Three Point Identities and Inequalities for n-time Differential Functions

Three Point Identities and Inequalities for n-time Differentiable Functions

P. Cerone and S.S. Dragomir

ABSTRACT. Identities and inequalities are obtained involving n-time differentiable functions in terms of evaluations at an interior and at the end points. It is shown how previous work is recaptured as particular instances of the current development. Generalised Taylor type series expansions are obtained and applications to numerical quadrature are demonstrated.

1. Introduction

Recently, Cerone and Dragomir [**2**] obtained the following three point identity for $f : [a, b] \to \mathbb{R}$ and $\alpha : [a, x] \to \mathbb{R}$, $\beta : (x, b] \to \mathbb{R}$ then

$$\begin{aligned} &\int_a^b f(t)\,dt - [(\beta(x) - \alpha(x)) f(x) + (\alpha(x) - a) f(a) + (b - \beta(x)) f(b)] \\ &= \int_a^b K(x,t)\,df(t), \end{aligned} \tag{1.1}$$

where

$$K(x,t) = \begin{cases} t - \alpha(x), & t \in [a, x] \\ t - \beta(x), & t \in (x, b]. \end{cases} \tag{1.2}$$

They obtained a variety of inequalities for f satisfying different conditions such as bounded variation, Lipschitzian or monotonic. For f absolutely continuous then the above Riemann-Stieltjes integral would be equivalent to a Riemann integral and again a variety of bounds were obtained for $f \in L_p[a, b]$, $p \geq 1$.

Inequalities of Grüss type and a number of premature variants were examined fully in the comprehensive article covering the situation in which f exhibits at most a first derivative. Applications to numerical quadrature were investigated covering rules of Newton-Cotes type containing the evaluation of the function at three possible points: the interior and extremities. The development included the midpoint, trapezoidal and Simpson type rules. However, unlike the classical rules, the results were not as restrictive in that the bounds are derived in terms of the behaviour of at most the first derivative and the Peano kernel (1.2).

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It is the aim of the current article to obtain inequalities for $f^{(n)} \in L_p[a,b]$, $p \geq 1$ where $f^{(n)}$ are again evaluated at most at an interior point x and the end points. Results that involve the evaluation only at an interior point are termed Ostrowski type and those that involve only the boundary points will be referred to as trapezoidal type. In the numerical analysis literature these are also termed as Open and Closed Newton-Cotes rules (Atkinson [**1**]) respectively.

In 1938, Ostrowski (see for example [**32**, p. 468]) proved the following integral inequality:

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on $\mathring{I}$ ($\mathring{I}$ is the interior of I), and let $a, b \in \mathring{I}$ with $a < b$. If $f' : (a,b) \to \mathbb{R}$ is bounded on (a,b), i.e., $\|f'\|_\infty := \sup_{t\in(a,b)} |f'(t)| < \infty$, then we have the inequality:

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \leq \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right](b-a)\|f'\|_\infty \tag{1.3}$$

for all $x \in [a,b]$.

The constant $\frac{1}{4}$ is sharp in the sense that it cannot be replaced by a smaller one.

For applications of Ostrowski's inequality to some special means and some numerical quadrature rules, we refer the reader to the recent paper [**24**] by S.S. Dragomir and S. Wang who used integration by parts from $\int_a^b p(x,t) f'(t)\,dt$ to prove Ostrowski's inequality (1.3) where $p(x,t)$ is a peano kernel given by

$$p(x,t) = \begin{cases} t-a, & t \in [a,x] \\ \\ t-b, & t \in (x,b]. \end{cases} \tag{1.4}$$

Fink [**27**] used the integral remainder from a Taylor series expansion to show that for $f^{(n-1)}$ absolutely continuous on $[a,b]$, then the identity

$$\int_a^b f(t)\,dt - \frac{1}{n}\left((b-a) f(x) + \sum_{k=1}^{n-1} F_k(x) \right) = \int_a^b K_F(x,t) f^{(n)}(t)\,dt \tag{1.5}$$

is shown to hold where

$$K_F(x,t) = \frac{(x-t)^{n-1}}{(n-1)!} \cdot \frac{p(x,t)}{n} \quad \text{with} \quad p(x,t) \text{ being given by (1.4)} \tag{1.6}$$

and

$$F_k(x) = \frac{n-k}{k!}\left[(x-a)^k f^{(k-1)}(a) + (-1)^{k-1}(b-x)^k f^{(k-1)}(b) \right].$$

Fink then proceeds to obtain a variety of bounds from (1.5), (1.6) for $f^{(n)} \in L_p[a,b]$. Milovanović and Pečarić [**31**] earlier obtained a result for $f^{(n)} \in L_\infty[a,b]$ although they did not use the integral form of the remainder. It may be noticed that (1.5) is again an identity that involves function evaluations at three points to approximate the integral from the resulting inequalities. See Mitrinović, Pečarić and Fink [**33**, Chapter XV] for further related results and papers [**19**], [**20**] and [**21**].

A number of other authors have obtained results in the literature that may be recaptured under the general formulation of the current paper. These will be highlighted throughout the article.

The paper is structured as follows.

A variety of identities are obtained in Section 2 for $f^{(n-1)}$ absolutely continuous for a generalisation of the kernel (1.2). Specific forms are highlighted and a generalised Taylor-like expansion is obtained. Inequalities are developed in Section 3 and perturbed results through Grüss inequalities and premature variants are discussed in Section 4. Section 5 demonstrates the applicability of the inequalities to numerical integration. Concluding remarks are given in Section 6.

2. Some Integral Identities

In this section identities are obtained involving n-time differentiable functions with evaluation at an interior point and at the end points.

THEOREM 1. *Let $f : [a,b] \to \mathbb{R}$ be a mapping such that $f^{(n-1)}$ is absolutely continuous on $[a,b]$. Further, let $\alpha : [a,x] \to \mathbb{R}$ and $\beta : (x,b] \to \mathbb{R}$. Then, for all $x \in [a,b]$ the following identity holds,*

$$(-1)^n \int_a^b K_n(x,t) f^{(n)}(t)\,dt = \int_a^b f(t)\,dt - \sum_{k=1}^{n} \frac{1}{k!}\left[R_k(x) f^{(k-1)}(x) + S_k(x)\right], \tag{2.1}$$

where the kernel $K_n : [a,b]^2 \to \mathbb{R}$ is given by

$$K_n(x,t) := \begin{cases} \frac{(t-\alpha(x))^n}{n!}, & t \in [a,x] \\ \\ \frac{(t-\beta(x))^n}{n!}, & t \in (x,b], \end{cases} \tag{2.2}$$

$$\begin{cases} R_k(x) = (\beta(x) - x)^k + (-1)^{k-1}(x-\alpha(x))^k \\ \text{and } \; S_k(x) = (\alpha(x)-a)^k f^{(k-1)}(a) + (-1)^{k-1}(b-\beta(x))^k f^{(k-1)}(b) \end{cases}. \tag{2.3}$$

PROOF. Let

$$I_n(x) = (-1)^n \int_a^b K_n(x,t) f^{(n)}(t)\,dt = (-1)^n J_n(a,x,b) \tag{2.4}$$

then from (2.3)

$$J_n(a,x,x) = \int_a^x \frac{(t-\alpha(x))^n}{n!} f^{(n)}(t)\,dt$$

giving, upon using integration by parts

$$\begin{aligned} J_n(a,x,x) &= \frac{(x-\alpha(x))^n}{n!} f^{(n-1)}(x) \\ &\quad + (-1)^{n-1} \frac{(\alpha(x)-a)^n f^{(n-1)}(a)}{n!} - J_{n-1}(a,x,x). \end{aligned} \tag{2.5}$$

Similarly,

$$\begin{aligned} J_n(x,x,b) &= (-1)^{n-1} \frac{(\beta(x)-x)^n}{n!} f^{(n-1)}(x) \\ &\quad + \frac{(b-\beta(x))^n}{n!} f^{(n-1)}(b) - J_{n-1}(x,x,b) \end{aligned}$$

and so upon adding to (2.5) gives from (2.4) the recurrence relation

$$I_n(x) - I_{n-1}(x) = -\omega_n(x), \tag{2.6}$$

where

$$n!\,\omega_n(x) = \left[R_n(x)\, f^{(n-1)}(x) + S_n(x)\right] \tag{2.7}$$

with $R_n(x)$ and $S_n(x)$ being given by (2.3).

It may easily be shown that

$$I_n(x) = -\sum_{k=1}^{n} \omega_k(x) + I_0(x) \tag{2.8}$$

is a solution of (2.6) and so the theorem is proven since (2.8) is equivalent to (2.1) and $I_n(x)$ is as given by (2.4). ∎

Remark 1. *If we take $n = 1$ then an identity obtained by Cerone and Dragomir* [**2**] *results. In the same paper Riemann-Stieltjes integrals were also considered.*

Remark 2. *If $\alpha(x) = a$ and $\beta(x) = b$ then $S_k(x) \equiv 0$ and the Ostrowski type results for n-time differentiable functions of Cerone et al.* [**9**] *are recaptured. Merkle* [**30**] *also obtains Ostrowski type results. For $\alpha(x) = \beta(x) = x$ then $R(x) \equiv 0$ and the generalized trapezoidal type rules for n-time differentiable functions of Cerone et al.* [**10**] *are obtained. Qi* [**36**] *used a Taylor series whose remainder was not expressed in integral form so that only the supremum norm was possible. If the integral form of the remainder were used, then similar to Fink* [**27**]*, the other $L_p(a,b)$ norms for $p \geq 1$ would be possible. However, this will not be pursued further here. For $\alpha(x)$ and $\beta(x)$ at their respective midpoints, then the identity*

$$\begin{aligned} &(-1)^n \int_a^b K_n(x,t)\, f^{(n)}(t)\,dt \\ &= \int_a^b f(t)\,dt - \sum_{k=1}^{n} \frac{2^{-k}}{k!}\left\{\left[(b-x)^k + (-1)^{k-1}(x-a)^k\right] f^{(k-1)}(x) \right. \\ &\quad \left. + \left[(x-a)^k f^{(k-1)}(a) + (-1)^{k-1}(b-x)^k f^{(k-1)}(b)\right]\right\} \end{aligned} \tag{2.9}$$

results, where

$$K_n(x,t) = \begin{cases} \dfrac{\left(t-\frac{a+x}{2}\right)^n}{n!}, & t \in [a,x] \\[2ex] \dfrac{\left(t-\frac{x+b}{2}\right)^n}{n!}, & t \in (x,b]. \end{cases} \tag{2.10}$$

As demonstrated in the above remarks, different choices of $\alpha(x)$ and $\beta(x)$ give a variety of identities. The following corollary allows for $\alpha(x)$ and $\beta(x)$ to be in the same relative position within their respective intervals.

