Inequalities for a Weighted Integral

INEQUALITIES FOR A WEIGHTED INTEGRAL

FENG QI

ABSTRACT. In the article, along with the procedure generalizing the mean value theorems for derivative from Rolle to Taylor, we generalize the Iyengar inequality for an integral to obtain some inequalities for weighted integrals involving bounded derivative of the integrands.

1. INTRODUCTION

The Iyengar inequality (see [7, 8, 10, 11]) involving bounded derivative of a function has been researched by many mathematicians from its appearance in 1938. For details please see the references [8, 10, 11]. Recently, it has aroused much interest again (see [1, 2, 3, 5, 9, 15]). The new results obtained in recent years are as follows.

In [4, 6, 12, 13, 14], using the mean value theorems for derivative, step by step, F. Qi and other coauthors generalized the Iyengar inequality to some inequalities for single and multiple integrals involving the bounded n-th derivatives of the integrands.

For given points $a = (a_1, \cdots, a_m)$, $b = (b_1, \cdots, b_m) \in \mathbb{R}^m$ and $a_i < b_i$, $i = 1, 2, \cdots, m$, denote the m-rectangles by

$$Q_m = \prod_{i=1}^{m} [a_i, b_i], \quad Q_m(t) = \prod_{i=1}^{m} \big[a_i, c_i(t)\big], \tag{1.1}$$

where $c_i(t) = (1-t)a_i + tb_i$, $i = 1, 2, \cdots, m$, $t \in [0, 1]$.

Let $\boldsymbol{\nu} = (\nu_1, \cdots, \nu_m)$ be a multi-index, that is, $\nu_i =$ integer $\geqslant 0$, with $|\boldsymbol{\nu}| = \sum\limits_{i=1}^{m} \nu_i$. Let f be a function of several variables defined on Q_m, and its partial derivatives of $(n+1)$-th order remain between the upper and lower bounds $M_{n+1}(\boldsymbol{\nu})$ and $N_{n+1}(\boldsymbol{\nu})$ as follows

$$N_{n+1}(\boldsymbol{\nu}) \leqslant D^{\boldsymbol{\nu}} f(x) \leqslant M_{n+1}(\boldsymbol{\nu}), \quad x \in Q_m, \tag{1.2}$$

where we define

$$D^{\boldsymbol{\nu}} f(x) = \partial^{n+1} f(x) \Big/ \prod_{i=1}^{m} \partial x_i^{\nu_i}. \tag{1.3}$$

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We introduce the following notations

$$A(\boldsymbol{\nu}) = \prod_{i=1}^{m} \frac{(b_i - a_i)^{\nu_i+1}}{(\nu_i+1)!} \cdot M_{n+1}(\boldsymbol{\nu}), \tag{1.4}$$

$$B\big(\boldsymbol{\nu}, f(x)\big) = \prod_{i=1}^{m} \left(\frac{(b_i - a_i)^{\nu_i+1}}{(\nu_i+1)!} \cdot \left(\frac{\partial}{\partial x_i}\right)^{\nu_i} \right) f(x), \tag{1.5}$$

$$C(\boldsymbol{\nu}) = \prod_{i=1}^{m} \frac{(b_i - a_i)^{\nu_i+1}}{(\nu_i+1)!} \cdot N_{n+1}(\boldsymbol{\nu}). \tag{1.6}$$

Theorem 1.1 ([14])**.** *Let $f \in C^{n+1}(Q_m)$ and $N_{n+1}(\boldsymbol{\nu}) \leqslant D^{\boldsymbol{\nu}} f(x) \leqslant M_{n+1}(\boldsymbol{\nu})$ hold for any $x \in Q_m$ and $|\boldsymbol{\nu}| = n+1$, where $M_{n+1}(\boldsymbol{\nu})$ and $N_{n+1}(\boldsymbol{\nu})$ are constants depending on n and $\boldsymbol{\nu}$, then, for any $t \in [0,1]$,*

(i) *when n is an even, we have*

$$\begin{aligned}
&\sum_{|\boldsymbol{\nu}|=n+1} C(\boldsymbol{\nu}) t^{m+n+1} + \sum_{|\boldsymbol{\nu}|=n+1} A(\boldsymbol{\nu}) T(\boldsymbol{\nu}, t) \\
&\leqslant \int_{Q_m} f(x)\,dx - \sum_{k=0}^{n} \sum_{|\boldsymbol{\nu}|=k} B\big(\boldsymbol{\nu}, f(a)\big) t^{m+k} + \sum_{k=0}^{n} (-1)^k \sum_{|\boldsymbol{\nu}|=k} B\big(\boldsymbol{\nu}, f(b)\big) T(\boldsymbol{\nu}, t) \\
&\leqslant \sum_{|\boldsymbol{\nu}|=n+1} A(\boldsymbol{\nu}) t^{m+n+1} + \sum_{|\boldsymbol{\nu}|=n+1} C(\boldsymbol{\nu}) T(\boldsymbol{\nu}, t);
\end{aligned} \tag{1.7}$$

