integrals in Sections 2, 3 and 4 for functions that are: of bounded variation, Lipschitzian and monotonic respectively. In Section 5, trapezoidal type rules are obtained for $f^{(n)} \in L_p[a,b]$, implying that

$$\left\|f^{(n)}\right\|_p := \left(\int_a^b \left|f^{(n)}(x)\right|^p dx\right)^{\frac{1}{p}} < \infty \quad \text{for } p \geq 1$$

and $\left\|f^{(n)}\right\|_\infty := \sup_{x\in[a,b]} \left|f^{(n)}(x)\right|$.

Perturbed trapezoidal type rules are obtained in Section 5.3 using what are termed as **premature** variants of Grüss, Chebychev and Lupaş inequalities. Atkinson [30] uses an asymptotic error estimate technique to obtain what he defines as a corrected trapezoidal rule. His approach, however, does not readily produce a bound on the error.

In Section 6, non-symmetric bounds are obtained for a trapezoidal type rule for functions whose derivative is bounded above and below. Section 7 utilises a Grüss type inequality to obtain trapezoidal rules whose bound relies on $f'(x) - S$ where $S = \frac{f(b)-f(a)}{b-a}$, the secant slope. Finally, in Section 8, trapezoidal rules whose error bound involves the second derivative belonging to a variety of norms are investigated. This allows for greater flexibility since either of them may be best for different functions.

The current work brings together results for trapezoidal type rules giving explicit error bounds, using Peano type kernels and results from the modern theory of inequalities. Although bounds through the use of Peano kernels have been obtained in some classical review books on numerical integration such as Stroud [35], Engels [34] and, Davis and Rabinowitz [33]. These do not seem to be utilised to perhaps the extent that they should be. So much so that even in the more recent comprehensive monograph by Krommer and Ueberhuber [36], a constructive approach is taken via Taylor or interpolating polynomials to obtain quadrature results. This approach does not readily provide explicit error bounds but rather gives the order of the approximation.

2. Estimates of the Remainder for Mappings of Bounded Variation

In this section we develop Trapezoidal type quadrature rules for functions that are of bounded variation. This covers a very large class of functions unlike the traditional Trapezoidal rule which relies on the second derivative of the function for its error approximation.

2.1. **Some Integral Inequalities.** Let us start with the following integral inequality for mappings of bounded variation [10]:

Theorem 1. *Let $f : [a,b] \to \mathbb{R}$ be a mapping of bounded variation.*

We then have the inequality:

$$\left| \int_a^b f(t)\,dt - [(x-a) f(a) + (b-x) f(b)] \right| \leq \left[\frac{1}{2}(b-a) + \left| x - \frac{a+b}{2} \right| \right] \bigvee_a^b (f) \tag{2.1}$$

holding for all $x \in [a,b]$, where $\bigvee_a^b (f)$ denotes the total variation of f on the interval $[a,b]$.

The constant $\frac{1}{2}$ is the best possible one.

Proof. Using the integration by parts formula for a Riemann-Stieltjes integral, we have

$$\int_a^b (x-t)\,df(t) = (x-t) f(t) \Big|_a^b + \int_a^b f(t)\,dt$$

from which we get the identity

$$\int_a^b f(t)\,dt = (x-a) f(a) + (b-x) f(b) + \int_a^b (x-t)\,df(t) \tag{2.2}$$

for all $x \in [a,b]$.

It is well known [31, p. 159] that if $g, v : [a,b] \to \mathbb{R}$ are such that g is continuous on $[a,b]$ and v is of bounded variation on $[a,b]$, then $\int_a^b g(t)\,dv(t)$ exists and [31, p. 177]

$$\left| \int_a^b g(t)\,dv(t) \right| \leq \sup_{t\in[a,b]} |g(t)| \bigvee_a^b (v). \tag{2.3}$$

Applying inequality (2.3) we can state that:

$$\left| \int_a^b (x-t)\,df(t) \right| \leq \sup_{t\in[a,b]} |x-t| \bigvee_a^b (f). \tag{2.4}$$

As

$$\sup_{t\in[a,b]} |x-t| = \max\{x-a, b-x\} = \frac{1}{2}(b-a) + \left| x - \frac{a+b}{2} \right|$$

then by (2.4) and (2.2) we get (2.1).

Now to prove that $\frac{1}{2}$ is the best possible. Suppose that (2.1) holds with a constant $c > 0$. That is,

$$\left| \int_a^b f(t)\,dt - [f(b)(b-x) + f(a)(x-a)] \right| \leq \left[c(b-a) + \left| x - \frac{a+b}{2} \right| \right] \bigvee_a^b (f) \tag{2.5}$$

for all $x \in [a,b]$. For $x = \frac{a+b}{2}$, we get

$$\left| \int_a^b f(t)\,dt - \frac{f(a)+f(b)}{2}(b-a) \right| \leq c\,(b-a) \bigvee_a^b (f). \tag{2.6}$$

Consider the mapping $f : [a,b] \to \mathbb{R}$ given by

$$f(x) = \begin{cases} 0 & \text{if } x = a \\ \\ 1 & \text{if } x \in (a,b) \\ \\ 0 & \text{if } x = b, \end{cases}$$

then f is of bounded variation on $[a,b]$ and

$$\int_a^b f(x)\,dx = b-a, \qquad \bigvee_a^b (f) = 2.$$

Hence, from inequality (2.6) applied for this particular mapping we have

$$(b-a) \leq 2c\,(b-a)$$

from which we get $c \geq \frac{1}{2}$ and thus showing that $\frac{1}{2}$ is the best constant in (2.1). ∎

Remark 1. *If we choose $x = \frac{a+b}{2}$, then we get (see also [1]):*

$$\left| \int_a^b f(t)\,dt - \frac{f(a)+f(b)}{2}(b-a) \right| \leq \frac{1}{2}(b-a) \bigvee_a^b (f) \tag{2.7}$$

which is the "trapezoid" inequality. Note that the trapezoid inequality (2.7) *is in a sense the best possible inequality we can get from* (2.1). *Also, the constant $\frac{1}{2}$ is the best possible, as shown earlier.*

If we assume that the mapping f is continuous and differentiable on $[a,b]$, then we get the following corollary.

Corollary 1. *Let $f \in C^{(1)}[a,b]$. Then we have the inequality*

$$\begin{aligned} & \left| \int_a^b f(t)\,dt - [(x-a)f(a) + (b-x)f(b)] \right| \\ \leq \; & \left[\frac{1}{2}(b-a) + \left| x - \frac{a+b}{2} \right| \right] \|f'\|_1 \end{aligned} \tag{2.8}$$

for all $x \in [a,b]$, where $\|\cdot\|_1$ is the L_1-norm, namely

$$\|f'\|_1 := \int_a^b |f'(t)|\,dt.$$

The following corollaries are also interesting.

Corollary 2. *Let $f : [a,b] \to \mathbb{R}$ be a Lipschitzian mapping with the constant $L > 0$. Then we have the inequality:*

$$\begin{aligned} & \left| \int_a^b f(t)\,dt - [(x-a)f(a) + (b-x)f(b)] \right| \\ \leq \; & \left[\frac{1}{2}(b-a) + \left| x - \frac{a+b}{2} \right| \right] (b-a)\,L \end{aligned} \tag{2.9}$$

for all $x \in [a,b]$ *.*

Proof. The mapping f being $L-$Lipschitzian on $[a,b]$ implies it is also of bounded variation on $[a,b]$, since

$$\begin{aligned}\bigvee_a^b (f) &= \sup_{I_n \in Div[a,b]} \sum_{i=0}^{n-1} |f(x_{i+1}) - f(x_i)| \\ &\leq L \sup_{I_n \in Div[a,b]} |x_{i+1} - x_i| = (b-a) L,\end{aligned}$$

and the inequality (2.9) is proved, upon using (2.1). ∎

The case of monotonic mappings is embodied in the following corollary which is a special case of Theorem 1.

Corollary 3. *Let* $f : [a,b] \to \mathbb{R}$ *be a monotonic mapping on* $[a,b]$ *. Then we have the inequality:*

$$\begin{aligned}&\left| \int_a^b f(t)\,dt - [(x-a) f(a) + (b-x) f(b)] \right| \\ &\leq \left[\frac{1}{2}(b-a) + \left| x - \frac{a+b}{2} \right| \right] |f(b) - f(a)|\end{aligned} \tag{2.10}$$

for all $x \in [a,b]$ *.*

Remark 2. *The following inequality is well known in the literature as the Hermite-Hadamard inequality (see for example* [17, p. 137]*):*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2} \tag{2.11}$$

where $f : [a,b] \to \mathbb{R}$ *is a convex mapping on* $[a,b]$ *.*

Using the above results, we are able to point out the following counterparts of the second Hermite-Hadamard inequality, namely from (2.7),

$$0 \leq \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(t)\,dt \leq \frac{1}{2} \bigvee_a^b (f) \tag{2.12}$$

provided that f is convex and of bounded variation on $[a,b]$.

If f is convex and Lipschitzian with the constant L on $[a,b]$, then we get from (2.9)

$$0 \leq \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(t)\,dt \leq \frac{1}{2}(b-a) L. \tag{2.13}$$

If f is convex and monotonic on $[a,b]$, then we have from (2.10)

$$0 \leq \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(t)\,dt \leq \frac{1}{2} |f(b) - f(a)|. \tag{2.14}$$

Finally, if f is continuous, differentiable and convex and $f' \in L_1(a,b)$, then, from (2.8),

$$0 \leq \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(t)\,dt \leq \frac{1}{2} \|f'\|_1. \tag{2.15}$$

2.2. **Applications for Quadrature Formulae.** Let $I_n : a = x_0 < x_1 < ... < x_{n-1} < x_n = b$ be an arbitrary division of $[a,b]$ and $\xi_i \in [x_i, x_{i+1}]$, $i = 0, ..., n-1$; be intermediate points. Put $h_i := x_{i+1} - x_i$ and define the sum:

$$T_P(f, I_n, \boldsymbol{\xi}) := \sum_{i=0}^{n-1} \left[(\xi_i - x_i) f(x_i) + (x_{i+1} - \xi_i) f(x_{i+1}) \right].$$

We have the following result concerning the approximation of the integral $\int_a^b f(x)\,dx$ in terms of T_P [10].

Theorem 2. *Let $f : [a,b] \to \mathbb{R}$ be a mapping of bounded variation on $[a,b]$, then we have*

$$\int_a^b f(x)\,dx = T_P(f, I_n, \boldsymbol{\xi}) + R_P(f, I_n, \boldsymbol{\xi}). \tag{2.16}$$

The remainder term $R_P(f, I_n, \boldsymbol{\xi})$ satisfies the estimate

$$\begin{aligned} &|R_P(f, I_n, \boldsymbol{\xi})| \\ &\leq \left[\frac{1}{2}\nu(h) + \max_{i=\overline{0,n-1}} \left| \xi_i - \frac{x_i + x_{i+1}}{2} \right| \right] \bigvee_a^b (f) \leq \nu(h) \bigvee_a^b (f) \end{aligned} \tag{2.17}$$

where $\nu(h) := \max\{h_i \mid i = \overline{0,n-1}\}$.

The constant $\frac{1}{2}$ is the best possible.

Proof. Apply Theorem 1 on the intervals $[x_i, x_{i+1}]$ $(i = 0, ..., n-1)$ to get

$$\begin{aligned} &\left| \int_{x_i}^{x_{i+1}} f(t)\,dt - \left[f(x_i)(\xi_i - x_i) + f(x_{i+1})(x_{i+1} - \xi_i) \right] \right| \\ \leq\ &\left[\frac{1}{2} h_i + \left| \xi_i - \frac{x_i + x_{i+1}}{2} \right| \right] \bigvee_{x_i}^{x_{i+1}} (f), \end{aligned}$$

for all $i \in \{0, ..., n-1\}$.

Using this and the generalized triangle inequality, we have successively

$$\begin{aligned} &|R_P(f, I_n, \boldsymbol{\xi})| \\ &\leq \sum_{i=0}^{n-1} \left| \int_{x_i}^{x_{i+1}} f(t)\,dt - \left[f(x_i)(\xi_i - x_i) + f(x_{i+1})(x_{i+1} - \xi_i) \right] \right| \\ &\leq \sum_{i=0}^{n-1} \left[\frac{1}{2} h_i + \left| \xi_i - \frac{x_i + x_{i+1}}{2} \right| \right] \bigvee_{x_i}^{x_{i+1}} (f) \\ &\leq \max_{i=\overline{0,n-1}} \left[\frac{1}{2} h_i + \left| \xi_i - \frac{x_i + x_{i+1}}{2} \right| \right] \sum_{i=0}^{n-1} \bigvee_{x_i}^{x_{i+1}} (f) \\ &\leq \left[\frac{1}{2}\nu(h) + \max_{i=\overline{0,n-1}} \left| \xi_i - \frac{x_i + x_{i+1}}{2} \right| \right] \bigvee_a^b (f) \end{aligned}$$

and the first inequality in (2.17) is proved.

For the second inequality, we observe that

$$\left|\xi_i - \frac{x_i + x_{i+1}}{2}\right| \leq \frac{1}{2} h_i, \qquad i = 0, ..., n-1;$$

and then

$$\max_{i=\overline{0,n-1}} \left|\xi_i - \frac{x_i + x_{i+1}}{2}\right| \leq \frac{1}{2} \nu(h).$$

Thus the theorem is proved. ∎

Remark 3. *If we choose* $\xi_i = \frac{x_i + x_{i+1}}{2}$, *then we get (see also* [1]*):*

$$\int_a^b f(x)\,dx = T(f, I_n) + R_T(f, I_n),$$

where $T(f, I_n)$ *is the "trapezoid rule", namely,*

$$T(f, I_n) := \frac{1}{2} \sum_{i=0}^{n-1} \left[f(x_i) + f(x_{i+1})\right]$$

and the remainder satisfies the estimate

$$|R_T(f, I_n)| \leq \frac{1}{2} \nu(h) \bigvee_a^b (f).$$

Note that, the trapezoid inequality is in a certain sense the best possible one we can get from Theorem 2.

The following corollaries can be useful in practice.

Corollary 4. *Let* $f : [a, b] \to \mathbb{R}$ *be a Lipschitzian mapping with the constant* $L > 0$, *and* I_n *is as above. Then we have the formula* (2.16) *and the remainder satisfies the estimate*

$$\begin{aligned} & |R_T(f, I_n, \xi)| \\ \leq\ & L\left[\frac{1}{2}\nu(h) + \max_{i=\overline{0,n-1}} \left|\xi_i - \frac{x_i + x_{i+1}}{2}\right|\right] \leq L\nu(h). \end{aligned} \tag{2.18}$$

Corollary 5. *Let* $f : [a, b] \to \mathbb{R}$ *be a monotonic mapping on* $[a, b]$. *Then we have the quadrature formula* (2.16) *and the remainder satisfies the inequality*

$$\begin{aligned} & |R_T(f, I_n, \xi)| \\ \leq\ & \left[\frac{1}{2}\nu(h) + \max_{i=\overline{0,n-1}} \left|\xi_i - \frac{x_i + x_{i+1}}{2}\right|\right] |f(b) - f(a)| \\ \leq\ & \nu(h) |f(b) - f(a)|. \end{aligned} \tag{2.19}$$

3. An Estimate of the Remainder for Monotonic Mappings

Some bounds were obtained in Corollary 2 for monotonic mappings as a particular instance in the development for functions of bounded variation. This section treats specifically monotonic mappings, enabling tighter bounds to be determined.

3.1. **Some Integral Inequalities.** We know that, from Corollary 3, for a monotonic mapping $f:[a,b]\to\mathbb{R}$, the following inequality holds

$$\begin{aligned}(3.1)\qquad & \left|\int_a^b f(t)\,dt-[(x-a)f(a)+(b-x)f(b)]\right| \\ \leq\ & \left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]|f(b)-f(a)|\end{aligned}$$

for all $x\in[a,b]$.

Using a firmer argument, we can improve this result by obtaining tighter bounds as follows:

Theorem 3. *Let $f:[a,b]\to\mathbb{R}$ be a monotonic nondecreasing mapping on $[a,b]$. Then we have the inequality*

$$\begin{aligned}(3.2)\qquad & \left|\int_a^b f(t)\,dt-[(x-a)f(a)+(b-x)f(b)]\right| \\ \leq\ & (b-x)f(b)-(x-a)f(a)+\int_a^b sgn(x-t)f(t)\,dt \\ \leq\ & (x-a)[f(x)-f(a)]+(b-x)[f(b)-f(x)] \\ \leq\ & \left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right][f(b)-f(a)],\end{aligned}$$

for all $x\in[a,b]$.
The above inequalities are sharp.

Proof. Using the integration by parts formula for a Riemann-Stieltjes integral, we have the identity as given by (2.2).