Corollary 1. *Let f satisfy the conditions as stated in Theorem 1. Then the following identity holds for any $\gamma \in [0,1]$ and $x \in [a,b]$. Namely,*

$$\begin{aligned} &(-1)^n \int_a^b C_n(x,t)\, f^{(n)}(t)\,dt \\ &= \int_a^b f(t)\,dt - \sum_{k=1}^{n} \frac{1}{k!}\left\{(1-\gamma)^k \left[(b-x)^k + (-1)^{k-1}(x-a)^k\right] f^{(k-1)}(x) \right. \\ &\quad \left. + \gamma^k \left[(x-a)^k f^{(k-1)}(a) + (-1)^{k-1}(b-x)^k f^{(k-1)}(b)\right]\right\}, \end{aligned} \tag{2.11}$$

where

$$C_n(x,t)=\begin{cases}\frac{[t-(\gamma x+(1-\gamma)a)]^n}{n!}, & t\in[a,x]\\ \\ \frac{[t-(\gamma x+(1-\gamma)b)]^n}{n!}, & t\in(x,b].\end{cases} \tag{2.12}$$

PROOF. Let

$$\alpha(x)=\gamma x+(1-\gamma)a \text{ and } \beta(x)=\gamma x+(1-\gamma)b, \tag{2.13}$$

then

$$\begin{cases} x-\alpha(x)=(1-\gamma)(x-a), & \alpha(x)-a=\gamma(x-a)\\ \text{and } \beta(x)-x=(1-\gamma)(b-x), & b-\beta(x)=\gamma(b-x)\end{cases} \tag{2.14}$$

so that from (2.3)

$$R_k(x)=(1-\gamma)^k\left[(b-x)^k+(-1)^{k-1}(x-a)^k\right]$$

and

$$S_k(x)=\gamma^k\left[(x-a)^k f^{(k-1)}(a)+(-1)^{k-1}(b-x)^k f^{(k-1)}(b)\right].$$

In addition, $C_n(x,t)$ is the same as $K_n(x,t)$ in (2.2) with $\alpha(x)$ and $\beta(x)$ as given by (2.13) and hence the corollary is proven. ∎

The following Taylor-like formula with integral remainder also holds.

COROLLARY 2. *Let $g:[a,y]\to\mathbb{R}$ be a mapping such that $g^{(n)}$ is absolutely continuous on $[a,y]$. Then for all $x\in[a,y]$ we have the identity*

$$\begin{aligned} g(y) &\\ = {} & g(a)+\sum_{k=1}^{n}\frac{1}{k!}\left\{\left[(\beta(x)-x)^k+(-1)^{k-1}(x-\alpha(x))^k\right]g^{(k)}(x)\right.\\ & \left.+\left[(\alpha(x)-a)^k g^{(k)}(a)+(-1)^{k-1}(y-\beta(x))^k g^{(k)}(y)\right]\right\}\\ & +(-1)^n\int_a^y \tau_n(x,t)\,g^{(n+1)}(t)\,dt \end{aligned} \tag{2.15}$$

where

$$\tau_n(x,t)=\begin{cases}\frac{(t-\alpha(x))^n}{n!}, & t\in[a,x]\\ \\ \frac{(t-\beta(x))^n}{n!}, & t\in(x,y].\end{cases} \tag{2.16}$$

PROOF. The proof is straight forward from Theorem 1 on taking $f\equiv g'$ and $b=y$ so that $\beta(x)\in(x,y]$ and $\tau_n(x,t)\equiv K_n(x,t)$ for $t\in[a,y]$. ∎

REMARK 3. *If $\alpha(x)=\beta(x)=x$ then we recapture the results of Cerone et al.* [**10**]*, a trapezoidal type series expansion. That is, an expansion involving the end points. For $\alpha(x)=a$, $\beta(x)=b$ then a Taylor-like expansion of Cerone et al.* [**9**] *is reproduced as are the results of Merkle* [**30**].

3. Integral Inequalities

In this section we develop some inequalities from using the identities obtained in Section 2.

THEOREM 2. *Let $f:[a,b]\to\mathbb{R}$ be a mapping such that $f^{(n-1)}$ is absolutely continuous on $[a,b]$ and, let $\alpha:[a,x]\to\mathbb{R}$ and $\beta:(x,b]\to\mathbb{R}$. Then the following inequalities hold for all $x\in[a,b]$*

$$|P_n(x)| := \left|\int_a^b f(t)\,dt-\sum_{k=1}^{n}\frac{1}{k!}\left[R_k(x)\,f^{(k-1)}(x)+S_k(x)\right]\right| \tag{3.1}$$

$$\leq\begin{cases}\frac{\|f^{(n)}\|_\infty}{n!}Q_n(1,x) & \text{if } f^{(n)}\in L_\infty[a,b],\\ \frac{\|f^{(n)}\|_p}{n!}\left[Q_n(q,x)\right]^{\frac{1}{q}} & \text{if } f^{(n)}\in L_p[a,b] \\ & \text{with } p>1,\ \frac{1}{p}+\frac{1}{q}=1,\\ \frac{\|f^{(n)}\|_1}{n!}M^n(x), & \text{if } f^{(n)}\in L_1[a,b],\end{cases}$$

where

$$Q_n(q,x) = \frac{1}{nq+1}\Big[(\alpha(x)-a)^{nq+1}+(x-\alpha(x))^{nq+1} + (\beta(x)-x)^{nq+1}+(b-\beta(x))^{nq+1}\Big], \tag{3.2}$$

$$M(x) = \frac{1}{2}\left\{\frac{b-a}{2}+\left|\alpha(x)-\frac{a+x}{2}\right|+\left|\beta(x)-\frac{x+b}{2}\right| +\left|x-\frac{a+b}{2}+\left|\alpha(x)-\frac{a+x}{2}\right|+\left|\beta(x)-\frac{x+b}{2}\right|\right|\right\}, \tag{3.3}$$

$R_k(x)$, $S_k(x)$ are given by (2.3), and

$$\left\|f^{(n)}\right\|_\infty := ess\sup_{t\in[a,b]}\left|f^{(n)}(t)\right|<\infty \quad and \quad \left\|f^{(n)}\right\|_p := \left(\int_a^b\left|f^{(n)}(t)\right|^p\right)^{\frac{1}{p}},\ p\geq 1.$$

PROOF. Taking the modulus of (2.1) then

$$|P_n(x)| = |I_n(x)|, \tag{3.4}$$

where $P_n(x)$ is as defined by the left hand side of (3.1) and

$$|I_n(x)| = \left|\int_a^b K_n(x,t)\,f^{(n)}(t)\,dt\right|, \tag{3.5}$$

with $K_n(x,t)$ given by (2.2).

Now, observe that

$$|I_n(x)| \leq \left\|f^{(n)}\right\|_\infty \|K_n(x,\cdot)\|_1 = \left\|f^{(n)}\right\|_\infty \int_a^b |K_n(x,t)|\,dt, \tag{3.6}$$

where, from (2.2),

$$\begin{aligned}(3.7)\quad \int_a^b |K_n(x,t)|\,dt &= \frac{1}{n!}\left\{\int_a^{\alpha(x)} |t-\alpha(x)|^n\,dt + \int_{\alpha(x)}^{x} |t-\alpha(x)|^n\,dt \right.\\ &\quad \left. + \int_x^{\beta(x)} |t-\beta(x)|^n\,dt + \int_{\beta(x)}^{b} |t-\beta(x)|^n\,dt\right\}\\ &= \frac{1}{(n+1)!}\Big[(\alpha(x)-a)^{n+1} + (x-\alpha(x))^{n+1}\\ &\quad + (\beta(x)-x)^{n+1} + (b-\beta(x))^{n+1}\Big].\end{aligned}$$

Thus, on combining (3.4), (3.6) and (3.7), the first inequality in (3.1) is obtained.

Further, using Hölder's integral inequality we have the result

$$\begin{aligned}(3.8)\qquad |I_n(x)| &\le \left\|f^{(n)}\right\|_p \|K_n(x,\cdot)\|_q \quad \text{where } \frac{1}{p}+\frac{1}{q}=1, \text{ with } p>1\\ &= \left\|f^{(n)}\right\|_p \left(\int_a^b |K_n(x,t)|^q\,dt\right)^{\frac{1}{q}}.\end{aligned}$$

Now,

$$\begin{aligned}(3.9)\ \int_a^b |K_n(x,t)|^q\,dt &= \frac{1}{n!}\left\{\int_a^{\alpha(x)} |t-\alpha(x)|^{nq}\,dt + \int_{\alpha(x)}^{x} |t-\alpha(x)|^{nq}\,dt \right.\\ &\quad \left. + \int_x^{\beta(x)} |t-\beta(x)|^{nq}\,dt + \int_{\beta(x)}^{b} |t-\beta(x)|^{nq}\,dt\right\}\\ &= \frac{1}{n!}Q_n(q,x),\end{aligned}$$

where $Q_n(q,x)$ is as given by (3.2).

