On Some Inequalities Arising from Montgomery's Identity

ON SOME INEQUALITIES ARISING FROM MONTGOMERY'S IDENTITY

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Abstract. An identity due to Montgomery is utilised to obtain other identities from which a number of novel inequalities are developed. The work also recaptures some of the existing results as special cases, such as the Mahajani inequality. Bounds are obtained for expressions involving moments, within a general framework.

1. Introduction

The following identity, attributed to Montgomery, is well known in the literature (see [9, Chapter XVII, p. 565])

$$f(x) = \mathfrak{M}(a,b;f) + \kappa(x), \tag{1.1}$$

where

$$\mathfrak{M}(a,b;f) = \frac{1}{b-a}\int_a^b f(t)\,dt \text{ is the integral mean,} \tag{1.2}$$

$$\kappa(x) = \int_a^b P(x,t) f'(t)\,dt, \tag{1.3}$$

and the Peano kernel $P(x,t)$ is given by

$$(b-a)P(x,t) := \begin{cases} t-a, & a \le t \le x, \\ \\ t-b, & x < t \le b. \end{cases} \tag{1.4}$$

Recently, Dragomir and Wang [5] utilized (1.1)-(1.4) to prove Ostrowski's inequality [6, p. 469]

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \le \frac{M}{b-a}\left[\left(\frac{b-a}{2}\right)^2 + \left(x - \frac{a+b}{2}\right)^2\right],$$

where $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is a differential mapping on $\mathring{I}$, the interior of I, and $|f'(x)| \le M$ for all $x \in [a,b]$, $a < b \in \mathring{I}$. Many Ostrowski type results applied to numerical integration and probability have appeared in the literature (see for example [5]-[1] and the references therein).

It is the intention of the current article to develop, through the framework of Montgomery's identity, a systematic study which produces some novel results and recaptures existing results as special cases. Bounds are obtained in terms of the Lebesgue norms of the first derivative.

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In Section 2, results are obtained for a generalised Chebychev functional involving the integral mean of functions over different intervals. In particular, bounds are obtained for the difference of means over two different intervals, producing a generalisation of Mahajani type inequalities. In Section 3, we study bounds involving moments about any general parameter producing results for central and moments about the origin as special cases. Bounds for the expectation and the variance are investigated, in particular, recapturing some earlier results and obtaining some previously unknown results.

2. Main Results and Some Ramifications

We start with the following theorem.

Theorem 1. *Let $f : [a,b] \to \mathbb{R}$ be an absolutely continuous mapping as is $u : [\alpha, \beta] \to \mathbb{R}$ with $[\alpha, \beta] \subseteq [a,b]$. The following inequalities hold. Namely,*

$$\left| \int_\alpha^\beta u(x) f(x)\, dx - \mathfrak{M}(a,b;f) \int_\alpha^\beta u(x)\, dx \right| \tag{2.1}$$

$$\leq \begin{cases} \frac{\|f'\|_\infty}{2(b-a)} \int_\alpha^\beta |u(x)| \left[(x-a)^2 + (b-x)^2 \right] dx, & f' \in L_\infty [a,b]; \\ \\ \frac{\|f'\|_p}{(q+1)^{\frac{1}{q}}(b-a)} \int_\alpha^\beta |u(x)| \left[(x-a)^{q+1} + (b-x)^{q+1} \right]^{\frac{1}{q}} dx, & f' \in L_p [a,b], \\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \\ \frac{\|f'\|_1}{b-a} \int_\alpha^\beta |u(x)| \left[\frac{b-a}{2} + \left| x - \frac{a+b}{2} \right| \right] dx, & f' \in L_1 [a,b], \end{cases}$$

where

$$\mathfrak{M}(a,b;f) = \frac{1}{b-a} \int_a^b f(t)\, dt \tag{2.2}$$

and $\|\cdot\|_p$, $p \geq 1$ are the usual Lebesgue norms on $[a,b]$. That is,

$$\|g\|_\infty := ess \sup_{t \in [a,b]} |g(t)| \quad and \quad \|g\|_p := \left(\int_a^b |g(t)|^p\, dt \right)^{\frac{1}{p}}, \quad 1 \leq p < \infty.$$

Proof. Using identity (1.1), we obtain, for $a \leq \alpha < \beta \leq b$,

$$\int_\alpha^\beta u(x) f(x)\, dx = \mathfrak{M}(a,b;f) \int_\alpha^\beta u(x)\, dx + \int_\alpha^\beta u(x) \kappa(x)\, dx, \tag{2.3}$$

and therefore

$$\left| \int_\alpha^\beta u(x) f(x)\, dx - \mathfrak{M}(a,b;f) \int_\alpha^\beta u(x)\, dx \right| = \left| \int_\alpha^\beta u(x) \kappa(x)\, dx \right|. \tag{2.4}$$

