Generalisations of the Grüss, Chebychev and Lupas inequalities for Integrals over Different Intervals

GENERALISATIONS OF THE GRÜSS, CHEBYCHEV AND LUPAŞ INEQUALITIES FOR INTEGRALS OVER DIFFERENT INTERVALS

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Abstract. Generalisations of the Grüss, Chebychev and Lupaş inequalities for integrals defined on two different intervals are given. Application in numerical analysis is demonstrated.

1. Introduction

For two measurable functions $f, g : [a, b] \to \mathbb{R}$, define the functional, which is known in the literature as Chebychev's functional

$$T(f, g; a, b) := \frac{1}{b-a} \int_a^b f(x) g(x) dx - \frac{1}{(b-a)^2} \int_a^b f(x) dx \cdot \int_a^b g(x) dx, \tag{1.1}$$

provided that the involved integrals exist.

The following inequality is well known in the literature as the Grüss inequality [9]

$$|T(f, g; a, b)| \leq \frac{1}{4} (M - m)(N - n), \tag{1.2}$$

provided that $m \leq f \leq M$ and $n \leq g \leq N$ a.e. on $[a, b]$, where m, M, n, N are real numbers. The constant $\frac{1}{4}$ in (1.2) is the best possible.

Another inequality of this type is due to Chebychev (see for example [1, p. 207]). Namely, if f, g are absolutely continuous on $[a, b]$ and $f', g' \in L_\infty [a, b]$ and $\|f'\|_\infty := ess \sup_{t \in [a,b]} |f'(t)|$, then

$$|T(f, g; a, b)| \leq \frac{1}{12} \|f'\|_\infty \|g'\|_\infty (b - a)^2 \tag{1.3}$$

and the constant $\frac{1}{12}$ is the best possible.

Finally, let us recall a result by Lupaş (see for example [1, p. 210]), which states that:

$$|T(f, g; a, b)| \leq \frac{1}{\pi^2} \|f'\|_2 \|g'\|_2 (b - a)^2, \tag{1.4}$$

provided f, g are absolutely continuous and $f', g' \in L_2 [a, b]$. The constant $\frac{1}{\pi^2}$ is the best possible here.

For other Grüss type inequalities, see the books [1] and [2], and the papers [3]-[10], where further references are given.

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In this note, we point out generalizations of the above results for integrals defined on two different intervals $[a,b]$ and $[c,d]$ and demonstrate their application in numerical analysis.

2. The Results

We define the functional (generalised Chebychev functional)

$$\begin{aligned}
&T(f,g;a,b,c,d) \\
:&= \frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx + \frac{1}{d-c}\int_c^d f(y)\,g(y)\,dy \\
&-\frac{1}{b-a}\int_a^b f(x)\,dx\cdot\frac{1}{d-c}\int_c^d g(y)\,dy - \frac{1}{b-a}\int_a^b g(x)\,dx\cdot\frac{1}{d-c}\int_c^d f(y)\,dy,
\end{aligned} \tag{2.1}$$

and the integral mean

$$M(f;a,b) := \frac{1}{b-a}\int_a^b f(x)\,dx. \tag{2.2}$$

The following result holds.

Theorem 1. *Let $f,g: I\subseteq\mathbb{R}\to\mathbb{R}$ be measurable on I and the intervals $[a,b],[c,d]\subset I$. Assume that the integrals involved in (2.1) exist. Then we have the inequality*

$$\begin{aligned}
&|T(f,g;a,b,c,d)| \\
\leq\ &\left[T(f;a,b)+T(f;c,d)+(M(f;a,b)-M(f;c,d))^2\right]^{\frac{1}{2}} \\
&\times\left[T(g;a,b)+T(g;c,d)+(M(g;a,b)-M(g;c,d))^2\right]^{\frac{1}{2}},
\end{aligned} \tag{2.3}$$

where

$$T(f;a,b) := \frac{1}{b-a}\int_a^b f^2(x)\,dx - \left(\frac{1}{b-a}\int_a^b f(x)\,dx\right)^2 \tag{2.4}$$

and the integrals involved in the right membership of (2.3) exist.

Proof. A simple calculation shows that:

$$T(f,g;a,b,c,d) = \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d (f(x)-f(y))(g(x)-g(y))\,dydx. \tag{2.5}$$

This is a generalisation of the classical identity due to Korkine.