Combing (3.9) with (3.8) gives the second inequality in (3.1).

Finally, let us observe that from (3.4)

$$\begin{aligned}(3.10)\qquad & |I_n(x)|\\ \le\ & \|K_n(x,\cdot)\|_\infty \left\|f^{(n)}\right\|_1 = \left\|f^{(n)}\right\|_1 \sup_{t\in[a,b]} |K_n(x,t)|\\ =\ & \frac{\left\|f^{(n)}\right\|_1}{n!} \max\left\{|a-\alpha(x)|^n, |x-\alpha(x)|^n, |b-\beta(x)|^n, |x-\beta(x)|^n\right\}\\ =\ & \frac{\left\|f^{(n)}\right\|_1}{n!} M^n(x),\end{aligned}$$

where

$$(3.11)\qquad M(x) = \max\{M_1(x), M_2(x)\}$$

with

$$M_1(x) = \max\{\alpha(x)-a, x-\alpha(x)\}$$

and

$$M_2(x) = \max\{\beta(x)-x, b-\beta(x)\}.$$

The well known identity

$$\max\{X,Y\} = \frac{X+Y}{2} + \left|\frac{X-Y}{2}\right| \tag{3.12}$$

may be used to give

$$\begin{aligned} M_1(x) &= \tfrac{x-a}{2} + \left|\alpha(x) - \tfrac{a+x}{2}\right|, \\ \text{and}\quad M_2(x) &= \tfrac{b-x}{2} + \left|\beta(x) - \tfrac{x+b}{2}\right|. \end{aligned} \tag{3.13}$$

Using the identity (3.12) again gives, from (3.11),

$$M(x) = \frac{M_1(x) + M_2(x)}{2} + \left|\frac{M_1(x) - M_2(x)}{2}\right|$$

which on substituting (3.13) gives (3.3) and so from (3.10) and (3.4) readily results in the third inequality in (3.1) and the theorem is completely proved. ∎

Remark 4. *Various choices of $\alpha(\cdot)$ and $\beta(\cdot)$ allow us to reproduce many of the earlier inequalities involving function and derivative evaluations at an interior point and/or boundary points. For other related results see Chapter XV of* [**32**].

If $\alpha(x) = a$ and $\beta(x) = b$ then $S_k(x) \equiv 0$ and Ostrowski type results for n-time differentiable functions of Cerone et al. [**9**] are reproduced (see also [**35**]). Further, taking $n = 1$ recaptures the results of Dragomir and Wang [**22**]-[**25**] and $n = 2$ gives the results of Cerone, Dragomir and Roumeliotis [**5**]-[**8**]. The $n = 2$ case is of importance since with $x = \frac{a+b}{2}$ the classic midpoint rule is obtained. However, here the bound is obtained for $f'' \in L_p[a,b]$ for $p \geq 1$ rather than the traditional $f'' \in L_\infty[a,b]$, see for example [**16**] and [**17**].

If $\alpha(x) = \beta(x) = x$ then $R(x) \equiv 0$ and inequalities are obtained for a generalised trapezoidal type rule in which functions are assumed to be n-time differentiable, recapturing the results in Cerone et al. [**10**]. Taking $n = 2$ the classic trapezoidal rule in which the bound involves the behaviour of the second derivative is recaptured as presented in Dragomir et al. [**18**].

Taking $\alpha(\cdot)$ and $\beta(\cdot)$ to be other than at the extremities results in three point inequalities for $n-$time differentiable functions. Cerone and Dragomir [**2**] presented results for functions that at most admit a first derivative.

Remark 5. *It should be noted that the bounds in (3.1) may themselves be bounded since $\alpha(\cdot)$, $\beta(\cdot)$ and x have not been explicitly specified.*

To demonstrate, consider the mappings, for $t \in [A,B]$,

$$\begin{cases} h_1(t) = (t-A)^\theta + (B-t)^\theta,\ \theta > 1 \\ \text{and } h_2(t) = \tfrac{B-A}{2} + \left|t - \tfrac{A+B}{2}\right|. \end{cases} \tag{3.14}$$

Now, both these functions attain their maximum values at the ends of the interval and their minimums at the midpoints. That is, they are symmetric and convex.

Thus,

$$(3.15)\qquad \begin{cases} \sup\limits_{t\in[A,B]} h_1(t) = h_1(A) = h_1(B) = (B-A)^{\theta}, \\ \\ \sup\limits_{t\in[A,B]} h_2(t) = h_2(A) = h_2(B) = B-A, \\ \\ \inf\limits_{t\in[A,B]} h_1(t) = h_1\left(\frac{A+B}{2}\right) = 2\left(\frac{B-A}{2}\right)^{\theta}, \\ \\ \text{and} \quad \inf\limits_{t\in[A,B]} h_2(t) = h_2\left(\frac{A+B}{2}\right) = \frac{B-A}{2}. \end{cases}$$

Using (3.14) and (3.15) then from (3.2) and (3.3), on taking $\alpha(\cdot)$ and $\beta(\cdot)$ at either of their extremities gives

$$Q_n(q,x) \le Q_n^U(q,x) = \frac{1}{nq+1}\left[(x-a)^{nq+1} + (b-x)^{nq+1}\right] \le \frac{(b-a)^{nq+1}}{nq+1}$$

and

$$M(x) \le M^U(x) = \frac{1}{2}\left[b-a+|x-a|\right] \le b-a,$$

where the coarsest bounds are obtained from taking x at its extremities.

The following corollary holds.

Corollary 3. *Let the conditions of Theorem 2 hold. Then the following result is valid for any $x\in[a,b]$. Namely,*

$$(3.16)\qquad \left|\int_a^b f(t)\,dt - \sum_{k=1}^{n}\frac{2^{-k}}{k!}\left\{\left[(b-x)^k + (-1)^{k-1}(x-a)^k\right] f^{(k-1)}(x) + \left[(x-a)^k f^{(k-1)}(a) + (-1)^{k-1}(b-x)^k f^{(k-1)}(b)\right]\right\}\right|$$

$$\le \begin{cases} \frac{\|f^{(n)}\|_\infty}{n!} 2^{-n}\left[(x-a)^{n+1} + (b-x)^{n+1}\right], & f^{(n)}\in L_\infty[a,b], \\ \\ \frac{\|f^{(n)}\|_p}{n!}\frac{2^{-n}}{(nq+1)^{\frac{1}{q}}}\left[(x-a)^{nq+1} + (b-x)^{nq+1}\right]^{\frac{1}{q}} & f^{(n)}\in L_p[a,b] \\ & \text{with } p>1,\ \frac{1}{p}+\frac{1}{q}=1, \\ \frac{\|f^{(n)}\|_1}{n!}2^{-n}\left[\frac{b-a}{2} + \left|x-\frac{a+b}{2}\right|\right]^n, & f^{(n)}\in L_1[a,b]. \end{cases}$$

Proof. Taking $\alpha(\cdot)$ and $\beta(\cdot)$ at their respective midpoints, namely $\alpha(x) = \frac{a+x}{2}$ and $\beta(x) = \frac{x+b}{2}$ in (3.1)-(3.3) and using (2.3) readily gives (3.16) ∎

Remark 6. *Corollary 3 could have equivalently been proven using (2.9) and (2.10) following essentially the same proof of Theorem 2 from using identity (2.9). The more general setting however, allows greater flexibility and, it is argued, is no more difficult to prove.*

Corollary 4. *Let the conditions on f of Theorem 2 hold. Then the following result for any $x \in [a,b]$, is valid. Namely, for any $\gamma \in [0,1]$*

$$\left| \int_a^b f(t)\,dt - \sum_{k=1}^{n} \frac{(-1)^k}{k!} \left[(1-\gamma)^k r_k(x) f^{(k-1)}(x) + \gamma^k s_k(x) \right] \right| \tag{3.17}$$

$$\leq \begin{cases} \frac{\left\| f^{(n)} \right\|_\infty}{(n+1)!} H_1(\gamma) G_1(x), & f^{(n)} \in L_\infty[a,b], \\ \frac{\left\| f^{(n)} \right\|_p}{n!(nq+1)^{\frac{1}{q}}} H_q^{\frac{1}{q}}(\gamma) G_q^{\frac{1}{q}}(x), & f^{(n)} \in L_p[a,b] \\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1, \\ \frac{\left\| f^{(n)} \right\|_1}{n!} \nu^n(x), & f^{(n)} \in L_1[a,b], \end{cases}$$

where

$$\begin{cases} H_q(\gamma) = \gamma^{nq+1} + (1-\gamma)^{nq+1}, \\ G_q(x) = (x-a)^{nq+1} + (b-x)^{nq+1}, \\ \nu(x) = \left[\frac{1}{2} + \left| \gamma - \frac{1}{2} \right| \right] \left[\frac{b-a}{2} + \left| x - \frac{a+b}{2} \right| \right], \\ r_k(x) = (b-x)^k + (-1)^{k-1}(x-a)^k, \\ \text{and } s_k(x) = (x-a)^k f^{(k-1)}(a) + (-1)^{k-1}(b-x)^k f^{(k-1)}(b). \end{cases} \tag{3.18}$$

Proof. Take $\alpha(\cdot)$ and $\beta(\cdot)$ to be a convex combination of their respective boundary points as given by (2.13) then from (3.1)-(3.3) and using (2.14) and (2.3) readily produces the stated result. We omit any further details. ∎

Remark 7. *It is instructive to note that the relative location of $\alpha(\cdot)$ and $\beta(\cdot)$ is the same in Corollary 4 and is determined through the parameter γ as defined in (2.13). Theorem 2 is much more general. From (2.13) it may be seen that $\alpha(x) = \beta(x) = x$ is equivalent to $\gamma = 1$, giving trapezoidal type rules while $\alpha(x) = a$, $\beta(x) = b$ corresponds to $\gamma = 0$ which produces interior point rules. Taking $\gamma = 0$ and $\gamma = 1$ reproduces the results of Cerone et al.* [**9**] *and* [**10**] *respectively.*

Taking $\gamma = \frac{1}{2}$ in (3.17) produces the optimal rule while keeping x general and thus reproducing the result of Corollary 3. Following the discussion in Remark 5 and as may be ascertained from (3.18) the optimal rules, in the sense of providing the tightest bounds, are obtained by taking γ and x at their respective midpoints.

