On Some Inequalities for the Expectation and Variance

ON SOME INEQUALITIES FOR THE EXPECTATION AND VARIANCE

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Abstract. A variety of bounds are obtained for the variance and expectation of a continuous random variable whose p.d.f. is defined over a finite interval. Previous results are shown to be particular cases of the current more general development.

1. Introduction

Let $f : [a,b] \to \mathbb{R}$ be the p.d.f. of the random variable T and M_n be the moments about the origin so that

$$M_n := \int_a^b t^n f(t)\,dt. \tag{1.1}$$

Further, the *expectation* of the random variable T

$$E(T) := M_1 \tag{1.2}$$

and the *variance* $\sigma^2(T)$ is defined as the second moment about the expectation so that

$$\sigma^2(T) := \int_a^b (t - E(T))^2 f(t)\,dt, \tag{1.3}$$

giving, on simplification and using (1.1),

$$\sigma^2(T) = M_2 - M_1^2. \tag{1.4}$$

Based on the identity

$$\sigma^2(T) + [x - E(T)]^2 = \int_a^b (x-t)^2 f(t)\,dt, \tag{1.5}$$

Barnett et al. [2] obtained a variety of bounds on the left hand side of (1.5). Bounds involving higher order derivatives were obtained in [2], by substituting a Taylor series expansion for $f(t)$ in (1.5), in terms of the $L_p[a,b]$ norms of the resulting double integral.

Barnett and Dragomir [3] obtained further results for the variance based on the identity

$$\sigma^2(T) + (E(T) - b)(E(T) - a) = \int_a^b (t-a)(t-b) f(t)\,dt. \tag{1.6}$$

The aim of this paper is to obtain a variety of bounds for the variance from an identity which regains (1.5) and (1.6) as special cases. Premature Grüss, Chebychev

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and Lupaş results are also obtained. Further, substitution of a Taylor expansion with integral remainder allows bounds to be obtained for the situation in which the p.d.f. is n-time differentiable. Taking a convex combination of expansions about two separate points allows for further generalisations and a number of novel results.

2. Integral Identities

Lemma 1. *Let $f:[a,b]\to\mathbb{R}$ be a p.d.f. of the random variable T. Then the following integral identity holds, involving the variance and expectation*

$$\sigma^2(T)+(E(T)-\alpha)(E(T)-\beta)=\int_a^b(t-\alpha)(t-\beta)f(t)\,dt, \tag{2.1}$$

where $\alpha,\ \beta\in[a,b]$ and $\alpha<\beta$.

Proof. A simple expansion gives

$$\int_a^b(t-\alpha)(t-\beta)f(t)\,dt=\int_a^b\left[t^2-(\alpha+\beta)t+\alpha\beta\right]f(t)\,dt,$$

which, upon using (1.4) together with (1.1), (1.2) and the fact that $f(\cdot)$ is a p.d.f. over $[a,b]$, gives

$$\begin{aligned}\int_a^b(t-\alpha)(t-\beta)f(t)\,dt &= \sigma^2(T)+M_1^2-(\alpha+\beta)M_1+\alpha\beta \\ &= \sigma^2(T)+(M_1-\alpha)(M_1-\beta)\end{aligned} \tag{2.2}$$

and hence (2.1) results on using (1.2). ∎

Remark 1. *If we take $\alpha=\beta=x$, then identity (1.5) is recaptured from (2.1). If further, $x=E(T)$, then (1.3) results. Taking $\alpha=a$ and $\beta=b$ in (2.1) gives the identity (1.6).*

Lemma 2. *Let T be a random variable whose p.d.f. $f:[a,b]\to\mathbb{R}$ is n-time differentiable and $f^{(n)}$ is absolutely continuous on $[a,b]$. Then the following identity holds for $z\in[a,b]$*

$$\begin{aligned}&\sigma^2(T)+(E(T)-\alpha)(E(T)-\beta)\\ &=\sum_{k=0}^{n}\left[U_{k+3}(b-z)-U_{k+3}(a-z)\right]\frac{f^{(k)}(z)}{k!}+R_{n+1}(z),\end{aligned} \tag{2.3}$$

where

$$\begin{aligned}&U_{r+1}(u)\\ &=\frac{u^{r-1}}{r(r^2-1)}\left\{r(r-1)u^2+2\left(r^2-1\right)\left[z-\frac{\alpha+\beta}{2}\right]u+r(r+1)\alpha\beta\right\}\end{aligned} \tag{2.4}$$

and

$$R_{n+1}(z)=\frac{1}{n!}\int_a^b(t-\alpha)(t-\beta)\rho_n(t,z)\,dt \tag{2.5}$$

with

$$\rho_n(t,z)=\int_z^t(t-s)^n f^{(n+1)}(s)\,ds. \tag{2.6}$$

Proof. Using Taylor's formula with integral remainder and expanding about $t = z$ gives

$$f(t) = \sum_{k=0}^{n} \frac{(t-z)^k}{k!} f^{(k)}(z) + \frac{1}{n!}\rho_n(t,z) \tag{2.7}$$

for all $t, z \in [a,b]$ with $\rho_n(t,z)$ being given by (2.6).

Substitution of (2.7) into (2.1) gives

$$\begin{aligned} &\sigma^2(T) + (E(T)-\alpha)(E(T)-\beta) \\ = &\int_a^b (t-\alpha)(t-\beta)\left\{\sum_{k=0}^{n} \frac{(t-z)^k}{k!} f^{(k)}(z) + \frac{1}{n!}\rho_n(t,z)\right\} dt \\ = &\sum_{k=0}^{n}\left[\int_a^b (t-\alpha)(t-\beta)(t-z)^k\,dt\right]\frac{f^{(k)}(z)}{k!} + R_{n+1}(z), \end{aligned} \tag{2.8}$$

where $R_{n+1}(z)$ is as given by (2.5).