Now,

$$\left| \int_\alpha^\beta u(x) \kappa(x)\, dx \right| \leq \int_\alpha^\beta |u(x)| |\kappa(x)|\, dx \tag{2.5}$$

and using the properties of modulus and Hölder's integral inequality

$$|\kappa(x)| \le \begin{cases} \|f'\|_\infty \int_a^b |P(x,t)|\,dt, & f' \in L_\infty[a,b]; \\ \|f'\|_p \left(\int_a^b |P(x,t)|^q\,dt\right)^{\frac{1}{q}}, & f' \in L_p[a,b], \\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \|f'\|_1 \sup\limits_{t\in[a,b]} |P(x,t)|, & f' \in L_1[a,b]. \end{cases}$$

That is,

$$|\kappa(x)| \le \begin{cases} \frac{\|f'\|_\infty}{2(b-a)}\left[(x-a)^2+(b-x)^2\right], & f' \in L_\infty[a,b]; \\ \frac{\|f'\|_p}{(q+1)^{\frac{1}{q}}(b-a)}\left[(x-a)^{q+1}+(b-x)^{q+1}\right]^{\frac{1}{q}}, & f' \in L_p[a,b], \\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{\|f'\|_1}{b-a}\max\{x-a,b-x\}, & f' \in L_1[a,b]. \end{cases} \tag{2.6}$$

Substituting (2.6) into (2.5) and using identity (2.4) gives (2.1) on noting that for $X, Y \in \mathbb{R}$,

$$\max\{X,Y\} = \frac{X+Y}{2} + \left|\frac{X-Y}{2}\right|.$$

Hence, the theorem is proved. ∎

Lemma 1. *Let f and u satisfy the conditions of Theorem 1. Then the following identity is valid*

$$\begin{aligned} &\int_\alpha^\beta u(x)\,f(x)\,dx \\ = {} & A(\alpha,\beta)\left\{\mathfrak{M}(a,b;f) + \int_a^\alpha \left(\frac{t-a}{b-a}\right) f'(t)\,dt - \int_\beta^b \left(\frac{b-t}{b-a}\right) f'(t)\,dt\right\} \\ & + \frac{1}{b-a}\int_\alpha^\beta \left[(t-a)\,A(t,\beta) - (b-t)\,A(\alpha,t)\right] f'(t)\,dt, \end{aligned} \tag{2.7}$$

where $A(\alpha,\beta) = \int_\alpha^\beta u(x)\,dx$ and $\mathfrak{M}(a,b;f)$ is defined in (2.2).

Proof. The proof is straight forward from identity (2.3) by an interchange of the order of integration of

$$\int_\alpha^\beta u(x)\,\kappa(x)\,dx,$$

where $\kappa(x)$ is defined by (1.3).

That is,

$$\begin{aligned}
&\int_{\alpha}^{\beta} u(x)\,\kappa(x)\,dx \qquad (2.8)\\
&= \int_{\alpha}^{\beta} u(x)\int_{a}^{x}\left(\frac{t-a}{b-a}\right) f'(t)\,dtdx + \int_{\alpha}^{\beta} u(x)\int_{x}^{b}\left(\frac{t-b}{b-a}\right) f'(t)\,dtdx\\
&= A(\alpha,\beta)\int_{a}^{\alpha}\left(\frac{t-a}{b-a}\right) f'(t)\,dt + \int_{\alpha}^{\beta}\left(\frac{t-a}{b-a}\right) A(t,\beta)\, f'(t)\,dt\\
&\quad + A(\alpha,\beta)\int_{\beta}^{b}\left(\frac{t-b}{b-a}\right) f'(t)\,dt + \int_{\alpha}^{\beta}\left(\frac{t-b}{b-a}\right) A(\alpha,t)\, f'(t)\,dt.
\end{aligned}$$

Substitution into (2.3) produces (2.7). ∎

Theorem 2. *Let $f:[a,b]\to\mathbb{R}$ be an absolutely continuous mapping as is also $u:[\alpha,\beta]\to\mathbb{R}$ with $[\alpha,\beta]\subseteq[a,b]$. The following inequalities are then valid. Namely*

$$\left|\int_{\alpha}^{\beta} u(x)\,f(x)\,dx - \mathfrak{M}(a,b;f)\,A(\alpha,\beta)\right| \tag{2.9}$$

$$\leq \begin{cases}
\frac{\|f'\|_\infty}{b-a}\left\{\frac{|A(\alpha,\beta)|}{2}\left[(\alpha-a)^2+(b-\beta)^2\right]+\int_\alpha^\beta|\phi(t)|\,dt\right\}, & f'\in L_\infty[a,b];\\[2ex]
\frac{\|f'\|_p}{b-a}\cdot\left\{\frac{|A(\alpha,\beta)|^q}{q+1}\left[(\alpha-a)^{q+1}+(b-\beta)^{q+1}\right] + \int_\alpha^\beta|\phi(t)|^q\,dt\right\}^{\frac{1}{q}}, & f'\in L_p[a,b],\ p>1,\ \frac{1}{p}+\frac{1}{q}=1;\\[2ex]
\frac{\|f'\|_1}{b-a}\max\left\{|A(\alpha,\beta)|\,\Theta,\ \sup_{t\in[\alpha,\beta]}|\phi(t)|\right\}, & f'\in L_1[a,b],
\end{cases}$$