Now, assume that $\Delta_n : a=x_0^{(n)}<x_1^{(n)}<...<x_{n-1}^{(n)}<x_n^{(n)}=b$ is a sequence of divisions with $\nu(\Delta_n)\to 0$ as $n\to\infty$, where $\nu(\Delta_n):=\max_{i=\overline{0,n-1}}\left(x_{i+1}^{(n)}-x_i^{(n)}\right)$ and $\xi_i^{(n)}\in\left[x_i^{(n)},x_{i+1}^{(n)}\right]$. If $p:[a,b]\to\mathbb{R}$ is a continuous mapping on $[a,b]$ and v is monotonic nondecreasing on $[a,b]$, then

$$\begin{aligned}(3.3)\qquad \left|\int_a^b g(t)\,dv(t)\right| &= \left|\lim_{\nu(\Delta_n)\to 0}\sum_{i=0}^{n-1}p\left(\xi_i^{(n)}\right)\left[\nu\left(x_{i+1}^{(n)}\right)-\nu\left(x_i^{(n)}\right)\right]\right| \\ &\leq \lim_{\nu(\Delta_n)\to 0}\sum_{i=0}^{n-1}\left|p\left(\xi_i^{(n)}\right)\right|\left|\nu\left(x_{i+1}^{(n)}\right)-\nu\left(x_i^{(n)}\right)\right| \\ &= \lim_{\nu(\Delta_n)\to 0}\sum_{i=0}^{n-1}\left|p\left(\xi_i^{(n)}\right)\right|\left(\nu\left(x_{i+1}^{(n)}\right)-\nu\left(x_i^{(n)}\right)\right) \\ &= \int_a^b |g(t)|\,dv(t).\end{aligned}$$

Applying the inequality (3.3), we can state that

$$\begin{aligned}
&\left|\int_a^b (x-t)\,df(t)\right| \\
\leq\ & \int_a^b |x-t|\,df(t) = \int_a^x (x-t)\,df(t) + \int_x^b (t-x)\,df(t) \\
=\ & (x-t)\,f(t)\,|_a^x + \int_a^x f(t)\,dt + (t-x)\,f(t)\,|_t^b + \int_x^b f(t)\,dt \\
=\ & -(x-a)\,f(a) + \int_a^x f(t)\,dt + (b-x)\,f(b) - \int_x^b f(t)\,dt \\
=\ & (b-x)\,f(b) - (x-a)\,f(a) + \int_a^b sgn(x-t)\,f(t)\,dt
\end{aligned}$$

and the first inequality in (3.2) is proved on utilising identity (1.2).

As f is monotonic nondecreasing on $[a,b]$, we can state that

$$\int_a^x f(t)\,dt \leq (x-a)\,f(x)$$

and

$$\int_x^b f(t)\,dt \geq (b-x)\,f(x)$$

and then

$$\begin{aligned}
\int_a^b sgn(x-t)\,f(t)\,dt &= \int_a^x f(t)\,dt - \int_x^b f(t)\,dt \\
&\leq (x-a)\,f(x) + (x-b)\,f(x).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&(b-x)\,f(b) - (x-a)\,f(a) + \int_a^b sgn(x-t)\,f(t)\,dt \\
\leq\ & (b-x)\,f(b) - (x-a)\,f(a) + (x-a)\,f(x) + (x-b)\,f(x) \\
=\ & (x-a)\,[f(x) - f(a)] + (b-x)\,[f(b) - f(x)],
\end{aligned}$$

which proves the second inequality in (3.2).

As f is monotonic nondecreasing on $[a,b]$, we have $f(a) \leq f(x) \leq f(b)$ for all $x \in [a,b]$ and so

$$\begin{aligned}
&(x-a)\,[f(x) - f(a)] + (b-x)\,[f(b) - f(x)] \\
\leq\ & \max\{x-a, b-x\}\,[f(x) - f(a) + f(b) - f(x)] \\
=\ & \left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right][f(b) - f(a)]
\end{aligned}$$

and the inequality (3.2) is completely proved.

Now to demonstrate the sharpness of the inequalities in (3.2), let $f_0 : [a,b] \to \mathbb{R}$ be given by

$$f_0(x) = \begin{cases} 0 & \text{if} \quad x \in [a,b) \\ 1 & \text{if} \quad x = b\,. \end{cases}$$

Then f_0 is monotonic nondecreasing on $[a,b]$ and realizes the equality in (3.2) for $x = \frac{a+b}{2}$, as, a simple calculation shows that

$$\begin{aligned}
&\left|\int_a^b f(t)\,dt - \frac{(b-a)}{2}\left(f(a)+f(b)\right)\right| \\
&= \frac{(b-a)}{2}\left(f(b)-f(a)\right) + \int_a^b sgn\left(\frac{a+b}{2}-t\right) f(t)\,dt \\
&= \frac{1}{2}(b-a)\left(f(b)-f(a)\right) = \frac{b-a}{2}.
\end{aligned}$$

The theorem is thus completely proved. ∎

Remark 4. *For a more general result containing both the Ostrowski inequality and Simpson's inequality, see the recent paper* [13].

Remark 5. *It we choose $x = \frac{a+b}{2}$ in (3.2), we have*

$$\begin{aligned}
&\left|\int_a^b f(t)\,dt - \frac{(b-a)}{2}\left[f(a)+f(b)\right]\right| \\
&\leq \frac{1}{2}(b-a)\left[f(b)-f(a)\right] + \int_a^b sgn\left(\frac{a+b}{2}-t\right) f(t)\,dt \\
&\leq \frac{1}{2}(b-a)\left[f(b)-f(a)\right],
\end{aligned} \tag{3.4}$$

which is the "trapezoid inequality".

Note that the trapezoid inequality (3.4) is, in a sense, the best possible inequality we can obtain from (3.2). Moreover, the constant $\frac{1}{2}$ is the best possible for both inequalities.

Remark 6. *The following inequality is well known in the literature as the Hermite-Hadamard inequality (see also (2.19))*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2}, \tag{3.5}$$

provided that $f:[a,b]\to\mathbb{R}$ is a convex mapping on $[a,b]$.

Using the above inequality (3.4), we can state that

$$\begin{aligned}
0 &\leq \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)\,dt \\
&\leq \frac{1}{2}\left[f(b)-f(a)\right] + \frac{1}{b-a}\int_a^b \left(\frac{a+b}{2}-t\right) f(t)\,dt \\
&\leq \frac{1}{2}\left[f(b)-f(a)\right],
\end{aligned} \tag{3.6}$$

provided that f is monotonic nondecreasing and convex on $[a,b]$.

3.2. **Applications for Quadrature Formulae.** Let $I_n : a = x_0 < x_1 < ... < x_{n-1} < x_n = b$ be an arbitrary division of $[a,b]$ and $\xi_i \in [x_i, x_{i+1}]$ $(i = 0, ..., n-1)$ be intermediate points. Put $h_i := x_{i+1} - x_i$ and consider the sum

$$T_P(f, I_n, \boldsymbol{\xi}) := \sum_{i=0}^{n-1} \left[(\xi_i - x_i) f(x_i) + (x_{i+1} - \xi_i) f(x_{i+1}) \right].$$

In Section 2, Corollary 5, we proved that for a monotonic mapping $f : [a,b] \to \mathbb{R}$ we have

$$\int_a^b f(t)\,dt = T_P(f, I_n, \boldsymbol{\xi}) + R_P(f, I_n, \boldsymbol{\xi}) \tag{3.7}$$

and the remainder $R_P(f, I_n, \boldsymbol{\xi})$ satisfies the bound as given by (2.19) where $\nu(h)$ is the norm of the division I_n, that is, $\nu(h) = \max_{i=\overline{0,n-1}} h_i$.

We can improve this result as follows.

Theorem 4. *Let $f : [a,b] \to \mathbb{R}$ be a monotonic nondecreasing mapping on $[a,b]$ and $I_n, \boldsymbol{\xi}$ as above. Then we have the formula (3.7) and the remainder $R_P(f, I_n, \boldsymbol{\xi})$ satisfies the estimate*

$$\begin{aligned}
&|R_P(f, I_n, \boldsymbol{\xi})| \\
\leq\ & \sum_{i=0}^{n-1} \left[(x_{i+1} - \xi_i) f(x_{i+1}) - (\xi_i - x_i) f(x_i) \right] \\
&+ \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} sgn(\xi_i - t) f(t)\,dt \\
\leq\ & \sum_{i=0}^{n-1} (\xi_i - x_i) \left[f(\xi_i) - f(x_i) \right] + \sum_{i=0}^{n-1} (x_{i+1} - \xi_i) \left[f(x_{i+1}) - f(\xi_i) \right] \\
\leq\ & \sum_{i=0}^{n-1} \left[\frac{1}{2} h_i + \left| \xi_i - \frac{x_{i+1} - x_i}{2} \right| \right] (f(x_{i+1}) - f(x_i)) \\
\leq\ & \left[\frac{1}{2} \nu(h) + \max_{i=\overline{0,n-1}} \left| \xi_i - \frac{x_{i+1} - x_i}{2} \right| \right] (f(b) - f(a)) \\
\leq\ & \nu(h) (f(b) - f(a)).
\end{aligned} \tag{3.8}$$

The proof is obvious by Theorem 3 applied on the intervals $[x_i, x_{i+1}]$ $(i = 0, ..., n)$. We omit the details.

Now, if we consider the classical trapezoidal formula

$$T(f, I_n) := \frac{1}{2} \sum_{i=1}^{n-1} \left[f(x_i) + f(x_{i+1}) \right] h_i,$$

then we can state the following corollary.

Corollary 6. *Let $f : [a,b] \to \mathbb{R}$ be a monotonic nondecreasing mapping on $[a,b]$. Then we have*

$$\int_a^b f(t)\,dt = T(f, I_n) + R_T(f, I_n), \tag{3.9}$$

where the remainder satisfies the estimate

$$
\begin{aligned}
(3.10)\qquad & \left|R_T(f,I_n)\right| \\
\leq\ & \frac{1}{2}\sum_{i=0}^{n-1} h_i\left[f(x_{i+1})-f(x_i)\right]+\sum_{i=0}^{n-1}\int_{x_i}^{x_{i+1}} sgn\left(\frac{x_i+x_{i+1}}{2}-t\right)f(t)\,dt \\
\leq\ & \frac{1}{2}\sum_{i=0}^{n-1} h_i\left[f(x_{i+1})-f(x_i)\right] \\
\leq\ & \frac{1}{2}\nu(h)\left(f(b)-f(a)\right).
\end{aligned}
$$

4. An Estimate of the Remainder for Lipschitzian Mappings

We know that, from Corollary 2, for a mapping $f:[a,b]\to\mathbb{R}$, which is $L-$Lipschitzian, so that $f(\cdot)$ satisfies

$$
|f(x)-f(y)|\leq L|x-y| \quad \text{for all } x,y\in[a,b], \tag{4.1}
$$

where $L>0$ is given, we have the inequality

$$
\begin{aligned}
(4.2)\qquad & \left|\int_a^b f(t)\,dt-\left[f(a)(x-a)+f(b)(b-x)\right]\right| \\
\leq\ & \left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right](b-a)L
\end{aligned}
$$

for all $x\in[a,b]$.

Using a firmer argument, we are able now to improve this result as follows.

4.1. Inequalities for Lipschitzian Mappings.

The following theorem holds.

Theorem 5. *Let $f:[a,b]\to\mathbb{R}$ be an $L-$Lipschitzian mapping on $[a,b]$. Then we have the inequality*

$$
\begin{aligned}
(4.3)\qquad & \left|\int_a^b f(t)\,dt-\left[(x-a)f(a)+(b-x)f(b)\right]\right| \\
\leq\ & \left[\frac{1}{4}(b-a)^2+\left(x-\frac{a+b}{2}\right)^2\right]L
\end{aligned}
$$

for all $x\in[a,b]$.

The constant $\frac{1}{4}$ is the best in (4.3).

Proof. As f is $L-$Lipschitzian and thus of bounded variation, the following Riemann-Stieltjes integral $\int_a^b (x-t)\,df(t)$ exists and (see (2.2))

$$
\int_a^b f(t)\,dt=(x-a)f(a)+(b-x)f(b)+\int_a^b (x-t)\,df(t).
$$

Now, assume that $\Delta_n := a=x_0^{(n)}<x_1^{(n)}<...<x_{n-1}^{(n)}<x_n^{(n)}=b$ is a sequence of divisions with $\nu(\Delta_n)\to 0$ as $n\to\infty$, where $\nu(\Delta_n):=\max_{i=\overline{0,n-1}}\left(x_{i+1}^{(n)}-x_i^{(n)}\right)$ and $\xi_i^{(n)}\in\left[x_i^{(n)},x_{i+1}^{(n)}\right]$.

If $p : [a,b] \to \mathbb{R}$ is continuous on $[a,b]$ and $\nu : [a,b] \to \mathbb{R}$ is $L-$Lipschitzian, then the Riemann-Stieltjes integral $\int_a^b p(x)\,dv(x)$ exists and

$$\begin{aligned}
&\left|\int_a^b p(x)\,dv(x)\right| \\
&= \left|\lim_{\nu(\Delta_n)\to 0}\sum_{i=0}^{n-1} p\left(\xi_i^{(n)}\right)\left[\nu\left(x_{i+1}^{(n)}\right)-\nu\left(x_i^{(n)}\right)\right]\right| \\
&\le \lim_{\nu(\Delta_n)\to 0}\sum_{i=0}^{n-1}\left|p\left(\xi_i^{(n)}\right)\right|\left(x_{i+1}^{(n)}-x_i^{(n)}\right)\left|\frac{\nu\left(x_{i+1}^{(n)}\right)-\nu\left(x_i^{(n)}\right)}{x_{i+1}^{(n)}-x_i^{(n)}}\right| \\
&\le L\lim_{\nu(\Delta_n)\to 0}\sum_{i=0}^{n-1}\left|p\left(\xi_i^{(n)}\right)\right|\left(x_{i+1}^{(n)}-x_i^{(n)}\right) \\
&= L\int_a^b |p(x)|\,dx.
\end{aligned} \tag{4.4}$$

Applying the inequality (4.4) for $p(t) = x - t$ and $v(t) = f(t)$, $t \in [a,b]$, we obtain

$$\begin{aligned}
&\left|\int_a^b (x-t)\,df(t)\right| \\
&\le L\int_a^b |x-t|\,dt = L\left[\int_a^x (x-t)\,dt + \int_x^b (t-x)\,dt\right] \\
&= L\left[\frac{(x-a)^2+(b-x)^2}{2}\right] \\
&= L\left[\frac{1}{4}(b-a)^2 + \left(x-\frac{a+b}{2}\right)^2\right],
\end{aligned}$$

and the inequality (4.3) is proved.

To prove the sharpness of the constant $\frac{1}{4}$, assume that (4.3) holds with a constant $C > 0$ instead. That is,

$$\begin{aligned}
&\left|\int_a^b f(t)\,dt - [(b-x)f(b)+(x-a)f(a)]\right| \\
&\le \left[C(b-a)^2 + \left(x-\frac{a+b}{2}\right)^2\right],
\end{aligned} \tag{4.5}$$

for any L-Lipschitzian mapping and any $x \in [a,b]$.

Consider the mapping $f_0 : [a,b] \to M$, $f_0(x) = \left|x - \frac{a+b}{2}\right|$. Then

$$|f(x)-f(y)| = \left|\left|x-\frac{a+b}{2}\right| - \left|y-\frac{a+b}{2}\right|\right| \le |x-y|$$

for all $x, y \in [a,b]$, which shows that f_0 is $L-$Lipschitzian with the constant $L = 1$.

We have

$$\begin{aligned}&\int_a^b f_0(x)\,dx-\frac{f_0(a)+f_0(b)}{2}(b-a)\\=&\frac{(b-a)^2}{4}-\frac{(b-a)^2}{2}=-\frac{(b-a)^2}{4}\end{aligned}$$

and

$$L(b-a)^2=(b-a)^2,$$

which shows that, for $x=\frac{a+b}{2}$, the inequality (4.5) becomes

$$\left|\int_a^b f(x)\,dx-\frac{f(a)+f(b)}{2}(b-a)\right|\le CL(b-a)^2,$$

which is equivalent to

$$\frac{(b-a)^2}{4}\le C(b-a)^2,$$

thus implying that $C\ge\frac{1}{4}$, and the theorem is thus proved. ∎

Remark 7. *If we choose $x=\frac{a+b}{2}$, then we have*

$$\left|\int_a^b f(t)\,dt-\frac{f(a)+f(b)}{2}(b-a)\right|\le\frac{1}{4}(b-a)^2L, \tag{4.6}$$

which is the "trapezoid inequality". Note that the trapezoid inequality (4.6) is, in a sense, the best possible inequality we can obtain from (4.3). In addition, the constant $\frac{1}{4}$ is the best possible one, providing the sharpest bound in the class.

