Some Elementary Inequalities for the Expectation and Variance of a Random Variable whose PDF is defined on a Finite Interval

SOME ELEMENTARY INEQUALITIES FOR THE EXPECTATION AND VARIANCE OF A RANDOM VARIABLE WHOSE PDF IS DEFINED ON A FINITE INTERVAL

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Abstract. Some elementary inequalities for the expectation and variance of a continuous random variable whose pdf is defined on a finite interval are obtained using some standard and recent results from the theory of inequalities.

1. Introduction

Let X be a continuous random variable having the probability density function f defined on a finite interval $[a,b]$.

By definition

$$E(X) := \int_a^b t f(t)\,dt$$

the *expectation* of X, and

$$\begin{aligned} \sigma^2(X) &:= \int_a^b (t - E(X))^2 f(t)\,dt \\ &= \int_a^b t^2 f(t)\,dt - [E(X)]^2 \end{aligned}$$

the *variance* of X.

Using some tools from the theory of inequalities, namely Hölder's inequality, pre-Grüss inequality, pre-Chebychev inequality, Taylor's formula with the integral remainder, we point out some elementary inequalities for the expectation and variance.

2. The Results

Theorem 1. *Let X be a continuous random variable defined on $[a,b]$ having p.d.f., f. Then*

(i) *we have the inequality*

$$0 \le \sigma(X) \le [b - E(X)]^{\frac{1}{2}} [E(X) - a]^{\frac{1}{2}} \le \frac{1}{2}(b-a) \tag{2.1}$$

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and

$$0 \leq [b-E(X)][E(X)-a]-\sigma^2(X) \leq \begin{cases} \frac{(b-a)^3}{6}\|f\|_\infty \\ \\ [B(q+1,q+1)]^{\frac{1}{q}}(b-a)^{2+\frac{1}{q}}\|f\|_p \\ \text{if } f\in L_p[a,b],\ p>1,\ \frac{1}{p}+\frac{1}{q}=1 \end{cases} \tag{2.2}$$

where $B(\cdot,\cdot)$ is Euler's Beta function.

(ii) *If $m \leq f \leq M$ a.e. on $[a,b]$, then*

$$\frac{m(b-a)^3}{6} \leq [b-E(X)][E(X)-a]-\sigma^2(X) \leq \frac{M(b-a)^3}{6} \tag{2.3}$$

and

$$\left|[b-E(X)][E(X)-a]-\sigma^2(X)-\frac{(b-a)^2}{6}\right| \leq \frac{\sqrt{5}(b-a)^3(M-m)}{60}. \tag{2.4}$$

Proof. Note that:-

$$\begin{aligned} &\int_a^b (b-t)(t-a)f(t)\,dt \\ = &\int_a^b [(b-E(X))+(E(X)-t)][(E(X)-a)+(t-E(X))]f(t)\,dt \\ = &(b-E(X))(E(X)-a)\int_a^b f(t)\,dt+(E(X)-a)\int_a^b (E(X)-t)f(t)\,dt \\ &+(b-E(X))\int_a^b (t-E(X))f(t)\,dt-\int_a^b (t-E(X))^2 f(t)\,dt \\ = &[b-E(X)][E(X)-a]-\sigma^2(X) \end{aligned} \tag{2.5}$$

since

$$\int_a^b f(t)\,dt = 1 \quad \text{and} \quad \int_a^b (t-E(X))f(t)\,dt = 0.$$

(i) Using the fact that

$$\int_a^b (t-a)(b-t)f(t)\,dt \geq 0,$$

it follows that

$$\sigma^2(X) \leq [b-E(X)][E(X)-a]$$

and so the first inequality in (2.1) is established.

The second inequality in (2.1) follows from the elementary result that

$$\alpha\beta \leq \frac{1}{4}(\alpha+\beta)^2,\ \alpha,\beta\in\mathbb{R}$$

where $\alpha = b - E(X)$, $\beta = E(X) - a$.
The first inequality in (2.2) follows, since

$$\begin{aligned} \int_a^b (t-a)(b-t) f(t)\,dt &\le \|f\|_\infty \int_a^b (t-a)(b-t)\,dt \\ &= \frac{(b-a)^3}{6} \|f\|_\infty . \end{aligned}$$

The second inequality is obvious by Hölder's integral inequality,

$$\begin{aligned} \int_a^b (t-a)(b-t) f(t)\,dt &\le \left(\int_a^b f^p(t)\,dt\right)^{\frac{1}{p}} \left(\int_a^b (t-a)^q (b-t)^q\,dt\right)^{\frac{1}{q}} \\ &= \|f\|_p (b-a)^{2+\frac{1}{q}} \left[B(q+1,q+1)\right]^{\frac{1}{q}} . \end{aligned}$$