Now

$$\begin{aligned} &\int_a^b (t-\alpha)(t-\beta)(t-z)^k\,dt \\ = &\int_{a-z}^{b-z} u^k(u+z-\alpha)(u+z-\beta)\,du \\ = &\int_{a-z}^{b-z} u^k\left[u^2 - 2\left[z - \frac{\alpha+\beta}{2}\right]u + \alpha\beta\right] du \\ = &\left.\frac{u^{k+3}}{k+3} - 2\left[z-\frac{\alpha+\beta}{2}\right]\frac{u^{k+2}}{k+2} + \alpha\beta\frac{u^{k+1}}{k+1}\right]_{a-z}^{b-z}, \end{aligned}$$

and therefore

$$\int_a^b (t-\alpha)(t-\beta)(t-z)^k\,dt = \left.U_{k+3}(u)\right]_{a-z}^{b-z}, \tag{2.9}$$

where, after some simplification, $U_{r+1}(u)$ is as given in (2.4). Substitution of (2.9) into (2.8) readily produces the result (2.3). ∎

Remark 2. *Taking $\alpha = \beta = z = x$ reproduces an identity obtained by Barnett et al.* [2]*. Placing $\alpha = a$ and $\beta = b$ with $z = x$ gives an n-time differentiable generalisation of identity (1.6) and is thus a generalisation of the result by Barnett and Dragomir* [3].

Lemma 3. *Let T be a random variable with p.d.f. $f : [a,b] \to \mathbb{R}$ being n-time differentiable and $f^{(n)}$ absolutely continuous on $[a,b]$. Then the following identity holds*

$$\begin{aligned} &\sigma^2(T) + (E(T)-\alpha)(E(T)-\beta) \\ = &\sum_{k=0}^{n}\Big\{\lambda\left[V_{k+3}(b-\alpha) - V_{k+3}(a-\alpha)\right] f^{(k)}(\alpha) \\ &+ (1-\lambda)\left[W_{k+3}(b-\beta) - W_{k+3}(a-\beta)\right] f^{(k)}(\beta)\Big\} \\ &+\lambda R_{n+1}(\alpha) + (1-\lambda) R_{n+1}(\beta), \end{aligned} \tag{2.10}$$

where

$$\begin{cases} V_{k+3}(u) = \frac{u^{k+2}}{(k+3)(k+2)}\left[(k+2)u-(\beta-\alpha)(k+3)\right], \\ \\ W_{k+3}(u) = \frac{u^{k+2}}{(k+3)(k+2)}\left[(k+2)u+(\beta-\alpha)(k+3)\right], \end{cases} \tag{2.11}$$

and $R_{n+1}(\cdot)$ is as given by (2.5).

Proof. From (2.6), on letting $z=\alpha$, we obtain

$$f(t) = \sum_{k=0}^{n} \frac{(t-\alpha)^k}{k!} f^{(k)}(\alpha) + \frac{1}{n!}\rho_n(t,\alpha), \tag{2.12}$$

where $\rho_n(t,\cdot)$ is as given in (2.6).

Additionally, taking $z=\beta$ in (2.6) produces

$$f(t) = \sum_{k=0}^{n} \frac{(t-\beta)^k}{k!} f^{(k)}(\beta) + \frac{1}{n!}\rho_n(t,\beta). \tag{2.13}$$

If we let $\lambda\in[0,1]$ and evaluate $\lambda\cdot(2.12)+(1-\lambda)\cdot(2.13)$, we obtain

$$\begin{aligned} f(t) &= \sum_{k=0}^{n}\left[\lambda p_k(t-\alpha) f^{(k)}(\alpha) + (1-\lambda) p_k(t-\beta) f^{(k)}(\beta)\right] \\ &\quad + \frac{\lambda}{n!}\rho_n(t,\alpha) + \frac{1-\lambda}{n!}\rho_n(t,\beta), \end{aligned} \tag{2.14}$$

where

$$p_k(u) = \frac{u^k}{k!} \tag{2.15}$$

and $\rho_n(t,\cdot)$ is as given by (2.6).

Substitution of (2.14) into (2.1) gives

$$\begin{aligned} &\sigma^2(T) + (E(T)-\alpha)(E(T)-\beta) \\ = &\int_a^b (t-\alpha)(t-\beta)\left\{\sum_{k=0}^{n}\left[\lambda p_k(t-\alpha) f^{(k)}(\alpha) + (1-\lambda) p_k(t-\beta) f^{(k)}(\beta)\right]\right. \\ &\left. + \frac{\lambda}{n!}\rho_n(t,\alpha) + \frac{1-\lambda}{n!}\rho_n(t,\beta)\right\} \\ = &\sum_{k=0}^{n}(k+1)\int_a^b \left[\lambda(t-\beta) p_{k+1}(t-\alpha) f^{(k)}(\alpha)\right. \\ &\left. + (1-\lambda)(t-\alpha) p_{k+1}(t-\beta) f^{(k)}(\beta)\right] dt + \lambda R_{n+1}(\alpha) + (1-\lambda) R_{n+1}(\beta), \end{aligned}$$

with $R_{n+1}(\cdot)$ as given by (2.5).

Now, using (2.15)

$$\begin{aligned} \int_a^b (t-\beta) p_{k+1}(t-\alpha)\, dt &= \frac{1}{(k+1)!}\int_a^b (t-\beta)(t-\alpha)^{k+1} dt \\ &= \frac{1}{(k+1)!}\int_{a-\alpha}^{b-\alpha} u^{k+1}\left[u-(\beta-\alpha)\right] du \end{aligned}$$

and so

$$\int_a^b (t-\beta)\, p_{k+1}(t-\alpha)\, dt = V_{k+3}(u)\Big]_{a-\alpha}^{b-\alpha},$$

where $V_{k+3}(u)$ is as given by (2.11).

Similarly, interchanging α and β,

$$\int_a^b (t-\alpha)\, p_{k+1}(t-\beta)\, dt = \frac{1}{(k+1)!}\int_{a-\beta}^{b-\beta} u^{k+1}\left[u+(\beta-\alpha)\right] du,$$

giving

$$\int_a^b (t-\alpha)\, p_{k+1}(t-\beta)\, dt = W_{k+3}(u)\Big]_{a-\beta}^{b-\beta},$$

where $W_{k+3}(u)$ is as given by (2.11).

The lemma is thus completely proved. ∎

Remark 3. *It may be noted that identity (2.10) is a generalisation of (2.4) if* $\alpha=\beta=z$.

3. Bounds Involving Lebesgue Norms of a Function and Premature Results

A number of bounds will now be derived using the identities developed in Section 2 in terms of a variety of norms. Here, $\|\cdot\|_p$, $1\le p\le\infty$ are the usual Lebesgue norms on $[a,b]$. Namely, $\|g\|_\infty := ess \sup_{t\in[a,b]} |g(t)|$ and $\|g\|_p := \left(\int_a^b |g(t)|^p\, dt\right)^{\frac{1}{p}}$, $1\le p<\infty$.