Corollary 7. *Let $f\in C^{(1)}[a,b]$. That is, f is differentiable on (a,b) and the derivative is continuous on (a,b), and put $\|f'\|_\infty=\sup_{t\in(a,b)}|f'(t)|<\infty$. Then we have the inequality*

$$\begin{aligned}&\left|\int_a^b f(t)\,dt-[(x-a)f(a)+(b-x)f(b)]\right|\\\le&\left[\frac{1}{4}(b-a)^2+\left(x-\frac{a+b}{2}\right)^2\right]\|f'\|_\infty\end{aligned} \tag{4.7}$$

for all $x\in[a,b]$.

Remark 8. *Now, if we assume that $f:[a,b]\to\mathbb{R}$ is differentiable convex on (a,b) and the derivative $f':(a,b)\to\mathbb{R}$ is bounded on (a,b), i.e., $\|f'\|_\infty<\infty$, then we have the following converse of the second Hermite-Hadamard inequality*

$$0\le\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\le\frac{1}{4}(b-a)\|f'\|_\infty. \tag{4.8}$$

4.2. **Applications for Quadrature Formulae.** Let us reconsider the generalised trapezoid quadrature formula

$$T_P(f, I_n, \xi) := \sum_{i=0}^{n-1} \left[(\xi_i - x_i) f(x_i) + (x_{i+1} - \xi_i) f(x_{i+1}) \right],$$

provided that $I_n : a = x_0 < x_1 < ... < x_{n-1} < x_n = b$ is an arbitrary division of $[a, b]$, $\xi_i \in [x_i, x_{i+1}]$ $(i = 0, ..., n-1)$ are intermediate points, $h_i := x_{i+1} - x_i$ are the step sizes and $\nu(h) := \max_{i=\overline{0,n}} \{h_i\}$ is the norm of the division.

We can improve Corollary 4 in the following manner.

Theorem 6. *Let $f : [a, b] \to \mathbb{R}$ be a L-Lipschitzian mapping and I_n, $\boldsymbol{\xi}$ be as above. Then we have the formula*

$$\int_a^b f(t)\,dt = T_P(f, I_n, \boldsymbol{\xi}) + R_P(f, I_n, \boldsymbol{\xi}), \tag{4.9}$$

where the remainder $R_P(f, I_n, \boldsymbol{\xi})$ is such that it satisfies the estimate

$$\begin{aligned} |R_P(f, I_n, \boldsymbol{\xi})| &\leq \frac{1}{4} L \sum_{i=0}^{n-1} h_i^2 + L \sum_{i=0}^{n-1} \left(\xi_i - \frac{x_{i+1} + x_i}{2} \right)^2 \\ &\leq \frac{1}{2} L \sum_{i=0}^{n-1} h_i^2 \leq \frac{1}{2} L (b-a) \nu(h). \end{aligned} \tag{4.10}$$

The proof follows by Theorem 5 applied on the interval $[x_i, x_{i+1}]$ $(i = 0, ..., n-1)$.

Remark 9. *If we choose $\xi_i = \frac{x_i + x_{i+1}}{2}$, then we obtain the trapezoid formula where the remainder $R_T(f, I_n)$ satisfies the estimate*

$$|R_T(f, I_n)| \leq \frac{1}{4} L \sum_{i=0}^{n-1} h_i^2 \leq \frac{1}{4} L (b-a) \nu(h), \tag{4.11}$$

where $\nu(h) = \max\{h_i | i = 0, 1, ..., n-1\}$.

5. A Generalization for Derivatives which are Absolutely Continuous

5.1. **Integral Identities.** We start with the following result [32].

Theorem 7. *Let $f : [a, b] \to \mathbb{R}$ be a mapping such that the derivative $f^{(n-1)}$ $(n \geq 1)$ is absolutely continuous on $[a, b]$. Then*

$$\begin{aligned} \int_a^b f(t)\,dt &= \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left[(x-a)^{k+1} f^{(k)}(a) + (-1)^k (b-x)^{k+1} f^{(k)}(b) \right] \\ &\quad + \frac{1}{n!} \int_a^b (x-t)^n f^{(n)}(t)\,dt, \end{aligned} \tag{5.1}$$

for all $x \in [a, b]$.

Proof. The proof is by mathematical induction.

For $n = 1$, we have to prove that

$$\int_a^b f(t)\,dt = (x-a) f(a) + (b-x) f(b) + \int_a^b (x-t) f^{(1)}(t)\,dt, \tag{5.2}$$

which is straightforward as may be seen by the integration by parts formula applied for the integral

$$\int_a^b (x-t) f^{(1)}(t)\,dt.$$

Assume that (5.1) holds for "n" and let us prove it for "$n+1$". That is, we wish to show that:

$$\begin{aligned}\int_a^b f(t)\,dt &= \sum_{k=0}^{n} \frac{1}{(k+1)!}\left[(x-a)^{k+1} f^{(k)}(a) + (-1)^k (b-x)^{k+1} f^{(k)}(b)\right] \\ &\quad + \frac{1}{(n+1)!}\int_a^b (x-t)^{n+1} f^{(n+1)}(t)\,dt. \end{aligned} \tag{5.3}$$

For this purpose, we apply formula (5.2) for the mapping $g(t) := (x-t)^n f^{(n)}(t)$, which is absolutely continuous on $[a,b]$, and then, we can write:

$$\begin{aligned} &\int_a^b (x-t)^n f^{(n)}(t)\,dt \\ &= (x-a)(x-a)^n f^{(n)}(a) + (b-x)(x-b)^n f^{(n)}(b) \\ &\quad + \int_a^b (x-t)\frac{d}{dt}\left[(x-t)^n f^{(n)}(t)\right] dt \\ &= \int_a^b (x-t)\left[-n(x-t)^{n-1} f^{(n)}(t) + (x-t)^n f^{(n+1)}(t)\right] dt \\ &\quad + (x-a)^{n+1} f^{(n)}(a) + (-1)^n (b-x)^{n+1} f^{(n)}(b) \\ &= -n\int_a^b (x-t)^n f^{(n)}(t)\,dt + \int_a^b (x-t)^{n+1} f^{(n+1)}(t)\,dt \\ &\quad + (x-a)^{n+1} f^{(n)}(a) + (-1)^n (b-x)^{n+1} f^{(n)}(b). \end{aligned} \tag{5.4}$$

From identity (5.4) we can get

$$\begin{aligned} &\int_a^b (x-t)^n f^{(n)}(t)\,dt \\ &= \frac{1}{n+1}\int_a^b (x-t)^{n+1} f^{(n+1)}(t)\,dt \\ &\quad + \frac{1}{n+1}\left[(x-a)^{n+1} f^{(n)}(a) + (-1)^n (b-x)^{n+1} f^{(n)}(b)\right]. \end{aligned}$$

Now, using the induction hypothesis, we have

$$\begin{aligned}\int_a^b f(t)\,dt &= \sum_{k=0}^{n-1} \frac{1}{(k+1)!}\left[(x-a)^{k+1} f^{(k)}(a) + (-1)^k (b-x)^{k+1} f^{(k)}(b)\right] \\ &\quad + \frac{1}{n!}\left[\frac{1}{n+1}\int_a^b (x-t)^{n+1} f^{(n+1)}(t)\,dt \right. \\ &\quad \left. + \frac{1}{n+1}\left[(x-a)^{n+1} f^{(n)}(a) + (b-x)^{n+1} f^{(n)}(b)\right]\right] \end{aligned}$$

$$= \sum_{k=0}^{n} \frac{1}{(k+1)!} \left[(x-a)^{k+1} f^{(k)}(a) + (-1)^k (b-x)^{k+1} f^{(k)}(b) \right] + \frac{1}{(n+1)!} \int_a^b (x-t)^{n+1} f^{(n+1)}(t)\, dt$$

and the identity (5.3) is obtained. This completes the proof. ∎

The following corollary is useful in practice.

Corollary 8. *With the above assumptions for f and R, we have the particular identities (which can also be obtained by using Taylor's formula with the integral remainder)*

$$\int_a^b f(t)\, dt = \sum_{k=0}^{n-1} \frac{(-1)^k}{(k+1)!} (b-a)^{k+1} f^{(k)}(b) + \frac{(-1)^n}{n!} \int_a^b (t-a)^n f^{(n)}(t)\, dt, \tag{5.5}$$

$$\int_a^b f(t)\, dt = \sum_{k=0}^{n-1} \frac{1}{(k+1)!} (b-a)^{k+1} f^{(k)}(a) + \frac{1}{n!} \int_a^b (b-t)^n f^{(n)}(t)\, dt, \tag{5.6}$$

and the identity (see also [15]*)*

$$\int_a^b f(t)\, dt = \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left(\frac{b-a}{2} \right)^{k+1} \left[f^{(k)}(a) + (-1)^k f^{(k)}(b) \right] + \frac{(-1)^n}{n!} \int_a^b \left(t - \frac{a+b}{2} \right)^n f^{(n)}(t)\, dt. \tag{5.7}$$

Here (5.5), (5.6) and (5.7) are obtained from (5.1) with $t = b,\ a,\ \frac{a+b}{2}$ respectively.

Remark 10. a) *For $n = 1$, we get the identity (5.2) which is a generalization of the trapezoid rule. Further, with $x = \frac{a+b}{2}$, we get (*[8]*), (5.7) with $n = 1$. Namely,*

$$\int_a^b f(t)\, dt = \frac{b-a}{2} (f(a) + f(b)) - \int_a^b \left(t - \frac{a+b}{2} \right) f'(t)\, dt \tag{5.8}$$

which is the "trapezoid rule".

b) *For $n = 2$, we get the identity:*

$$\int_a^b f(t)\, dt = (x-a) f(a) + (b-x) f(b) + \frac{1}{2} \left[(x-a)^2 f'(a) + (b-x)^2 f'(b) \right] + \frac{1}{2} \int_a^b (x-t)^2 f''(t)\, dt. \tag{5.9}$$

Further, with $x = \frac{a+b}{2}$, we capture the "perturbed trapezoid rule" [15], *(or, equivalently, $n = 2$ in (5.7))*

$$\begin{aligned} \int_a^b f(t)\,dt &= \frac{b-a}{2}\left[f(a)+f(b)\right]+\frac{(b-a)^2}{8}\left(f'(a)-f'(b)\right) \\ &\quad +\frac{1}{2}\int_a^b \left(t-\frac{a+b}{2}\right)^2 f''(t)\,dt. \end{aligned} \tag{5.10}$$

5.2. **Integral Inequalities.** Using the integral representation by Theorem 7, we can prove the following inequality [32]

Theorem 8. *Let $f : [a,b] \to \mathbb{R}$ be a mapping so that the derivative $f^{(n-1)}$ $(n \geq 1)$ is absolutely continuous on $[a,b]$. Then*

$$\begin{aligned} &\left|\int_a^b f(t)\,dt - \sum_{k=0}^{n-1}\frac{1}{(k+1)!}\left[(x-a)^{k+1} f^{(k)}(a)\right.\right. \\ &\left.\left. + (-1)^k (b-x)^{k+1} f^{(k)}(b)\right]\right| \\ &\leq \begin{cases} \frac{\left\|f^{(n)}\right\|_\infty}{(n+1)!}\left[(x-a)^{n+1}+(b-x)^{n+1}\right] & \text{if} \quad f^{(n)} \in L_\infty[a,b], \\ \frac{\left\|f^{(n)}\right\|_p}{n!}\left[\frac{(x-a)^{nq+1}+(b-x)^{nq+1}}{nq+1}\right]^{\frac{1}{q}} & \text{if} \quad p>1,\ \frac{1}{p}+\frac{1}{q}=1 \\ & \text{and} \quad f^{(n)} \in L_p[a,b], \\ \frac{\left\|f^{(n)}\right\|_1}{n!}\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]^n & \text{if} \quad f^{(n)} \in L_1[a,b] \end{cases} \end{aligned} \tag{5.11}$$

for all $x \in [a,b]$.

Proof. From equation (5.1) and the properties of the modulus, we have

$$\begin{aligned} &\left|\int_a^b f(t)\,dt - \sum_{k=0}^{n-1}\frac{1}{(k+1)!}\left[(x-a)^{k+1} f^{(k)}(a) + (-1)^k (b-x)^{k+1} f^{(k)}(b)\right]\right| \\ \leq\ & \frac{1}{n!}\int_a^b |x-t|^n \left|f^{(n)}(t)\right| dt =: R(x). \end{aligned}$$

Observe that

$$\begin{aligned} R(x) &\leq \left[\frac{1}{n!}\int_a^b |x-t|^n\,dt\right]\left\|f^{(n)}\right\|_\infty \\ &= \frac{\left\|f^{(n)}\right\|_\infty}{n!}\left[\int_a^x (x-t)^n\,dt + \int_x^b (t-x)^n\,dt\right] \\ &= \frac{\left\|f^{(n)}\right\|_\infty}{n!}\left[\frac{(x-a)^{n+1}+(b-x)^{n+1}}{n+1}\right] \end{aligned}$$

and the first inequality in (5.11) is proved.

Using Hölder's integral inequality, we also have

$$\begin{aligned} R(x) &\leq \frac{1}{n!}\left(\int_a^b \left|f^{(n)}(t)\right|^p dt\right)^{\frac{1}{p}} \left(\int_a^b |x-t|^{nq} dt\right)^{\frac{1}{q}} \\ &= \frac{1}{n!}\left\|f^{(n)}\right\|_p \left[\frac{(x-a)^{nq+1}+(b-x)^{nq+1}}{nq+1}\right]^{\frac{1}{q}}, \end{aligned}$$

which proves the second inequality in (5.11).

Finally, let us observe that

$$\begin{aligned} R(x) &\leq \frac{1}{n!}\sup_{t\in[a,b]}|x-t|^n \int_a^b \left|f^{(n)}(t)\right| dt \\ &= \frac{1}{n!}\left[\sup_{t\in[a,b]}|x-t|\right]^n \left\|f^{(n)}\right\|_1 \\ &= \frac{1}{n!}\left[\max(x-a,b-x)\right]^n \left\|f^{(n)}\right\|_1 \\ &= \frac{1}{n!}\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]^n \left\|f^{(n)}\right\|_1 \end{aligned}$$

and the theorem is completely proved. ∎

The following corollary is useful in practice.

Corollary 9. *With the above assumptions for f and n, we have the particular inequalities (see also* [15]*)*

$$\begin{aligned} &\left|\int_a^b f(t)\,dt - \sum_{k=0}^{n-1}\frac{1}{(k+1)!}\left(\frac{b-a}{2}\right)^{k+1}\left[f^{(k)}(a)+(-1)^k f^{(k)}(b)\right]\right| \\ (5.12)\quad &\leq \begin{cases} \frac{\left\|f^{(n)}\right\|_\infty}{2^n(n+1)!}(b-a)^{n+1} & \text{if} \quad f^{(n)}\in L_\infty[a,b], \\ \frac{\left\|f^{(n)}\right\|_p}{2^n n!(nq+1)^{1/q}}(b-a)^{n+\frac{1}{q}} & \text{if} \quad p>1,\ \frac{1}{p}+\frac{1}{q}=1 \\ & \text{and} \quad f^{(n)}\in L_p[a,b], \\ \frac{\left\|f^{(n)}\right\|_1}{2^n n!}(b-a)^n; & \text{if} \quad f^{(n)}\in L_1[a,b] \end{cases} \end{aligned}$$

respectively which are the sharpest possible from (5.11) with $x=\frac{a+b}{2}$.

Remark 11. *If we put $n=1$ in (5.11), we capture the inequality*

$$\begin{aligned} &\left|\int_a^b f(t)\,dt - \left[(x-a)f(a)+(b-x)f(b)\right]\right| \\ (5.13)\quad &\leq \begin{cases} \left[\frac{1}{4}(b-a)^2+\left(x-\frac{a+b}{2}\right)^2\right]\left\|f^{(1)}\right\|_\infty & \text{if} \quad f'\in L_\infty[a,b], \\ \left\|f'\right\|_p\left[\frac{(x-a)^{q+1}+(b-x)^{q+1}}{q+1}\right]^{\frac{1}{q}} & \text{if} \quad p>1,\ \frac{1}{p}+\frac{1}{q}=1 \\ & \text{and} \quad f'\in L_p[a,b], \\ \left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]\left\|f'\right\|_1; & \text{if } f'\in L_1[a,b] \end{cases} \end{aligned}$$

for all $x \in [a,b]$, and, in particular, the "trapezoid" inequality

$$\left|\int_a^b f(t)\,dt - \frac{b-a}{2}[f(a)+f(b)]\right| \tag{5.14}$$
$$\leq \begin{cases} \frac{\|f'\|_\infty}{4}(b-a)^2 & \text{if } f' \in L_\infty[a,b], \\ \frac{\|f'\|_p}{2(q+1)^{1/q}}(b-a)^{1+\frac{1}{q}} & \text{if } f' \in L_p[a,b], \\ \frac{\|f'\|_1}{2}(b-a) & \text{if } f' \in L_1[a,b] \end{cases}$$

is obtained by taking $x = \frac{a+b}{2}$.