(ii) The inequality (2.3) is obvious, taking into account that if $m \le f \le M$ a.e. on $[a,b]$, then $m(t-a)(b-t) \le (t-a)(b-t) f(t) \le M(t-a)(b-t)$ a.e. on $[a,b]$, and by integrating over $[a,b]$.
To prove (2.4), we use the following *"pre-Grüss"* inequality established in [1]

$$\begin{aligned} &\left| \frac{1}{b-a}\int_a^b h(t)g(t)\,dt - \frac{1}{b-a}\int_a^b h(t)\,dt \cdot \frac{1}{b-a}\int_a^b g(t)\,dt \right| \\ &\le \frac{1}{2}(\phi-\gamma)\left[\frac{1}{b-a}\int_a^b g^2(t)\,dt - \left(\frac{1}{b-a}\int_a^b g(t)\,dt\right)^2\right]^{\frac{1}{2}}, \end{aligned} \tag{2.6}$$

provided that the mappings $h, g : [a,b] \to \mathbb{R}$ are measurable, all the integrals involved in (2.6) exist and are finite and $\gamma \le h \le \phi$ a.e. on $[a,b]$.
Choose in (2.6), $h(t) = f(t)$ and $g(t) = (t-a)(b-t)$, which then gives

$$\begin{aligned} &\left| \frac{1}{b-a}\int_a^b (t-a)(b-t) f(t)\,dt \right. \\ &\left. -\frac{1}{b-a}\int_a^b (t-a)(b-t)\,dt \cdot \frac{1}{b-a}\int_a^b f(t)\,dt \right| \\ \le\ & \frac{1}{2}(M-m)\left[\frac{1}{b-a}\int_a^b (t-a)^2(b-t)^2\,dt \right. \\ &\left. -\left(\frac{1}{b-a}\int_a^b (t-a)(b-t)\,dt\right)^2\right]^{\frac{1}{2}} . \end{aligned} \tag{2.7}$$

However,

$$\begin{aligned} \int_a^b (t-a)(b-t)\,dt &= \frac{(b-a)^3}{6}, \quad \int_a^b f(t)\,dt = 1, \\ \int_a^b (t-a)^2(b-t)^2\,dt &= (b-a)^5 \int_0^1 t^2(1-t)^2\,dt = \frac{(b-a)^5}{30} \end{aligned}$$

and

$$\frac{1}{b-a}\int_a^b (t-a)^2(b-t)^2\,dt - \left(\frac{1}{b-a}\int_a^b (t-a)(b-t)\,dt\right)^2$$
$$= \frac{(b-a)^4}{30} - \frac{(b-a)^4}{36} = \frac{(b-a)^4}{180}.$$

Consequently, by (2.7), we deduce that

$$\begin{aligned}\left|\int_a^b (t-a)(b-t) f(t)\,dt - \frac{(b-a)^2}{6}\right| &\leq \frac{1}{2}(b-a)(M-m)\left[\frac{(b-a)^4}{180}\right]^{\frac{1}{2}} \\ &= \frac{(b-a)^3(M-m)}{12\sqrt{5}}.\end{aligned}$$

Using (2.5), we deduce (2.4).

∎

Remark 1. *For a different proof of the inequality (2.1) see* [2].

With additional information about the derivative of f, we can state the following result which complements (2.4).

Theorem 2. *Assume that the p.d.f. of X is absolutely continuous on $[a,b]$.*

(i) *If $f' \in L_\infty[a,b]$, then we have:*

$$\left|[b-E(X)][E(X)-a] - \sigma^2(X) - \frac{(b-a)^2}{6}\right| \leq \frac{\sqrt{30}}{720}\|f'\|_\infty (b-a)^3. \tag{2.8}$$

(ii) *If $f' \in L_2[a,b]$, then we have:*

$$\left|[b-E(X)][E(X)-a] - \sigma^2(X) - \frac{(b-a)^2}{6}\right| \leq \frac{\sqrt{5}}{60\pi}\|f'\|_2 (b-a)^3. \tag{2.9}$$

Proof. (i) Use is made of the following "pre-Chebychev" inequality proved in [1],

$$\left|\frac{1}{b-a}\int_a^b h(t) g(t)\,dt - \frac{1}{b-a}\int_a^b h(t)\,dt \cdot \frac{1}{b-a}\int_a^b g(t)\,dt\right| \tag{2.10}$$
$$\leq \frac{1}{2\sqrt{3}}\|h'\|_\infty \left[\frac{1}{b-a}\int_a^b g^2(t)\,dt - \left(\frac{1}{b-a}\int_a^b g(t)\,dt\right)^2\right]^{\frac{1}{2}}.$$