(ii) *when n is an odd,*

$$\begin{aligned}
&\sum_{|\boldsymbol{\nu}|=n+1} C(\boldsymbol{\nu}) \left(t^{m+n+1} + T(\boldsymbol{\nu}, t)\right) \\
&\leqslant \int_{Q_m} f(x)\,dx - \sum_{k=0}^{n} \sum_{|\boldsymbol{\nu}|=k} B\big(\boldsymbol{\nu}, f(a)\big) t^{m+k} + \sum_{k=0}^{n} (-1)^k \sum_{|\boldsymbol{\nu}|=k} B\big(\boldsymbol{\nu}, f(b)\big) T(\boldsymbol{\nu}, t) \\
&\leqslant \sum_{|\boldsymbol{\nu}|=n+1} A(\boldsymbol{\nu}) \left(t^{m+n+1} + T(\boldsymbol{\nu}, t)\right).
\end{aligned} \tag{1.8}$$

Where $T(\boldsymbol{\nu}, t) = \prod_{i=1}^{m} \left\{1 - (1-t)^{\nu_i+1}\right\} - 1$, $t \in [0,1]$.

If we take $m = 1$ in Theorem 1.1, we could get the following

Corollary 1.1 ([13])**.** *Let $f(x)$ be a differentiable function of $C^{n+1}([a,b])$ satisfying $N \leqslant f^{(n+1)}(x) \leqslant M$. Denote*

$$S_n(u,v,w) = \sum_{k=1}^{n-1} \frac{(-1)^k}{k!} \cdot u^k f^{(k-1)}(v) + \frac{w}{n!} \cdot (-1)^n u^n, \tag{1.9}$$

$$\frac{\partial^k S_n}{\partial u^k} = S_n^{(k)}(u,v,w), \tag{1.10}$$

then, for any $t \in [a,b]$, when n is an odd we have

$$\sum_{i=0}^{n+2} \frac{(-1)^i}{i!} \left(S_{n+2}^{(i)}(a,a,N) - S_{n+2}^{(i)}(b,b,N) \right) t^i \leqslant \int_a^b f(x)\,dx \leqslant \sum_{i=0}^{n+2} \frac{(-1)^i}{i!} \left(S_{n+2}^{(i)}(a,a,M) - S_{n+2}^{(i)}(b,b,M) \right) t^i; \tag{1.11}$$

when n is an even we have

$$\sum_{i=0}^{n+2} \frac{(-1)^i}{i!} \left(S_{n+2}^{(i)}(a,a,N) - S_{n+2}^{(i)}(b,b,M) \right) t^i \leqslant \int_a^b f(x)\,dx \leqslant \sum_{i=0}^{n+2} \frac{(-1)^i}{i!} \left(S_{n+2}^{(i)}(a,a,M) - S_{n+2}^{(i)}(b,b,N) \right) t^i. \tag{1.12}$$

If we take $n = 1$ in Corollary 1.1 and choose suitable t, we obtain

Corollary 1.2 ([13])**.** *Let $f(x)$ be a 2-times differentiable function satisfying $N \leqslant f''(x) \leqslant M$, then*

$$\begin{aligned} \frac{N(b^3-a^3)}{6} + \frac{\left\{f(a) - f(b) + bf'(b) - af'(a) + N(a^2-b^2)/2\right\}^2}{2\left((a-b)N - f'(a) + f'(b)\right)} &\leqslant \int_a^b f(x)\,dx - bf(b) + af(a) + \frac{b^2 f'(b) - a^2 f'(a)}{2} \\ &\leqslant \frac{M(b^3-a^3)}{6} + \frac{\left\{f(a) - f(b) + bf'(b) - af'(a) + M(a^2-b^2)/2\right\}^2}{2\left((a-b)M - f'(a) + f'(b)\right)}. \end{aligned} \tag{1.13}$$

Proof. In inequality (1.11), taking $n = 1$ and

$$t = \frac{f(a) - f(b) - af'(a) + bf'(b) + N(a^2-b^2)/2}{(a-b)N + f'(b) - f'(a)} \in [a,b], \tag{1.14}$$

the value of the left-hand side of (1.11) takes a maximum, and then the left-hand side inequality of (1.13) follows. Using the symmetry of inequality (1.11), the right-hand side inequality of (1.13) follows easily. ∎