Remark 12. *If we put $n = 2$ in (5.11), we get the inequality*

$$\left|\int_a^b f(t)\,dt - [(x-a)f(a)+(b-x)f(b)] - \frac{1}{2}\left[(x-a)^2 f'(a) - (b-x)^2 f'(b)\right]\right| \tag{5.15}$$
$$\leq \begin{cases} \frac{\|f''\|_\infty}{6}\left[(b-a)^3+(b-x)^3\right] & \text{if} \quad f'' \in L_\infty[a,b]; \\ \frac{\|f''\|_p}{2}\left[\frac{(x-a)^{2q+1}+(b-x)^{2q+1}}{2q+1}\right]^{\frac{1}{q}} & \text{if} \quad p>1,\ \frac{1}{p}+\frac{1}{q}=1 \\ & \text{and} \quad f'' \in L_p[a,b]; \\ \frac{\|f''\|_1}{2}\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]^2 & \text{if} \quad f'' \in L_1[a,b] \end{cases}$$

for all $x \in [a,b]$, and, in particular: the "perturbed trapezoid" inequality

$$\left|\int_a^b f(t)\,dt - \frac{b-a}{2}[f(a)+f(b)] + \frac{(b-a)^2}{8}\left(f'(b)-f'(a)\right)\right| \tag{5.16}$$
$$\leq \begin{cases} \frac{\|f''\|_\infty}{24}(b-a)^3 & \text{if } f'' \in L_\infty[a,b]; \\ \frac{\|f''\|_p}{8(2q+1)^{1/q}}(b-a)^{2+\frac{1}{q}} & \text{if } f'' \in L_p[a,b]; \\ \frac{\|f''\|_1}{8}(b-a)^2 & \text{if } f'' \in L_1[a,b] \end{cases}$$

is obtained on taking $x = \frac{a+b}{2}$.

In practice the perturbed trapezoid inequality only involves the evaluation of the derivatives at the boundary points for a uniform partition of the interval.

5.3. **A Perturbed Version.** A *premature* Grüss inequality is embodied in the following theorem which was considered and applied for the first time in the paper [18] by Matić, Pečarić and Ujević.

Theorem 9. *Let h, g be integrable functions defined on $[a,b]$ and let $d \leq g(t) \leq D$. Then*

$$|T(h,g)| \leq \frac{D-d}{2}[T(h,h)]^{\frac{1}{2}}, \tag{5.17}$$

where

$$T(h,g)=\frac{1}{b-a}\int_a^b h(t)\,g(t)\,dt-\frac{1}{b-a}\int_a^b h(t)\,dt\cdot\frac{1}{b-a}\int_a^b g(t)\,dt.$$

Remark 13. *For some applications of this result for three-point quadrature formulae see* [16].

Using the above theorem, the following result may be stated [32].

Theorem 10. *Let $f:[a,b]\to\mathbb{R}$ so that the derivative $f^{(n-1)}$, $n\geq 1$ is absolutely continuous on $[a,b]$. Assume that there exist constants $\gamma,\Gamma\in\mathbb{R}$ such that $\gamma\leq f^{(n)}(t)\leq\Gamma$ a.e on $[a,b]$. Then, the following inequality holds*

$$\begin{aligned}(5.18)\qquad |P_T(x)| &:= \Bigg|\int_a^b f(t)\,dt-\sum_{k=0}^{n-1}\left(\frac{1}{(k+1)!}\times\right.\\ &\left.\left[(x-a)^{k+1}f^{(k)}(a)+(-1)^k(b-x)^{k+1}f^{(k)}(b)\right]\right)\\ &-\frac{(x-a)^{n+1}+(-1)^n(b-x)^{n+1}}{(n+1)!}\left[\frac{f^{(n-1)}(b)-f^{(n-1)}(a)}{b-a}\right]\Bigg|\\ &\leq\frac{\Gamma-\gamma}{2}\cdot\frac{1}{n!}I(x,n)\\ &\leq\frac{\Gamma-\gamma}{2}\cdot\frac{n}{n+1}\cdot\frac{(b-a)^{n+1}}{\sqrt{2n+1}},\end{aligned}$$

where

$$\begin{aligned}(5.19)\qquad & I(x,n)\\ =\ &\frac{1}{(n+1)\sqrt{2n+1}}\left\{n^2(b-a)\left[(x-a)^{2n+1}+(b-x)^{2n+1}\right]\right.\\ &\left.+(2n+1)(x-a)(b-x)\left[(x-a)^n-(x-b)^n\right]^2\right\}^{\frac{1}{2}}.\end{aligned}$$

Proof. Applying the premature Grüss result (5.17) on $(x-t)^n$ and $f^{(n)}(t)$, we have

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b(x-t)^n f^{(n)}(t)\,dt-\frac{1}{b-a}\int_a^b(x-t)^n\,dt\cdot\frac{1}{b-a}\int_a^b f^{(n)}(t)\,dt\right|\\ \leq\ &\frac{\Gamma-\gamma}{2}\left\{\frac{1}{b-a}\int_a^b(x-t)^{2n}\,dt-\left[\frac{1}{b-a}\int_a^b(x-t)^n\,dt\right]^2\right\}^{\frac{1}{2}}.\end{aligned}$$

Therefore,

$$\begin{aligned}&\Bigg|\frac{1}{b-a}\int_a^b(x-t)^n f^{(n)}(t)\,dt\\ &-\frac{(x-a)^{n+1}+(-1)^n(b-x)^{n+1}}{(n+1)(b-a)}\cdot\frac{f^{(n-1)}(b)-f^{(n-1)}(a)}{b-a}\Bigg|\end{aligned}$$

$$\leq \frac{\Gamma-\gamma}{2}\left\{\frac{(x-a)^{2n+1}+(b-x)^{2n+1}}{(2n+1)(b-a)}\right.$$
$$\left.-\left[\frac{(x-a)^{n+1}+(-1)^n(b-x)^{n+1}}{(b-a)(n+1)}\right]^2\right\}^{\frac{1}{2}}.$$

Further, simplification of the above result by multiplying throughout by $\frac{b-a}{n!}$ gives

$$\begin{aligned}(5.20)\qquad &\left|\frac{1}{n!}\int_a^b (x-t)^n f^{(n)}(t)\,dt\right.\\ &\left.-\frac{(x-a)^{n+1}+(-1)^n(b-x)^{n+1}}{(n+1)!}\cdot\left[\frac{f^{(n-1)}(b)-f^{(n-1)}(a)}{b-a}\right]\right|\\ \leq\ &\frac{\Gamma-\gamma}{2}\cdot\frac{1}{n!}J(x,n),\end{aligned}$$

where

$$\begin{aligned}(5.21)\qquad &J^2(x,n)\\ =\ &\frac{1}{(2n+1)(n+1)^2}\left\{(n+1)^2(A+B)\left(A^{2n+1}+B^{2n+1}\right)\right.\\ &\left.-(2n+1)\left(A^{n+1}+(-1)^n B^{n+1}\right)^2\right\}\end{aligned}$$

with $A=x-a$, $B=b-x$.

Now, from (5.21),

$$\begin{aligned}&(2n+1)(n+1)^2 J^2(x,n)\\ =\ &n^2(A+B)\left(A^{2n+1}+B^{2n+1}\right)\\ &+(2n+1)\left[(A+B)\left(A^{2n+1}+B^{2n+1}\right)-\left(A^{n+1}+(-1)^n B^{n+1}\right)^2\right]\\ =\ &n^2(A+B)\left(A^{2n+1}+B^{2n+1}\right)\\ &+(2n+1)\left[AB\left(A^{2n}+B^{2n}\right)-2A^{n+1}\cdot(-1)^n B^{n+1}\right]\\ =\ &n^2(A+B)\left[A^{2n+1}+B^{2n+1}\right]+(2n+1)AB\left[A^n-(-B)^n\right]^2.\end{aligned}$$

Now, substitution of $A=x-a$, $B=b-x$ and the fact that $A+B=b-a$ gives $I(x,n)=\frac{J(x,n)}{(n+1)\sqrt{2n+1}}$, as presented in (5.19). Substitution of identity (5.1) into (5.20) gives (5.18) and the first part of the theorem is thus proved.

The upper bound is obtained by taking either $I(a,n)$ or $I(b,n)$ since $I(x,n)$ is convex and symmetric. Hence the theorem is completely proved. ∎

Corollary 10. *Let the conditions of Theorem 10 hold. Then the following result holds*

$$\left|\int_a^b f(t)\,dt - \sum_{k=0}^{n-1} \frac{1}{(k+1)!}\left(\frac{b-a}{2}\right)^{k+1}\left[f^{(k)}(a) + (-1)^k f^{(k)}(b)\right] - \left(\frac{b-a}{2}\right)^{n+1}\frac{[1+(-1)^n]}{(n+1)!}\left[\frac{f^{(n-1)}(b) - f^{(n-1)}(a)}{b-a}\right]\right|$$

$$\leq \frac{\Gamma-\gamma}{2}\cdot\frac{1}{n!}\left(\frac{b-a}{2}\right)^{n+1}\cdot\frac{1}{\sqrt{2n+1}}\cdot\begin{cases}\frac{2n}{n+1}, & n \text{ even}\\ 2, & n \text{ odd}\end{cases}. \tag{5.22}$$

Proof. Taking $x = \frac{a+b}{2}$ in (5.18) gives (5.22), where

$$I\left(\frac{a+b}{2}, n\right) = \frac{1}{(n+1)\sqrt{2n+1}}\left(\frac{b-a}{2}\right)^{n+1}\left\{4n^2 + (2n+1)\left[1+(-1)^n\right]^2\right\}^{\frac{1}{2}}.$$

Examining the above expression for n even or n odd readily gives the result (5.22). ∎

Remark 14. *For n odd, then the third term in the modulus sign vanishes and thus there is no perturbation to the trapezoidal rule (5.22).*

Theorem 11. *Let the conditions of Theorem 10 be satisfied. Further, suppose that $f^{(n)}$ is differentiable and be such that*

$$\left\|f^{(n+1)}\right\|_\infty := \sup_{t\in[a,b]}\left|f^{n+1}(t)\right| < \infty.$$

Then

$$|P_T(x)| \leq \frac{b-a}{\sqrt{12}}\left\|f^{(n+1)}\right\|_\infty \cdot \frac{1}{n!} I(x,n), \tag{5.23}$$

where $P_T(x)$ is the perturbed trapezoidal type rule given by the left hand side of (5.18) and $I(x,n)$ is as given by (5.19).

Proof. Let $h, g : [a,b] \to \mathbb{R}$ be absolutely continuous and h', g' be bounded. Then Chebychev's inequality holds (see [17, p. 207])

$$|T(h,g)| \leq \frac{(b-a)^2}{12}\sup_{t\in[a,b]}|h'(t)| \cdot \sup_{t\in[a,b]}|g'(t)|.$$

Matić, Pečarić and Ujević [18] using a *premature* Grüss type argument proved that

$$|T(h,g)| \leq \frac{(b-a)}{\sqrt{12}}\sup_{t\in[a,b]}|g'(t)|\sqrt{T(h,h)}. \tag{5.24}$$

Thus, associating $f^{(n)}(\cdot)$ with $g(\cdot)$ and $(x-t)^n$ with h in (5.24) readily produces (5.23) where $I(x,n)$ is as given by (5.19). ∎

Theorem 12. *Let the conditions of Theorem 10 be satisfied. Further, suppose that $f^{(n)}$ is locally absolutely continuous on (a,b) and let $f^{(n+1)} \in L_2(a,b)$. Then*

$$|P_T(x)| \leq \frac{b-a}{\pi}\left\|f^{(n+1)}\right\|_2 \cdot \frac{1}{n!} I(x,n), \tag{5.25}$$

where $P_T(x)$ is the perturbed trapezoidal type rule given by the left hand side of (5.18) and $I(x,n)$ is as given in (5.19).

Proof. The following result was obtained by Lupaş (see [17, p. 210]). For $h,g:(a,b)\to\mathbb{R}$ locally absolutely continuous on (a,b) and $h',g'\in L_2(a,b)$, then

$$|T(h,g)|\leq\frac{(b-a)^2}{\pi^2}\|h'\|_2\|g'\|_2,$$

where

$$\|h\|_2:=\left(\frac{1}{b-a}\int_a^b|h(t)|^2\right)^{\frac{1}{2}}\quad\text{for}\quad h\in L_2(a,b).$$

Matić, Pečarić and Ujević [18] further show that

$$|T(h,g)|\leq\frac{(b-a)}{\pi}\|g'\|_2\sqrt{T(h,h)}. \tag{5.26}$$

Now, associating $f^{(n)}(\cdot)$ with $g(\cdot)$ and $(x-t)^n$ with h in (5.26) gives (5.25), where $I(x,n)$ is found in (5.19). ∎

Remark 15. *Results (5.23) and (5.25) are not readily comparable to that obtained in Theorem 10 since the bound now involves the behaviour of $f^{(n+1)}(\cdot)$ rather than $f^{(n)}(\cdot)$.*

5.3.1. *Application In Numerical Integration.* Consider the partition $I_m:a=x_0<x_1<...<x_{m-1}<x_m=b$ of the interval $[a,b]$ and the intermediate points $\xi=(\xi_0,...,\xi_{m-1})$, where $\xi_j\in[x_j,x_{j+1}]$ $(j=0,...,m-1)$. Put $h_j:=x_{j+1}-x_j$ and $\nu(h)=\max\{h_j|j=0,...,m-1\}$.

In [7], the authors considered the following generalization of the trapezoid formula

$$T_{m,n}(f,I_m):=\sum_{j=0}^{m-1}\sum_{k=0}^{n-1}\frac{h_j^{k+1}}{(k+1)!}\left[\frac{f^{(k)}(x_j)+(-1)^kf^{(k)}(x_{j+1})}{2}\right] \tag{5.27}$$

and proved the following theorem:

Theorem 13. *Let $f:[a,b]\to\mathbb{R}$ be such that it's derivative $f^{(n-1)}$ is absolutely continuous on $[a,b]$. Then we have*

$$\int_a^bf(t)\,dt=T_{m,n}(f,I_m)+R_{m,n}(f,I_m), \tag{5.28}$$

where the reminder $R_{m,n}(f,I_m)$ satisfies the estimate

$$|R_{m,n}(f,I_m)|\leq\frac{C_n}{(n+1)!}\left\|f^{(n)}\right\|_\infty\sum_{j=0}^{m-1}h_j^{n+1}, \tag{5.29}$$

and

$$C_n:=\begin{cases}1 & \text{if } n=2r\\ \frac{2^{2r+1}-1}{2^{2r}} & \text{if } n=2r+1.\end{cases}$$

Now, let us define the even more generalized quadrature formula

$$\begin{aligned}\tilde{T}_{m,n}(f,\boldsymbol{\xi},I_m)\quad:\quad&=\sum_{j=0}^{m-1}\sum_{k=0}^{n-1}\frac{1}{(k+1)!}\left[\left(\xi_j-x_j\right)^{k+1}f^{(k)}(x_j)\right.\\&\left.+(-1)^k\left(x_{j+1}-\xi_j\right)^{k+1}f^{(k)}(x_{j+1})\right],\end{aligned}$$

where x_j,ξ_j $(j=0,...,m-1)$ are as above.

The following theorem holds [32].