Provided that $h,g:[a,b]\to\mathbb{R}$ are measurable on $[a,b]$, the integrals involved in (2.10) exist and are finite, h is absolutely continuous and $h' \in L_\infty[a,b]$. Now, if we choose $h(t)=f(t)$, $g(t)=(t-a)(b-t)$ in (2.10), we get

$$\begin{aligned}\left|\int_a^b (t-a)(b-t) f(t)\,dt - \frac{(b-a)^2}{6}\right| &\leq \frac{\|h'\|_\infty (b-a)}{2\sqrt{3}} \cdot \frac{(b-a)^2}{12\sqrt{5}} \\ &= \frac{(b-a)^3\|h'\|_\infty}{24\sqrt{30}}.\end{aligned}$$

Using (2.5), we deduce (2.8).

(ii) For the second part of the theorem, we use the following "pre-Lupaş" inequality as stated in [1]

$$\left|\frac{1}{b-a}\int_a^b h(t)\,g(t)\,dt-\frac{1}{b-a}\int_a^b h(t)\,dt\cdot\frac{1}{b-a}\int_a^b g(t)\,dt\right| \tag{2.11}$$
$$\leq \frac{b-a}{\pi}\left\|h'\right\|_2\left[\frac{1}{b-a}\int_a^b g^2(t)\,dt-\left(\frac{1}{b-a}\int_a^b g(t)\,dt\right)^2\right]^{\frac{1}{2}},$$

provided that g,h are as above and $h'\in L_2[a,b]$.
Now if we choose in (2.11) $h(t)=f(t)$, $g(t)=(t-a)(b-t)$, we obtain the desired inequality (2.9). The details are omitted.

∎

Theorem 3. *Let X be a random variable and $f:[a,b]\to\mathbb{R}$ its p.d.f. If f is such that $f^{(n)}$ $(n\geq 0)$ is absolutely continuous on $[a,b]$, then we have the inequality*

$$\left|[E(X)-a][b-E(X)]-\sigma^2(X)-\sum_{k=0}^{n}\frac{(k+1)(b-a)^{k+3}f^{(k)}(a)}{(k+3)!}\right| \tag{2.12}$$
$$\leq\begin{cases}\frac{\left\|f^{(n+1)}\right\|_\infty}{(n+1)!(n+3)(n+4)}(b-a)^{n+4} & \text{if } f^{(n+1)}\in L_\infty[a,b]\\[2ex] \frac{\left\|f^{(n+1)}\right\|_p(b-a)^{n+3+\frac{1}{q}}}{n!(nq+1)^{\frac{1}{q}}\left(n+2+\frac{1}{q}\right)\left(n+3+\frac{1}{q}\right)} & \text{if } f^{(n+1)}\in L_p[a,b],\ p>1\\[2ex] \frac{\left\|f^{(n+1)}\right\|_1(b-a)^{n+3}}{n!(n+2)(n+3)} & \text{if } f^{(n+1)}\in L_1[a,b]\end{cases}$$

where $\|\cdot\|_p$ $(1\leq p\leq\infty)$ are the usual Lebesgue norms on $[a,b]$, i.e.,

$$\|g\|_\infty:=ess\sup_{t\in[a,b]}|g(t)|,\quad \|g\|_p:=\left(\int_a^b|g(t)|^p\,dt\right)^{\frac{1}{p}}\quad(p\geq 1).$$

Proof. The following Taylor's formula with integral remainder is well known in the literature (see for example [3]):

$$f(t)=\sum_{k=0}^{n}\frac{(t-a)^k}{k!}f^{(k)}(a)+\frac{1}{n!}\int_a^t(t-s)^n f^{(n+1)}(s)\,ds \tag{2.13}$$

for all $t\in[a,b]$.
Since

$$[E(X)-a][b-E(X)]-\sigma^2(X)=\int_a^b(t-a)(b-t)f(t)\,dt, \tag{2.14}$$

then we have

$$
\begin{aligned}
(2.15)\quad & [E(X)-a][b-E(X)]-\sigma^2(X) \\
&= \int_a^b (t-a)(b-t)\left[\sum_{k=0}^{n}\frac{(t-a)^k}{k!}f^{(k)}(a)+\frac{1}{n!}\int_a^t (t-s)^n f^{(n+1)}(s)\,ds\right]dt \\
&= \sum_{k=0}^{n}\frac{f^{(k)}(a)}{k!}\int_a^b (t-a)^{k+1}(b-t)\,dt \\
&\quad +\frac{1}{n!}\int_a^b\left[(t-a)(b-t)\int_a^t (t-s)^n f^{(n+1)}(s)\,ds\right]dt.
\end{aligned}
$$