The following two corollaries may thus be stated.

Corollary 5. *Let the conditions on f of Theorem 2 be valid. Then for any $\gamma \in [0,1]$, the following inequalities hold*

$$\left| \int_a^b f(t)\,dt - \sum_{k=1}^{n} \frac{(-1)^k}{k!} \left[(1-\gamma)^k r_k\left(\frac{a+b}{2}\right) f^{(k-1)}\left(\frac{a+b}{2}\right) + \gamma^k s_k\left(\frac{a+b}{2}\right) \right] \right| \tag{3.19}$$
$$\leq \begin{cases} \frac{\|f^{(n)}\|_\infty}{(n+1)!} H_1(\gamma) G_1\left(\frac{a+b}{2}\right), & f^{(n)} \in L_\infty[a,b], \\ \frac{\|f^{(n)}\|_p}{n!(nq+1)^{\frac{1}{q}}} H_q^{\frac{1}{q}}(\gamma) G_q^{\frac{1}{q}}\left(\frac{a+b}{2}\right), & f^{(n)} \in L_p[a,b] \\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1, \\ \frac{\|f^{(n)}\|_1}{n!} \nu^n\left(\frac{a+b}{2}\right), & f^{(n)} \in L_1[a,b], \end{cases}$$

where

$$\begin{cases} H_q(\gamma) \text{ is as given by (3.18)}, \\ G_q\left(\frac{a+b}{2}\right) = 2\left(\frac{b-a}{2}\right)^{nq+1}, \\ \nu\left(\frac{a+b}{2}\right) = \left(\frac{b-a}{2}\right)\left(\frac{1}{2} + \left|\gamma - \frac{1}{2}\right|\right), \\ r_k\left(\frac{a+b}{2}\right) = \left(\frac{b-a}{2}\right)^k \left[1 + (-1)^{k-1}\right] \\ \text{and } s_k\left(\frac{a+b}{2}\right) = \left(\frac{b-a}{2}\right)^k \left[f^{(k-1)}(a) + (-1)^{k-1} f^{(k-1)}(b)\right]. \end{cases} \tag{3.20}$$

Proof. The proof is trivial. Taking $x = \frac{a+b}{2}$ in (3.17)-(3.18) readily produces the result. ∎

Remark 8. *It is of interest to note from (3.20) that*

$$r_k\left(\frac{a+b}{2}\right) = \begin{cases} 2\left(\frac{b-a}{2}\right)^k, & k \text{ odd} \\ 0, & k \text{ even} \end{cases} \tag{3.21}$$

so that only the evaluation of even order derivatives are involved in (3.19). Further, for $f^{(k-1)}(a) = f^{(k-1)}(b)$ then

$$s_k\left(\frac{a+b}{2}\right) = f^{(k-1)}(a)\, r_k\left(\frac{a+b}{2}\right) = f^{(k-1)}(b)\, r_k\left(\frac{a+b}{2}\right) \tag{3.22}$$

so that only evaluation of even order derivatives at the end points are present.

Corollary 6. *Let the conditions on f of Theorem 2 hold. Then the following inequalities are valid*

$$(3.23)\quad \left|\int_a^b f(t)\,dt-\sum_{k=1}^{n}\frac{(-1)^k}{k!}2^{-k}\left[r_k\left(\frac{a+b}{2}\right)f^{(k-1)}\left(\frac{a+b}{2}\right)+s_k\left(\frac{a+b}{2}\right)\right]\right|$$

$$\leq\begin{cases}\frac{\|f^{(n)}\|_\infty}{(n+1)!}\cdot 2^{-(n-1)}\left(\frac{b-a}{2}\right)^{n+1}, & f^{(n)}\in L_\infty[a,b],\\[2ex] \frac{\|f^{(n)}\|_p}{n!(nq+1)^{\frac{1}{q}}}\cdot 2^{-n}\left(\frac{b-a}{nq+1}\right)^{\frac{1}{q}}\left(\frac{b-a}{2}\right)^n, & f^{(n)}\in L_p[a,b]\\ & \text{with } p>1,\ \frac{1}{p}+\frac{1}{q}=1,\\ \frac{\|f^{(n)}\|_1}{n!}\left(\frac{b-a}{4}\right)^n, & f^{(n)}\in L_1[a,b],\end{cases}$$

where $r_k\left(\frac{a+b}{2}\right)$ and $s_k\left(\frac{a+b}{2}\right)$ are as given by (3.20).

Proof. Taking $\gamma=\frac{1}{2}$ in Corollary 5 will produce inequalities with the tightest bounds as given in (3.23). Alternatively, taking $\gamma=\frac{1}{2}$ and $x=\frac{a+b}{2}$ in Corollary 4 will produce the results (3.23). ∎

The results (3.21) and (3.22) together with the discussion in Remark 8 are also valid for Corollary 6.

The following are Taylor-like inequalities which are of interest (see [**12**] and [**14**] for related results).

Corollary 7. *Let $g:[a,y]\to\mathbb{R}$ be a mapping such that $g^{(n)}$ is absolutely continuous on $[a,y]$. Then for all $x\in[a,y]$*

$$(3.24)\quad \Bigg|g(y)-g(a)-\sum_{k=1}^{n}\frac{1}{k!}\Bigg\{\left[(\beta(x)-x)^k+(-1)^{k-1}(x-\alpha(x))^k\right]g^{(k)}(x)$$
$$+\left[(\alpha(x)-a)^k g^{(k)}(a)+(-1)^{k-1}(y-\beta(x))^k g^{(k)}(y)\right]\Bigg\}\Bigg|$$

$$\leq\begin{cases}\frac{\|g^{(n+1)}\|_\infty}{n!}\tilde{Q}_n(1,x), & g^{(n+1)}\in L_\infty[a,b],\\[2ex] \frac{\|g^{(n+1)}\|_p}{n!}\left[\tilde{Q}_n(q,x)\right]^{\frac{1}{q}}, & g^{(n+1)}\in L_p[a,b]\\ & \text{with } p>1,\ \frac{1}{p}+\frac{1}{q}=1,\\ \frac{\|g^{(n+1)}\|_1}{n!}\tilde{M}^n(x), & g^{(n+1)}\in L_1[a,b],\end{cases}$$

where

$$\begin{aligned}\tilde{Q}_n(q,x) &= \frac{1}{nq+1}\Big[(\alpha(x)-a)^{nq+1}+(x-\alpha(x))^{nq+1}\\ &\quad+(\beta(x)-a)^{nq+1}+(y-\beta(x))^{nq+1}\Big],\\ \tilde{M}(x) &= \frac{1}{2}\left\{\frac{y-a}{2}+\left|\alpha(x)-\frac{a+x}{2}\right|+\left|\beta(x)-\frac{x+y}{2}\right|\right.\\ &\quad\left.+\left|x-\frac{a+y}{2}+\left|\alpha(x)-\frac{a+x}{2}\right|+\left|\beta(x)-\frac{x+y}{2}\right|\right|\right\}.\end{aligned}$$

Proof. The proof follows from Theorem 2 on taking $f(\cdot) \equiv g'(\cdot)$ and $b = y$ so that $\beta(x) \in (x, y]$. Alternatively, starting from (2.15) and (2.16) and, following the proof of Theorem 2 with b replaced by y and $f(\cdot)$ replaced by $g'(\cdot)$ readily produces the results shown and the corollary is thus proven. ∎

Remark 9. *Similar corollaries to 3, 4, 7 and 8 could be determined from the Taylor-like inequalities given in Corollary 6. This would simply be done by taking specific forms of $\alpha(\cdot)$, $\beta(\cdot)$ or values of x as appropriate.*

Remark 10. *If in particular we take $\alpha(x) = a$ and $\beta(x) = y$ in (3.24) then for any $x \in [a, y]$*

$$\left|g(y) - g(a) - \sum_{k=1}^{n} \frac{1}{k!}\left[(y-x)^k + (-1)^{k-1}(x-a)^k\right] g^{(k)}(x)\right| \tag{3.25}$$

$$\leq e_n(x, y)$$

$$:= \begin{cases} \frac{\|g^{(n+1)}\|_\infty}{(n+1)!}\left[(x-a)^{n+1} + (y-x)^{n+1}\right], & g^{(n+1)} \in L_\infty[a, y], \\ \frac{\|g^{(n+1)}\|_p}{n!(nq+1)^{\frac{1}{q}}}\left[(x-a)^{nq+1} + (y-x)^{nq+1}\right]^{\frac{1}{q}}, & g^{(n+1)} \in L_p[a, y] \\ & \textit{with } p > 1,\ \frac{1}{p} + \frac{1}{q} = 1, \\ \frac{\|g^{(n+1)}\|_1}{n!}\left[\frac{x+y}{2} - a\right]^n, & g^{(n+1)} \in L_1[a, y]. \end{cases}$$

Merkle [**30**] effectively obtains the first bound in (3.25).