Theorem 14. *Let f be as in Theorem 13. Then we have the formula*

$$\int_a^b f(t)\,dt = \tilde{T}_{m,n}(f,\boldsymbol{\xi},I_m) + \tilde{R}_{m,n}(f,\boldsymbol{\xi},I_m), \tag{5.30}$$

where the remainder satisfies the estimate

$$\left|\tilde{R}_{m,n}(f,\boldsymbol{\xi},I_m)\right| \tag{5.31}$$
$$\leq \begin{cases} \frac{1}{(n+1)!}\left\|f^{(n)}\right\|_\infty \sum\limits_{j=0}^{m-1}\left[\left(\xi_j - x_j\right)^{n+1} + \left(x_{j+1}-\xi_j\right)^{n+1}\right], \\ \\ \frac{1}{n!(nq+1)^{1/q}}\left\|f^{(n)}\right\|_p \left[\sum\limits_{j=0}^{m-1}\left(\xi_j - x_j\right)^{nq+1} + \sum\limits_{j=0}^{m-1}\left(x_{j+1}-\xi_j\right)^{nq+1}\right]^{\frac{1}{q}}, \\ \\ \frac{1}{n!}\left\|f^{(n)}\right\|_1 \left[\frac{1}{2}\nu(h) + \max\limits_{j=0,\dots,m-1}\left|\xi_j - \frac{x_j+x_{j+1}}{2}\right|\right]^n. \end{cases}$$

Proof. Apply the inequality (5.11) on the subinterval $[x_j, x_{j+1}]$ to get

$$\Bigg|\int_{x_j}^{x_{j+1}} f(t)\,dt - \sum_{k=0}^{n-1}\frac{1}{(k+1)!}$$
$$\times\left[\left(\xi_j - x_j\right)^{k+1} f^{(k)}(x_j) + (-1)^k\left(x_{j+1}-\xi_j\right)^{k+1} f^{(k)}(x_{j+1})\right]\Bigg|$$
$$\leq \begin{cases} \frac{1}{(n+1)!}\sup\limits_{t\in[x_j,x_{j+1}]}\left|f^{(n)}(t)\right|\left[\left(\xi_j - x_j\right)^{n+1} + \left(x_{j+1}-\xi_j\right)^{n+1}\right], \\ \\ \frac{1}{n!}\left(\int_{x_j}^{x_{j+1}}\left|f^{(n)}(s)\right|^p ds\right)^{\frac{1}{p}}\left[\frac{\left(\xi_j - x_j\right)^{nq+1} + \left(x_{j+1}-\xi_j\right)^{nq+1}}{nq+1}\right]^{\frac{1}{q}}, \\ \\ \frac{1}{n!}\left(\int_{x_j}^{x_{j+1}}\left|f^{(n)}(s)\right| ds\right)\left[\frac{1}{2}h_j + \left|\xi_j - \frac{x_j+x_{j+1}}{2}\right|\right]^n. \end{cases}$$

Summing over j from 0 to $m-1$ and using the generalized triangle inequality, we have

$$\left|\tilde{R}_{m,n}(f,\boldsymbol{\xi},I_m)\right|$$
$$\leq \Bigg|\sum_{j=0}^{m-1}\int_{x_j}^{x_{j+1}} f(t)\,dt - \sum_{k=0}^{n-1}\frac{1}{(k+1)!}$$
$$\times\left[\left(\xi_j - x_j\right)^{k+1} f^{(k)}(x_j) + (-1)^k\left(x_{j+1}-\xi_j\right)^{k+1} f^{(k)}(x_{j+1})\right]\Bigg|$$

$$:= \begin{cases} \frac{1}{(n+1)!}\sum_{j=0}^{m-1}\sup_{t\in[x_j,x_{j+1}]}\left|f^{(n)}(t)\right|\left[\left(\xi_j-x_j\right)^{n+1}+\left(x_{j+1}-\xi_j\right)^{n+1}\right], \\ \frac{1}{n!}\sum_{j=0}^{m-1}\left(\int_{x_j}^{x_{j+1}}\left|f^{(n)}(s)\right|^p ds\right)^{\frac{1}{p}}\left[\frac{\left(\xi_j-x_j\right)^{nq+1}+\left(x_{j+1}-\xi_j\right)^{nq+1}}{nq+1}\right]^{\frac{1}{q}}, \\ \frac{1}{n!}\sum_{j=0}^{m-1}\left(\int_{x_j}^{x_{j+1}}\left|f^{(n)}(s)\right|ds\right)\left[\frac{1}{2}h_j+\left|\xi_j-\frac{x_j-x_{j+1}}{2}\right|\right]^n. \end{cases}$$

As $\sup_{t\in[x_j,x_{j+1}]}\left|f^{(n)}(t)\right|\le\left\|f^{(n)}\right\|_\infty$, the first inequality in (5.31) readily follows.

Now, using the discrete Hölder inequality, we have

$$\begin{aligned} &\frac{1}{(nq+1)^{1/q}}\sum_{j=0}^{m-1}\left(\int_{x_j}^{x_{j+1}}\left|f^{(n)}(s)\right|^p ds\right)^{\frac{1}{p}}\left[\left(\xi_j-x_j\right)^{nq+1}+\left(x_{j+1}-\xi_j\right)^{nq+1}\right]^{\frac{1}{q}} \\ \le\;& \frac{1}{(nq+1)^{1/q}}\left[\sum_{j=0}^{m-1}\left[\left(\int_{x_j}^{x_{j+1}}\left|f^{(n)}(s)\right|^p ds\right)^{\frac{1}{p}}\right]^p\right]^{\frac{1}{p}} \\ &\times\left[\sum_{j=0}^{m-1}\left[\left[\left(\xi_j-x_j\right)^{nq+1}+\left(x_{j+1}-\xi_j\right)^{nq+1}\right]^{\frac{1}{q}}\right]^q\right]^{\frac{1}{q}} \\ =\;& \frac{1}{(nq+1)^{1/q}}\left\|f^{(n)}\right\|_p\left[\sum_{j=0}^{m-1}\left(\xi_j-x_j\right)^{nq+1}+\sum_{j=0}^{m-1}\left(x_{j+1}-\xi_j\right)^{nq+1}\right]^{\frac{1}{q}} \end{aligned}$$

and thus the second inequality in (5.31) is proved.

Finally, let us observe that

$$\begin{aligned} &\frac{1}{n!}\sum_{j=0}^{m-1}\left(\int_{x_j}^{x_{j+1}}\left|f^{(n)}(s)\right|ds\right)\left[\frac{1}{2}h_j+\left|\xi_j-\frac{x_j+x_{j+1}}{2}\right|\right]^n \\ \le\;& \max_{j=0,\ldots,m-1}\left[\frac{1}{2}h_j+\left|\xi_j-\frac{x_j+x_{j+1}}{2}\right|\right]^n\sum_{j=0}^{m-1}\left(\int_{x_j}^{x_{j+1}}\left|f^{(n)}(s)\right|ds\right) \\ \le\;& \left[\frac{1}{2}h_j+\max_{j=0,\ldots,m-1}\left|\xi_j-\frac{x_j+x_{j+1}}{2}\right|\right]^n\left\|f^{(n)}\right\|_1 \end{aligned}$$

and the last part of (5.31) is proved. ∎

Remark 16. *As $(x-a)^\alpha+(b-x)^\alpha\le(b-a)^\alpha$ for $\alpha\ge1$, $x\in[a,b]$, then we remark that the first branch of (5.31) can be bounded by*

$$\frac{1}{(n+1)!}\left\|f^{(n)}\right\|_\infty\sum_{j=0}^{m-1}h_j^{n+1}. \tag{5.32}$$

The second branch can be bounded by

$$\frac{1}{n!(nq+1)^{1/q}}\left\|f^{(n)}\right\|_p\left[\sum_{j=0}^{m-1}h_j^{nq+1}\right]^{\frac{1}{q}} \tag{5.33}$$

and finally, the last branch in (5.31) can be bounded by

$$\frac{1}{n!}\left[\nu(h)\right]^{n}\left\|f^{(n)}\right\|_{1}. \tag{5.34}$$

Note that all the bounds provided by (5.32)-(5.34) are uniform bounds for $\tilde{R}_{m,n}(f,\xi,I_m)$ in terms of the intermediate points ξ.

A further inequality that we can obtain from (5.31) is the one that results from taking $\xi_j = \frac{x_j+x_{j+1}}{2}$. Consequently, we can state the following corollary (see also [15]):

Corollary 11. *Let f be as in Theorem 14. Then we have the formula*

$$\int_a^b f(t)\,dt = \tilde{T}_{m,n}(f,I_m) + \tilde{R}_{m,n}(f,I_m), \tag{5.35}$$

where

$$\tilde{T}_{m,n}(f,I_m) = \sum_{j=0}^{m-1}\sum_{k=0}^{n-1}\frac{1}{2^{k+1}(k+1)!}\left[f^{(k)}(x_j) + (-1)^k f^{(k)}(x_{j+1})\right]h_j^{n+1} \tag{5.36}$$

and the remainder $\tilde{R}$ satisfies the estimate

$$\left|\tilde{R}_{m,n}(f,I_m)\right| \le \begin{cases} \frac{1}{2^n(n+1)!}\left\|f^{(n)}\right\|_\infty \sum\limits_{j=0}^{m-1} h_j^{n+1}, \\ \\ \frac{1}{2^n n!(nq+1)^{1/q}}\left\|f^{(n)}\right\|_p \left[\sum\limits_{j=0}^{m-1} h_j^{n+1}\right]^{\frac{1}{q}}, \\ \\ \frac{1}{2^n n!}\left[\nu(h)\right]^n \left\|f^{(n)}\right\|_1. \end{cases}$$

Remark 17. *Similar results can be stated by using the "perturbed" versions embodied in Theorems 10, 11 and 12, but we omit the details.*

6. Trapezoidal Type Rules for Functions whose Derivative is Bounded Above and Below

6.1. **Introduction.**

In 1938, Iyengar proved the following theorem obtaining bounds for a trapezoidal quadrature rule for functions whose derivative is bounded (see for example [21, p. 471]).

Theorem 15. *Let f be a differentiable function on (a,b) and assume that there is a constant $M>0$ such that $|f'(x)| \le M$, for all $x \in (a,b)$. Then we have*

$$\left|\int_a^b f(x)\,dx - \frac{b-a}{2}\left[f(a)+f(b)\right]\right| \le \frac{M(b-a)^2}{4} - \frac{1}{4M}\left(f(a)-f(b)\right)^2. \tag{6.1}$$

Using the classical inequality due to Hayashi (see for example, [20, pp. 311-312]), Agarwal and Dragomir proved in [19] the following generalization of Theorem 15 involving the Trapezoidal rule.

Theorem 16. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping in $\mathring{I}$, the interior of I, and let $a, b \in \mathring{I}$ with $a < b$. Let $M = \sup_{x\in[a,b]} f'(x) < \infty$ and $m = \inf_{x\in[a,b]} f'(x) > -\infty$. If $m < M$ and f' is integrable on $[a,b]$, then we have*

$$\left|\int_a^b f(t)\,dt - \frac{b-a}{2}[f(a)+f(b)]\right| \leq \frac{[f(b)-f(a)-m(b-a)][M(b-a)-f(b)+f(a)]}{2(M-m)}. \tag{6.2}$$

Thus, by placing $m = -M$ in (6.2) then Iyengar's result (6.1) is recovered.

In this section we point out further results in connection to the trapezoid inequality.

6.2. **Integral Inequalities.** The following theorem due to Hayashi [20, pp. 311-312] will be required and thus it is stated for convenience.

Theorem 17. *Let $h : [a,b] \longrightarrow \mathbb{R}$ be a nonincreasing mapping on $[a,b]$ and $g : [a,b] \longrightarrow \mathbb{R}$ an integrable mapping on $[a,b]$ with*

$$0 \leq g(x) \leq A, \text{ for all } x \in [a,b],$$

then

$$A\int_{b-\lambda}^b h(x)\,dx \leq \int_a^b h(x)\,g(x)\,dx \leq A\int_a^{a+\lambda} h(x)\,dx \tag{6.3}$$

where

$$\lambda = \frac{1}{A}\int_a^b g(x)\,dx.$$

Using this result we can state the following trapezoid inequality.

Theorem 18. *Let $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be a differentiable mapping on $\mathring{I}$ ($\mathring{I}$ is the interior of I) and $[a,b] \subset \mathring{I}$ with $M = \sup_{x\in[a,b]} f'(x) < \infty$, $m = \inf_{x\in[a,b]} f'(x) > -\infty$ and $M > m$. If f' is integrable on $[a,b]$, then the following inequalities hold:*

$$\left|\int_a^b f(x)\,dx - \frac{b-a}{2}[f(a)+f(b)]\right|$$
$$\leq \frac{(b-a)^2}{2(M-m)}(S-m)(M-S) \tag{6.4}$$
$$\leq \frac{M-m}{2}\left(\frac{b-a}{2}\right)^2 \tag{6.5}$$

where $S = \frac{f(b)-f(a)}{b-a}$.

Proof. Let $h(x) = \theta - x$, $\theta \in [a,b]$ and $g(x) = f'(x) - m$. Then, from Hayashi's inequality (6.3)

$$L \leq I \leq U \tag{6.6}$$

where

$$I = \int_a^b (\theta - x)(f'(x) - m)\,dx,$$

$$\lambda = \frac{1}{M-m}\int_a^b \left(f'(x) - m\right) dx,$$

and

$$L = (M-m)\int_{b-\lambda}^{b} (\theta - x)\, dx,$$

$$U = (M-m)\int_{a}^{a+\lambda} (\theta - x)\, dx.$$

It is now a straight-forward matter to evaluate and simplify the above expansions to give

$$\begin{aligned} I &= \int_a^b f(u)\, du \\ &\quad - \left[m(b-a)\left(\theta - \frac{b+a}{2}\right) + (b-\theta) f(b) + (\theta - a) f(a)\right], \end{aligned} \tag{6.7}$$

$$\lambda = \frac{1}{M-m}\left[f(b) - f(a) - m(b-a)\right] = \frac{b-a}{M-m}(S-m), \tag{6.8}$$

$$L = \frac{(M-m)}{2}\lambda\left[\lambda + 2(\theta - b)\right], \tag{6.9}$$

and

$$U = \frac{(M-m)}{2}\lambda\left[2(\theta - a) - \lambda\right]. \tag{6.10}$$

In addition, it may be noticed from (6.6), that

$$\left|I - \frac{U+L}{2}\right| \le \frac{U-L}{2}, \tag{6.11}$$

where, upon using (6.9) and (6.10),

$$\frac{U+L}{2} = (M-m)\lambda\left(\theta - \frac{b+a}{2}\right) \tag{6.12}$$

and

$$\frac{U-L}{2} = \frac{(M-m)}{2}\lambda(b-a-\lambda). \tag{6.13}$$

Equation (6.11) is then, (6.4) upon using (6.7), (6.8), (6.12) and (6.13) together with some routine simplification.

Now, for inequality (6.5). Consider the right hand side of (6.4). Completing the square gives

$$\begin{aligned} &\frac{(b-a)^2}{2(M-m)}(S-m)(M-S) \\ &= \frac{2}{M-m}\left(\frac{b-a}{2}\right)^2 \times \left[\left(\frac{M-m}{2}\right)^2 - \left(S - \frac{M+m}{2}\right)^2\right] \end{aligned} \tag{6.14}$$

and (6.5) is readily determined by neglecting the negative term. ∎

Remark 18. *The above theorem was proved independently of the value of* θ. *Agarwal and Dragomir* [19] *proved an equivalent result with effectively* $\theta = a$. *It may be noticed from the above development however, that if* $\theta = \frac{a+b}{2}$ *then there is some simplification for* I *and* $\frac{U+L}{2} = 0$.

Remark 19. *For* $\|f'\|_\infty = \sup_{x\in[a,b]} |f'(x)| < \infty$ *and let* $m = -\|f'\|_\infty$, $M = \|f'\|_\infty$ *in* (6.4). *Then the result obtained by Iyengar* [21, p. 471] *using geometrical means, is recovered. It should also be noted that if either both* m *and* M *are positive or both negative, then the bound obtained here is tighter than that of Iyengar as given by* (6.1).

Bounds for the generalized trapezoidal rule will now be developed in the following theorem.