where

$$\phi(t) = (t-a)\,A(t,\beta) - (b-t)\,A(\alpha,t) = \begin{vmatrix} t-a & b-t\\ A(\alpha,t) & A(t,\beta)\end{vmatrix} \tag{2.10}$$

and

$$\Theta = \frac{b-a}{2} - \frac{\beta-\alpha}{2} + \left|\frac{b+a}{2} - \frac{\beta+\alpha}{2}\right|. \tag{2.11}$$

Proof. From identity (2.7)

$$\int_{\alpha}^{\beta} u(x)\,f(x)\,dx - \mathfrak{M}(a,b;f)\,A(\alpha,\beta) = R, \tag{2.12}$$

where

$$\begin{aligned}
R &= A(\alpha,\beta)\left\{\int_{a}^{\alpha}\left(\frac{t-a}{b-a}\right) f'(t)\,dt + \int_{\beta}^{b}\left(\frac{t-b}{b-a}\right) f'(t)\,dt\right\} \qquad (2.13)\\
&\quad + \frac{1}{b-a}\int_{\alpha}^{\beta}\phi(t)\,f'(t)\,dt,
\end{aligned}$$

with $\phi(t)$ being as given by (2.10).

Now, taking the modulus of (2.12) and using the triangle inequality gives, from (2.13),

$$
\begin{aligned}
|R| &\leq |A(\alpha,\beta)| \left\{ \sup_{t\in[a,\alpha]} |f'(t)| \cdot \frac{1}{b-a} \cdot \frac{(\alpha-a)^2}{2} \right. \\
&\quad + \sup_{t\in[\beta,b]} |f'(t)| \cdot \frac{1}{b-a} \cdot \frac{(b-\beta)^2}{2} \\
&\quad \left. + \sup_{t\in(\alpha,\beta)} |f'(t)| \frac{1}{b-a} \int_\alpha^\beta \phi(t)\,dt \right\} \\
&\leq \frac{\|f'\|_\infty}{b-a} \left\{ \left[\frac{(\alpha-a)^2 + (b-\beta)^2}{2} \right] + \int_\alpha^\beta |\phi(t)|\,dt \right\}.
\end{aligned}
\tag{2.14}
$$

Substitution of (2.14) into (2.12) produces the first inequality in (2.9).

Further, from (2.12), using Hölder's integral inequality gives

$$
|R| \leq \frac{\|f'\|_p}{b-a} \left\{ |A(\alpha,\beta)|^q \left[\int_a^\alpha (t-a)^q\,dt + \int_\beta^b (b-t)^q\,dt \right] + \int_\alpha^\beta |\phi(t)|^q\,dt \right\}^{\frac{1}{q}},
$$

which, upon some simple calculations and substitution into the identity (2.11), gives the second inequality in (2.9).

For the last inequality in (2.9), from (2.13)

$$
\begin{aligned}
|R| &\leq \frac{|A(\alpha,\beta)|}{b-a} \left[(\alpha-a) \int_a^\alpha |f'(t)|\,dt + (b-\beta) \int_\beta^b |f'(t)|\,dt \right] \\
&\quad + \frac{1}{b-a} \sup_{t\in[\alpha,\beta]} |\phi(t)| \int_\alpha^\beta |f'(t)|\,dt \\
&\leq \frac{\|f'\|_1}{b-a} \max\left\{ |A(\alpha,\beta)|\,\Theta, \sup_{t\in[\alpha,\beta]} |\phi(t)| \right\},
\end{aligned}
$$

where $\Theta = \max\{\alpha-a, b-\beta\}$.

On substitution of the last inequality into (2.12) and, using the fact that $\max\{X,Y\} = \frac{X+Y}{2} + \left|\frac{X-Y}{2}\right|$, $X, Y \in \mathbb{R}$, gives the final inequality in (2.9). Hence, the theorem is now proved. ∎

Remark 1. *The bound for the left hand side of (2.12) in terms of R as given by (2.13) was used so that a comparison could be made with the bounds obtained from Theorem 1. In Corollary 1, the first two terms in (2.13) disappear on using particular choices of the parameters, and in Theorem 3, the terms constitute part of the expression to be approximated.*

The following corollary gives an estimate of the error for the difference between weighted and unweighted integral means.