On taking $n = 0$ in Corollary 1.1 and choosing suitable t, we have

Corollary 1.3 ([12])**.** *Let $f(x)$ be continuous in $[a,b]$ and differentiable for $x \in (a,b)$. Suppose that $f(x)$ is not a constant, and that $N \leqslant f'(x) \leqslant M$ for $x \in (a,b)$. Then*

$$\frac{NM(b-a)^2 + 2(b-a)[Mf(a) - Nf(b)] + [f(a)-f(b)]^2}{2(M-N)} \leqslant \int_a^b f(x)\,dx \leqslant -\frac{NM(b-a)^2 + 2(b-a)[Nf(a) - Mf(b)] + [f(a)-f(b)]^2}{2(M-N)}. \tag{1.15}$$

It is noted that the inequalities in (1.15), obtained by F. Qi in [12] using the mean value theorems for the derivative, and those obtained by R. P. Agarwal and S. S. Dragomir in [1], using a classical inequality due to Hayashi (see [8, 10, 11]), are the same. We remark that the proof of the inequalities in (1.15) by F. Qi in [12] is much simpler and more natural.

When substituting $N = -M$ in (1.15), the Iyengar inequality is recovered. As a special case of Corollary 1.3, we further obtain

Corollary 1.4 ([12])**.** *Let $f(x)$ be a continuous function in $[a,b]$ and differentiable for $x \in (a,b)$. Suppose that $f(a) = f(b) = 0$, and that $N \leqslant f'(x) \leqslant M$ for $x \in (a,b)$. If $f(x)$ is not identically zero, then $N < 0 < M$, and we have*

$$\left|\int_a^b f(x)\,dx\right| \leqslant \frac{NM(b-a)^2}{2(N-M)}. \tag{1.16}$$

Taking $N = -M = -\sup_{x\in(a,b)} |f'(x)|$ reduces inequality (1.16) *to*

$$\left|\int_a^b f(x)\,dx\right| \leqslant \frac{(b-a)^2}{4} \cdot \sup_{x\in(a,b)} |f'(x)|\,. \tag{1.17}$$

It is well-known that inequality (1.17) always appears in standard text books, for instance see [16], and in mathematical contests or examinations for undergraduate students. In [6], B.-N. Guo and F. Qi gave many proofs of the inequality (1.17).

In [3], using Hayashi's inequality, P. Cerone and S. S. Dragomir obtained some inequalities for weighted integrals to give trapezoidal type quadrature rules, which may be looked upon as a generalization of inequalities in (1.15) and the Iyengar inequality.

In this article, along with the procedure generalizing the mean value theorems for derivatives from Rolle to Taylor, we will generalize inequalities (1.7)–(1.17) for an integral to obtain some inequalities for weighted integrals involving the bounded derivative of the integrands.

2. Main Results

In this section we will consider the weighted integral $\int_a^b w(x)f(x)\,dx$, where the weight $w(x)$ is a non-negative integrable function on $[a,b]$, $f(x)$ is a continuous function on $[a,b]$ with indicated derivatives in (a,b). Define

$$h_{s,k}(t) = \int_s^t (x-s)^k w(x)\,dx, \quad s,t \in [a,b], \quad k \in \mathbb{N}. \tag{2.1}$$

Theorem 2.1. *Let $f(x)$ be a continuous function on $[a,b]$ and differentiable in (a,b). Suppose that $f(a) = f(b) = 0$, and that $N = \inf_{x\in(a,b)} f'(x) > -\infty$, $M = \sup_{x\in(a,b)} f'(x) < \infty$. Let $w(x) \geqslant 0$ for all $x \in [a,b]$ and $h_{s,k}(t) = \int_s^t (x-s)^k w(x)\,dx$, $s,t \in [a,b]$, $k \in \mathbb{N}$. If $f(x)$ and $w(x)$ are not identically zero, then $N < 0 < M$, and*

$$Nh_{a,1}(t_1) - Mh_{b,1}(t_1) \leqslant \int_a^b w(x)f(x)\,dx \leqslant Mh_{a,1}(t_0) - Nh_{b,1}(t_0), \tag{2.2}$$

where $t_0 = \frac{aM-bN}{M-N} \in (a,b)$, $t_1 = \frac{aN-bM}{N-M} \in (a,b)$.