Theorem 19. *Let* f *satisfy the conditions of Theorem 18, then the following result holds*

$$\beta_L \le \int_a^b f(x)\,dx - (b-a)\left[\left(\frac{\theta-a}{b-a}\right)f(a) + \left(\frac{b-\theta}{b-a}\right)f(b)\right] \le \beta_U \tag{6.15}$$

where

$$\beta_U = \frac{(b-a)^2}{2(M-m)}\left[S(2\gamma_U - S) - mM\right], \tag{6.16}$$

$$\beta_L = \frac{(b-a)^2}{2(M-m)}\left[S(S-2\gamma_L) + mM\right], \tag{6.17}$$

$$\gamma_U = \left(\frac{\theta-a}{b-a}\right)M + \left(\frac{b-\theta}{b-a}\right)m,\ \ \gamma_L = M + m - \gamma_U, \tag{6.18}$$

and

$$S = \frac{f(b)-f(a)}{b-a}. \tag{6.19}$$

Proof. From (6.6) and (6.7) it may be readily seen that

$$\beta_U = U + m(b-a)\left(\theta - \frac{a+b}{2}\right) \tag{6.20}$$

and

$$\beta_L = L + m(b-a)\left(\theta - \frac{a+b}{2}\right). \tag{6.21}$$

Now, from (6.20) and using (6.10), (6.8) gives

$$\begin{aligned}
\beta_U &= \frac{1}{2(M-m)}\Big\{(b-a)(S-m)\left[2(M-m)(\theta-a) - (b-a)(S-m)\right] \\
&\quad + 2m(b-a)(M-m)\left(\theta - \frac{a+b}{2}\right)\Big\} \\
&= \frac{(b-a)^2}{2(M-m)}\left\{(S-m)\left[S-m+2(M-m)\left(\frac{\theta-a}{b-a}\right)\right]\right. \\
&\quad \left. + 2m\left(\frac{M-m}{b-a}\right)\left(\theta - \frac{a+b}{2}\right)\right\}.
\end{aligned}$$

Expanding in powers of S and after simplification we produce the expression (6.16). In a similar fashion, (6.17) may be derived from (6.21) and using (6.9), (6.8) gives

$$\begin{aligned}\beta_L &= \frac{1}{2(M-m)}\{(b-a)(S-m)[(b-a)(S-m)+2(M-m)(\theta-b)] \\ &\quad + 2m(b-a)(M-m)\left(\theta-\frac{a+b}{2}\right)\Big\} \\ &= \frac{(b-a)^2}{2(M-m)}\left\{(S-m)\left[S-m+2(M-m)\left(\frac{\theta-b}{b-a}\right)\right]\right. \\ &\quad \left. + 2m\left(\frac{M-m}{b-a}\right)\left(\theta-\frac{a+b}{2}\right)\right\}.\end{aligned}$$

Again, expanding in powers of S produces (6.17) after some algebra and thus the proof of the theorem is complete. ∎

Remark 20. *Allowing* $\theta = \frac{a+b}{2}$ *gives*

$$\beta_L = -\beta_U = \frac{(b-a)^2}{2(M-m)}(S-m)(M-S),$$

thus reproducing the result of Theorem 18.

Remark 21. *It may be shown from* (6.16) *and* (6.17) *that for any* $\theta \in [a,b]$, *the size of the bound interval for the generalized trapezoidal rule is:*

$$\beta_U - \beta_L = \frac{(b-a)^2}{(M-m)}\left[\left(\frac{M-m}{2}\right)^2 - \left(S-\frac{M+m}{2}\right)^2\right].$$

This is the same size as that for the symmetric bounds for the trapezoidal rule of Theorem 18 which seems, at first, surprising though on observing (6.20) *and* (6.21) *may be less so.*

Remark 22. *The difference between the upper and lower bounds is always positive since*

$$\beta_U - \beta_L = \frac{(b-a)^2}{M-m}(S-m)(M-S) \geq 0$$

where S, from (6.19), *is the slope of the secant and* $m \leq S \leq M$.

Remark 23. *For* $\|f'\|_\infty = \sup_{x\in[a,b]}|f'(x)| < \infty$, *let* $m = -\|f'\|_\infty$ *and* $M = \|f'\|$ *in* $(6.15)-(6.19)$ *then an Iyengar type result for the generalized trapezoidal rule will be obtained.*

Corollary 12. *Let f satisfy the conditions of Theorems 18 and 19. Then*

$$\begin{aligned}&\frac{(b-a)^2}{2(M-m)}\left(mM-\gamma_L^2\right) \\ \leq\ & \int_a^b f(u)\,du - (b-a)\left[\left(\frac{\theta-a}{b-a}\right)f(a) + \left(\frac{b-\theta}{b-a}\right)f(b)\right] \\ \leq\ & \frac{(b-a)^2}{2(M-m)}\left[\gamma_U^2 - mM\right]\end{aligned} \tag{6.22}$$

where γ_U *and* γ_L *are as defined in* (6.18).

Proof. From (6.15) and (6.16) it may be shown by completing the square that

$$\beta_U = \frac{(b-a)^2}{2(M-m)}\left[\gamma_U^2 - mM - (S-\gamma_U)^2\right]$$

and

$$\beta_L = \frac{(b-a)^2}{2(M-m)}\left[(S-\gamma_L)^2 + mM - \gamma_L^2\right].$$

The result (6.22) follows from neglecting the negative term from β_U and the positive term from β_L. ∎

Remark 24. *The results obtained in this section could also be implemented by constructing composite quadrature rules as previously. This, however, will not be pursued further here.*

7. Grüss Type Bounds

In 1935, G. Grüss (see for example [20, p. 296]), proved the following integral inequality which gives an approximation for the integral of a product in terms of the product of integrals:

Theorem 20. *Let $f, g : [a,b] \to \mathbb{R}$ be two integrable mappings so that $\varphi \le h(x) \le \Phi$ and $\gamma \le g(x) \le \Gamma$ for all $x \in [a,b]$, where $\varphi, \Phi, \gamma, \Gamma$ are real numbers. Then we have:*

$$\begin{aligned} |T(h,g)| \quad : \quad &= \left|\frac{1}{b-a}\int_a^b h(x)\,g(x)\,dx\right. \\ &\quad \left. -\frac{1}{b-a}\int_a^b h(x)\,dx \cdot \frac{1}{b-a}\int_a^b g(x)\,dx\right| \\ &\le \frac{1}{4}(\Phi-\varphi)(\Gamma-\gamma) \end{aligned} \tag{7.1}$$

and the inequality is sharp, in the sense that the constant $\frac{1}{4}$ can not be replaced by a smaller one.

For a simple proof of this fact as well as for extensions, generalizations, discrete variants and other associated material, see [20, p. 296], and the papers [24]-[29] where further references are given.

In this section, we point out a different Grüss type inequality and apply it for trapezoid formula.

7.1. A Grüss type Result and Applications for the Trapezoid Inequality.

We start with the following result of Grüss type [12].

Theorem 21. *Let $h, g : [a,b] \to \mathbb{R}$ be two integrable mappings. Then we have the following Grüss type inequality:*

$$\begin{aligned} &\left|\frac{1}{b-a}\int_a^b h(x)\,g(x)\,dx - \frac{1}{b-a}\int_a^b h(x)\,dx \cdot \frac{1}{b-a}\int_a^b g(x)\,dx\right| \\ &\le \frac{1}{b-a}\int_a^b \left|\left(h(x) - \frac{1}{b-a}\int_a^b h(y)\,dy\right)\cdot\left(g(x) - \frac{1}{b-a}\int_a^b g(y)\,dy\right)\right| dx. \end{aligned} \tag{7.2}$$

The inequality (7.2) is sharp.

Proof. First of all, let us observe that

$$
\begin{aligned}
I \quad : & = \frac{1}{b-a}\int_a^b \left(h(x)-\frac{1}{b-a}\int_a^b h(y)\,dy\right)\cdot\left(g(x)-\frac{1}{b-a}\int_a^b g(y)\,dy\right)dx \\
& = \frac{1}{b-a}\int_a^b \Bigg(h(x)\,g(x)-g(x)\cdot\frac{1}{b-a}\int_a^b h(y)\,dy-h(x)\cdot\frac{1}{b-a}\int_a^b g(y)\,dy \\
& \quad +\frac{1}{b-a}\int_a^b h(y)\,dy\cdot\frac{1}{b-a}\int_a^b g(y)\,dy\Bigg)dx \\
& = \frac{1}{b-a}\int_a^b h(x)\,g(x)\,dx-\frac{1}{b-a}\int_a^b g(x)\,dx\cdot\frac{1}{b-a}\int_a^b h(y)\,dy \\
& \quad -\frac{1}{b-a}\int_a^b h(x)\,dx\cdot\frac{1}{b-a}\int_a^b g(y)\,dy+\frac{1}{b-a}\int_a^b h(y)\,dy\cdot\frac{1}{b-a}\int_a^b g(y)\,dy \\
& = \frac{1}{b-a}\int_a^b h(x)\,g(x)\,dx-\frac{1}{b-a}\int_a^b g(x)\,dx\cdot\frac{1}{b-a}\int_a^b h(x)\,dx.
\end{aligned}
$$

On the other hand, by the use of modulus properties, we have

$$
|I| \le \frac{1}{b-a}\int_a^b \left|\left(h(x)-\frac{1}{b-a}\int_a^b h(y)\,dy\right)\cdot\left(g(x)-\frac{1}{b-a}\int_a^b g(y)\,dy\right)\right|dx
$$

and the inequality (7.2) is proved.
Choosing $h(x)=g(x)=sgn\left(x-\frac{a+b}{2}\right)$, equality is satisfied in (7.2). ∎

The following corollaries follow immediately.

Corollary 13. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) having the first derivative $f':(a,b)\to\mathbb{R}$ bounded on (a,b). Then we have the inequality:*

$$
\begin{aligned}
& \left|\frac{b-a}{2}\left[f(a)+f(b)\right]-\int_a^b f(x)\,dx\right| \\
\le\; & \frac{(b-a)^2}{4}\max_{x\in(a,b)}\left|f'(x)-\frac{f(b)-f(a)}{b-a}\right|
\end{aligned}
\tag{7.3}
$$

Proof. A simple integration by parts gives that:

$$
\frac{b-a}{2}\left[f(a)+f(b)\right]-\int_a^b f(x)\,dx=\int_a^b\left(x-\frac{a+b}{2}\right)f'(x)\,dx. \tag{7.4}
$$

Applying the inequality (7.2) we find that:

$$\begin{aligned}&\left|\int_a^b \frac{1}{b-a}\left(x-\frac{a+b}{2}\right) f'(x)\,dx\right.\\&\left.-\frac{1}{b-a}\int_a^b\left(x-\frac{a+b}{2}\right)dx\cdot\frac{1}{b-a}\int_a^b f'(x)\,dx\right|\\ \le\ & \frac{1}{b-a}\int_a^b\left|\left(x-\frac{a+b}{2}-\frac{1}{b-a}\int_a^b\left(y-\frac{a+b}{2}\right)dy\right)\right.\\&\left.\cdot\left(f'(x)-\frac{1}{b-a}\int_a^b f'(y)\,dy\right)\right|dx.\end{aligned}$$

As

$$\int_a^b\left(x-\frac{a+b}{2}\right)dx=0,$$

we obtain

$$\begin{aligned}&\left|\int_a^b\left(x-\frac{a+b}{2}\right)f'(x)\,dx\right| \\ \le\ & \int_a^b\left|\left(x-\frac{a+b}{2}\right)\left(f'(x)-\frac{f(b)-f(a)}{b-a}\right)\right|dx\\ \le\ & \max_{x\in(a,b)}\left|f'(x)-\frac{f(b)-f(a)}{b-a}\right|\int_a^b\left|\left(x-\frac{a+b}{2}\right)\right|dx\\ =\ & \frac{(b-a)^2}{4}\max_{x\in(a,b)}\left|f'(x)-\frac{f(b)-f(a)}{b-a}\right|.\end{aligned}\tag{7.5}$$

Now, using the identity (7.4), the inequality(7.5) becomes the desired result (7.3) . ∎

Corollary 14. *Suppose $p,q>1$ satisfy $\frac{1}{p}+\frac{1}{q}=1$. Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) having the first derivative $f':(a,b)\to\mathbb{R}$ being p-integrable on (a,b). Then we have the inequality:*

$$\begin{aligned}&\left|\frac{b-a}{2}[f(a)+f(b)]-\int_a^b f(x)\,dx\right|\\ \le\ & \frac{1}{2}\left(\frac{b-a}{q+1}\right)^{\frac{1}{q}+1}\left(\int_a^b\left|f'(x)-\frac{f(b)-f(a)}{b-a}\right|^p dx\right)^{\frac{1}{p}}.\end{aligned}\tag{7.6}$$

Proof. Using Hölder's inequality, we have that:

$$\begin{aligned}&\int_a^b\left|\left(x-\frac{a+b}{2}\right)\left(f'(x)-\frac{f(b)-f(a)}{b-a}\right)\right|dx\\ \le\ & \left(\int_a^b\left|x-\frac{a+b}{2}\right|^q dx\right)^{\frac{1}{q}}\left(\int_a^b\left|f'(x)-\frac{f(b)-f(a)}{b-a}\right|^p dx\right)^{\frac{1}{p}}.\end{aligned}$$

A simple computation shows that

$$\begin{aligned}\int_a^b \left|x-\frac{a+b}{2}\right|^q dx &= \int_a^{\frac{a+b}{2}} \left(\frac{a+b}{2}-x\right)^q dx + \int_{\frac{a+b}{2}}^b \left(x-\frac{a+b}{2}\right)^q dx \\ &= \frac{(b-a)^{q+1}}{(q+1)\,2^q}\end{aligned}$$

and so

$$\begin{aligned}&\int_a^b \left|\left(x-\frac{a+b}{2}\right)\left(f'(x)-\frac{f(b)-f(a)}{b-a}\right)\right| dx \\ \leq\ & \frac{(b-a)^{\frac{1}{q}+1}}{2\,(q+1)^{\frac{1}{q}}}\left(\int_a^b \left|f'(x)-\frac{f(b)-f(a)}{b-a}\right|^p dx\right)^{\frac{1}{p}}.\end{aligned}$$

Now, using the first part of (7.5) and the identity (7.4), we obtain the desired result (7.6). ∎

The following result also holds.

Corollary 15. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) and suppose that $f':(a,b)\to\mathbb{R}$ is integrable on (a,b). Then we have the inequality:*

$$\begin{aligned}&\left|\frac{b-a}{2}\left[f(a)+f(b)\right]-\int_a^b f(x)\,dx\right| \\ \leq\ & \frac{b-a}{2}\int_a^b \left|f'(x)-\frac{f(b)-f(a)}{b-a}\right| dx\end{aligned} \tag{7.7}$$

Proof. We have

$$\begin{aligned}&\int_a^b \left|\left(x-\frac{a+b}{2}\right)\left(f'(x)-\frac{f(b)-f(a)}{b-a}\right)\right| dx \\ \leq\ & \max_{x\in(a,b)}\left|x-\frac{a+b}{2}\right|\times\int_a^b \left|f'(x)-\frac{f(b)-f(a)}{b-a}\right| dx \\ =\ & \frac{b-a}{2}\int_a^b \left|f'(x)-\frac{f(b)-f(a)}{b-a}\right| dx.\end{aligned}$$

Using the first part of (7.5) and the identity (7.4), we obtain the desired result (7.7). ∎

Remark 25. *The results of this section may be compared with those as presented in equation (5.14). Either may be tighter depending on the specific function $f(\cdot)$.*

Remark 26. *Theorem 21 may be utilised with $h(t)=\frac{(x-t)^n}{n!}$ and $g(t)=f^{(n)}(t)$ to obtain perturbed generalised trapezoidal type rules. However, this will not be pursued further here.*
Taking $x=\frac{a+b}{2}$ and $n=1$ would produce the results of this section.

8. Trapezoidal Rules with an Error Bound Involving the Second Derivative

In this section, via the use of some classical results from the Theory of Inequalities (Hölder's inequality, Grüss inequality and the Hermite-Hadamard inequality), we produce some *quasi-trapezoid quadrature formulae* for which the remainder term is smaller than the classical one.

This section focuses on the trapezoidal rule in which the error bound involves the behaviour of the second derivative in terms of a variety of norms. Section 5, on the other hand, examined the generalised trapezoidal rule in which the bound on the error involved the $\left\|f^{(n)}\right\|_{\infty}$ norm.

For other results in connection with trapezoid inequalities, see Chapter XV of the recent book by Mitrinović et al. [21].

8.1. **Some Integral Inequalities.** We shall start with the following theorem which is also interesting in its own right [3].

Theorem 22. *Let $f:[a,b]\to\mathbb{R}$ be a twice differentiable mapping on (a,b). Then we have the estimation*

$$\left|\int_a^b f(x)\,dx-\frac{b-a}{2}\left[f(a)+f(b)\right]\right| \tag{8.1}$$
$$\le\begin{cases}\frac{\|f''\|_{\infty}}{12}(b-a)^3 & \text{if } f''\in L_{\infty}[a,b]\\ \frac{1}{2}\|f''\|_p\left[B(q,q)\right]^{\frac{1}{q}}(b-a)^{2+\frac{1}{q}}, & \frac{1}{p}+\frac{1}{q}=1,p>1\\ & \text{if } f''\in L_p[a,b]\\ \frac{\|f''\|_1}{8}(b-a)^2 & \text{if } f''\in L_1[a,b]\end{cases},$$

where

$$\begin{aligned}\|f''\|_{\infty} &:= \sup_{t\in[a,b]}|f''(t)|,\\ \|f''\|_1 &:= \int_a^b|f''(t)|\,dt,\\ \|f''\|_p &:= \left(\int_a^b|f''(t)|^p\,dt\right)^{\frac{1}{p}},\ p>1\end{aligned}$$

and B is the Beta function of Euler, that is,

$$B(r,s):=\int_0^1 t^{r-1}(1-t)^{s-1}\,dt,\ r,s>0.$$

Proof. Integrating by parts we can state that:

$$\begin{aligned}
&\int_a^b (x-a)(b-x) f''(x)\,dx \\
&= [(x-a)(b-x) f'(x)]_a^b - \int_a^b [(a+b)-2x] f'(x)\,dx \\
&= \int_a^b [2x-(a+b)] f'(x)\,dx \\
&= f(x)[2x-(a+b)]_a^b - 2\int_a^b f(x)\,dx \\
&= (b-a)(f(a)+f(b)) - 2\int_a^b f(x)\,dx,
\end{aligned}$$

from which we get the well known identity

$$\begin{aligned}
\int_a^b f(x)\,dx &= \frac{b-a}{2}[f(a)+f(b)] \\
&\quad -\frac{1}{2}\int_a^b (x-a)(b-x) f''(x)\,dx.
\end{aligned} \tag{8.2}$$

Thus, using properties of the modulus gives

$$\begin{aligned}
&\left|\int_a^b f(x)\,dx - \frac{b-a}{2}[f(a)+f(b)]\right| \\
&\le \frac{1}{2}\int_a^b (x-a)(b-x)\left|f''(x)\right|dx.
\end{aligned} \tag{8.3}$$

Now, firstly, let us observe that

$$\begin{aligned}
&\int_a^b (x-a)(b-x)\left|f''(x)\right|dx \\
&\le \left\|f''\right\|_\infty \int_a^b (x-a)(b-x)\,dx \\
&= \frac{\left\|f''\right\|_\infty}{6}(b-a)^3.
\end{aligned}$$

Thus, by (8.3), we get the first inequality in (8.1).

