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AN IMPROVEMENT OF THE REMAINDER ESTIMATE IN THE GENERALISED TAYLOR FORMULA

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ABSTRACT. In this note we point out an estimate for the remainder in the generalised Taylor formula which improves the recent result by Matić, Pečarić and Ujević [2].

1. INTRODUCTION

In the recent paper [2], M. Matić, J. E. Pečarić and N. Ujević proved the following generalised Taylor formula.

Theorem 1. *Let $\{P_n\}_{n \in \mathbb{N}}$ be a harmonic sequence of polynomials, that is, $P'_n(t) = P_{n-1}(t)$ for $n \geq 1$, $n \in \mathbb{N}$, $P_0(t) = 1$, $t \in \mathbb{R}$. Further, let $I \subset \mathbb{R}$ be a closed interval and $a \in I$. If $f : I \rightarrow \mathbb{R}$ is a function such that for some $n \in \mathbb{N}$, $f^{(n)}$ is absolutely continuous, then*

$$(1.1) \quad f(x) = \tilde{T}_n(f; a, x) + \tilde{R}_n(f; a, x), \quad x \in I,$$

where

$$(1.2) \quad \tilde{T}_n(f; a, x) = f(a) + \sum_{k=1}^n (-1)^{k+1} \left[P_k(x) f^{(k)}(x) - P_k(a) f^{(k)}(a) \right]$$

and

$$(1.3) \quad \tilde{R}_n(f; a, x) = (-1)^n \int_a^x P_n(t) f^{(n+1)}(t) dt.$$

For some particular instances of harmonic sequences, they obtained the following Taylor-like expansions:

$$(1.4) \quad f(x) = T_n^{(M)}(f; a, x) + R_n^{(M)}(f; a, x), \quad x \in I,$$

where

$$(1.5) \quad T_n^{(M)}(f; a, x) = f(a) + \sum_{k=1}^n \frac{(x-a)^k}{2^k k!} \left[f^{(k)}(a) + (-1)^{k+1} f^{(k)}(x) \right],$$

$$(1.6) \quad R_n^{(M)}(f; a, x) = \frac{(-1)^n}{n!} \int_a^x \left(t - \frac{a+x}{2} \right)^n f^{(n+1)}(t) dt;$$

and

$$(1.7) \quad f(x) = T_n^{(B)}(f; a, x) + R_n^{(B)}(f; a, x), \quad x \in I,$$

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where

$$(1.8) \quad T_n^{(B)}(f; a, x) = f(a) + \frac{x-a}{2} [f'(x) + f'(a)] - \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \frac{(x-a)^{2k}}{(2k)!} B_{2k} [f^{(2k)}(x) - f^{(2k)}(a)],$$

and $[r]$ is the integer part of r . Here, B_{2k} are the Bernoulli numbers, and

$$(1.9) \quad R_n^{(B)}(f; a, x) = (-1)^n \frac{(x-a)^n}{n!} \int_a^x B_n \left(\frac{t-a}{x-a} \right) f^{(n+1)}(t) dt,$$

where $B_n(\cdot)$ are the Bernoulli polynomials, respectively.

In addition, they proved that

$$(1.10) \quad f(x) = T_n^{(E)}(f; a, x) + R_n^{(E)}(f; a, x), \quad x \in I,$$

where

$$(1.11) \quad T_n^{(E)}(f; a, x) = f(a) + 2 \sum_{k=1}^{\lfloor \frac{n+1}{2} \rfloor} \frac{(x-a)^{2k-1} (4^k - 1)}{(2k)!} B_{2k} [f^{(2k-1)}(x) + f^{(2k-1)}(a)]$$

and

$$(1.12) \quad R_n^{(E)}(f; a, x) = (-1)^n \frac{(x-a)^n}{n!} \int_a^x E_n \left(\frac{t-a}{x-a} \right) f^{(n+1)}(t) dt,$$

where $E_n(\cdot)$ are the Euler polynomials.

In [1], S.S. Dragomir was the first author to introduce the perturbed Taylor formula

$$(1.13) \quad f(x) = T_n(f; a, x) + \frac{(x-a)^{n+1}}{(n+1)!} [f^{(n)}; a, x] + G_n(f; a, x),$$

where

$$(1.14) \quad T_n(f; a, x) = \sum_{k=0}^n \frac{(x-a)^k}{k!} f^{(k)}(a)$$

and

$$[f^{(n)}; a, x] := \frac{f^{(n)}(x) - f^{(n)}(a)}{x-a};$$

and had the idea to estimate the remainder $G_n(f; a, x)$ by using Grüss and Chebyshev type inequalities.