Proof. Firstly, $N < 0 < M$ is an immediate consequence of the Rolle's mean value theorem. The idea now is to apply the Lagrange mean value theorem again in order to estimate the weighted integral. Let θ be a parameter satisfying $\theta \in (a,b)$, and write

$$\int_a^b w(x)f(x)\,\mathrm{d}x = \int_a^\theta w(x)[f(x)-f(a)]\,\mathrm{d}x + \int_\theta^b w(x)[f(x)-f(b)]\,\mathrm{d}x$$
$$= \int_a^\theta w(x)(x-a)f'(\xi_1)\,\mathrm{d}x + \int_\theta^b w(x)(x-b)f'(\xi_2)\,\mathrm{d}x,$$

where $a<\xi_1<\theta<\xi_2<b$. From $f'(\xi_1)\leqslant M$ and $f'(\xi_2)\geqslant N$, it now follows that

$$\int_a^b w(x)f(x)\,\mathrm{d}x \leqslant M\int_a^\theta (x-a)w(x)\,\mathrm{d}x + N\int_\theta^b (x-b)w(x)\,\mathrm{d}x$$
$$= Mh_{a,1}(\theta)-Nh_{b,1}(\theta).$$

Since

$$\frac{\mathrm{d}h_{s,k}(t)}{\mathrm{d}t} = (t-s)^k w(t),$$
$$\frac{\mathrm{d}\big(Mh_{a,1}(\theta)-Nh_{b,1}(\theta)\big)}{\mathrm{d}\theta} = [(M-N)\theta+(bN-aM)]w(\theta),$$

it is easy to see that the upper bound has its minimum value at the point $\theta = \frac{aM-bN}{M-N}\in(a,b)$.

Similarly, we have

$$\int_a^b w(x)f(x)\,\mathrm{d}x \geqslant N\int_a^\theta (x-a)w(x)\,\mathrm{d}x + M\int_\theta^b (x-b)w(x)\,\mathrm{d}x$$
$$= Nh_{a,1}(\theta)-Mh_{b,1}(\theta),$$

and, on maximising this with respect to θ, we find that the lower bound has its maximum value at the point $\theta=\frac{aN-bM}{N-M}\in(a,b)$.

The proof is complete. ∎

Theorem 2.2. *Let $f(x)$ be a continuous function on $[a,b]$ and differentiable in (a,b). Suppose that $f(x)$ is not a constant, and that $N=\inf_{x\in(a,b)} f'(x)>-\infty$, $M=\sup_{x\in(a,b)} f'(x)<\infty$. Let $w(x)\geqslant 0$ for all $x\in[a,b]$ and $h_{s,k}(t)=\int_s^t (x-s)^k w(x)\,dx$, $s,t\in[a,b]$, $k\in\mathbb{N}$. If $w(x)$ is not identically zero, then*

$$\left[\frac{f(b)-f(a)}{b-a}-M\right]\cdot h_{b,1}(t_3) - \left[\frac{f(b)-f(a)}{b-a}-N\right]\cdot h_{a,1}(t_3) \tag{2.3}$$
$$\leqslant \int_a^b w(x)f(x)\,dx - f(a)h_{a,0}(b) - \frac{f(b)-f(a)}{b-a}\cdot h_{a,1}(b)$$
$$\leqslant \left[\frac{f(b)-f(a)}{b-a}-N\right]\cdot h_{b,1}(t_2) - \left[\frac{f(b)-f(a)}{b-a}-M\right]\cdot h_{a,1}(t_2)$$

where $t_2=\frac{aM-bN+f(b)-f(a)}{M-N}\in(a,b)$, $t_3=\frac{aN-bM+f(b)-f(a)}{N-M}\in(a,b)$.

Proof. For $x\in[a,b]$ we set

$$\phi(x) = [f(x)-f(a)](b-a)-[f(b)-f(a)](x-a),$$

so that $\phi(a)=\phi(b)=0$. We also have

$$\phi'(x) = (b-a)f'(x)-f(b)+f(a),$$

and hence

$$(b-a)N-f(b)+f(a)\leqslant \phi'(x)\leqslant (b-a)M-f(b)+f(a).$$

The required result (2.3) now follows from Theorem 2.1 applied to $\phi(x)$, on noting that

$$\begin{aligned}\int_a^b w(x)\phi(x)\,\mathrm{d}x &= (b-a)\int_a^b w(x)[f(x)-f(a)]\,\mathrm{d}x - [f(b)-f(a)]\int_a^b (x-a)w(x)\,\mathrm{d}x\\ &= (b-a)\left[\int_a^b w(x)f(x)\,\mathrm{d}x - f(a)h_{a,0}(b)\right] - [f(b)-f(a)]h_{a,1}(b).\end{aligned}$$