Further, by Hölder's integral inequality we obtain:

$$\begin{aligned}
&\int_a^b (x-a)(b-x)\left|f''(x)\right|dx \\
&\le \left(\int_a^b (x-a)^q (b-x)^q\,dx\right)^{\frac{1}{q}} \left\|f''\right\|_p,
\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1,\ p>1$ and $\left\|f''\right\|_p$ is as given above.
Now, using the transformation $x=(1-t)a+tb,\ t\in[0,1]$, we get

$$\begin{aligned}
(x-a)^q (b-x)^q &= (b-a)^{2q} t^q (1-t)^q, \\
dx &= (b-a)\,dt
\end{aligned}$$

and thus

$$\begin{aligned}\int_a^b (x-a)^q (b-x)^q \, dx &= (b-a)^{2q+1} \int_0^1 t^q (1-t)^q \, dt \\ &= (b-a)^{2q+1} B(q+1, q+1), \end{aligned} \tag{8.4}$$

where B is the Beta function of Euler; and the second inequality in (8.1) is proved. Finally, we have that

$$\int_a^b (x-a)(b-x) |f''(x)| \, dx \leq \max_{x \in [a,b]} [(x-a)(b-x)] \|f''\|_1$$

and, since

$$\max_{x \in [a,b]} [(x-a)(b-x)] = \frac{(b-a)^2}{4},$$

at $x = \frac{a+b}{2}$, we deduce the last part of (8.1). ∎

Remark 27. *The inequalities in* (8.1) *provided a variety of norms involving the second derivative give flexibility since any of them may be tighter depending on the function which we wish to approximate (see* [3]*) for further details).*

The following theorem is of interest since it provides another integral inequality in connection with the trapezoid formula, giving a perturbed rule.

Theorem 23. *Let* $f : [a,b] \to \mathbb{R}$ *be a twice differentiable mapping on* (a,b) *and assume that*

$$\gamma := \inf_{x \in (a,b)} f''(x) > -\infty \text{ and } \Gamma := \sup_{x \in (a,b)} f''(x) < \infty. \tag{8.5}$$

Then, we have the estimation

$$\begin{aligned} &\left| \int_a^b f(x) \, dx - \frac{b-a}{2} [f(a) + f(b)] + \frac{(b-a)^2}{12} (f'(b) - f'(a)) \right| \\ &\leq \frac{(b-a)^3 (\Gamma - \gamma)}{32}. \end{aligned} \tag{8.6}$$

Proof. We shall apply the celebrated Grüss inequality as given by (7.1).

Now, if we choose in (7.1), $h(x) = (x-a)(b-x)$, $g(x) = f''(x)$, $x \in [a,b]$, we get:

$$\phi = 0, \ \Phi = \frac{(b-a)^2}{4},$$

and we can state that

$$\begin{aligned} &\left| \frac{1}{b-a} \int_a^b (x-a)(b-x) f''(x) \, dx \right. \\ &\left. - \frac{1}{b-a} \int_a^b (x-a)(b-x) \, dx \cdot \frac{1}{b-a} \int_a^b f''(x) \, dx \right| \\ &\leq \frac{(b-a)^2 (\Gamma - \gamma)}{16}. \end{aligned} \tag{8.7}$$

A simple calculation gives us that

$$\int_a^b (x-a)(b-x)\,dx = \frac{(b-a)^3}{6} \text{ and } \int_a^b f''(x)\,dx = f'(b) - f'(a),$$

then, from (8.7),

$$\begin{aligned} & \left| \int_a^b (x-a)(b-x) f''(x)\,dx - \frac{(b-a)^2}{6} \left(f'(b) - f'(a) \right) \right| \\ \leq \; & \frac{(b-a)^3 (\Gamma - \gamma)}{16}. \end{aligned}$$

Finally, using the identity (8.2) gives

$$\begin{aligned} & \left| \frac{f(a)+f(b)}{2}(b-a) - \int_a^b f(x)\,dx - \frac{(b-a)^2}{12} \left(f'(b) - f'(a) \right) \right| \\ \leq \; & \frac{(b-a)^3 (\Gamma - \gamma)}{32} \end{aligned}$$

and the theorem is proved. ∎

Theorem 24. *Let f have the properties of Theorem 23. Then the estimation*

$$\begin{aligned} & \left| \int_a^b f(x)\,dx - \frac{b-a}{2}\left[f(a) + f(b) \right] + \frac{(b-a)^2}{12} \left(f'(b) - f'(a) \right) \right| \\ \leq \; & \frac{1}{24\sqrt{5}} \cdot (b-a)^3 (\Gamma - \gamma) \end{aligned} \tag{8.8}$$

holds with γ, Γ as given by (8.5).

Proof. The proof utilises the premature Grüss inequality as given by Theorem 9, equation (5.17) rather than the Grüss inequality given by Theorem 20, equation (7.1). Only the bound varies from that of Theorem 23 and so taking $h(x) = (x-a)(b-x)$, $g(x) = f''(x)$, $x \in [a,b]$, we have that the bound as given by (5.17)

$$\frac{b-a}{2} \cdot \frac{\Gamma - \gamma}{2} \left[T(h,h) \right]^{\frac{1}{2}}, \tag{8.9}$$

where

$$T(h,h) = \frac{1}{b-a} \int_a^b h^2(x)\,dx - \left(\frac{1}{b-a} \int_a^b h(x)\,dx \right)^2.$$

Utilising (8.4), we have that

$$\begin{aligned} T(h,h) &= (b-a)^4 B(3,3) - \left[(b-a)^2 B(2,2) \right]^2 \\ &= \frac{(b-a)^4}{180}, \end{aligned} \tag{8.10}$$

since B is Euler's Beta function.

Thus, using (8.9), (8.10) gives from (5.17), (8.8) and the theorem is proved. ∎

Remark 28. *A comparison of the bounds in* (8.6) *and* (8.8) *shows that the premature Grüss inequality is 1.26% better than that obtained using the Grüss inequality.*

Remark 29. *Atkinson* [30] *terms the quadrature rule in* (8.6) *or* (8.8) *as a corrected trapezoidal rule and obtains it using an asymptotic error estimate approach which does not provide an expression for the error bound. He does state that the corrected trapezoidal rule is* $O\left(h^4\right)$ *compared with* $O\left(h^2\right)$ *for the trapezoidal rule. Atkinson does subsequently find an explicit bound using the Euler-Maclaurin summation formula. The bound given in* (8.6) *depends only on the bounds for* $f'(\cdot)$ *and not the fourth derivative.*

Remark 30. *From equation* (5.22) *of Corollary 10, an alternate premature Grüss inequality may be obtained with* $n = 2$, *giving, after some simplification,*

$$\begin{aligned} & \left|\int_a^b f(t)\,dt - \frac{b-a}{2}\left[f(a)+f(b)\right] + \left(f'(b)-f'(a)\right)\right| \\ \leq \;& \frac{1}{40\sqrt{5}} \cdot (b-a)^3 (\Gamma - \gamma)\,. \end{aligned} \tag{8.11}$$

This is superior to (8.8) *obtained from a different kernel. A premature Grüss inequality was used to obtain a generalised trapezoidal type rule containing an unspecified* $x \in [a,b]$. *Equation* (5.22) *is obtained from taking* $x = \frac{a+b}{2}$ *to give the tightest bound. Equation* (8.8) *was obtained directly without the extra degree of freedom, which may explain its inferiority.*
In comparing results (8.6), (8.8) and (8.11), the natural question that may be asked, which is to our best knowledge an open problem, is: what should the best constant C *be that satisfies*

$$\begin{aligned} & \left|\int_a^b f(x)\,dx - \frac{b-a}{2}\left[f(a)+f(b)\right] + \frac{(b-a)^2}{12}\left[f'(b)-f'(a)\right]\right| \\ \leq \;& C (b-a)^3 (\Gamma - \gamma)\,, \end{aligned}$$

where γ, Γ *are as given by (8.5).*
It may further be observed that the first inequality in (5.16) gives a different perturbed trapezoidal type rule with a bound involving $\|f''\|_\infty$. *Now, since* $0 \leq \Gamma - \gamma \leq 2\|f''\|_\infty$, *then (8.8) and (8.11) are all tighter bounds, while (8.6) is not by this coarse bound, however, it may be in practice. The fact that the constant of the perturbation is different has little consequence in practice since the perturbation only affects the end points in a composite rule.*

Finally, using a classical result on convex functions due to Hermite and Hadamard we have the following theorem concerning a double integral inequality [3].

Theorem 25. *Let* $f : [a,b] \to \mathbb{R}$ *be a twice differentiable mapping on* (a,b) *and suppose that* $-\infty < \gamma \leq f''(x) \leq \Gamma < \infty$ *for all* $x \in (a,b)$.
Then we have the double inequality

$$\begin{aligned} \frac{\gamma}{12}(b-a)^2 &\leq \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx \\ &\leq \frac{\Gamma}{12}(b-a)^2\,, \end{aligned} \tag{8.12}$$

and the estimation

$$
\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}(b-a)-\frac{\Gamma+\gamma}{24}(b-a)^2-\int_a^b f(x)\,dx\right| \\
\leq\ & \frac{(\Gamma-\gamma)(b-a)^3}{24}.
\end{aligned}
\tag{8.13}
$$

Proof. We shall use the following inequality for convex mappings $g:[a,b]\to\mathbb{R}$:

$$
\frac{1}{b-a}\int_a^b g(x)\,dx \leq \frac{g(a)+g(b)}{2},
\tag{8.14}
$$

which is well known in the literature as the Hermite-Hadamard inequality.
Let us choose firstly $g:[a,b]\to\mathbb{R}, g(x)=f(x)-\frac{\gamma}{2}x^2$. Then g is twice differentiable on $[a,b]$ and

$$
g'(x)=f'(x)-\gamma x,\ g''(x)=f''(x)-\gamma\geq 0 \text{ on } (a,b),
$$

hence, g is convex on $[a,b]$. Thus, we can apply (8.14) for g to get

$$
\frac{1}{b-a}\int_a^b\left(f(x)-\frac{\gamma}{2}x^2\right)dx \leq \frac{f(a)+f(b)}{2}-\frac{\gamma}{4}\left(a^2+b^2\right),
$$

giving on rearrangement

$$
\frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2}-\frac{\gamma}{12}(b-a)^2,
$$

which is ostensibly identical to the first inequality in (8.12).
The second part in (8.12) follows by (8.14) applied for the convex (and twice differentiable mapping) $h:[a,b]\to\mathbb{R},\ h(x)=\frac{\Gamma}{2}x^2-f(x)$.
Now, it is straightforward to see that, for $\alpha\leq t\leq\beta$, $\left|t-\frac{\alpha+\beta}{2}\right|\leq\frac{\beta-\alpha}{2}$. On taking $\alpha=\frac{\gamma}{12}(b-a)^2$ and $\beta=\frac{\Gamma}{12}(b-a)^2$ we get the desired estimation (8.13). ∎

8.2. **Some Trapezoid Quadrature Rules.** We now consider applications of the integral inequalities developed in 8.1 [3].

Theorem 26. *Let $f:[a,b]\to\mathbb{R}$ be as in Theorem 22.*
If $I_n: a=x_0<x_1<...<x_{n-1}<x_n=b$ is a partition of the interval $[a,b]$, then we have:

$$
\int_a^b f(x)\,dx = A_T(f,I_n)+R_T(f,I_n),
\tag{8.15}
$$

where

$$
A_T(f,I_n):=\frac{1}{2}\sum_{i=0}^{n-1}\left[f(x_i)+f(x_{i+1})\right]h_i
$$

is the trapezoid quadrature rule and the remainder $R_T(f,I_n)$ satisfies the relation:

$$
\begin{aligned}
&|R_T(f,I_n)| \\
\leq\ & \begin{cases}
\frac{1}{2}\|f''\|_\infty\sum_{i=0}^{n-1}h_i^3 \\[2ex]
\frac{1}{2}\|f''\|_p\left[B(q+1,q+1)\right]^{\frac{1}{q}}\left(\sum_{i=0}^{n-1}h_i^{2q+1}\right)^{\frac{1}{q}} & ,\ \dfrac{1}{p}+\dfrac{1}{q}=1,\ q>1, \\[2ex]
\frac{1}{8}\|f''\|_1\nu^2(I_n)
\end{cases}
\end{aligned}
\tag{8.16}
$$

where $h_i := x_{i+1} - x_i,\ i = 0, ..., n-1$ *and* $\nu(I_n) = \max\{h_i|,\ i = 0, ..., n-1\}$.

Proof. Applying the first inequality, (8.1), we get

$$\left|\int_{x_i}^{x_{i+1}} f(x)\,dx - \frac{f(x_i) + f(x_{i+1})}{2}(x_{i+1} - x_i)\right| \le \frac{\|f''\|_\infty}{12} h_i^3$$

for all $i \in \{0, ..., n-1\}$.
Summing over i from 0 to $n-1$ we get the first part of (8.16).
The second inequality in (8.1) gives us:

$$\begin{aligned} &\left|\int_{x_i}^{x_{i+1}} f(x)\,dx - \frac{f(x_i) + f(x_{i+1})}{2}(x_{i+1} - x_i)\right| \\ \le\ & \frac{1}{2} h_i^{2+\frac{1}{q}} [B(q+1, q+1)]^{\frac{1}{p}} \left(\int_{x_i}^{x_{i+1}} |f''(t)|^p\,dt\right)^{\frac{1}{p}}, \end{aligned}$$

for all $i = 0, ..., n-1$.
Summing and using Hölder's discrete inequality, we get:

$$\begin{aligned} &\left|\int_a^b f(x)\,dx - A_T(f, I_n)\right| \\ \le\ & \frac{1}{2}[B(q+1, q+1)]^{\frac{1}{q}} \sum_{i=0}^{n-1} h_i^{\frac{2q+1}{q}} \left(\int_{x_i}^{x_{i+1}} |f''(t)|^p\,dt\right)^{\frac{1}{p}} \\ \le\ & \frac{1}{2}[B(q+1, q+1)]^{\frac{1}{q}} \left[\sum_{i=0}^{n-1} \left(h_i^{\frac{2q+1}{q}}\right)^q\right]^{\frac{1}{q}} \left[\sum_{i=0}^{n-1} \left[\left(\int_{x_i}^{x_{i+1}} |f''(t)|^p\,dt\right)^{\frac{1}{p}}\right]^p\right]^{\frac{1}{p}} \\ =\ & \frac{1}{2}[B(q+1, q+1)]^{\frac{1}{q}} \left(\sum_{i=0}^{n-1} h^{2q+1}\right)^{\frac{1}{q}} \sum_{i=0}^{n-1} \left(\int_{x_i}^{x_{i+1}} |f''(t)|^p\,dt\right)^{\frac{1}{p}} \\ =\ & \frac{1}{2}[B(q+1, q+1)]^{\frac{1}{q}} \|f''\|_p \left(\sum_{i=0}^{n-1} h^{2q+1}\right)^{\frac{1}{q}}, \end{aligned}$$

and the second inequality in (8.16) is proved.
In the last part, we have by (8.1), that:

$$\begin{aligned} |R_T(f, I_n)| &\le \frac{1}{8} \sum_{i=0}^{n-1} \left(\int_{x_i}^{x_{i+1}} |f''(t)|\,dt\right) h_i^2 \\ &\le \frac{1}{8} \max_{i=\overline{0,n-1}} h_i^2 \sum_{i=0}^{n-1} \left(\int_{x_i}^{x_{i+1}} |f''(t)|\,dt\right) \\ &= \frac{1}{8}\nu^2(I_n)\|f''\|_1, \end{aligned}$$

and the theorem is proved. ∎

Remark 31. *We would like to note that in every book on numerical integration, encountered by the authors, only the first estimation in* (8.16) *is used. Sometimes, when* $\|f''\|_p$ $(p > 1)$ *or* $\|f''\|_1$ *are easier to compute, it would perhaps be more appropriate to use the second or the third estimation.*

We shall now investigate the case where we have an equidistant partitioning of $[a,b]$ given by:

$$I_n : x_i = a + \frac{b-a}{n} \cdot i, \; i = 0, 1, ..., .n.$$

The following result is a consequence of Theorem 26.