Theorem 1. *Let* $f:[a,b]\to\mathbb{R}_+$ *be the p.d.f. of the random variable* T. *Then*

$$\left|\sigma^2(T)+(E(T)-\alpha)(E(T)-\beta)\right| \tag{3.1}$$
$$\le \begin{cases} \left\{\frac{1}{3}\left[(\alpha-a)^3+(b-\beta)^3\right]+\frac{\beta-\alpha}{6}\left[3(\alpha-a)^2+(b-\beta)^2\right]\right\}\|f\|_\infty, \\ \text{for} \quad f\in L_\infty[a,b]; \\ \left[\psi_q(\alpha-a)+B_q(\beta-\alpha)+\psi_q(b-\beta)\right]^{\frac{1}{q}}\|f\|_p, \\ \text{if} \quad f\in L_p[a,b],\ p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \theta(a,\alpha,\beta,b)\|f\|_1, \quad f\in L_1[a,b], \end{cases}$$

where $\alpha,\beta\in[a,b]$ *and* $\alpha\le\beta$,

$$\psi_q(X)=\int_0^X u^q(u+\beta-\alpha)^q\, du, \quad B_q(X)=\int_0^X u^q(\beta-\alpha-u)^q\, du \tag{3.2}$$

and

$$\theta(a,\alpha,\beta,b)=\max\left\{(\alpha-a)(\beta-a),\left(\frac{\beta-\alpha}{2}\right)^2,(b-\alpha)(b-\beta)\right\}. \tag{3.3}$$

Proof. From identity (2.1), let

$$R_0(a,\alpha,\beta,b)=\int_a^b (t-\alpha)(t-\beta)f(t)\,dt \tag{3.4}$$

and thus taking the modulus gives

$$|R_0(a,\alpha,\beta,b)|\le \|f\|_\infty \int_a^b |(t-\alpha)(t-\beta)|\,dt. \tag{3.5}$$

Now

$$\begin{aligned}
&\int_a^b |(t-\alpha)(t-\beta)|\,dt \qquad (3.6)\\
=&\int_a^\alpha (\alpha-t)(\beta-t)\,dt+\int_\alpha^\beta (t-\alpha)(\beta-t)\,dt+\int_\beta^b (t-\alpha)(t-\beta)\,dt\\
=&\int_0^{\alpha-a} u(u+\beta-\alpha)\,du+\int_0^{\beta-\alpha} u(\beta-\alpha-u)\,du+\int_0^{b-\beta} u(u+\beta-\alpha)\,du\\
=&\frac{1}{3}\left[(\alpha-a)^3+(b-\beta)^3\right]+\frac{\beta-\alpha}{2}\left[(\alpha-a)^2+(b-\beta)^2\right]+\frac{(\beta-\alpha)^3}{6}.
\end{aligned}$$

A simple rearrangement of (3.6) and using (3.5) and (2.1) readily produces the first inequality in (3.1).

From (3.2), by Hölder's integral inequality, we obtain

$$\begin{aligned}
|R_0(a,\alpha,\beta,b)| &\le \|f\|_p \left(\int_a^b |(t-\alpha)(t-\beta)|^q\,dt\right)^{\frac{1}{q}} \qquad (3.7)\\
&:=\|f\|_p E_q^{\frac{1}{q}}(a,\alpha,\beta,b).
\end{aligned}$$

Then,

$$\begin{aligned}
&E_q(a,\alpha,\beta,b)\\
=&\int_a^\alpha (\alpha-t)^q(\beta-t)^q\,dt+\int_\alpha^\beta (t-\alpha)^q(\beta-t)^q\,dt+\int_\beta^b (t-\alpha)^q(t-\beta)^q\,dt\\
=&\int_0^{\alpha-a} [u(u+\beta-\alpha)]^q\,du+\int_0^{\beta-\alpha} [u(\beta-\alpha-u)]^q\,du\\
&+\int_0^{b-\beta} [u(u+\beta-\alpha)]^q\,du.
\end{aligned}$$

Hence, from (3.7), the second inequality in (3.1) results, where $\psi_q(\cdot)$ and $B_q(\cdot)$ are as defined in (3.2).

Now, for the last inequality in (3.1). From identity (2.1) and the inequality

$$|R_0(a,\alpha,\beta,b)|\le \sup_{t\in[a,b]} |(t-\alpha)(t-\beta)|\,\|f\|_1,$$

we have

$$\begin{aligned}
&\sup_{t\in[a,b]} |(t-\alpha)(t-\beta)| \\
= &\max\left\{ \sup_{t\in[a,\alpha)} (\alpha-t)(\beta-t)\,, \sup_{t\in(\alpha,\beta)} (t-\alpha)(\beta-t)\,, \sup_{t\in(\beta,b]} (t-\alpha)(t-\beta) \right\} \\
= &\max\left\{ (\alpha-a)(\beta-a)\,, \left(\frac{\beta-\alpha}{2}\right)^2, (b-\alpha)(b-\beta) \right\} \\
= &\theta(a,\alpha,\beta,b)
\end{aligned}$$

as given by (3.4) and hence the theorem is completely proved. ∎

Remark 4. *If $\alpha=\beta=x$ is taken in (3.1), then the results of Barnett et al.* [2] *based around the identity (1.5) are recaptured. In addition, if $x=E(T)$, then the bounds are on the variance alone. Taking $\alpha=a$ and $\beta=b$, the results of Barnett and Dragomir* [3] *are obtained. Some simplifications occur that have not as yet been developed, such as the result obtained from taking $\alpha=a$ and $\beta=x$.*

Remark 5. *The Euclidean norm is of special interest so that if $p=2$ and $f\in L_2[a,b]$, then from (3.1),*

$$\begin{aligned}
&\left|\sigma^2(T)+(E(T)-\alpha)(E(T)-\beta)\right| \\
\leq\ &\|f\|_2\left[\psi_2(\alpha-a)+B_2(\beta-\alpha)+\psi_2(b-\beta)\right]^{\frac{1}{2}},
\end{aligned}$$

where, from (3.2),

$$\psi_2(X)=\frac{X^3}{30}\left[6X^2+15(\beta-\alpha)X+10(\beta-\alpha)^2\right]$$

and

$$B_2(X)=\frac{X^3}{30}\left[6X^2-15(\beta-\alpha)X+10(\beta-\alpha)^2\right].$$

In addition, if we take $\alpha=\beta=x$, we obtain

$$\left|\sigma^2(T)+(E(T)-x)^2\right|\leq\frac{1}{\sqrt{5}}\left[(x-a)^5+(b-x)^5\right]^{\frac{1}{2}}\|f\|_2$$

and, for $x=E(T)$,

$$\sigma^2(T)\leq\frac{1}{\sqrt{5}}\left[(E(T)-a)^5+(b-E(T))^5\right]^{\frac{1}{2}}\|f\|_2.$$

Taking $\alpha=a$, $\beta=b$ gives

$$\left|\sigma^2(T)+(E(T)-a)(b-E(T))\right|\leq\frac{(b-a)^{\frac{5}{2}}}{\sqrt{30}}\|f\|_2.$$

A premature Grüss inequality is embodied in the following theorem. It provides a sharper bound than the Grüss inequality (see [5] for a statement of the Grüss inequality).