Using the Cauchy-Buniakowski-Schwartz inequality for double integrals, we have

$$\begin{aligned}
&|T(f,g;a,b,c,d)|^2 \\
=\ &\left[\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d (f(x)-f(y))(g(x)-g(y))\,dydx\right]^2 \\
\leq\ &\left[\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d (f(x)-f(y))^2\,dydx\right] \\
&\times\left[\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d (g(x)-g(y))^2\,dydx\right] \\
=\ &T(f,f;a,b,c,d)\,T(g,g;a,b,c,d).
\end{aligned} \tag{2.6}$$

We also remark that

$$
\begin{aligned}
(2.7) T(f,f;a,b,c,d) &= \frac{1}{b-a}\int_a^b f^2(x)\,dx + \frac{1}{d-c}\int_c^d f^2(y)\,dy \\
&\quad -2\cdot\frac{1}{b-a}\int_a^b f(x)\,dx\cdot\frac{1}{d-c}\int_c^d f(y)\,dy \\
&= T(f;a,b)+T(f;c,d)+(M(f;a,b)-M(f;c,d))^2
\end{aligned}
$$

and a similar identity holds for g.

Thus, using (2.6), we obtain the desired inequality (2.3). ∎

Remark 1. *If $c=a$, $d=b$, then it may be noticed from (2.5) that*

$$T(f,g;a,b,a,b)=2T(f,g;a,b),$$

where $T(f,g;a,b)$ is the classical Chebychev functional (1.1).

Corollary 1. *Assume that f and g are as in Theorem 1 and, in addition*

$$
\begin{aligned}
(2.8)\qquad m_1 &\le f\le M_1, \text{ a.e. on } [a,b] \text{ and } m_2\le f\le M_2 \text{ a.e. on } [c,d], \\
(2.9)\qquad n_1 &\le g\le N_1, \text{ a.e. on } [a,b] \text{ and } n_2\le g\le N_2 \text{ a.e. on } [c,d].
\end{aligned}
$$

Then we have the inequality

$$
\begin{aligned}
(2.10)\qquad & |T(f,g;a,b,c,d)| \\
\le & \left[\frac{1}{4}(M_1-m_1)^2+\frac{1}{4}(M_2-m_2)^2+(M(f;a,b)-M(f;c,d))^2\right]^{\frac{1}{2}} \\
& \times\left[\frac{1}{4}(N_1-n_1)^2+\frac{1}{4}(N_2-n_2)^2+(M(g;a,b)-M(g;c,d))^2\right]^{\frac{1}{2}}.
\end{aligned}
$$

Proof. The proof follows by (2.3), taking into account that, by the Grüss inequality (1.2),

$$
\begin{aligned}
T(f;a,b) &\le \frac{1}{4}(M_1-m_1)^2, \\
T(f;c,d) &\le \frac{1}{4}(N_1-n_1)^2
\end{aligned}
$$

and the similar inequalities for the mapping g. ∎

Remark 2. *If $a=c$, $b=d$, then $m_1=m_2=:m$, $M_1=M_2=:M$, $M(f;a,b)=M(f;c,d)$ and $T(f,g;a,b,c,d)=2T(f,g;a,b)$, and then from (2.10), we recapture the Grüss inequality (1.2).*

Corollary 2. *Assume that f and g are absolutely continuous on $\bar{I}$. In addition, assume that $f',g'\in L_\infty(\mathring{I})$ and $[a,b],[c,d]\subseteq\bar{I}$ ($\bar{I}$ is the closure of I). Then we have* *the inequality*

$$
\begin{aligned}
&\left|T\left(f,g;a,b,c,d\right)\right| \\
\leq &\left[\frac{1}{12}\left\|f'\right\|_{\infty,[a,b]}^{2}\left(b-a\right)^{2}+\frac{1}{12}\left\|f'\right\|_{\infty,[c,d]}^{2}\left(d-c\right)^{2}\right. \\
&\left.+\left(M\left(f;a,b\right)-M\left(f;c,d\right)\right)^{2}\right]^{\frac{1}{2}} \\
&\times\left[\frac{1}{12}\left\|g'\right\|_{\infty,[a,b]}^{2}\left(b-a\right)^{2}+\frac{1}{12}\left\|g'\right\|_{\infty,[c,d]}^{2}\left(d-c\right)^{2}\right. \\
&\left.+\left(M\left(g;a,b\right)-M\left(g;c,d\right)\right)^{2}\right]^{\frac{1}{2}},
\end{aligned}
\tag{2.11}
$$

where

$$\left\|f'\right\|_{\infty,[a,b]}:=ess\sup_{t\in[a,b]}\left|f'\left(t\right)\right|.$$

The proof follows by (2.3) and utilising the fact that, by the Chebychev inequality (1.3),

$$T\left(f;a,b\right)\leq\frac{1}{12}\left\|f'\right\|_{\infty}^{2}\left(b-a\right)^{2}.$$

Remark 3. *Similarly, if $a=c$, $b=d$, then by (2.11) we deduce Chebychev's inequality (1.3).*

Finally, by the inequality due to Lupaş (1.4), we may state the following corollary.