Corollary 1. *Let $f,u:[a,b]\to\mathbb{R}$ be absolutely continuous mappings on $[a,b]$. Then*

$$\left|\frac{\int_a^b u(x)f(x)\,dx}{\int_a^b u(x)\,dx}-\frac{\int_a^b f(x)\,dx}{b-a}\right| \tag{2.15}$$
$$\leq\begin{cases}\frac{\|f'\|_\infty}{b-a}\int_a^b|\Phi(t)|\,dt, & f'\in L_\infty[a,b];\\ \frac{\|f'\|_p}{b-a}\left[\int_a^b|\Phi(t)|^q\,dt\right]^{\frac{1}{q}}, & f'\in L_p[a,b],\ p>1,\ \frac{1}{p}+\frac{1}{q}=1;\\ \frac{\|f'\|_1}{b-a}\sup\limits_{t\in[a,b]}|\Phi(t)|, & f'\in L_1[a,b],\end{cases}$$

where

$$\Phi(t)=(t-a)H(t,b)-(b-t)H(a,t) \tag{2.16}$$

with

$$H(a,t)=\frac{\int_a^t u(x)\,dx}{\int_a^b u(x)\,dx}\quad\text{and}\quad H(t,b)=1-H(a,t). \tag{2.17}$$

Proof. Setting $\alpha=a$ and $\beta=b$ in Theorem 2 produces the result (2.15) after some minor rearrangements. ∎

Theorem 3. *Let $f:[a,b]\to\mathbb{R}$ be an absolutely continuous mapping as is $u:[\alpha,\beta]\to\mathbb{R}$ with $[\alpha,\beta]\subseteq[a,b]$. Then the following inequalities are valid. That is,*

$$|\mathcal{T}| := \left|\int_\alpha^\beta u(t)f(t)\,dt - A(\alpha,\beta)\{[1-(\lambda_1+\lambda_2)]\mathfrak{M}(\alpha,\beta;f)+\lambda_1 f(\alpha)+\lambda_2 f(\beta)\}\right| \tag{2.18}$$
$$\leq\begin{cases}\frac{\|f'\|_\infty}{b-a}\int_\alpha^\beta|\phi(t)|\,dt, & f'\in L_\infty[\alpha,\beta];\\ \frac{\|f'\|_p}{b-a}\left(\int_\alpha^\beta|\phi(t)|^q\,dt\right)^{\frac{1}{q}}, & f'\in L_p[\alpha,\beta],\ p>1,\\ & \frac{1}{p}+\frac{1}{q}=1;\\ \frac{\|f'\|_1}{b-a}\sup\limits_{t\in[\alpha,\beta]}|\phi(t)|, & f'\in L_1[\alpha,\beta],\end{cases}$$

where

$$A(\alpha,\beta)=\int_\alpha^\beta u(t)\,dt,\ \lambda_1=\frac{\alpha-a}{b-a},\quad \lambda_2=\frac{b-\beta}{b-a}, \tag{2.19}$$

and $\phi(t)$ is as defined by (2.10).

Proof. From identity (2.7), we have

$$
\begin{aligned}
(2.20)\qquad & \int_{\alpha}^{\beta} u(t) f(t)\,dt - A(\alpha,\beta)\left\{\mathfrak{M}(a,b;f) + \int_{a}^{\alpha}\left(\frac{t-a}{b-a}\right) f'(t)\,dt \right. \\
& \left. - \int_{\beta}^{b}\left(\frac{b-t}{b-a}\right) f'(t)\,dt\right\} \\
= {} & \frac{1}{b-a}\int_{\alpha}^{\beta} \phi(t) f'(t)\,dt,
\end{aligned}
$$

where $A(\alpha,\beta)$ is as given by (2.19), $\mathfrak{M}(a,b;f)$ is as defined in (2.2) and $\phi(t)$ is as given by (2.10).

Simple integration by parts gives

$$
\int_{a}^{\alpha}\left(\frac{t-a}{b-a}\right) f'(t)\,dt = \left(\frac{\alpha-a}{b-a}\right) f(\alpha) - \frac{1}{b-a}\int_{a}^{\alpha} f(t)\,dt
$$

and

$$
-\int_{\beta}^{b}\left(\frac{b-t}{b-a}\right) f'(t)\,dt = \left(\frac{b-\beta}{b-a}\right) f(\beta) - \frac{1}{b-a}\int_{\beta}^{b} f(t)\,dt,
$$

which, upon substitution into (2.20) produces the identity

$$
\begin{aligned}
(2.21)\qquad & \int_{\alpha}^{\beta} u(t) f(t)\,dt - \frac{A(\alpha,\beta)}{b-a}\left[\int_{\alpha}^{\beta} f(t)\,dt + (\alpha-a) f(\alpha) + (b-\beta) f(\beta)\right] \\
= {} & \frac{1}{b-a}\int_{\alpha}^{\beta} \phi(t) f'(t)\,dt.
\end{aligned}
$$