The term *premature* is used to denote the fact that the result is obtained from not completing the proof of the Grüss inequality if one of the functions is known explicitly. The following theorem was proven in [1].

Theorem 2. *Let h, g be integrable functions defined on $[a,b]$ and let $m \leq g(t) \leq M$. Then*

$$|T(h,g)| \leq \frac{M-m}{2} [T(h,h)]^{\frac{1}{2}}, \tag{3.8}$$

where the Chebychev functional,

$$T(h,g) = \mathcal{M}(hg) - \mathcal{M}(h)\mathcal{M}(g) \tag{3.9}$$

with

$$\mathcal{M}(f) = \frac{1}{b-a}\int_a^b f(x)\,dx. \tag{3.10}$$

Theorem 3. *Let $f : [a,b] \to \mathbb{R}_+$ be a p.d.f. of the random variable T be such that for $m \leq f \leq M$, then*

$$\begin{aligned} |\mathcal{T}_p| \quad : \quad &= \Bigg| \sigma^2(T) + (E(T)-\alpha)(E(T)-\beta) \\ &\quad - \left[\frac{(b-a)^2}{3} - \left(\frac{\alpha+\beta}{2} - a\right)(b-a) + (\alpha-a)(\beta-a)\right]\Bigg| \\ &\leq \frac{M-m}{2} I(a,\alpha,\beta,b), \end{aligned} \tag{3.11}$$

where

$$\begin{aligned} &I(a,\alpha,\beta,b) \\ &= \frac{(b-a)^2}{\sqrt{3}}\left[\frac{4}{15}(b-a)^2 - \left(\frac{\alpha+\beta}{2} - a\right)\left(b - \frac{\alpha+\beta}{2}\right)\right]^{\frac{1}{2}}. \end{aligned} \tag{3.12}$$

Proof. Applying the *premature* Grüss result (3.8) by associating $f(t)$ with $g(t)$ and taking

$$h(t) = (t-\alpha)(t-\beta) \tag{3.13}$$

gives, on noting that $\mathcal{M}(f) = \frac{1}{b-a}$ since f is a p.d.f.,

$$\begin{aligned} &\left|\int_a^b (t-\alpha)(t-\beta) f(t)\,dt - \mathcal{M}(h)\right| \\ &\leq (b-a)\frac{M-m}{2}[T(h,h)]^{\frac{1}{2}}, \end{aligned} \tag{3.14}$$

where, from (3.9),

$$T(h,h) = \mathcal{M}(h^2) - [\mathcal{M}(h)]^2. \tag{3.15}$$

Now, from (3.13) and (3.15)

$$\begin{aligned} \mathcal{M}(h) &= \tfrac{1}{b-a}\textstyle\int_a^b (t-\alpha)(t-\beta)\,dt \\ &= \tfrac{1}{D}\textstyle\int_0^D (u-A)(u-B)\,du, \end{aligned} \tag{3.16}$$

$$\text{where } \; u = t-a,\; D = b-a,\; A = \alpha - a,\; B = \beta - a.$$

That is,

$$\mathcal{M}(h) = \frac{D^2}{3} - \frac{A+B}{2}D + AB. \tag{3.17}$$

Further, following a similar argument to the above,

$$
\begin{aligned}
(3.18)\qquad \mathcal{M}\left(h^2\right) &= \frac{1}{D}\int_0^D (u-A)^2 (u-B)^2\,du \\
&= \frac{1}{D}\int_0^D \left[u^2-(A+B)\,u+AB\right]^2 du \\
&= \frac{1}{D}\int_0^D \left\{u^4+(A+B)^2 u^2+(AB)^2\right. \\
&\quad \left.+2\left[ABu^2-AB\,(A+B)\,u-(A+B)\,u^3\right]\right\}du \\
&= \frac{1}{D}\int_0^D \left\{u^4-2\,(A+B)\,u^3+\left[(A+B)^2+2AB\right]u^2\right. \\
&\quad \left.-AB\,(A+B)\,u+(AB)^2\right\}du \\
&= \frac{D^4}{5}-\frac{(A+B)}{2}D^3+\left[(A+B)^2+2AB\right]\frac{D^2}{3} \\
&\quad -AB\frac{(A+B)}{2}D+(AB)^2 .
\end{aligned}
$$

Thus, from (3.14), (3.16) and (3.17), we have, after some algebra

$$
T\,(h,h)=\frac{D^2}{3}\left[\frac{4}{15}D^2-\frac{A+B}{2}D+\left(\frac{A+B}{2}\right)^2\right].
$$

Using the definitions (3.15), the inequality (3.13) and the identity (2.1), gives the result (3.11) and, after some algebra, the theorem is thus proved. ∎

Remark 6. *Taking $\alpha=a$, $\beta=b$ in (3.11)-(3.12) recaptures the results obtained by Barnett and Dragomir* [3] *while allowing $\alpha=\beta=x$ reproduces the results in Barnett et al.* [2]. *Note from (3.12) that $I\,(a,\alpha,\beta,b)\le\frac{2(b-a)^3}{3\sqrt{5}}$. In addition, note that if $\frac{\alpha+\beta}{2}=\frac{a+b}{2}$ in (3.12), then*

$$
I\,(a,\alpha,\beta,b)=\frac{(b-a)^3}{6\sqrt{3}},
$$

which is 4 times better.

Theorem 4. *Let $f:[a,b]\to\mathbb{R}$ and suppose that $f\,(\cdot)$ is differentiable and is such that*

$$
\|f'\|_\infty:=\sup_{t\in[a,b]}|f'\,(t)|<\infty.
$$

Then

$$
(3.19)\qquad |\mathcal{T}_p|\le\frac{b-a}{\sqrt{12}}\,\|f'\|_\infty\,I\,(a,\alpha,\beta,b)\,,
$$

where $\mathcal{T}_p$ is the perturbed result given by the left hand side of (3.11) and $I\,(a,\alpha,\beta,b)$ is as given by (3.12).