In [2], the authors generalised and improved the results from [1]. We mention here the following result obtained via a pre-Grüss inequality (see [2, Theorem 3]).

Theorem 2. *Let $\{P_n\}_{n \in \mathbb{N}}$ be a harmonic sequence of polynomials. Let $I \subset \mathbb{R}$ be a closed interval and $a \in I$. Suppose $f : I \rightarrow \mathbb{R}$ is as in Theorem 1. Then for all $x \in I$ we have the perturbed generalised Taylor formula:*

$$(1.15) \quad f(x) = \tilde{T}_n(f; a, x) + (-1)^n [P_{n+1}(x) - P_{n+1}(a)] [f^{(n)}; a, x] + \tilde{G}_n(f; a, x).$$

For $x \geq a$, the remainder $\tilde{G}(f; a, x)$ satisfies the estimate

$$(1.16) \quad \left| \tilde{G}_n(f; a, x) \right| \leq \frac{x-a}{2} \sqrt{T(P_n, P_n)} [\Gamma(x) - \gamma(x)],$$

provided that $f^{(n+1)}$ is bounded and

$$(1.17) \quad \Gamma(x) := \sup_{t \in [a, x]} f^{(n+1)}(t) < \infty, \quad \gamma(x) := \inf_{t \in [a, x]} f^{(n+1)}(t) > -\infty,$$

where $T(\cdot, \cdot)$ is the Chebychev functional on the interval $[a, x]$, that is, we recall

$$(1.18) \quad T(g, h) := \frac{1}{x-a} \int_a^x g(t) h(t) dt - \frac{1}{x-a} \int_a^x g(t) dt \cdot \frac{1}{x-a} \int_a^x h(t) dt.$$

The main aim of the present note is to improve the inequality (1.16) as follows.

2. THE RESULTS

The following result holds.

Theorem 3. Assume that $\{P_n\}_{n \in \mathbb{N}}$ is a sequence of harmonic polynomials and $f : I \rightarrow \mathbb{R}$ is such that $f^{(n)}$ is absolutely continuous and $f^{(n+1)} \in L_2(I)$. If $x \geq a$, then we have the inequality

$$(2.1) \quad \left| \tilde{G}_n(f; a, x) \right| \leq (x-a) [T(P_n, P_n)]^{\frac{1}{2}} \left[\frac{1}{x-a} \left\| f^{(n+1)} \right\|_2^2 - \left([f^{(n)}; a, x] \right)^2 \right]^{\frac{1}{2}} \left(\leq \frac{x-a}{2} [T(P_n, P_n)]^{\frac{1}{2}} [\Gamma(x) - \gamma(x)], \quad \text{if } f^{(n+1)} \in L_\infty[a, x] \right),$$

where $\|\cdot\|_2$ is the usual Euclidean norm on $[a, x]$, i.e.,

$$\left\| f^{(n+1)} \right\|_2 = \left(\int_a^x \left| f^{(n+1)}(t) \right|^2 dt \right)^{\frac{1}{2}}.$$

Proof. Recall Korkine's identity for the mappings h, g , which can be easily proved by direct computation:

$$(2.2) \quad T(h, g) := \frac{1}{2(x-a)^2} \int_a^x \int_a^x (h(t) - h(s))(g(t) - g(s)) dt ds.$$

Using (2.2), we have

$$(2.3) \quad \begin{aligned} & (-1)^n \int_a^x P_n(t) f^{(n+1)}(t) dt - \frac{(-1)^n}{x-a} \int_a^x f^{(n+1)}(t) dt \cdot \int_a^x P_n(t) dt \\ &= \frac{(-1)^n}{2} \cdot \frac{1}{x-a} \int_a^x \int_a^x (P_n(t) - P_n(s)) \left(f^{(n+1)}(t) - f^{(n+1)}(s) \right) dt ds \end{aligned}$$

and then, by the equalities (1.1) and (1.15), we have the following representation of the remainder in the perturbed formula (1.15)

$$(2.4) \quad \begin{aligned} & \tilde{G}_n(f; a, x) \\ &= \frac{(-1)^n}{2} \cdot \frac{1}{x-a} \int_a^x \int_a^x (P_n(t) - P_n(s)) \left(f^{(n+1)}(t) - f^{(n+1)}(s) \right) dt ds, \end{aligned}$$

which is an identity that is interesting in itself as well. Using now the Cauchy-Buniakowsky-Schwartz integral inequality for double integrals, we have