Corollary 16. *Let $f : [a,b] \to \mathbb{R}$ be a twice differentiable mapping and $\|f''\|_\infty < \infty$. Then we have*

$$\int_a^b f(x)\,dx = A_{T,n}(f) + R_{T,n}(f),$$

where

$$A_{T,n}(f) = \frac{b-a}{2n} \sum_{i=0}^{n-1} \left[f\left(a + \frac{b-a}{n} i\right) + f\left(a + \frac{b-a}{n}(i+1)\right)\right]$$

and the remainder $R_{T,n}(f)$ satisfies the estimation

$$|R_{T,n}(f)| \le \begin{cases} \frac{(b-a)^3 \|f''\|_\infty}{12n^2} \\ \frac{(b-a)^2 [B(q+1,q+1)]^{2+\frac{1}{q}} \|f''\|_p}{2n^2} & , \; \frac{1}{p} + \frac{1}{q} = 1, \; p > 1, \\ \frac{(b-a)^2 \|f''\|_1}{8n^2} \end{cases}$$

for all $n \ge 1$.

The following theorem gives a perturbed-trapezoid formula using Theorem 24, which is sometimes more appropriate.

Theorem 27. *Let $f : [a,b] \to \mathbb{R}$ be as in Theorem 24 and I_n be an arbitrary partition of the interval $[a,b]$. Then we have:*

$$\int_a^b f(x)\,dx = A_T(f, f', I_n) + \tilde{R}_T(f, I_n), \tag{8.17}$$

where

$$A_T(f, f', I_n) := \frac{1}{2} \sum_{i=0}^{n-1} [f(x_i) + f(x_{i+1})] h_i + \frac{1}{12} \sum_{i=0}^{n-1} (f'(x_{i+1}) - f'(x_i)) h_i^2$$

is a perturbed trapezoidal rule and the remainder term $\tilde{R}_T(f, I_k)$ satisfies the estimation:

$$\left|\tilde{R}_T(f, I_k)\right| \le \frac{1}{24\sqrt{5}} (\Gamma - \gamma) \sum_{i=0}^{n-1} h_i^3, \tag{8.18}$$

where the h_i are as above.

Proof. Writing the inequality (8.6) on the intervals $[x_i, x_{i+1}]$ $(i = 0, ..., n-1)$ we get:

$$\begin{aligned} & \left| \int_{x_i}^{x_{i+1}} f(x)\,dx - \frac{f(x_i) + f(x_{i+1})}{2} \cdot h_i + \frac{1}{12} (f'(x_{i+1}) - f'(x_i)) h_i^2 \right| \\ \le \; & \frac{1}{24\sqrt{5}} (\Gamma - \gamma) \cdot h_i^3 \end{aligned}$$

for all $i = 0, ..., n-1$.
Summing over i from 0 to $n-1$, we deduce the desired estimation (8.18). ∎

It should be noted that similar results could be obtained for (8.6) or (8.11). All that would change would be the coefficient for the bound.

Remark 32. *As*

$$0 \leq \Gamma - \gamma \leq 2 \left\| f'' \right\|_{\infty},$$

then

$$\frac{1}{24\sqrt{5}} (\Gamma - \gamma) \leq \frac{\left\| f'' \right\|_{\infty}}{12\sqrt{5}} < \frac{\left\| f'' \right\|_{\infty}}{12},$$

and so the approximation of the integral $\int_a^b f(x)\,dx$ by the use of $A_T(f, f', I_n)$ is better than that provided by the classical trapezoidal formulae $A_T(f, I_n)$ for every partition I_n of the interval $[a, b]$. Atkinson [30] *calls this the corrected trapezoidal rule. However, only the classical $\left\| f'' \right\|_{\infty}$ norm is used as the bound on the error. Atkinson* [30] *uses the idea of an asymptotic error estimate rather than the inequality by Grüss.*

The following corollary of Theorem 27 holds:

Corollary 17. *Let $f : [a, b] \to \mathbb{R}$ be as in Theorem 23. Thus we have*

$$\int_a^b f(x)\,dx = A_{T,n}(f, f') + \tilde{R}_{T,n}(f),$$

where

$$\begin{aligned} A_{T,n}(f, f') &= \frac{b-a}{2n} \sum_{i=0}^{n-1} \left[f\left(a + \frac{b-a}{n} \cdot i \right) + f\left(a + \frac{b-a}{n} \cdot (i+1) \right) \right] \\ &\quad + \frac{(b-a)^2}{12n^2} \left(f'(b) - f'(a) \right), \end{aligned}$$

and the remainder $\tilde{R}_T(f)$ satisfies the estimation:

$$\left| \tilde{R}_{T,n}(f) \right| \leq \frac{(\Gamma - \gamma)(b-a)^3}{24\sqrt{5}n^2},$$

for all $n \geq 1$.

Now, if we apply Theorem 25, we can state the following quadrature formulae which is a quasi-trapezoid formula or a perturbed trapezoid formula.

Theorem 28. *Let f be a in Theorem 25. If I_n is a partition of the interval $[a, b]$, then we have:*

$$\int_a^b f(x)\,dx = A_{T,\gamma,\Gamma}(f, I_n) + R_{T,\gamma,\Gamma}(f, I_n), \tag{8.19}$$

where

$$A_{T,\gamma,\Gamma}(f, I_n) = \sum_{i=0}^{n-1} \frac{f(x_i) + f(x_{i+1})}{2} \cdot h_i - \frac{\Gamma + \gamma}{24} \sum_{i=0}^{n-1} h_i^3$$

and

$$|R_{T,\gamma,\Gamma}(f,I_n)| \leq \frac{\Gamma+\gamma}{24}\sum_{i=0}^{n-1} h_i^3. \tag{8.20}$$

Proof. Applying the inequality (8.13) on $[x_i, x_{i+1}]$, we get:

$$\left|\int_{x_i}^{x_{i+1}} f(x)\,dx - \frac{f(x_i)+f(x_{i+1})}{2}\cdot h_i + \frac{\Gamma+\gamma}{24}\cdot h_i^2\right| \leq \frac{\Gamma+\gamma}{24}\cdot h_i^3$$

for all $i \in \{0,...,n-1\}$.
Summing over i from 0 to $n-1$ we get the representation (8.19) over the estimation (8.20). ∎

Corollary 18. *Let f be as above. Then we have:*

$$\int_a^b f(x)\,dx = A_{T,\gamma,\Gamma,n}(f) + R_{T,\gamma,\Gamma,n}(f),$$

where

$$\begin{aligned} A_{T,\gamma,\Gamma,n}(f) &= \frac{1}{2}\sum_{i=0}^{n-1}\left[f\left(a+i\cdot\frac{b-a}{n}\right) - f\left(a+(i+1)\cdot\frac{b-a}{n}\right)\right] \\ &\quad + \frac{\Gamma+\gamma}{12}\cdot\frac{(b-a)^2}{n} \end{aligned}$$

and the remainder term $R_{T,\gamma,\Gamma,n}(f)$ satisfies

$$|R_{T,\gamma,\Gamma,n}(f)| \leq \frac{(\Gamma-\gamma)(b-a)^3}{24n^2}.$$

Remark 33. *As $0 \leq \Gamma-\gamma \leq 2\|f''\|_\infty$, then the approximation given by $A_{T,\gamma,\Gamma,n}(f)$ to the integral $\int_a^b f(x)\,dx$ is better than the classical trapezoidal rule.*

9. Concluding Remarks

The current work has demonstrated the development of trapezoidal type rules. Identities are obtained by using a Peano kernel approach which enables us, through the use of the modern theory of inequalities, to obtain bounds in terms of a variety of norms. This is useful in practice as the behaviour of the function would necessitate the use of one norm over another. Although not all inequalities have been developed into composite quadrature rules, we believe that enough demonstrations have been given that would enable the reader to proceed further.

Rules have been developed that do not necessarily require the second derivative to be well behaved or indeed, exist, thus allowing the treatment of a much larger class of functions. Rules have been developed by examining the Riemann-Stieltjes integral. Additionally, the rules also allow for a non-uniform partition, thus giving the user the option of choosing a partition that minimises the bound or enabling the calculation of the bound given a particular partition.

If we wish to approximate the integral $\int_a^b f(x)\,dx$ using a quadrature rule $Q(f, I_n)$ with bound $B(n)$, where I_n is a uniform partition for example, with an accuracy of $\varepsilon > 0$, then we will need $n_\varepsilon \in \mathbb{N}$ where

$$n_\varepsilon \geq \left[B^{-1}(\varepsilon)\right] + 1$$

with $[x]$ denoting the integer part of x.

This approach enables the user to predetermine the partition required to *assure* that the result is within a certain tolerance rather than utilizing the commonly used method of halving the mesh size and comparing the resulting estimation.

References

[1] S.S. DRAGOMIR, On the trapezoid formula for mappings of bounded variation and applications, *submitted.* `http://rgmia.vu.edu.au/severdvi/TRAPB.dvi`[1]

[2] D.S. MITRINOVIĆ, J.E. PEČARIĆ and A.M. FINK, *Inequalities for Functions and Their Integrals and Derivatives*, Kluwer Academic Publisher, 1994.

[3] S.S. DRAGOMIR, P. CERONE and A. SOFO, Some remarks on the trapezoid rule in numerical integration, *Indian J. of Pure and Appl. Math.,* (in press). *RGMIA Res. Rep. Coll.*, 5, **2**(1999), Article 1.

[4] S.S. DRAGOMIR and T.C. PEACHEY, New estimation of the remainder in the trapezoidal formula with applications, *submitted.* `http://rgmia.vu.edu.au/v1n2/trap1.dvi`

[5] S.S. DRAGOMIR, On the trapezoid quadrature formula and applications, *submitted.*

[6] S.S. DRAGOMIR, On the trapezoid quadrature formula for Lipschitzian mappings and applications, *Tamkang J. of Math.*, (in press). `http://rgmia.vu.edu.au/severdvi/TRAPL.dvi`

[7] P. CERONE, S. S. DRAGOMIR and J. ROUMELIOTIS, Some Ostrowski type inequalities for n-time differentiable mappings and applications , *Preprint. RGMIA Res. Rep Coll., Vol.1, No.1 (1998), 51-66.* `http://rgmia.vu.edu.au`

[8] S.S. DRAGOMIR, On the trapezoid inequality for absolutely continuous mappings, *submitted.*

[9] S.S. DRAGOMIR, On the trapezoid quadrature formula for mappings of bounded variation and applications, `http://rgmia.vu.edu.au/severdvi/TRAPB.dvi`.

[10] S.S. DRAGOMIR, P. CERONE and C. E. M. PEARCE, Generalizations of the trapezoid inequality for mappings of bounded variation and applications, *submitted.*

[11] N. S. BARNETT, S.S. DRAGOMIR and C. E. M. PEARCE, A quasi-trapezoid inequality for double integrals, *submitted.*

[12] S.S. DRAGOMIR and A. McANDREW, On trapezoid inequality via a Grüss type result and applications, *RGMIA Res. Rep. Coll.*, 2, **2**(1999), Article 6.

[13] S.S. DRAGOMIR, J.E. PEČARIĆ and S. WANG, The Unified treatment of trapezoid, Simpson and Ostrowski type inequality for monotonic mappings and applications, *RGMIA Res. Rep. Coll.*, 4, **2**(1999), Article 3.

[14] V. ČULJAK, C.E.M. PEARCE and J. E. PEČARIĆ, The unified treatment of some inequalities of Ostrowski and Simpson's type, *RGMIA Res. Rep. Coll.*, 4, **2**(1999), Article 8.

[15] S.S. DRAGOMIR, A Taylor like formula and application in numerical integration, *submitted.*

[16] P. CERONE and S. S. DRAGOMIR, Three point quadrature rules involving, at most, a first derivative, *submitted, RGMIA Res. Rep. Coll.*, 4, **2**(1999), Article 8.

[17] J.E. PEČARIĆ, F. PROSCHAN and Y.L. TONG, *Convex Functions, Partial Orderings, and Statistical Applications*, Academic Press, 1992.

[18] M. MATIĆ, J.E. PEČARIĆ and N. UJEVIĆ, On New estimation of the remainder in Generalised Taylor's Formula, *M.I.A.,* Vol. **2** No. 3 (1999), 343-361.

[19] R.P. AGARWAL and S.S. DRAGOMIR, An application of Hayashi's inequality for differentiable functions, *Computers Math. Appl.*, **32**(6) (1996), 95-99.

[20] D.S. MITRINOVIĆ, J.E. PEČARIĆ and A.M. FINK, *Classical and New Inequalities in Analysis,* Kluwer Academic Publishers, 1993.

[21] D.S. MITRINOVIĆ, J.E. PEČARIĆ and A.M. FINK, *Inequalities for Functions and Their Integrals and Derivatives,* Kluwer Academic Publishers, 1994.

[22] S.S. DRAGOMIR and S. WANG, Applications of Iyengar's type inequalities to the estimation of error bounds for the trapezoidal quadrature rule, *Tamkang Journal of Mathematics*, **29**(1) (1998), 55-58.

[23] S.S. DRAGOMIR, Y.J. CHO and S.S. KIM, Some remarks on the Milovanović-Pečarić Inequality and in Applications for special means and numerical integration*., Tamkang Journal of Mathematics,* accepted.

---

[1]Submitted articles may be found at
http://matilda.vu.edu.au/~rgmia

[24] S.S. DRAGOMIR, Some integral inequalities of Grüss type, *Italian J. of Pure and Appl. Math.,* accepted.

[25] S.S. DRAGOMIR, Grüss inequality in inner product spaces, *The Australian Math. Gazette, 2,* **26** (1999), 66-70.

[26] S.S. DRAGOMIR, New estimation of the remainder in Taylor's formula using Grüss' type inequalities and applications, *Mathematical Inequalities and Applications, 2,* **2** (1999), 183-194.

[27] S.S. DRAGOMIR and I. FEDOTOV, An inequality of Grüss type for Riemann-Stieltjes integral and applications for special means, *Tamkang J. of Maths.*, (4) **29** (1998), 287-292.

[28] I. FEDOTOV and S.S. DRAGOMIR, Another approach to quadrature methods, *Indian Journal of Pure and Applied Mathematics*, (in press).

[29] S.S. DRAGOMIR, A Grüss type integral inequality for mappings of r-Hölder's type and applications for trapezoid formula, *Tamkang Journal of Mathematics,* accepted.

[30] K.E. ATKINSON, *An Introduction to Numerical Analysis,* Wiley and Sons, Second Edition, 1989.

[31] T. M. APOSTOL, *Mathematical Analysis,* Second Edition, Addison-Wesley Publishing Company, 1975.

[32] P. CERONE, S.S. DRAGOMIR, J. ROUMELIOTIS and J. SUNDE, A new generalization of the trapezoid formula for n-time differentiable mappings and applications, *RGMIA Res. Rep. Coll.,* 5, **2** (1999), Article 7.

[33] P.J. DAVIS and P. RABINOWITZ, *Methods of Numerical Integration*, 2nd Ed., Academic Press, New York, 1994.

[34] H. ENGELS, *Numerical Quadrature and Cubature*, Academic Press, New York, 1980.

[35] A.H. STROUD, *Numerical Quadrature and Solution of Ordinary Differential Equations*, Springer-Verlag, New York, 1974.

[36] A.R. KROMMER and C.W. UEBERHUBER, *Numerical Integration on Advanced Computer Systems*, Lecture Notes in Computer Science, 848, Springer-Verlag, Berlin, 1994.

[37] W.H. PRESS, B.P. FLANNERY, S.A. TELKOLSKY and W.T. VERLING, *Numerical Recipes: The Art of Scientific Computing*, Cambridge University Press, Cambridge, 1986.

School of Communications and Informatics, Victoria University of Technology, P.O. Box 14428, Melbourne City MC, Victoria, Australia. 8001

E-mail address: pc@matilda.vu.edu.au

E-mail address: sever.dragomir@vu.edu.au