Now, allowing λ_1, λ_2 to be as given in (2.19), then, from (2.21),

$$
\begin{aligned}
(2.22)\qquad & \int_{\alpha}^{\beta} u(t) f(t)\,dt \\
& - A(\alpha,\beta)\left\{[1-(\lambda_1+\lambda_2)]\frac{1}{\beta-\alpha}\int_{\alpha}^{\beta} f(t)\,dt + \lambda_1 f(\alpha) + \lambda_2 f(\beta)\right\} \\
= {} & \frac{1}{b-a}\int_{\alpha}^{\beta} \phi(t) f'(t)\,dt.
\end{aligned}
$$

Taking the modulus of (2.22) and using the results from the proof of Theorem 2 involving the modulus and integral and, Hölder's inequality, produces the results as stated in (2.18) involving the Lebesgue norms, for $f' \in L_p[\alpha,\beta]$, $p \geq 1$. ∎

Remark 2. *The left hand side of (2.18) may be written in the form*

$$
\begin{aligned}
(2.23)\ \mathcal{T} = {} & (\beta-\alpha) T(u,f) \\
& + A(\alpha,\beta)\left\{\lambda_1[\mathfrak{M}(\alpha,\beta;f) - f(\alpha)] + \lambda_2[\mathfrak{M}(\alpha,\beta;f) - f(\beta)]\right\},
\end{aligned}
$$

where $T(g,h)$ is the Chebychev functional (see for example [8]*) given by*

$$
(2.24)\qquad T(g,h) = \mathcal{M}(gh) - \mathcal{M}(g)\mathcal{M}(h),
$$

where $\mathcal{M}(\cdot)$ is the mean over some interval. Hence, the bounds of Theorem 3 may be viewed as bounds for a perturbed Chebychev functional. If $\lambda_1 = \lambda_2 = 0$, then

there is no perturbation.
If $\lambda_1 = \lambda_2 = \lambda$, say, then from (2.23)

$$(\beta - \alpha) T(u, f) + 2\lambda A(\alpha, \beta) \left[\mathfrak{M}(\alpha, \beta; f) - \frac{f(\alpha) + f(\beta)}{2} \right],$$

where the perturbation to the Chebychev functional involves the difference between the mean and the trapezoidal approximation of a function $f(\cdot)$.
If $\lambda_1 = \lambda_2 = \frac{1}{2}$, then, from the left hand side of (2.18), on division by $\beta - \alpha$ gives the difference between the average of the product of two functions and the average of one by the difference between the average and the trapezoidal approximation of another. If $\lambda_1 = 0$ $(\alpha = a)$ and $\lambda_2 = \frac{b-x}{b-a}$ $(\beta = x)$, then from (2.18)

$$\begin{aligned} \frac{\mathcal{T}}{\beta - \alpha} &= \mathfrak{M}(a, x; uf) - \\ & \mathfrak{M}(a, x; u) \left[\left(\frac{x-a}{b-a} \right) \mathfrak{M}(a, x; f) + \left(\frac{b-x}{b-a} \right) f(x) \right], \end{aligned} \tag{2.25}$$

giving a convex combination between the mean of $f(\cdot)$ and evaluation at only one end point.
As a matter of fact, from (2.18)

$$\begin{aligned} \frac{\mathcal{T}}{\beta - \alpha} &= \mathfrak{M}(\alpha, \beta; uf) - \\ & \mathfrak{M}(\alpha, \beta; u) \{ [1 - (\lambda_1 + \lambda_2)] \mathfrak{M}(\alpha, \beta; f) + \lambda_1 f(\alpha) + \lambda_2 f(\beta) \}, \end{aligned}$$

giving a comparison between the mean of a product of two functions and the product of the mean of a function and a convex combination of the mean and the end point function evaluations of the other function.

The following corollary gives bounds for the difference between the mean of a function compared to its mean over a subinterval.

Corollary 2. *Let the conditions of Theorem 1 hold. Then, for $[\alpha, \beta] \subseteq [a, b]$*

$$\begin{aligned} & |\mathfrak{M}(\alpha, \beta; f) - \mathfrak{M}(a, b; f)| \\ & \leq \begin{cases} \frac{\|f'\|_\infty}{2(b-a)(\beta-\alpha)} [M_2(\alpha - a, \beta - a) + M_2(b - \beta, b - \alpha)], & f' \in L_\infty[a, b]; \\ \frac{\|f'\|_p}{(q+1)^{\frac{1}{q}}(b-a)(\beta-\alpha)} [M_{q+1}(\alpha - a, \beta - a) + M_{q+1}(b - \beta, b - \alpha)]^{\frac{1}{q}}, & \\ \qquad f' \in L_p[a, b],\ p > 1,\ \frac{1}{p} + \frac{1}{q} = 1; & \\ \frac{\|f'\|_1}{b-a} \cdot \left[\frac{(b-a)(\beta-\alpha)}{2} + \int_{\alpha - \frac{a+b}{2}}^{\beta - \frac{a+b}{2}} |u| \, du \right], & f' \in L_1[a, b], \end{cases} \end{aligned} \tag{2.26}$$

where $\mathfrak{M}(\cdot, \cdot; f)$ is as defined by (2.2) and

$$M_r(x_1, x_2) = \int_{x_1}^{x_2} u^r du = \frac{x_2^{r+1} - x_1^{r+1}}{r+1}. \tag{2.27}$$