Corollary 3. *Assume that f and g are absolutely continuous on $\bar{I}$, $f',g'\in L_2\left(\bar{I}\right)$ and $[a,b],[c,d]\subset\bar{I}$. Then we have the inequality*

$$
\begin{aligned}
&\left|T\left(f,g;a,b,c,d\right)\right| \\
\leq &\left[\frac{1}{\pi^{2}}\left\|f'\right\|_{2,[a,b]}^{2}\left(b-a\right)^{2}+\frac{1}{\pi^{2}}\left\|f'\right\|_{2,[c,d]}^{2}\left(d-c\right)^{2}\right. \\
&\left.+\left(M\left(f;a,b\right)-M\left(f;c,d\right)\right)^{2}\right]^{\frac{1}{2}} \\
&\times\left[\frac{1}{\pi^{2}}\left\|g'\right\|_{2,[a,b]}^{2}\left(b-a\right)^{2}+\frac{1}{\pi^{2}}\left\|g'\right\|_{2,[c,d]}^{2}\left(d-c\right)^{2}\right. \\
&\left.+\left(M\left(g;a,b\right)-M\left(g;c,d\right)\right)^{2}\right]^{\frac{1}{2}},
\end{aligned}
\tag{2.12}
$$

where

$$\left\|f'\right\|_{2,[a,b]}:=\left(\int_a^b\left|f'\left(t\right)\right|^{2}dt\right)^{\frac{1}{2}}.$$

Remark 4. *If $a=c$, $b=d$, then by (2.12), we recapture the inequality (1.4) due to Lupaş.*

We define another generalised Chebychev functional involving the mean of the product of two functions, and the product of the means of each of the functions, where one is over a different interval, by

$$\mathfrak{T}\left(f,g;a,b,c,d\right):=M\left(fg;a,b\right)-M\left(f;a,b\right)M\left(g;c,d\right),\tag{2.13}$$

where $M(\cdot,\cdot,\cdot)$ is as defined in (2.2). It may be noticed from (2.13) and (1.1) that $\mathfrak{T}(f,g;a,b;a,b) = T(f,g;a,b)$.

It may be noticed that (2.13) is related to (1.1) by the identity

$$T(f,g;a,b,c,d) = \mathfrak{T}(f,g;a,b,c,d) + \mathfrak{T}(g,f;c,d,a,b). \tag{2.14}$$

Theorem 2. *Let $f,g : I \subseteq \mathbb{R} \to \mathbb{R}$ be measurable on I and the intervals $[a,b],[c,d] \subset I$. In addition, let $m_1 \le f \le M_1$ and $n_1 \le g \le N_1$ a.e. on $[a,b]$ with $n_2 \le g \le N_2$ a.e. on $[c,d]$. Then the following inequalities hold*

$$\begin{aligned} & |\mathfrak{T}(f,g;a,b,c,d)| \\ \le\ & \left[T(f;a,b) + M^2(f;a,b)\right]^{\frac{1}{2}} \\ & \times \left[T(g;a,b) + T(g;c,d) + \left(M(g;a,b) - M(g;c,d)\right)^2\right]^{\frac{1}{2}} \\ \le\ & \left[\left(\frac{M_1 - m_1}{2}\right)^2 + M^2(f;a,b)\right]^{\frac{1}{2}} \\ & \times \left[\left(\frac{N_1 - n_1}{2}\right)^2 + \left(\frac{N_2 - n_2}{2}\right)^2 + \left(M(g;a,b) - M(g;c,d)\right)^2\right]^{\frac{1}{2}}, \end{aligned} \tag{2.15}$$

where $T(f;a,b)$ is as given by (2.4) and $M(f;a,b)$ by (2.2).