Proof. Let $h,g:[a,b]\to\mathbb{R}$ be absolutely continuous and h', g' be bounded. Then Chebychev's inequality holds (see [5])

$$
|T\,(h,g)|\le\frac{(b-a)^2}{\sqrt{12}}\sup_{t\in[a,b]}|h'\,(t)|\cdot\sup_{t\in[a,b]}|g'\,(t)|\,.
$$

Matić et al. [1], using a *premature* Grüss type argument proved that

$$|T(h,g)| \leq \frac{b-a}{\sqrt{12}} \sup_{t\in[a,b]} |g'(t)| \sqrt{T(h,h)}.$$

Thus, associating $f(\cdot)$ with $g(\cdot)$ and $h(\cdot)$ with (3.13) produces (3.19) where $I(a,\alpha,\beta,b)$ is as given by (3.12). ∎

Theorem 5. *Let $f:[a,b]\to\mathbb{R}$ and suppose $[\alpha,\beta]\subseteq[a,b]$. Further, suppose that f is locally absolutely continuous on (a,b) and let $f'\in L_2(a,b)$. Then*

$$|\mathcal{T}_p| \leq \frac{b-a}{\pi} \|f'\|_2 I(a,\alpha,\beta,b), \tag{3.20}$$

where $\mathcal{T}_p$ is the perturbed result given by the left hand side of (3.11) and $I(a,\alpha,\beta,b)$ is as given by (3.12).

Proof. The following result was obtained by Lupaş (see [5]). For $h,g:(a,b)\to\mathbb{R}$ locally absolutely continuous on (a,b) and $h',g'\in L_2(a,b)$, then,

$$|T(h,g)| \leq \frac{(b-a)^2}{\pi^2} \|h'\|_2 \|g'\|_2,$$

where

$$\|k\|_2 := \left(\frac{1}{b-a}\int_a^b |k(t)|^2\,dt\right)^{\frac{1}{2}} \quad \text{for } k\in L_2(a,b).$$

Moreover, Matić et al. [1] showed that

$$|T(h,g)| \leq \frac{b-a}{\pi} \|g'\|_2 \sqrt{T(h,h)}.$$

Now, associating $f(\cdot)$ with $g(\cdot)$ and $h(\cdot)$ as given by (3.13) produces (3.20) where $I(a,\alpha,\beta,b)$ is as found in (3.12). ∎

4. Bounds Involving Lebesgue Norms of the $n-$th Derivative of a Function

In this section, bounds are obtained for $f^{(n)}\in L_p[a,b]$, $p\geq 1$ and n a non-negative integer.

Theorem 6. *Let T be a random variable whose p.d.f. $f:[a,b]\to\mathbb{R}$ is n-time differentiable and $f^{(n)}$ is absolutely continuous on $[a,b]$. The following inequalities* *hold for $z \in [a,b]$,*

$$
\begin{aligned}
(4.1)\ \mathcal{T}_n \quad : \quad &= \Bigg| \sigma^2(T) + (E(T)-\alpha)(E(T)-\beta) \\
&\quad - \sum_{k=0}^{n} \left[U_{k+3}(b-z) - U_{k+3}(a-z)\right] \frac{f^{(k)}(z)}{k!} \Bigg| \\
&\leq |R_{n+1}(z)| \\
&\leq \begin{cases}
\left[\phi_{n+1}(a,\alpha,z) - \phi_{n+1}(\alpha,\beta,z) + \phi_{n+1}(\beta,b,z)\right] \frac{\|f^{(n+1)}\|_\infty}{(n+1)!}, \\
f^{(n+1)} \in L_\infty[a,b]; \\
\left[\phi_{n+\frac{1}{q}}(a,\alpha,z) - \phi_{n+\frac{1}{q}}(\alpha,\beta,z) + \phi_{n+\frac{1}{q}}(\beta,b,z)\right] \frac{\|f^{(n+1)}\|_p}{n!(nq+1)^{\frac{1}{q}}}, \\
f^{(n+1)} \in L_p[a,b],\ p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\
\left[\phi_n(a,\alpha,z) - \phi_n(\alpha,\beta,z) + \phi_n(\beta,b,z)\right] \frac{\|f^{(n+1)}\|_1}{n!}, \\
f^{(n+1)} \in L_1[a,b],
\end{cases}
\end{aligned}
$$

where $U_{k+3}(\cdot)$ are as defined by (2.4),

$$
(4.2) \qquad \phi_{n+\gamma}(x_1,x_2,z) = \int_{x_1-z}^{x_2-z} |u|^{n+\gamma}(u+z-\alpha)(u+z-\beta)\,du, \quad x_1 \leq x_2.
$$

Proof. From identity (2.3), on taking the modulus, we have

$$
(4.3) \qquad \mathcal{T}_n = |R_{n+1}(z)|,
$$

where $R_{n+1}(z)$ is as given by (2.5) and (2.6).
Now

$$
\begin{aligned}
(4.4) \qquad & |R_{n+1}(z)| \\
&\leq \frac{1}{n!}\int_a^b |(t-\alpha)(t-\beta)\rho_n(t,z)|\,dt \\
&\leq \frac{1}{n!}\left\{\int_a^\alpha (\alpha-t)(\beta-t)|\rho_n(t,z)|\,dt + \int_\alpha^\beta (t-\alpha)(\beta-t)|\rho_n(t,z)|\,dt \right. \\
&\qquad \left. + \int_\beta^b (t-\alpha)(t-\beta)|\rho_n(t,z)|\,dt\right\}.
\end{aligned}
$$

Further, using properties relating to the modulus and integral, and Hölder's integral inequality, gives

$$\left|\rho_n(t,z)\right| \leq \begin{cases} \sup\limits_{s\in[z,t]} \left|f^{(n+1)}(s)\right| \left|\int_z^t |t-s|^n\, ds\right|, \\ \\ \left|\int_z^t \left|f^{(n+1)}(s)\right|^p ds\right|^{\frac{1}{p}} \left|\int_z^t |t-s|^{nq}\, ds\right|^{\frac{1}{q}}, \\ \\ |t-z|^n \left|\int_z^t \left|f^{(n+1)}(s)\right| ds\right| \end{cases}$$

and hence

$$\left|\rho_n(t,z)\right| \leq \begin{cases} \sup\limits_{s\in[z,t]} \left|f^{(n+1)}(s)\right| \frac{|t-z|^{n+1}}{n+1} \\ \\ \left|\int_z^t \left|f^{(n+1)}(s)\right|^p ds\right|^{\frac{1}{p}} \left(\frac{|t-z|^{nq+1}}{nq+1}\right)^{\frac{1}{q}}, \\ \\ \left|\int_z^t \left|f^{(n+1)}(s)\right| ds\right| |t-z|^n. \end{cases} \tag{4.5}$$