It is well known (see for example, Dragomir [**14**]) that the classical Taylor expansion around a point satisfies the inequality

$$\left|g(y) - \sum_{k=1}^{n} \frac{(y-a)^k}{k!} g^{(k)}(a)\right| \leq \left|\frac{1}{n!}\int_a^y (y-u)^n g^{(n+1)}(u)\, du\right| := E_n(y), \tag{3.26}$$

where

$$E_n(y) \leq \begin{cases} \frac{(y-a)^{n+1}}{(n+1)!}\left\|g^{(n+1)}\right\|_\infty, & g^{(n+1)} \in L_\infty[a, y], \\ \frac{(y-a)^{n+\frac{1}{q}}}{n!(nq+1)^{\frac{1}{q}}}\left\|g^{(n+1)}\right\|_p, & g^{(n+1)} \in L_p[a, y] \\ & \text{with } p > 1,\ \frac{1}{p} + \frac{1}{q} = 1, \\ \frac{(y-a)^n}{n!}\left\|g^{(n+1)}\right\|_1, & g^{(n+1)} \in L_1[a, y], \end{cases} \tag{3.27}$$

for $y \geq a$ and $y \in I \subset \mathbb{R}$.

Now, it may readily be noticed that if $x = a$ in (3.25), then the classical result as given by (3.26) is regained. As discussed in Remark 5 the bounds are convex so that a coarse bound is obtained at the end points and the best at the midpoint.

Thus, taking $x = \frac{a+y}{2}$ gives

$$\left| g(y) - g(a) - \sum_{k=1}^{n} \frac{\left[1 + (-1)^{k-1}\right]}{k!} 2^{-k} (y-a)^k g^{(k)}\left(\frac{a+y}{2}\right) \right| \tag{3.28}$$

$$\leq e_n\left(\frac{a+y}{2}, y\right)$$

$$= \begin{cases} \frac{\left\|g^{(n+1)}\right\|_\infty}{(n+1)!} 2^{-n} (y-a)^{n+1}, & g^{(n+1)} \in L_\infty[a,y], \\ \frac{\left\|g^{(n+1)}\right\|_p}{n!(nq+1)^{\frac{1}{q}}} 2^{-n} (y-a)^{n+\frac{1}{q}}, & g^{(n+1)} \in L_p[a,y] \\ & \text{with } p > 1,\ \frac{1}{p} + \frac{1}{q} = 1, \\ \frac{\left\|g^{(n+1)}\right\|_1}{n!} \left(\frac{3}{4}\right)^n (y-a)^n, & g^{(n+1)} \in L_1[a,y]. \end{cases}$$

The above inequalities (3.28) show that for $g \in C^\infty[a,b]$ the series

$$g(a) + \sum_{k=1}^{\infty} \frac{\left[1 + (-1)^{k-1}\right]}{k! 2^k} (y-a)^k g^{(k)}\left(\frac{a+y}{2}\right)$$

converges more rapidly to $g(y)$ than the usual one

$$\sum_{k=0}^{\infty} \frac{(y-a)^k}{k!} g^{(k)}(a),$$

which comes from Taylor's expansion (3.26). It should further be noted that (3.27) only involves the odd derivatives of $g(\cdot)$ evaluated at the midpoint of the interval under consideration.

Remark 11. *If $\alpha(x) = \beta(x) = x$ in (3.24), then for any $x \in [a,y]$*

$$\left| g(y) - g(a) - \sum_{k=1}^{n} \frac{1}{k!} \left[(x-a)^k g^{(k)}(a) + (-1)^{k-1} (y-x)^k g^{(k)}(y)\right] \right| \tag{3.29}$$

$$\leq e_n(x,y),$$

where $e_n(x,y)$ is as defined by (3.25). See Cerone et al. [**10**] *for related results.*

4. Perturbed Rules Through Grüss Type Inequalities

In 1935, G. Grüss (see for example [**32**]), proved the following integral inequality which gives an approximation for the integral of a product in terms of the product of integrals.

Theorem 3. *Let $f, g : [a,b] \to \mathbb{R}$ be two integrable mappings so that $\phi \leq h(x) \leq \Phi(x)$ and $\gamma \leq g(x) \leq \Gamma$ for all $x \in [a,b]$, where $\phi, \Phi, \gamma, \Gamma$ are real numbers. Then we have*

$$|T(h,g)| \leq \frac{1}{4} (\Phi - \phi)(\Gamma - \gamma), \tag{4.1}$$

where

$$T(h,g) = \frac{1}{b-a} \int_a^b h(x) g(x)\, dx - \frac{1}{b-a} \int_a^b h(x)\, dx \cdot \frac{1}{b-a} \int_a^b g(x)\, dx \tag{4.2}$$

and the inequality is sharp, in the sense that the constant $\frac{1}{4}$ cannot be replaced by a smaller one.

For a simple proof of this fact as well as for extensions, generalisations, discrete variants and other associated material, see [**32**], and the papers [**6**], [**11**], [**13**], [**15**] and [**22**] where further references are given.

A *premature* Grüss inequality is embodied in the following theorem which was proved in the paper [**29**]. It provides a sharper bound than the above Grüss inequality. The term *premature* is used to denote the fact that the result is obtained from not completing the proof of the Grüss inequality if one of the functions is known explicitly. See also [**2**] for further details.

THEOREM 4. *Let h, g be integrable functions defined on $[a,b]$ and let $d \le g(t) \le D$. Then*

$$|T(h,g)| \le \frac{D-d}{2}[T(h,h)]^{\frac{1}{2}}, \tag{4.3}$$

where $T(h,g)$ is as defined in (4.2).

The above Theorem 4 will now be used to provide a perturbed generalised three point rule.

4.1. Perturbed Rules From Premature Inequalities. We start with the following result.

THEOREM 5. *Let $f : [a,b] \to \mathbb{R}$ be such that the derivative $f^{(n-1)}$, $n \ge 1$ is absolutely continuous on $[a,b]$. Assume that there exist constants $\gamma, \Gamma \in \mathbb{R}$ such that $\gamma \le f^{(n)}(t) \le \Gamma$ a.e. on $[a,b]$. Then the following inequality holds*

$$\begin{aligned} |\rho_n(x)| \quad : \quad &= \left| \int_a^b f(t)\,dt - \sum_{k=1}^{n} \frac{1}{k!}\left[R_k(x) f^{(k-1)}(x) + S_k(x)\right] \right. \\ &\quad \left. - (-1)^n \frac{\theta_n(x)}{n+1} \cdot \frac{f^{(n-1)}(b) - f^{(n-1)}(a)}{b-a} \right| \\ &\le \frac{\Gamma-\gamma}{2} \cdot \frac{1}{n!} I(x,n) \\ &\le \frac{\Gamma-\gamma}{\sqrt{2}} \cdot \frac{n}{(n+1)!} \cdot \frac{(b-a)^{n+1}}{\sqrt{2n+1}}, \end{aligned} \tag{4.4}$$

where

$$\begin{aligned} I(x,n) &= \frac{1}{(n+1)\sqrt{2n+1}} \left\{ n^2 (b-a) \hat{Q}_n(2,x) \right. \\ &\quad \left. + (2n+1) \sum_{\substack{i=1 \\ j>i}}^{4} z_i z_j \left[z_i^n - (-z_j)^n\right]^2 \right\} \end{aligned} \tag{4.5}$$

$$\begin{aligned} Z &= \{\alpha(x) - a, x - \alpha(x), \beta(x) - x, b - \beta(x)\}, \ z_i \in Z, \ i = 1, ..., 4, \\ \hat{Q}_n(\cdot, x) &= (2n+1) Q_n(\cdot, x) \ \text{ with } Q_n(\cdot, x) \text{ being as defined in (3.2)}, \\ \theta_n(x) &= (-1)^n z_1^{n+1} + z_2^{n+1} + (-1)^n z_3^{n+1} + z_4^{n+1}, \end{aligned}$$

and $R_k(x)$, $S_k(x)$ are as given by (2.3).

PROOF. Applying the premature Grüss result (4.3) by associating $f^{(n)}(t)$ with $g(t)$ and $h(t)$ with $K_n(x,t)$, from (2.2) gives

$$\begin{aligned} & \left|(-1)^n \int_a^b K_n(x,t) f^{(n)}(t)\,dt \right. \\ & \left. - \left((-1)^n \int_a^b K_n(x,t)\,dt\right) \frac{f^{(n-1)}(b) - f^{(n-1)}(a)}{b-a}\right| \\ \le\ & (b-a)\frac{\Gamma-\gamma}{2}\left[T(K_n,K_n)\right]^{\frac{1}{2}}, \end{aligned} \tag{4.6}$$

where from (4.2)

$$T(K_n,K_n) = \frac{1}{b-a}\int_a^b K_n^2(x,t)\,dt - \left(\frac{1}{b-a}\int_a^b K_n(x,t)\,dt\right)^2.$$

Now, from (2.2),

$$\begin{aligned} & \frac{1}{b-a}\int_a^b K_n(x,t)\,dt \\ =\ & \frac{1}{b-a}\left[\int_a^x \frac{(t-\alpha(x))^n}{n!}dt + \int_x^b \frac{(t-\beta(x))^n}{n!}dt\right] \\ =\ & \frac{1}{(b-a)(n+1)!}\left[(x-\alpha(x))^{n+1} + (-1)^n(\alpha(x)-a)^{n+1}\right. \\ & \left.(b-\beta(x))^{n+1} + (-1)^n(\beta(x)-x)^{n+1}\right] \\ :=\ & \frac{1}{(b-a)(n+1)!}\theta_n(x) \end{aligned} \tag{4.7}$$

and

$$\begin{aligned} & \frac{1}{b-a}\int_a^b K_n^2(x,t)\,dt \\ =\ & \frac{1}{(b-a)(n!)^2}\left[\int_a^x (t-\alpha(x))^{2n}\,dt + \int_x^b (t-\beta(x))^{2n}\,dt\right] \\ =\ & \frac{1}{(b-a)(n!)^2(2n+1)}\left[(x-\alpha(x))^{2n+1} + (\alpha(x)-a)^{2n+1}\right. \\ & \left.(b-\beta(x))^{2n+1} + (\beta(x)-x)^{2n+1}\right] \\ =\ & \frac{1}{(b-a)(n!)^2(2n+1)}\hat{Q}_n(2,x) \end{aligned} \tag{4.8}$$

on using (3.2).