Proof. The proof follows closely that of Theorem 1 and Corollary 1.
From (2.14) and (2.5), or directly from (2.13), it may be shown that

$$\mathfrak{T}(f,g;a,b,c,d) = \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x)\left(g(x) - g(y)\right) dydx,$$

and thus, using the Cauchy-Buniakowski-Schwartz inequality for double integrals gives

$$\begin{aligned} & |\mathfrak{T}(f,g;a,b,c,d)|^2 \\ =\ & \left[\frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x)\left(g(x) - g(y)\right) dydx\right]^2 \\ \le\ & \left(\frac{1}{b-a}\int_a^b f^2(x)\,dx\right)\left(\frac{1}{(b-a)(d-c)} \int_a^b \int_c^d \left(g(x) - g(y)\right)^2 dydx\right) \\ =\ & M\left(f^2;a,b\right) T(g,g;a,b,c,d), \end{aligned}$$

which, upon using (2.4) and (2.7) produces the first inequality in (2.15).

Using the Grüss inequality (1.2), identity (2.4) together with the bounds on f and g over the respective intervals produces the second inequality, and the theorem is proved. ∎

Remark 5. *If the Chebychev and Lupaş bounds were used with the first inequality in (2.15), then similar results to those obtained in Corollaries 2 and 3 for $T(g,f;a,b,c,d)$ can be obtained for $\mathfrak{T}(f,g;a,b,c,d)$ as defined in (2.13).*

3. Applications in Numerical Analysis

If in (2.3) we assume that $c = z - \varepsilon$, $d = z + \varepsilon$ where $z \in (a,b)$ and $\varepsilon > 0$ is chosen accordingly such that $c, d \in (a,b)$, then, letting $\varepsilon \to 0+$ we get from (2.3) and, using (2.1), (2.2) and (2.4), the following inequality

$$\begin{aligned} & \left| \frac{1}{b-a} \int_a^b f(x) g(x) \, dx - g(z) \frac{1}{b-a} \int_a^b f(x) \, dx \right. \\ & \left. - f(z) \frac{1}{b-a} \int_a^b g(x) \, dx + f(z) g(z) \right| \\ \leq & \left[\frac{1}{b-a} \int_a^b f^2(x) \, dx - \left(\frac{1}{b-a} \int_a^b f(x) \, dx \right)^2 \right. \\ & \left. + \left(\frac{1}{b-a} \int_a^b f(x) \, dx - f(z) \right)^2 \right]^{\frac{1}{2}} \\ & \times \left[\frac{1}{b-a} \int_a^b g^2(x) \, dx - \left(\frac{1}{b-a} \int_a^b g(x) \, dx \right)^2 \right. \\ & \left. + \left(\frac{1}{b-a} \int_a^b g(x) \, dx - g(z) \right)^2 \right]^{\frac{1}{2}} \\ : & = K. \end{aligned} \tag{3.1}$$

If, for example, we assume that f and g satisfy the assumptions of Corollary 2, then we have:

$$\begin{aligned} K \leq & \left[\frac{1}{12} \|f'\|_{\infty,[a,b]}^2 (b-a)^2 + \left(\frac{1}{b-a} \int_a^b f(x) \, dx - f(z) \right)^2 \right]^{\frac{1}{2}} \\ & \times \left[\frac{1}{12} \|g'\|_{\infty,[a,b]}^2 (b-a)^2 + \left(\frac{1}{b-a} \int_a^b g(x) \, dx - g(z) \right)^2 \right]^{\frac{1}{2}}. \end{aligned} \tag{3.2}$$

Similar bounds for K can be obtained if we use Corollary 1 or Corollary 3 as indeed for $\mathfrak{T}(f,g;a,b,c,d)$ of Theorem 2. For a given division $\boldsymbol{\delta} : a = x_0 < x_1 < ... < x_{n-1} < x_n = b$ of $[a,b]$ having the intermediate points $\xi_i \in [x_i, x_{i+1}]$, $i = 0, ..., n-1$, we can construct the formula

$$\begin{aligned} & A(f,g,\boldsymbol{\delta},\boldsymbol{\xi}) \\ : & = \sum_{i=0}^{n-1} g(\xi_i) \int_{x_i}^{x_{i+1}} f(t) \, dt + \sum_{i=0}^{n-1} f(\xi_i) \int_{x_i}^{x_{i+1}} g(t) \, dt - \sum_{i=0}^{n-1} h_i f(\xi_i) g(\xi_i), \end{aligned} \tag{3.3}$$

where $h_i := x_{i+1} - x_i$ $(i = 0, ..., n-1)$.

We may state the following result which gives an approximation of the integral of the product, namely, $\int_a^b f(x) g(x) \, dx$ in terms of the sum $A(f,g,\boldsymbol{\delta},\boldsymbol{\xi})$.