For $f^{(n+1)} \in L_\infty[a,b]$ using (4.5) and (4.4) gives

$$|R_{n+1}(z)| \leq \frac{\left\|f^{(n+1)}\right\|_\infty}{(n+1)!}\left[\phi_{n+3}(a,\alpha,z) - \phi_{n+3}(\alpha,\beta,z) + \phi_{n+3}(\beta,b,z)\right],$$

where

$$\phi_{n+1}(x_1,x_2,z) = \int_{x_1}^{x_2} (t-\alpha)(t-\beta)|t-z|^{n+1}\, dt,$$

which, on substitution of $u = t - z$, produces (4.2) with $\gamma = 1$ and so the first inequality in (4.1) is obtained.

Now, for the second inequality in (4.1). Substitution of the second inequality from (4.5) into (4.4) gives, after substitution of $u = t - z$,

$$|R_{n+1}(z)| \leq \frac{\left\|f^{(n+1)}\right\|_p}{n!\,(nq+1)^{\frac{1}{q}}}\left[\phi_{n+\frac{1}{q}}(a,\alpha,z) - \phi_{n+\frac{1}{q}}(\alpha,\beta,z) + \phi_{n+\frac{1}{q}}(\beta,b,z)\right],$$

where ϕ is as defined in (4.2).

Finally, the third inequality in (4.1) is obtained by placing the third inequality in (4.5) into (4.4). In the above, we have used the fact that the respective norms over any subinterval, as represented in (4.5), is less than or equal to the equivalent norm over $[a,b]$. ∎

Remark 7. *Result (4.1) is very general, containing three parameters α, β and z to be specified besides the degree of differentiability of the p.d.f. f.*

Perturbed results on $\mathcal{T}_n$ as defined by (4.1) will now be obtained.

Theorem 7. *Let $f : [a,b] \to \mathbb{R}_+$, a p.d.f. of the random variable T, be such that $d_{n+1} \le f^{(n+1)}(t) \le D_{n+1}$ for $t \in [a,b]$. Then*

$$\begin{aligned}
&\Bigg|\sigma^2(T) + (E(T)-\alpha)(E(T)-\beta) \\
&- \sum_{k=0}^{n} [U_{k+3}(b-z) - U_{k+3}(a-z)] \frac{f^{(k)}(z)}{k!} \\
&+ (-1)^n \mathcal{M}(h) \left[1 - \sum_{k=0}^{n} \frac{(b-z)^{k+1} + (-1)^k (x-a)^{k+1}}{(k+1)!}\right] f^{(k)}(z)\Bigg| \\
\le\; & \frac{\theta_n(z)}{2} \cdot I(a,\alpha,\beta,b), \qquad (4.6)
\end{aligned}$$

where

$$\mathcal{M}(h) = \tfrac{(b-a)^2}{3} - \left(\tfrac{\alpha+\beta}{2} - a\right)(b-a) + (\alpha-a)(\beta-a),$$

$U_{k+3}(\cdot)$ *are as defined in (2.4),*

$$I(a,\alpha,\beta,b) \text{ is as given by (3.12),} \qquad (4.7)$$

and

$$\theta_n(z) = \begin{cases} \frac{D_{n+1}}{(n+1)!}\left[(z-a)^{n+1} + (b-z)^{n+1}\right], & n \text{ even} \\ \\ \frac{1}{(n+1)!}\max\left\{(z-a)^{n+1} d_{n+1}, (b-z)^{n+1} D_{n+1}\right\}, & n \text{ odd} \end{cases}.$$

Proof. Applying the *premature* Grüss result (3.8) and associating $\frac{1}{n!}\rho_n(t,z)$ as given by (2.6) with $g(t)$ and taking $h(t)$ as defined in (3.13), gives

$$\begin{aligned}
&\left|\int_a^b (t-\alpha)(t-\beta)\frac{\rho_n(t,z)}{n!}dt - \mathcal{M}(h) \cdot \frac{1}{n!}\mathcal{M}(\rho_n(\cdot,z))\right| \\
\le\; & \frac{\Gamma(z) - \gamma(z)}{2}(b-a)[T(h,h)]^{\frac{1}{2}}, \qquad (4.8)
\end{aligned}$$

where $T(h,h)$ is as defined in (3.15) and

$$\gamma(z) \le \frac{\rho_n(t,z)}{n!} \le \Gamma(z) \text{ for } t \in [a,b]. \qquad (4.9)$$

Further, $\mathcal{M}(h)$ is as given by (3.17) with $A = \alpha - a$, $B = \beta - a$ and $D = b - a$.

Now,

$$\begin{aligned}
&\frac{(b-a)}{n!}\mathcal{M}(\rho_n(\cdot,z)) \\
=\; & \frac{1}{n!}\int_a^b \int_z^t (t-s)^n f^{(n+1)}(s)\,dsdt \\
=\; & \frac{1}{n!}\left[\int_a^z \int_z^t (t-s)^n f^{(n+1)}(s)\,dsdt + \int_z^b \int_z^t (t-s)^n f^{(n+1)}(s)\,dsdt\right] \\
=\; & \frac{1}{n!}\left[-\int_a^z \int_a^s (t-s)^n f^{(n+1)}(s)\,dtds + \int_z^b \int_s^b (t-s)^n f^{(n+1)}(s)\,dtds\right]
\end{aligned}$$

$$= \frac{1}{n!}\left[(-1)^{n+1}\int_a^z \frac{(s-a)^{n+1}}{n+1} f^{(n+1)}(s)\,ds + \int_z^b \frac{(b-s)^{n+1}}{n+1} f^{(n+1)}(s)\,ds\right]$$

$$= (-1)^{n+1}\left[\int_a^b f(t)\,dt - \sum_{k=0}^{n} \frac{(b-z)^{k+1} + (-1)^k (x-a)^{k+1}}{(k+1)!}\right] f^{(k)}(z), \tag{4.10}$$

where, to obtain the last result, we have used an identity obtained in Cerone et al. [6] (Lemma 2.1, equation (2.1)) involving an Ostrowski result for n-time differentiable functions.