The proof is complete. ∎

Theorem 2.3. *Let $f(x)$ be a differentiable function in $C^{n-1}([a,b])$ with the n-th derivative $f^{(n)}(x)$ in (a,b) such that $N \leqslant f^{(n)}(x) \leqslant M$ for all $x \in (a,b)$. Let $w(x) \geqslant 0$ for $x \in [a,b]$ and $h_{s,k}(t) = \int_s^t (x-s)^k w(x)\,dx$, $s,t \in [a,b]$, $k \in \mathbb{N}$. If $w(x)$ is not identically zero, then, for any $t \in (a,b)$, when n is an odd, we have*

$$\begin{aligned}&\frac{Nh_{a,n}(t)-Mh_{b,n}(t)}{n!}\\ &\qquad\leqslant \int_a^b w(x)f(x)\,dx + \sum_{i=0}^{n-1}\frac{f^{(i)}(b)h_{b,i}(t)-f^{(i)}(a)h_{a,i}(t)}{i!}\\ &\qquad\qquad\leqslant \frac{Mh_{a,n}(t)-Nh_{b,n}(t)}{n!};\end{aligned}\tag{2.4}$$

when n is an even we have

$$\begin{aligned}&\frac{N(h_{a,n}(t)-h_{b,n}(t))}{n!}\\ &\qquad\leqslant \int_a^b w(x)f(x)\,dx + \sum_{i=0}^{n-1}\frac{f^{(i)}(b)h_{b,i}(t)-f^{(i)}(a)h_{a,i}(t)}{i!}\\ &\qquad\qquad\leqslant \frac{M(h_{a,n}(t)-h_{b,n}(t))}{n!}.\end{aligned}\tag{2.5}$$

Proof. Let t be a parameter such that $a<t<b$, and write

$$\int_a^b w(x)f(x)\,\mathrm{d}x = \int_a^t w(x)f(x)\,\mathrm{d}x + \int_t^b w(x)f(x)\,\mathrm{d}x.\tag{2.6}$$

The well-known Taylor's formula states that

$$f(x) = \sum_{i=0}^{n-1}\frac{f^{(i)}(a)}{i!}(x-a)^i + \frac{f^{(n)}(\xi)}{n!}(x-a)^n, \quad \xi \in (a,x),\tag{2.7}$$

$$f(x) = \sum_{i=0}^{n-1}\frac{f^{(i)}(b)}{i!}(x-b)^i + \frac{f^{(n)}(\eta)}{n!}(x-b)^n, \quad \eta \in (x,b).\tag{2.8}$$

Integrating on both sides of (2.7) over $[a,t]$ yields

$$\int_a^t w(x)f(x)\,\mathrm{d}x = \sum_{i=0}^{n-1}\frac{f^{(i)}(a)}{i!}h_{a,i}(t) + \frac{f^{(n)}(\xi)}{n!}h_{a,n}(t).\tag{2.9}$$

Since $N \leqslant f^{(n)}(x) \leqslant M$ for $x \in (a,b)$, then

$$\frac{Nh_{a,n}(t)}{n!} \leqslant \int_a^t w(x)f(x)\,\mathrm{d}x - \sum_{i=0}^{n-1}\frac{f^{(i)}(a)}{i!}h_{a,i}(t) \leqslant \frac{Mh_{a,n}(t)}{n!}.\tag{2.10}$$

Integrating on both sides of (2.8) over $[t, b]$ leads to

$$\int_t^b w(x)f(x)\,\mathrm{d}x = -\sum_{i=0}^{n-1} \frac{f^{(i)}(b)}{i!} h_{b,i}(t) - \frac{f^{(n)}(\eta)}{n!} h_{b,n}(t). \tag{2.11}$$

When n is even, from (2.11), it follows that

$$-\frac{N h_{b,n}(t)}{n!} \leqslant \int_t^b w(x)f(x)\,\mathrm{d}x + \sum_{i=0}^{n-1} \frac{f^{(i)}(b) h_{b,i}(t)}{i!} \leqslant -\frac{M h_{b,n}(t)}{n!}; \tag{2.12}$$

when n is odd, the reversed inequalities of (2.12) hold.

Substituting of inequalities in (2.10) and (2.12) into (2.6) leads to the inequalities in (2.4) and (2.5) respectively. The proof is complete. ∎

Remark 2.1. If we take $w(x) = 1$ for all $x \in [a, b]$ in Theorems 2.1, 2.2 and 2.3, we can obtain Corollaries 1.4, 1.3 and 1.1 respectively, and then the Iyengar inequality (1.15) is recovered again.

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Department of Mathematics, Jiaozuo Institute of Technology, #142, Mid-Jiefang Road, Jiaozuo City, Henan 454000, The People's Republic of China

E-mail address: `qifeng@jzit.edu.cn`