Theorem 3. *Assume that the mapping f and g satisfy the assumptions in Corollary 2. Then we have, for $\boldsymbol{\delta}$ and $\boldsymbol{\xi}$ as above:*

$$\int_a^b f(x)\,g(x)\,dx = A(f,g,\boldsymbol{\delta},\boldsymbol{\xi}) + R(f,g,\boldsymbol{\delta},\boldsymbol{\xi}) \tag{3.4}$$

where the remainder $R(f,g,\boldsymbol{\delta},\boldsymbol{\xi})$ satisfies the bound

$$
\begin{aligned}
&|R(f,g,\boldsymbol{\delta},\boldsymbol{\xi})| \qquad (3.5)\\
\leq\ & \sum_{i=0}^{n-1} h_i \left[\frac{1}{12}\|f'\|_{\infty,[a,b]}^2 h_i^2 + \left(\frac{1}{h_i}\int_{x_i}^{x_{i+1}} f(x)\,dx - f(\xi_i)\right)^2\right]^{\frac{1}{2}} \\
&\times \left[\frac{1}{12}\|g'\|_{\infty,[a,b]}^2 h_i^2 + \left(\frac{1}{h_i}\int_{x_i}^{x_{i+1}} g(x)\,dx - g(\xi_i)\right)^2\right]^{\frac{1}{2}} \\
\leq\ & \left[\frac{1}{12}\|f'\|_{\infty,[a,b]}^2 \sum_{i=0}^{n-1} h_i^3 + \sum_{i=0}^{n-1} h_i \left(\frac{1}{h_i}\int_{x_i}^{x_{i+1}} f(x)\,dx - f(\xi_i)\right)^2\right]^{\frac{1}{2}} \\
&\times \left[\frac{1}{12}\|g'\|_{\infty,[a,b]}^2 \sum_{i=0}^{n-1} h_i^3 + \sum_{i=0}^{n-1} h_i \left(\frac{1}{h_i}\int_{x_i}^{x_{i+1}} g(x)\,dx - g(\xi_i)\right)^2\right]^{\frac{1}{2}} \\
\leq\ & \left[\frac{1}{12}\|f'\|_{\infty,[a,b]}^2 \sum_{i=0}^{n-1} h_i^3 + \|f'\|_{\infty,[a,b]} \sum_{i=0}^{n-1} \left(\frac{1}{4} + \frac{\left(\xi_i - \frac{x_i+x_{i+1}}{2}\right)^2}{h_i^2}\right)^2 h_i^3\right]^{\frac{1}{2}} \\
&\times \left[\frac{1}{12}\|g'\|_{\infty,[a,b]}^2 \sum_{i=0}^{n-1} h_i^3 + \|g'\|_{\infty,[a,b]} \sum_{i=0}^{n-1} \left(\frac{1}{4} + \frac{\left(\xi_i - \frac{x_i+x_{i+1}}{2}\right)^2}{h_i^2}\right)^2 h_i^3\right]^{\frac{1}{2}} \\
\leq\ & \frac{1}{3}\|f'\|_{\infty,[a,b]}\|g'\|_{\infty,[a,b]} \sum_{i=0}^{n-1} h_i^3 \to 0 \quad \text{as} \quad \nu(\boldsymbol{\delta}) := \max_{i=\overline{0,n-1}} h_i \to 0.
\end{aligned}
$$

Proof. The first inequality in (3.4) follows by applying (3.1) and (3.2) on the intervals $[x_i, x_{i+1}]$ with $\xi_i \in [x_i, x_{i+1}]$ and summing the resulting inequalities.

The second inequality follows by the Cauchy-Schwartz inequality with the positive weight h_i $(i = 0, ..., n-1)$.

The third inequality results from using Ostrowski's inequality (see [2, p. 468])

$$\left|\frac{1}{h_i}\int_{x_i}^{x_{i+1}} f(x)\,dx - f(\xi_i)\right| \leq \left[\frac{1}{4} + \frac{\left(\xi_i - \frac{x_i+x_{i+1}}{2}\right)^2}{h_i^2}\right] \|f'\|_{\infty,[a,b]}\, h_i$$

for all $i = 0, ..., n-1$, and the last inequality is obtained on using the fact that

$$\left(\xi_i - \frac{x_i + x_{i+1}}{2}\right)^2 \leq \frac{1}{4}h_i^2, \quad i = 0, ..., n-1.$$

∎

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