We need to obtain the bounds on $\rho_n(t,z)$ for all $t \in [a,b]$. We are given that

$$d_{n+1} \le f^{(n+1)}(t) \le D_{n+1}. \tag{4.11}$$

For the case $t \ge z$, from (4.11) we have

$$d_{n+1}\int_z^t \frac{(t-s)^n}{n!}ds \le \frac{\rho_n(t,z)}{n!} \le D_{n+1}\int_z^t \frac{(t-s)^n}{n!}ds.$$

That is,

$$\frac{(t-z)^{n+1}}{(n+1)!}d_{n+1} \le \frac{\rho_n(t,z)}{n!} \le D_{n+1}\frac{(t-z)^{n+1}}{(n+1)!}, \quad t \in [z,b]$$

and so for $t \ge z$,

$$0 \le \frac{\rho_n(t,z)}{n!} \le D_{n+1}\frac{(b-z)^{n+1}}{(n+1)!}. \tag{4.12}$$

For the situation $t < z$, two separate cases need to be considered. Namely, whether n is even or odd.

From (4.11) we have

$$d_{n+1}\int_t^z \frac{(t-s)^n}{n!}ds \le -\frac{\rho_n(t,z)}{n!} \le D_{n+1}\int_t^z \frac{(t-s)^n}{n!}ds, \tag{4.13}$$

and so for n even

$$\begin{aligned}\frac{(z-t)^{n+1}}{(n+1)!}d_{n+1} &\le -\frac{\rho_n(t,z)}{n!} \le \frac{(z-t)^{n+1}}{(n+1)!}D_{n+1},\\ -\frac{(z-t)^{n+1}}{(n+1)!}D_{n+1} &\le \frac{\rho_n(t,z)}{n!} \le -\frac{(z-t)^{n+1}}{(n+1)!}d_{n+1}, \quad t \in [a,z]\end{aligned}$$

giving for any $t \le z$ and n even

$$-\frac{(z-a)^{n+1}}{(n+1)!}D_{n+1} \le \frac{\rho_n(t,z)}{n!} \le 0. \tag{4.14}$$

If n is odd, then from (4.13)

$$-\frac{(z-t)^{n+1}}{(n+1)!}d_{n+1} \le -\frac{\rho_n(t,z)}{n!} \le -\frac{(z-t)^{n+1}}{(n+1)!}D_{n+1},$$

giving

$$\frac{(z-t)^{n+1}}{(n+1)!}D_{n+1} \le \frac{\rho_n(t,z)}{n!} \le \frac{(z-t)^{n+1}}{(n+1)!}d_{n+1}, \quad t \in [a,z]$$

and so for $t < z$ and n odd

$$0 \leq \frac{\rho_n(t,z)}{n!} \leq \frac{(z-a)^{n+1}}{(n+1)!} d_{n+1}. \tag{4.15}$$

Thus, for n even, from (4.12) and (4.14) for all $t \in [a,b]$

$$-\frac{(z-a)^{n+1}}{(n+1)!} D_{n+1} \leq \frac{\rho_n(t,z)}{n!} \leq \frac{(b-z)^{n+1}}{(n+1)!} D_{n+1}. \tag{4.16}$$

For n odd, from (4.12) and (4.15) for all $t \in [a,b]$

$$0 \leq \frac{\rho_n(t,z)}{n!} \leq \frac{1}{(n+1)!} \max\left\{(z-a)^{n+1} d_{n+1}, (b-z)^{n+1} D_{n+1}\right\}. \tag{4.17}$$

Using (4.16) and (4.17) gives, from (4.8) and (4.9), $\theta_n(z) = \Gamma(z) - \gamma(z)$ as defined in (4.7). Substitution of identity (2.3) into (4.8) and using the fact that $I(a,\alpha,\beta,b) = (b-a)[T(h,h)]^{\frac{1}{2}}$, where h is as defined by (3.15), produces (4.6). We have further, in (4.10), used the fact that f is a p.d.f. ∎

Remark 8. *Chebychev and Lupaş of Theorems 3 and 4 could be obtained here in a straight forward fashion for the expressions on the left of (4.6). The bound would be different and involve the behaviour of $f^{(n+2)}(\cdot)$ instead of $f^{(n+1)}(\cdot)$. This however will not be pursued further.*

Theorem 8. *Let T be a random variable with p.d.f. $f:[a,b] \to \mathbb{R}$ being $n-$time differentiable and $f^{(n)}$ is absolutely continuous on $[a,b]$. The following inequality holds*

$$\begin{aligned} \kappa_n &:= \Bigg|\sigma^2(T) + (E(T)-\alpha)(E(T)-\beta) \\ &\quad - \sum_{k=0}^{n}\Bigg\{\lambda\left[V_{k+3}(b-\alpha) - V_{k+3}(a-\alpha)\right] f^{(k)}(\alpha) \\ &\quad + (1-\lambda)\left[W_{k+3}(b-\beta) - W_{k+3}(a-\beta)\right] f^{(k)}(\beta)\Bigg\}\Bigg| \end{aligned} \tag{4.18}$$

$$\leq \lambda\left|R_{n+1}(\alpha)\right| + (1-\lambda)\left|R_{n+1}(\beta)\right|$$

$$\leq \begin{cases} \left[\lambda Y_{n+1}(\alpha) + (1-\lambda) Y_{n+1}(\beta) + \zeta_{n+1}(\beta-\alpha)\right] \frac{\left\|f^{(n+1)}\right\|_\infty}{(n+1)!}, \\ f^{(n+1)} \in L_\infty[a,b]; \\ \left[\lambda Y_{n+\frac{1}{q}}(\alpha) + (1-\lambda) Y_{n+\frac{1}{q}}(\beta) + \zeta_{n+\frac{1}{q}}(\beta-\alpha)\right] \frac{\left\|f^{(n+1)}\right\|_p}{n!(nq+1)^{\frac{1}{q}}}, \\ f^{(n+1)} \in L_p[a,b],\ p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \left[\lambda Y_n(\alpha) + (1-\lambda) Y_n(\beta) + \zeta_n(\beta-\alpha)\right] \frac{\left\|f^{(n+1)}\right\|_1}{n!}, \\ f^{(n+1)} \in L_1[a,b]; \end{cases} \tag{4.19}$$

where $V_{k+3}(\cdot)$, $W_{k+3}(\cdot)$ are as defined in (2.11) and

$$Y_{n+\gamma}(\cdot) = A(\cdot - a) + B(b - \cdot) \tag{4.20}$$

with

$$\begin{cases} A(u) = \frac{u^{n+\gamma+2}}{(n+\gamma+3)(n+\gamma+2)}\left[(n+\gamma+2)u + (\beta-\alpha)(n+\gamma+3)\right], \\ \\ B(u) = \frac{u^{n+\gamma+2}}{(n+\gamma+3)(n+\gamma+2)}\left[(n+\gamma+2)u - (\beta-\alpha)(n+\gamma+3)\right], \\ \qquad\qquad \textit{and} \\ \qquad \zeta_{n+\gamma}(\beta-\alpha) = -2B(\beta-\alpha) = \frac{-2(\beta-\alpha)^{n+\gamma+2}}{(n+\gamma+3)(n+\gamma+2)}. \end{cases} \tag{4.21}$$

Proof. Rearranging identity (2.10) and using the triangle inequality produces inequality (4.18).