Proof. The proof follows from Theorem 1. Placing $u \equiv 1$ gives the above results after some straight forward algebra and noting that from (2.27)

$$\int_\alpha^\beta [(x-a)^r + (b-x)^r] \, dx = M_r(\alpha - a, \beta - a) + M_r(b - \alpha, b - \beta).$$

∎

Corollary 3. *Let the conditions of Theorem 1 hold. Then*

$$\left|\int_a^x f(u)\,du - \left(\frac{x-a}{b-a}\right)\int_a^b f(u)\,du\right| \tag{2.28}$$

$$\leq \begin{cases} \frac{\|f'\|_\infty}{6(b-a)}\left[(x-a)^3+(b-a)^3-(b-x)^3\right], & f'\in L_\infty[a,b]; \\ \frac{\|f'\|_p}{[(q+2)(q+1)]^{\frac{1}{q}}(b-a)}\left[(x-a)^{q+2}+(b-a)^{q+2}-(b-x)^{q+2}\right]^{\frac{1}{q}}, & \\ \qquad f'\in L_p[a,b],\ p>1,\ \frac{1}{p}+\frac{1}{q}=1; & \\ \frac{\|f'\|_1}{b-a}(x-a)\left[\frac{(b-a)(x-a)}{2}+\int_{-\frac{b-a}{2}}^{x-\frac{a+b}{2}}|u|\,du\right], & f'\in L_1[a,b]. \end{cases}$$

Proof. Taking $u \equiv 1$ in Theorem 1 and placing $\beta = x$ and $\alpha = a$ or taking $\beta = x$ and $\alpha = a$ in Corollary 2 produces the results stated after some simplification. Namely, from (2.26), using (2.27),

$$M_r(0,x-a)+M_r(b-x,b-a)=\frac{(x-a)^{r+1}+(b-a)^{r+1}-(b-x)^{r+1}}{r+1}$$

gives the results as stated in (2.28). ∎

Remark 3. *An upper bound may be obtained from Corollary 2 when $x_1 \equiv 0$. That is, if $\alpha = a$ and $\beta = b$. Taking $x = \frac{a+b}{2}$ on the right hand side of Corollary 3 produces the result*

$$\left|\int_a^{\frac{a+b}{2}} f(u)\,du - \frac{1}{2}\int_a^b f(u)\,du\right|$$

$$\leq \begin{cases} \frac{(b-a)^2}{6}\|f'\|_\infty, & f'\in L_\infty[a,b]; \\ \frac{(b-a)^{\frac{2}{q}}}{[(q+2)(q+1)]^{\frac{1}{q}}}\|f'\|_p, & f'\in L_p[a,b],\ p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{3}{4}\left(\frac{b-a}{2}\right)^2\|f'\|_1, & f'\in L_1[a,b]. \end{cases}$$

Furthermore, if f is a probability density function such that $f:[a,b]\to\mathbb{R}_+$ and $\int_a^b f(t)\,dt = 1$, then $\int_a^x f(u)\,du = F(x)$, the cumulative density function and so Corollary 3 may be viewed as a first order approximation for F.

Corollary 4. *Let the conditions of Theorem 2 hold. Then for $[\alpha,\beta]\subset[a,b]$, the following inequalities are valid. Namely*

$$|\mathfrak{M}(\alpha,\beta;f)-\mathfrak{M}(a,b;f)| \tag{2.29}$$

$$\leq \begin{cases} \frac{\|f'\|_\infty}{2[b-a-(\beta-\alpha)]}\left[(\alpha-a)^2+(b-\beta)^2\right], & f'\in L_\infty[a,b]; \\ \frac{\|f'\|_p}{b-a}\left[\frac{(\alpha-a)^{q+1}+(b-\beta)^{q+1}}{(q+1)(1-\rho)}\right]^{\frac{1}{q}}, & f'\in L_p[a,b],\ p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{\|f'\|_1}{b-a}\Theta, & f'\in L_1[a,b], \end{cases}$$

where $\rho = \frac{\beta-\alpha}{b-a}$ and $\Theta = \max\{\alpha-a, b-\beta\}$ as given by (2.11).