Hence, substitution of (4.7) and (4.8) into (4.6) gives

$$\begin{aligned} & \left|\int_a^b K_n(x,t) f^{(n)}(t)\,dt - (-1)^n \frac{\theta_n(x)}{(n+1)!}\cdot\frac{f^{(n-1)}(b) - f^{(n-1)}(a)}{b-a}\right| \\ \le\ & \frac{\Gamma-\gamma}{2}\cdot\frac{1}{n!}J(x,n), \end{aligned} \tag{4.9}$$

where

$$(2n+1)(n+1)^2 J^2(x,n) = (n+1)^2 (b-a)\hat{Q}_n(2,x) - (2n+1)\theta_n^2(x). \tag{4.10}$$

Now, let

$$A = \alpha(x) - a,\ X = x - \alpha(x),\ Y = \beta(x) - x \text{ and } B = b - \beta(x), \tag{4.11}$$

then (4.7) and (4.8) imply that

$$\hat{Q}_n(2,x) = A^{2n+1} + X^{2n+1} + Y^{2n+1} + B^{2n+1}$$

and

$$\theta_n(x) = (-1)^n A^{n+1} + X^{n+1} + (-1)^n Y^{n+1} + B^{n+1}.$$

Hence, from (4.10) and using the fact that $b-a = A+X+Y+B$,

$$\begin{aligned} &(n+1)^2 (b-a)\hat{Q}_n(2,x) - (2n+1)\theta_n^2(x) \\ = \ & n^2\hat{Q}_n(2,x) + (2n+1)\left[(A+X+Y+B)Q_n(2,x) - \theta_n^2(x)\right] \\ = \ & n^2\hat{Q}_n(2,x) + (2n+1)\sum_{\substack{i=1\\ j>i}}^{4} z_i z_j \left[z_i^n - (-z_j)^n\right]^2 \end{aligned} \tag{4.12}$$

after some straight forward algebra, where $Z = \{A, X, Y, B\}$, $z_i \in Z$, $i = 1, ..., 4$.

Substitution of (4.12) into (4.10) gives $I(x,n) = \frac{J(x,n)}{(n+1)\sqrt{2n+1}}$ as presented by (4.5). Utilising identity (2.1) in (4.6) gives (4.4) and the first part of the theorem is proved.

The upper bound is obtained by taking $\alpha(\cdot)$, $\beta(\cdot)$, x at their end points since $I(x,n)$ is convex and symmetric. The second term for $I(x,n)$ is then zero and $\hat{Q}_n(2,x) < 2(b-a)^{2n+1}$ and hence after some simplification, the theorem is completely proven. ∎

Corollary 8. *Let the conditions of Theorem 3 hold. Then the following result is valid,*

$$\begin{aligned} &\left|\int_a^b f(t)\,dt - \sum_{k=1}^{n} \frac{2^{-k}}{k!}\left[r_k(x) f^{(k-1)}(x) + s_k(x)\right] \right. \\ &\left. -2^{-n}\left[1+(-1)^n\right](A^n + B^n)\cdot\frac{f^{(n-1)}(b) - f^{(n-1)}(a)}{b-a}\right| \\ \le\ & \frac{\Gamma-\gamma}{2}\cdot\frac{2^{-2(n+1)}}{n!}\left\{\left[4n^2 + \left(1+(-1)^{n-1}\right)(2n+1)\right]\left[A^{2(n+1)} + B^{2(n+1)}\right]\right. \\ &\left. + \left[4n^2 + 2(2n+1)\right]AB\left(A^{2n} + B^{2n}\right) + 4(2n+1)(-1)^{n-1}(A-B)^{n+1}\right\}, \end{aligned} \tag{4.13}$$

where $r_m(x)$ and $s_m(x)$ are as given by (3.18) and $A = x - a$, $B = b - x$.

Proof. Let $\alpha(x) = \frac{a+x}{2}$ and $\beta(x) = \frac{x+b}{2}$ in (4.4), readily giving the left hand side of (4.13). Now, for the right hand side. Taking $A = x - a$, $B = b - x$, we have

$$\hat{Q}_n(2,x) = 2^{-2n}\left[A^{2n+1} + B^{2n+1}\right]$$

and

$$\sum_{\substack{i=1\\j>i}}^{4} z_i z_j \left[z_i^n - (-z_j)^n\right]^2 = 2^{-2(n+1)}\left[A^{2(n+1)} + B^{2(n+1)}\right]\left(1 + (-1)^{n-1}\right) + 2AB\left(A^n + (-1)^{n-1} B^n\right)^2$$

so that from (4.5) and using the fact that $b - a = A + B$,

$$\begin{aligned}(4.14)\quad & (n+1)\sqrt{2n+1} I(x,n) \\ = \; & 2^{-2(n+1)}\Big\{4n^2 (A+B)\left[A^{2n+1} + B^{2n+1}\right] \\ & + (2n+1)\Big[\left(A^{2(n+1)} + B^{2(n+1)}\right)\left(1 + (-1)^{n-1}\right) \\ & + 2AB\left(A^n + (-1)^{n-1} B^n\right)^2\Big]\Big\} \\ = \; & 2^{-2(n+1)}\Big\{\left[4n^2 + \left(1 + (-1)^{n-1}\right)(2n+1)\right]\left[A^{2(n+1)} + B^{2(n+1)}\right] \\ & + \left[4n^2 + 2(2n+1)\right] AB\left(A^{2n} + B^{2n}\right) + 4(2n+1)(-1)^{n-1}(AB)^{n+1}\Big\}.\end{aligned}$$

A simple substitution in (4.4) of (4.14) completes the proof. ∎

COROLLARY 9. *Let the conditions of Theorem 3 and Corollary 8 hold. Then the following inequality results,*

$$\begin{aligned}(4.15)\quad & \Bigg|\int_a^b f(t)\,dt - \sum_{k=1}^{n} \frac{2^{-k}}{k!}\left[r_k\left(\frac{a+b}{2}\right) f^{(k-1)}\left(\frac{a+b}{2}\right) + s_k\left(\frac{a+b}{2}\right)\right] \\ & -2 \cdot 4^{-n}\left(1 + (-1)^n\right)\left[f^{(n-1)}(b) - f^{(n-1)}(a)\right]\Bigg| \\ \le \; & \frac{\Gamma - \gamma}{n!}\left(\frac{b-a}{4}\right)^{2(n+1)}\left[8n^2 + 3(2n+1)\left(1 + (-1)^{n-1}\right)\right],\end{aligned}$$

where $r_m\left(\frac{a+b}{2}\right)$ and $s_m\left(\frac{a+b}{2}\right)$ are as given in (3.20).

PROOF. The proof follows directly from (4.13) with $x = \frac{a+b}{2}$ so that $A = B = \frac{b-a}{2}$, giving for the braces on the right hand side

$$2\left(\frac{b-a}{4}\right)^{2(n+1)}\left[8n^2 + 3(2n+1)\left(1 + (-1)^{n-1}\right)\right].$$

Some straight forward simplification produces the result (4.15). ∎

REMARK 12. *It may be noticed (See also Remark 8) that only even order degrees are involved, in (4.15), at the midpoint while this is only the case at the endpoints if the further restriction $f^{(k-1)}(a) = f^{(k-1)}(b)$ is imposed. Further, if n is odd, then there is no perturbation arising from the Grüss type result (4.15).*

THEOREM 6. *Let the conditions of Theorem 3 be satisfied. Further, suppose that $f^{(n)}$ is absolutely continuous and is such that*

$$\left\|f^{(n+1)}\right\|_\infty := ess \sup_{t\in[a,b]}\left|f^{(n+1)}(t)\right| < \infty.$$

Then

$$|\rho_n(x)| \leq \frac{b-a}{\sqrt{12}} \left\| f^{(n+1)} \right\|_\infty \cdot \frac{1}{n!} I(x,n), \tag{4.16}$$

where $\rho_n(x)$ *is the perturbed interior point rule given by the left hand side of (4.4) and* $I(x,n)$ *is as given by (4.5).*

Proof. Let $h, g : [a,b] \to \mathbb{R}$ be absolutely continuous and h', g' be bounded. Then Chebychev's inequality holds (see [**34**, p. 207])

$$|T(h,g)| \leq \frac{(b-a)^2}{\sqrt{12}} \sup_{t\in[a,b]} |h'(t)| \cdot \sup_{t\in[a,b]} |g'(t)|.$$