Now, from the right hand side of (4.1), let

$$\begin{aligned} X_{n+\gamma}(\alpha) &= \chi_{n+\gamma}(a,\alpha,\beta,b,\alpha) \\ &= \phi_{n+\gamma}(a,\alpha,\alpha) - \phi_{n+\gamma}(\alpha,\beta,\alpha) + \phi_{n+\gamma}(\beta,b,\alpha). \end{aligned} \tag{4.22}$$

From (4.2),

$$\begin{aligned} \phi_{n+\gamma}(a,\alpha,\alpha) &= \int_{a-\alpha}^{0} |u|^{n+\gamma} u(u-(\beta-\alpha))\,du \\ &= \int_{0}^{a-\alpha} u^{n+\gamma+1}(u+\beta-\alpha)\,du = A(\alpha-a), \\ \phi_{n+\gamma}(\alpha,\beta,\alpha) &= \int_{0}^{\beta-\alpha} u^{n+\gamma+1}(u-(\beta-\alpha))\,du = B(\beta-\alpha), \end{aligned}$$

and

$$\phi_{n+\gamma}(\beta,b,\alpha) = \int_{\beta-\alpha}^{b-\alpha} u^{n+\gamma+1}(u-(\beta-\alpha))\,du = B(b-\alpha) - B(\beta-\alpha).$$

Hence, substitution into (4.22) gives

$$X_{n+\gamma}(\alpha) = A(\alpha-a) - 2B(\beta-\alpha) + B(b-\alpha)$$

and so

$$X_{n+\gamma}(\alpha) = Y_{n+\gamma}(\alpha) + \zeta_{n+\gamma}(\beta-\alpha), \tag{4.23}$$

as defined in (4.20) and (4.21).

Again, from the right hand side of (4.1) and (4.20), let

$$\begin{aligned} X_{n+\gamma}(\beta) &= \chi_{n+\gamma}(a,\alpha,\beta,b,\beta) \\ &= \phi_{n+\gamma}(a,\alpha,\beta) - \phi_{n+\gamma}(\alpha,\beta,\beta) + \phi_{n+\gamma}(\beta,b,\beta). \end{aligned} \tag{4.24}$$

From (4.2)

$$\begin{aligned} \phi_{n+\gamma}(a,\alpha,\beta) &= \int_{a-\beta}^{\alpha-\beta} \left|u^{n+\gamma}\right| u(u+\beta-\alpha)\,du \\ &= \int_{\beta-a}^{\beta-\alpha} u^{n+\gamma+1}(\beta-\alpha-u)\,du \\ &= \int_{\beta-\alpha}^{\beta-a} u^{n+\gamma+1}(u-(\beta-\alpha))\,du \\ &= B(\beta-a) - B(\beta-\alpha), \end{aligned}$$

$$
\begin{aligned}
\phi_{n+\gamma}(\alpha,\beta,\beta) &= \int_{\alpha-\beta}^{0}\left|u^{n+\gamma}\right| u\left(u+\beta-\alpha\right) du \\
&= \int_{0}^{\beta-\alpha} u^{n+\gamma+1}\left(u-(\beta-\alpha)\right) du = B\left(\beta-\alpha\right)
\end{aligned}
$$

and

$$
\phi_{n+\gamma}(\beta,b,\beta) = \int_{0}^{b-\beta} |u|^{n+\gamma+1}\left(u+\beta-\alpha\right) du = A\left(b-\beta\right).
$$

Hence, substitution into (4.24) gives

$$
X_{n+\gamma}(\beta) = B(\beta-a) - 2B(\beta-\alpha) + A(b-\beta)
$$

and so

$$
X_{n+\gamma}(\beta) = Y_{n+\gamma}(\beta) + \zeta_{n+\gamma}(\beta-\alpha). \tag{4.25}
$$

On using (4.1) and (4.18), we have, from (4.23) and (4.25),

$$
\begin{aligned}
& \lambda X_{n+\gamma}(\alpha) + (1-\lambda) X_{n+\gamma}(\beta) \\
= \; & \lambda Y_{n+\gamma}(\alpha) + (1-\lambda) Y_{n+\gamma}(\beta) + \zeta_{n+\gamma}(\beta-\alpha)
\end{aligned}
$$

and so (4.19) is obtained for $\gamma = 1, \frac{1}{q}$ and 0 respectively. ∎

Remark 9. *Perturbed results on κ_n as defined in (4.19) may be obtained here in a similar fashion to those of Theorem 7. This, however will not be pursued further here.*

References

[1] M. MATIĆ, J.E. PEČARIĆ and N. UJEVIĆ, On new estimation of the remainder in generalized Taylor's formula, *Mathematical Inequalities and Applications*, **2** (3) (1999), 343-361.

[2] N.S. BARNETT, P. CERONE, S.S. DRAGOMIR and J. ROUMELIOTIS, Some inequalities for the dispersion of a random variable whose pdf is defined on a finite interval, *submitted.*

[3] N.S. BARNETT and S.S. DRAGOMIR, Some elementary inequalities for the expectation and variance of a random variable whose pdf is defined on a finite interval, *submitted.*

[4] N.S. BARNETT, P. CERONE, S.S. DRAGOMIR and J. ROUMELIOTIS, Some inequalities for the expectation and variance of a random variable whose pdf is n-time differentiable, *submitted.*

[5] D.S. MITRINOVIĆ, J.E. PEČARIĆ and A.M. FINK, *Classical and New Inequalities in Analysis,* Kluwer Academic Publishers, Dordrecht, 1993.

[6] P. CERONE, S.S. DRAGOMIR and J. ROUMELIOTIS, Some Ostrowski type inequalities for $n-$time differentiable mappings and applications, *Demonstratio Mathematica,* **XXXII** (4) (1999), 697-712.

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