Proof. Taking $u \equiv 1$ in Theorem 2 gives $A(\alpha,\beta) = \beta - \alpha$ and for $k \geq 1$,

$$\begin{aligned} \int_\alpha^\beta |\Phi(t)|^k \, dt &= \int_\alpha^\beta |(t-a)(\beta-t) - (b-t)(t-\alpha)|^k \, dt \\ &= \int_\alpha^\beta |\gamma t - c|^k \, dt, \end{aligned} \tag{2.30}$$

where

$$\begin{cases} & \gamma = b - a - (\beta - a) \\ \text{and} & \\ & c = \alpha b - a\beta. \end{cases} \tag{2.31}$$

Thus, from (2.30),

$$\begin{aligned} \int_\alpha^\beta |\Phi(t)|^k \, dt &= \gamma^k \int_\alpha^{\frac{c}{\gamma}} \left(\frac{c}{\gamma} - t\right)^k dt + \int_{\frac{c}{\gamma}}^\beta \left(t - \frac{c}{\gamma}\right)^k dt \\ &= \frac{(c - \alpha\gamma)^{k+1} + (\beta\gamma - c)^{k+1}}{(q+1)\gamma}. \end{aligned}$$

Substituting for γ and c from (2.31) gives

$$\int_\alpha^\beta |\Phi(t)|^k \, dt = \frac{(\beta - a)^{k+1}}{[b - a - (\beta - \alpha)]} \left[\frac{(\alpha - a)^{k+1} + (b - \beta)^{k+1}}{k+1}\right]$$

and thus, from (2.9), after a little algebra, we obtain the first two inequalities for $k = 1$ and q respectively.

Now, for the third inequality, from (2.9) and (2.30),

$$\begin{aligned} \sup_{t \in [\alpha,\beta]} |\Phi(t)| &= \max\{|\gamma\alpha - c|, |\gamma\beta - c|\} \\ &= (\beta - \alpha)\max\{\alpha - a, b - \beta\} = (\beta - \alpha)\Theta, \end{aligned}$$

where Θ is as given by (2.11) and hence the corollary is proved. ∎

Corollary 5. *Let the conditions of Theorem 2 hold. Then*

$$\begin{aligned} &\left|\int_a^x f(u)\,du - \left(\frac{x-a}{b-a}\right)\int_a^b f(u)\,du\right| \\ &\leq \begin{cases} \|f'\|_\infty \frac{(x-a)(b-x)}{2}, & f' \in L_\infty[a,b]; \\ \frac{\|f'\|_p}{(q+1)^{\frac{1}{q}}} \cdot \frac{(x-a)(b-x)}{(b-a)^{1-\frac{1}{q}}}, & f' \in L_p[a,b],\ p > 1,\ \frac{1}{p} + \frac{1}{q} = 1; \\ \|f'\|_1 \frac{(x-a)(b-x)}{b-a}, & f' \in L_1[a,b]. \end{cases} \end{aligned} \tag{2.32}$$

Proof. Take $u \equiv 1$ in Theorem 2 with $\beta = x$ and $\alpha = a$, or, alternatively, and perhaps the easier route, take $\beta = x$ and $\alpha = a$ in Corollary 4. This produces the result (2.32) on multiplication by $x - a$. ∎

Remark 4. *The tightest bound from (2.32) is obtained by taking $x = \frac{a+b}{2}$ to give*

$$\left| \int_a^{\frac{a+b}{2}} f(u)\,du - \frac{1}{2}\int_a^b f(u)\,du \right|$$

$$\leq \begin{cases} \frac{(b-a)^2}{8}\left\|f'\right\|_\infty, & f' \in L_\infty[a,b]; \\ \frac{(b-a)^{1+\frac{1}{q}}}{4(q+1)^{\frac{1}{q}}}\left\|f'\right\|_p, & f' \in L_p[a,b],\ p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{b-a}{4}\left\|f'\right\|_1, & f' \in L_1[a,b]. \end{cases}$$

It may be noticed that these bounds are sharper than those of Remark 3. As a matter of fact, it may be shown that the bounds given by Corollary 5 are better than those of (2.32) except for the case $f' \in L_1[a,b]$ for $b-a < \frac{4}{3}$.

Remark 5. *If we allow $\int_a^b f(u)\,du = 0$, then Corollary 5 reproduces the results of a comprehensive article by Fink* [6] *dealing with Ostrowski, Mahajani and Iyengar type inequalities, In fact, the first inequality in (2.32) with $\int_a^b f(u)\,du = 0$ is superior to the Mahajani inequality*

$$\left| \int_a^x f(x)\,dx \right| \leq \frac{(b-a)^2}{8}\left\|f'\right\|_\infty \tag{2.33}$$

except at $x = \frac{a+b}{2}$.

It is important to note that the Mahajani inequality (2.33) and the Mahajani type generalisations of Fink [6], which are recaptured as a special case of (2.32) $\left(\int_a^b f(u)\,du = 0\right)$ effectively involve obtaining bounds on the area over a specific subinterval $[a,x]$ of $[a,b]$. The following corollary may be viewed as Mahajani type inequalities over **any** subinterval.