Matić, Pečarić and Ujević [**29**] using a *premature* Grüss type argument proved that

$$|T(h,g)| \leq \frac{(b-a)}{\sqrt{12}} \sup_{t\in[a,b]} |g'(t)| \sqrt{T(h,h)}. \tag{4.17}$$

Thus, associating $f^{(n)}(\cdot)$ with $g(\cdot)$ and $K(x,\cdot)$, from (2.2), with $h(\cdot)$ in (4.17) produces (4.16) where $I(x,n)$ is as given by (4.5). ∎

Theorem 7. *Let the conditions of Theorem 3 be satisfied. Further, suppose that* $f^{(n)}$ *is locally absolutely continuous on* (a,b) *and let* $f^{(n+1)} \in L_2(a,b)$. *Then*

$$|\rho_n(x)| \leq \frac{b-a}{\pi} \left\| f^{(n+1)} \right\|_2 \cdot \frac{1}{n!} I(x,n), \tag{4.18}$$

where $\rho_n(x)$ *is the perturbed generalised interior point rule given by the left hand side of (4.4) and* $I(x,n)$ *is as given in (4.5).*

Proof. The following result was obtained by Lupaş (see [**34**, p. 210]). For $h, g : (a,b) \to \mathbb{R}$ locally absolutely continuous on (a,b) and $h', g' \in L_2(a,b)$, then

$$|T(h,g)| \leq \frac{(b-a)^2}{\pi^2} \|h'\|_2 \|g'\|_2,$$

where

$$\|k\|_2 := \left(\frac{1}{b-a} \int_a^b |k(t)|^2 \right)^{\frac{1}{2}} \quad \text{for} \quad k \in L_2(a,b).$$

Matić, Pečarić and Ujević [**29**] further show that

$$|T(h,g)| \leq \frac{b-a}{\pi} \|g'\|_2 \sqrt{T(h,h)}. \tag{4.19}$$

Now, associating $f^{(n)}(\cdot)$ with $g(\cdot)$ and $K(x,\cdot)$, from (2.2) with $h(\cdot)$ in (4.19) gives (4.18) where $I(x,n)$ is as found in (4.5). ∎

Remark 13. *Results (4.16) and (4.18) are not readily comparable to that obtained in Theorem 3 since the bound now involves the behaviour of* $f^{(n+1)}(\cdot)$ *rather than* $f^{(n)}(\cdot)$.

Remark 14. *Premature results presented in this section may also be obtained, producing bounds for generalized Taylor-like series expansion by taking* $f \equiv g'$ *and* $b = y$. *See also Matić et al.* [**29**] *for related results.*

5. Applications in Numerical Integration

Any of the inequalities in Sections 3 and 4 may be utilised for numerical implementation. Here we illustrate the procedure by giving details for the implementation of Corollary 4.

Consider the partition $I_m : a = x_0 < x_1 < ... < x_{m-1} < x_m = b$ of the interval $[a,b]$ and let the intermediate points $\boldsymbol{\xi} = (\xi_0, ..., \xi_{m-1})$ where $\xi_j \in [x_j, x_{j+1}]$ for $j = 0, 1, ..., m-1$. Define the formula for $\gamma \in [0,1]$,

$$\begin{aligned} \mathcal{A}_{m,n}(f, I_m, \boldsymbol{\xi}) &= \sum_{j=0}^{m-1}\sum_{k=1}^{n} \frac{(-1)^k}{k!}\Big\{ (1-\gamma)^k r_k(\xi_j) f^{(k-1)}(\xi_j) \\ &\quad + \gamma^k \left[A_j^k f^{(k-1)}(x_j) + (-1)^{k-1} B_j^k f^{(k-1)}(x_{j+1})\right] \Big\}, \end{aligned} \tag{5.1}$$

where

$$\begin{cases} r_k(\xi_j) = B_j^k + (-1)^{k-1} A_j^k \\ A_j = \xi_j - x_j, \quad B_j = x_{j+1} - \xi_j, \\ \text{and} \quad h_j = A_j + B_j = x_{j+1} - x_j \quad \text{for} \quad j = 0, 1, ..., m-1. \end{cases} \tag{5.2}$$

The following theorem holds involving (5.1).

THEOREM 8. *Let $f : [a,b] \to \mathbb{R}$ be a mapping such that $f^{(n-1)}$ is absolutely continuous on $[a,b]$ and I_m be a partition of $[a,b]$ as described above. Then the following quadrature rule holds. Namely,*

$$\int_a^b f(x)\,dx = \mathcal{A}_{m,n}(f, I_m, \boldsymbol{\xi}) + \mathcal{R}_{m,n}(f, I_m, \boldsymbol{\xi}), \tag{5.3}$$

where $\mathcal{A}_{m,n}$ is as defined by (5.1)-(5.2) and the remainder $\mathcal{R}_{m,n}(f, I_m, \boldsymbol{\xi})$ satisfies the estimation

$$|\mathcal{R}_{m,n}(f, I_m, \boldsymbol{\xi})| \tag{5.4}$$

$$\leq \begin{cases} \frac{\|f^{(n)}\|_\infty}{(n+1)!} H_1(\gamma) \sum_{j=0}^{m-1} \left(A_j^{n+1} + B_j^{n+1}\right), & \text{for} \quad f^{(n)} \in L_\infty[a,b], \\ \frac{\|f^{(n)}\|_p}{n!} \frac{H_q(\gamma)}{(nq+1)^{\frac{1}{q}}} \left[\sum_{j=0}^{m-1} \left(A_j^{nq+1} + B_j^{nq+1}\right)\right]^{\frac{1}{q}}, & \text{for} \quad f^{(n)} \in L_p[a,b], \\ & p > 1,\ \frac{1}{p} + \frac{1}{q} = 1, \\ \frac{\|f^{(n)}\|_1}{n!} \left(\frac{1}{2} + \left|\gamma - \frac{1}{2}\right|\right)^n \times & \\ \quad \left[\frac{\nu(h)}{2} + \max_{j=0,...,m-1}\left|\xi_j - \frac{x_j + x_{j+1}}{2}\right|\right]^n, & \text{for} \quad f^{(n)} \in L_1[a,b], \end{cases}$$

where $H_q(\gamma)$ is given by (3.18), $\nu(h) = \max\{h_j | j = 0, ..., m-1\}$, and the rest of the terms are as given in (5.2).

PROOF. Apply Corollary 4 on the interval $[x_j, x_{j+1}]$ to give

$$\left|\int_{x_j}^{x_{j+1}} f(t)\,dt - \sum_{k=1}^{n} \frac{(-1)^k}{k!}\left\{ (1-\gamma)^k r_k(\xi_j) f^{(k-1)}(\xi_j) + \gamma^k \left[A_j^k f^{(k-1)}(x_j) + (-1)^{k-1} B_j^k f^{(k-1)}(x_{j+1})\right]\right\}\right| \tag{5.5}$$

$$\leq \begin{cases} \frac{H_1(\gamma)}{(n+1)!} \sup_{t\in[x_j,x_{j+1}]} \left|f^{(n)}(t)\right| \left(A_j^{n+1} + B_j^{n+1}\right), \\ \\ \frac{H_q(\gamma)}{n!} \left[\int_{x_j}^{x_{j+1}} \left|f^{(n)}(u)\right|^p du\right]^{\frac{1}{p}} \left(\frac{A_j^{nq+1}+B_j^{nq+1}}{nq+1}\right)^{\frac{1}{q}}, \\ \\ \frac{\left(\frac{1}{2}+\left|\gamma-\frac{1}{2}\right|\right)^n}{n!} \left[\int_{x_j}^{x_{j+1}} \left|f^{(n)}(u)\right| du\right] \left(\frac{h_j}{2} + \left|\xi_j - \frac{x_j+x_{j+1}}{2}\right|\right)^n, \end{cases}$$

where the parameters are as defined in (5.2) and $H_q(\gamma)$ is as given in (3.18). Summing over j from 0 to $m-1$ and using the generalised triangle inequality gives

$$\left|\mathcal{R}_{m,n}(f, I_m, \boldsymbol{\xi})\right| \tag{5.6}$$

$$\leq \begin{cases} \frac{H_1(\gamma)}{(n+1)!} \sum_{j=0}^{m-1} \sup_{t\in[x_j,x_{j+1}]} \left|f^{(n)}(t)\right| \left(A_j^{n+1} + B_j^{n+1}\right), \\ \\ \frac{H_q(\gamma)}{n!} \sum_{j=0}^{m-1} \left(\int_{x_j}^{x_{j+1}} \left|f^{(n)}(u)\right|^p du\right)^{\frac{1}{p}} \left(\frac{A_j^{nq+1}+B_j^{nq+1}}{nq+1}\right)^{\frac{1}{q}}, \\ \\ \frac{\left(\frac{1}{2}+\left|\gamma-\frac{1}{2}\right|\right)^n}{n!} \sum_{j=0}^{m-1} \left(\int_{x_j}^{x_{j+1}} \left|f^{(n)}(u)\right| du\right) \left(\frac{h_j}{2} + \left|\xi_j - \frac{x_j+x_{j+1}}{2}\right|\right)^n. \end{cases}$$

Now, since $\sup_{t\in[x_j,x_{j+1}]} \left|f^{(n)}(t)\right| \leq \left\|f^{(n)}\right\|_\infty$, the first inequality in (5.4) readily follows.