Using the transform, $t=(1-u)a+ub$, we have

$$
\int_a^b (t-a)^{k+1}(b-t)\,dt=(b-a)^{k+3}\int_0^1 u^{k+1}(1-u)\,du=\frac{1}{(k+2)(k+3)}
$$

and by (2.15), we deduce that

$$
\begin{aligned}
& \left|[E(X)-a][b-E(X)]-\sigma^2(X)-\sum_{k=0}^{n}\frac{(k+1)(b-a)^{k+3}f^{(k)}(a)}{(k+3)!}\right| \\
\leq\ & \frac{1}{n!}\int_a^b (t-a)(b-t)\left|\int_a^t (t-s)^n f^{(n+1)}(s)\,ds\right|dt =: M(a,b).
\end{aligned}
$$

However, for all $t\in[a,b]$ we have

$$
\begin{aligned}
\left|\int_a^t (t-s)^n f^{(n+1)}(s)\,ds\right| &\leq \int_a^t |t-s|^n\left|f^{(n+1)}(s)\right|ds \\
&\leq \sup_{s\in[a,b]}\left|f^{(n+1)}(s)\right|\int_a^t (t-s)^n\,ds \\
&\leq \left\|f^{(n+1)}\right\|_\infty\frac{(t-a)^{n+1}}{n+1}.
\end{aligned}
$$

By Hölder's integral inequality we have,

$$
\begin{aligned}
& \left|\int_a^t (t-s)^n f^{(n+1)}(s)\,ds\right| \\
\leq\ & \left(\int_a^t \left|f^{(n+1)}(s)\right|^p ds\right)^{\frac{1}{p}}\left(\int_a^t (t-s)^{nq}\,ds\right)^{\frac{1}{q}} \\
\leq\ & \left\|f^{(n+1)}\right\|_p\left[\frac{(t-a)^{nq+1}}{nq+1}\right]^{\frac{1}{q}},\quad \frac{1}{p}+\frac{1}{q}=1,\ p>1
\end{aligned}
$$

for all $t\in[a,b]$.

Finally, we observe that

$$
\begin{aligned}
\left|\int_a^t (t-s)^n f^{(n+1)}(s)\,ds\right| &\leq \int_a^t (t-s)^n\left|f^{(n+1)}(s)\right|ds \\
&\leq (t-a)^n\int_a^t \left|f^{(n+1)}(s)\right|ds \\
&\leq (t-a)^n\left\|f^{(n+1)}\right\|_1
\end{aligned}
$$

for all $t \in [a,b]$.

Consequently,

$$M(a,b) \leq \frac{1}{n!} \times \begin{cases} \frac{\left\|f^{(n+1)}\right\|_\infty}{n+1} \int_a^b (t-a)^{n+2}(b-t)\,dt \\ \\ \frac{\left\|f^{(n+1)}\right\|_p}{(nq+1)^{\frac{1}{q}}} \int_a^b (t-a)^{n+1+\frac{1}{q}}(b-t)\,dt \\ \\ \left\|f^{(n+1)}\right\|_1 \int_a^b (t-a)^{n+1}(b-t)\,dt \end{cases}$$

$$= \begin{cases} \frac{\left\|f^{(n+1)}\right\|_\infty}{n+1} (b-a)^{n+4} \int_0^1 u^{n+2}(1-u)\,du \\ \\ \frac{\left\|f^{(n+1)}\right\|_p}{(nq+1)^{\frac{1}{q}}} (b-a)^{n+3+\frac{1}{q}} \int_0^1 u^{n+1+\frac{1}{q}}(1-u)\,du \\ \\ \left\|f^{(n+1)}\right\|_1 (b-a)^{n+3} \int_0^1 u^{n+1}(1-u)\,du \end{cases}$$

and as

$$\begin{aligned} \int_0^1 u^{n+2}(1-u)\,du &= \frac{1}{(n+3)(n+4)}, \\ \int_0^1 u^{n+1+\frac{1}{q}}(1-u)\,du &= \frac{1}{\left(n+2+\frac{1}{q}\right)\left(n+3+\frac{1}{q}\right)} \quad \text{and} \\ \int_0^1 u^{n+1}(1-u)\,du &= \frac{1}{(n+2)(n+3)}, \end{aligned}$$

the inequality (2.12) is proved. ∎

Remark 2. *A similar result can be obtained if use is made of a Taylor expansion around the point b.*

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