$$\begin{aligned}
& \left| \tilde{G}_n(f; a, x) \right| \\
& \leq \frac{1}{2(x-a)} \left[\int_a^x \int_a^x (P_n(t) - P_n(s))^2 dt ds \right]^{\frac{1}{2}} \\
& \quad \times \left[\int_a^x \int_a^x \left(f^{(n+1)}(t) - f^{(n+1)}(s) \right)^2 dt ds \right]^{\frac{1}{2}} \\
& = (x-a) [T(P_n, P_n)]^{\frac{1}{2}} \left[\frac{1}{x-a} \|f^{(n+1)}\|_2^2 - \left(\frac{1}{x-a} \int_a^x f^{(n+1)}(t) dt \right)^2 \right]^{\frac{1}{2}},
\end{aligned}$$

and the first inequality in (2.1) is proved.

The second inequality is obvious by the Grüss inequality

$$\begin{aligned}
(2.5) \quad & \frac{1}{x-a} \int_a^x \left[f^{(n+1)}(t) \right]^2 dt - \left(\frac{1}{x-a} \int_a^x f^{(n+1)}(t) dt \right)^2 \\
& \leq \frac{1}{4} [\Gamma(x) - \gamma(x)]^2,
\end{aligned}$$

and the theorem is proved. ■

Remark 1. If $f^{(n+1)}$ is unbounded on (a, x) but $f^{(n+1)} \in L_2(a, x)$, then the first inequality in (2.1) can still be applied, but not the Matić-Pečarić-Ujević result (1.16) which requires the boundedness of the derivative $f^{(n+1)}$.

The following corollary improves Corollary 3 of [2], which deals with the estimation of the remainder for the particular perturbed Taylor-like formulae (1.4), (1.7) and (1.10).

Corollary 1. With the assumptions in Theorem 3, we have the following inequalities

$$(2.6) \quad \left| \tilde{G}_n^{(M)}(f; a, x) \right| \leq \frac{(x-a)^{n+1}}{n!2^n\sqrt{2n+1}} \times \sigma\left(f^{(n+1)}; a, x\right),$$

$$(2.7) \quad \left| \tilde{G}_n^{(B)}(f; a, x) \right| \leq (x-a)^{n+1} \left[\frac{|B_{2n}|}{(2n)!} \right]^{\frac{1}{2}} \times \sigma\left(f^{(n+1)}; a, x\right),$$

$$\begin{aligned}
(2.8) \quad & \left| \tilde{G}_n^{(E)}(f; a, x) \right| \\
& \leq 2(x-a)^{n+1} \left[\frac{(4^{n+1}-1)|B_{2n+2}|}{(2n+2)!} - \left[\frac{2(2^{n+2}-1)B_{n+2}}{(n+1)!} \right]^2 \right]^{\frac{1}{2}} \\
& \quad \times \sigma\left(f^{(n+1)}; a, x\right),
\end{aligned}$$

and

$$(2.9) \quad |G_n(f; a, x)| \leq \frac{n(x-a)^{n+1}}{(n+1)!\sqrt{2n+1}} \times \sigma\left(f^{(n+1)}; a, x\right),$$

where, as in [2],

$$\begin{aligned}\tilde{G}_n^{(M)}(f; a, x) &= f(x) - T_n^M(f; a, x) - \frac{(x-a)^{n+1} [1 + (-1)^n]}{(n+1)! 2^{n+1}} [f^{(n)}; a, x]; \\ \tilde{G}_n^{(B)}(f; a, x) &= f(x) - T_n^B(f; a, x); \\ \tilde{G}_n^{(E)}(f; a, x) &= f(x) - \frac{4(-1)^n (x-a)^{n+1} (2^{n+2} - 1) B_{n+2}}{(n+2)!} [f^{(n)}; a, x], \\ G_n(f; a, x) &\text{ is as defined by (1.13),}\end{aligned}$$

$$(2.10) \quad \sigma(f^{(n+1)}; a, x) := \left[\frac{1}{x-a} \|f^{(n+1)}\|_2^2 - \left([f^{(n+1)}; a, x] \right)^2 \right]^{\frac{1}{2}},$$

and $x \geq a$, $f^{(n+1)} \in L_2[a, x]$.

Note that for all the examples considered in [1] and [2] for f , the quantity $\sigma(f^{(n+1)}; a, x)$ can be completely computed and then those particular inequalities may be improved accordingly. We omit the details.

Remark 2. *Theorem 3 is an implicit refinement of Theorem 4 from [2] which uses Chebychev's inequality to estimate $\sigma(f^{(n+1)}; a, x)$ and of Theorem 5 from [2] which uses Lupaş's inequality to upper bound the same term $\sigma(f^{(n+1)}; a, x)$.*

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