Further, using the discrete Hölder inequality, we have

$$\begin{aligned} &\sum_{j=0}^{m-1} \left(\int_{x_j}^{x_{j+1}} \left|f^{(n)}(u)\right|^p du\right)^{\frac{1}{p}} \left(\frac{A_j^{nq+1}+B_j^{nq+1}}{nq+1}\right)^{\frac{1}{q}} \\ \leq\ & \left(\frac{1}{nq+1}\right)^{\frac{1}{q}} \left[\sum_{j=0}^{m-1} \left[\left(\int_{x_j}^{x_{j+1}} \left|f^{(n)}(u)\right|^p du\right)^{\frac{1}{p}}\right]^p\right]^{\frac{1}{p}} \\ & \times \left[\sum_{j=0}^{m-1} \left[\left(A_j^{nq+1} + B_j^{nq+1}\right)^{\frac{1}{q}}\right]^q\right]^{\frac{1}{q}} \\ =\ & \frac{\left\|f^{(n)}\right\|_p}{(nq+1)^{\frac{1}{q}}} \left[\sum_{j=0}^{m-1} \left(A_j^{nq+1} + B_j^{nq+1}\right)\right]^{\frac{1}{q}} \end{aligned}$$

and thus the second inequality in (5.4) is proven.

Finally, let us observe from (5.5) that

$$\begin{aligned}
&\sum_{j=0}^{m-1}\left(\int_{x_j}^{x_{j+1}}\left|f^{(n)}(u)\right|du\right)\left(\frac{h_j}{2}+\left|\xi_j-\frac{x_j+x_{j+1}}{2}\right|\right)^n \\
\leq\ & \max_{j=0,\dots,m-1}\left(\frac{h_j}{2}+\left|\xi_j-\frac{x_j+x_{j+1}}{2}\right|\right)^n\sum_{j=0}^{m-1}\int_{x_j}^{x_{j+1}}\left|f^{(n)}(u)\right|du \\
\leq\ & \left(\frac{\nu(h)}{2}+\max_{j=0,\dots,m-1}\left|\xi_j-\frac{x_j+x_{j+1}}{2}\right|\right)^n\left\|f^{(n)}\right\|_1 .
\end{aligned}$$

Hence, the theorem is completely proved. ∎

Remark 15. *Following the discussion in Remark 5, coarser upper bounds to those in (5.4) are obtained by taking ξ_j at either extremity of its interval, giving*

$$\frac{\left\|f^{(n)}\right\|_\infty}{(n+1)!}H_1(\gamma)\sum_{j=0}^{m-1}h_j^{n+1},\quad \frac{\left\|f^{(n)}\right\|_p}{n!}\frac{H_q(\gamma)}{(nq+1)^{\frac{1}{q}}}\left(\sum_{j=0}^{m-1}h_j^{nq+1}\right)^{\frac{1}{q}},$$
$$\frac{\left\|f^{(n)}\right\|_1}{n!}\nu^n(h)$$

for $f^{(n)}$ belonging to the obvious $L_p[a,b]$, $1\leq p\leq\infty$. These are uniform bounds relative to the intermediate points $\boldsymbol{\xi}$.

Corollary 10. *Let the conditions of Theorem 8 hold. Then we have*

$$\int_a^b f(x)\,dx=\mathcal{A}_{m,n}(f,I_m)+\mathcal{R}_{m,n}(f,I_m),$$

where

$$\begin{aligned}
\mathcal{A}_{m,n}(f,I_m) &= \sum_{j=0}^{m-1}\sum_{k=1}^{n}\frac{(-1)^k}{k!}\Big\{(1-\gamma)^k r_k(\delta_j)f^{(k-1)}(\delta_j) \\
&\quad +\gamma^k h_j^k\left[f^{(k-1)}(x_j)+(-1)^{k-1}f^{(k-1)}(x_{j+1})\right]\Big\},
\end{aligned}$$

with

$$\delta_j=\frac{x_j+x_{j+1}}{2}\quad\text{and}\quad r_k(\delta_j)=\frac{h_j^k}{2}\left(1+(-1)^{k-1}\right)$$

and the remainder $\mathcal{R}_{m,n}(f,I_m)$ satisfies the inequality

$$\begin{aligned}
&|\mathcal{R}_{m,n}(f,I_m)| \\
\leq\ & \begin{cases}
\frac{\left\|f^{(n)}\right\|_\infty}{(n+1)!}2\sum_{j=0}^{m-1}\left(\frac{h_j}{2}\right)^k, & \text{for } f^{(n)}\in L_\infty[a,b], \\
\frac{\left\|f^{(n)}\right\|_p}{n!}\frac{H_q(\gamma)}{(nq+1)^{\frac{1}{q}}}\left[\sum_{j=0}^{m-1}2\left(\frac{h_j}{2}\right)^{nq+1}\right]^{\frac{1}{q}}, & \text{for } f^{(n)}\in L_p[a,b] \\
& p>1,\ \frac{1}{p}+\frac{1}{q}=1, \\
\frac{\left\|f^{(n)}\right\|_1}{n!}\left(\frac{1}{2}+\left|\gamma-\frac{1}{2}\right|\right)^n\left(\frac{\nu(h)}{2}\right)^n, & \text{for } f^{(n)}\in L_1[a,b].
\end{cases}
\end{aligned}$$

Proof. The proof is trivial from Theorem 8. Taking $\xi_j = \frac{x_j + x_{j+1}}{2}$ gives $A_j = B_j = \frac{h_j}{2}$ and the results stated follow. ∎

6. Concluding Remarks

Taking $\gamma = 0$ in Corollary 1 gives a generalised Ostrowski type identity which has bounds given by Corollary 4 with $\gamma = 0$, reproducing the results of Cerone and Dragomir [**4**]. This gives a coarse upper bound as discussed in Remark 5 since the bound is convex and symmetric in both γ and x. Let the identity be denoted by $M_n(x)$ which is produced from taking a Peano kernel of

$$k_M(x,t) = \begin{cases} \frac{(t-a)^n}{n!}, & t \in [a,x] \\ \frac{(t-b)^n}{n!}, & t \in (x,b]. \end{cases} \tag{6.1}$$

Further, taking $\gamma = 1$ in Corollary 1 gives a generalised Trapezoidal type identity with bounds given by Corollary 4 with $\gamma = 1$ reproducing the results of Cerone and Dragomir [**3**]. This choice of γ again gives the coarsest bound as discussed in Remark 5. Let the resulting identity be denoted by $T_n(x)$, which results from taking a Peano kernel of

$$k_T(x,t) = \frac{(x-t)^n}{n!}. \tag{6.2}$$

It was shown in Cerone and Dragomir [**4**] that $\|k_M(x,t)\|_q = \|k_T(x,t)\|_q$. Let

$$I_L(x) = \lambda M_n(x) + (1-\lambda) T_n(x)$$

which is obtained from the kernel

$$k(x,t) = \lambda k_M(x,t) + (1-\lambda) k_T(x,t)$$

where $k_M(x,t)$ and $k_T(x,t)$ are given by (6.1) and (6.2) respectively. The best one can do for $q > 1$, $q \neq 2$ with such a kernel when determining bounds is to use the triangle inequality and so

$$\begin{aligned} \|k(x,t)\|_q &\le \lambda \|k_M(x,t)\|_q + (1-\lambda)\|k_T(x,t)\|_q \\ &= \|k_M(x,t)\|_q = \|k_T(x,t)\|_q. \end{aligned} \tag{6.3}$$

The results thus obtained would be given by, for $f^{(n)} \in L_p[a,b]$, $p \ge 1$,

$$\left| \int_a^b f(t)\,dt - \lambda \sum_{k=1}^{n} (-1)^k r_k(x) f^{(k-1)}(x) - (1-\lambda) \sum_{k=1}^{n} (-1)^k s_k(x) \right|$$

$$\le \begin{cases} \frac{\|f^{(n)}\|_\infty}{(n+1)!} G_1(x), & \text{for } f^{(n)} \in L_\infty[a,b], \\ \frac{\|f^{(n)}\|_p}{n!(nq+1)^{\frac{1}{q}}} G_q(x), & \text{for } f^{(n)} \in L_p[a,b], \\ & p > 1,\ \frac{1}{p} + \frac{1}{q} = 1, \\ \frac{\|f^{(n)}\|_1}{n!} \left[\frac{b-a}{2} + \left| x - \frac{a+b}{2} \right| \right], & \text{for } f^{(n)} \in L_1[a,b], \end{cases}$$

where $r_k(x)$, $s_k(x)$, $G_q(x)$ are given by (3.18) and it should be noted that the bound is independent of λ. This result would be no more difficult to implement than (3.17) in Corollary 4, with the best bounds resulting from $\gamma = \frac{1}{2}$. For $q = 1$, 2 or infinity, $\|k(x,t)\|_q$ may be evaluated explicitly without using the triangle inequality at which stage comparison with the results of Corollary 4 would be less conclusive. This will not be discussed further here.

In the application of the current work to quadrature, if we wished to approximate the integral $\int_a^b f(x)\,dx$ using a rule $Q(f, I_m)$ with bound $E(m)$, where I_m is a uniform partition for example, with an accuracy of $\varepsilon > 0$, then we require $m_\varepsilon \in \mathbb{N}$ where

$$m_\varepsilon \geq \left[E^{-1}(\varepsilon)\right] + 1,$$

with $[w]$ denoting the integer part of $w \in \mathbb{R}$.

The approach thus described enables the user to predetermine the partition required to *assure* the result to be within a certain error tolerance. This approach is somewhat different from that commonly used of systematic mesh refinement followed by a comparison of successive approximations which forms the basis of a stopping rule. See [**1**], [**26**] and [**28**] for a comprehensive treatment of traditional methods.

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School of Communications and Informatics, Victoria University of Technology, PO Box 14428, Melbourne City MC, Victoria 8001, Australia
E-mail address: pc@matilda.vu.edu.au

E-mail address: sever.dragomir@vu.edu.au
URL: http://melba.vu.edu.au/~rgmia/SSDragomirWeb.html