Corollary 6. *Let $f : [a,b] \to \mathbb{R}$ be an absolutely continuous mapping with $[\alpha,\beta] \subseteq [a,b]$ and $\int_a^b f(u)\,du = 0$. Then*

$$\left| \int_\alpha^\beta f(u)\,du \right| \tag{2.34}$$

$$\leq \begin{cases} \frac{\rho}{2[1-\rho]}\left[(\alpha-a)^2+(b-\beta)^2\right]\left\|f'\right\|_\infty, & f' \in L_\infty[a,b]; \\ \rho\left[\frac{(\alpha-a)^{q+1}+(b-\beta)^{q+1}}{(q+1)(1-\rho)}\right]^{\frac{1}{q}}\left\|f'\right\|_p, & f' \in L_p[a,b],\ p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \rho\left[\frac{b-a}{2}-\frac{\beta-\alpha}{2}+\left|\frac{b+a}{2}-\frac{\beta+\alpha}{2}\right|\right]\left\|f'\right\|_1, & f' \in L_1[a,b], \end{cases}$$

where $\rho = \frac{\beta-\alpha}{b-a}$.

Proof. From Corollary 4 putting $\int_a^b f(u)\,du = 0$ and multiplying both sides by $\beta-\alpha$ readily produces the result (2.34). ∎

Remark 6. *If $\mathfrak{M}(a,b;f)$ is taken to be zero in any of the earlier results, then they may be looked upon as weighted Mahajani type inequalities over arbitrary subintervals $[\alpha,\beta]$. Further, the condition of $\int_a^b f(u)\,du = 0$ may be done away with* *if we consider a function shifted by its mean. That is, taking* $f(x) = g(x) - \frac{1}{b-a}\int_a^b g(u)\,du$.

The following theorem provides bounds in terms of Lebesgue norms over a subinterval.

Theorem 4. *Let* $f : [a,b] \to \mathbb{R}$ *be an absolutely continuous mapping as is also* $u : [\alpha,\beta] \subseteq [a,b]$. *Then*

$$\left|\int_\alpha^\beta u(x) f(x)\,dx - \mathfrak{M}(a,b;f)\int_\alpha^\beta u(x)\,dx\right| \tag{2.35}$$
$$\leq \begin{cases} \|S(f)\|_{\infty,s}\int_\alpha^\beta |u(x)|\,dx, & f \in L_\infty[\alpha,\beta]; \\ \|S(f)\|_{p,s}\left(\int_\alpha^\beta |u(x)|^q\,dx\right)^{\frac{1}{q}}, & f \in L_p[\alpha,\beta],\ p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \|S(f)\|_{1,s}\sup\limits_{x\in[\alpha,\beta]}|u(x)|, & f \in L_1[\alpha,\beta], \end{cases}$$

where $\mathfrak{M}(a,b;f)$ *is as given by (2.2),*

$$S(f(x)) = f(x) - \mathfrak{M}(a,b;f) \tag{2.36}$$

and $\|\cdot\|_{p,s}$, $1 \leq p \leq \infty$ *are the Lebesgue norms on the* ***s****ubinterval* $[\alpha,\beta]$. *That is,*

$$\|S(g)\|_{\infty,s} \; := ess\sup_{t\in[\alpha,\beta]}|S(g(t))|$$
$$\text{and}\quad \|S(f)\|_{p,s} \; := \left(\int_\alpha^\beta |S(g(t))|^p\,dt\right)^{\frac{1}{p}},\ 1 \leq p < \infty.$$

Proof. From (1.1) and (2.3) we obtain the identity

$$\int_\alpha^\beta u(x) f(x)\,dx - \mathfrak{M}(a,b;f)\int_\alpha^\beta u(x)\,dx = \int_\alpha^\beta u(x) S(f(x))\,dx, \tag{2.37}$$

where $S(f(\cdot))$ is a shift operator as defined by (2.36). From (2.37), using the properties of modulus and integral together with Hölder's integral inequality gives

$$\left|\int_\alpha^\beta u(x) S(f(x))\,dx\right|$$
$$\leq \begin{cases} \|S(f)\|_{\infty,s}\int_\alpha^\beta |u(x)|\,dx, & f \in L_\infty[\alpha,\beta]; \\ \|S(f)\|_{p,s}\left(\int_\alpha^\beta |u(x)|^q\,dx\right)^{\frac{1}{q}}, & f \in L_p[\alpha,\beta],\ p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \|S(f)\|_{1,s}\sup\limits_{x\in[\alpha,\beta]}|u(x)|, & f \in L_1[\alpha,\beta]. \end{cases}$$

That is, substitution into the right hand side of the modulus of (2.37) gives (2.35) and the theorem is proved. ∎

Remark 7. *The equivalent of the shifted norms has appeared in the work of Dragomir and McAndrew* [4] *in which they obtained bounds for perturbed trapezoidal rules in terms of the norms of functions shifted by their average. That is, the Lebesgue norms of (2.36).*

3. Results Involving Moments

In this section we investigate inequalities involving moments. Let

$$\text{(3.1)} \qquad \begin{cases} m_n(\gamma) = \int_\alpha^\beta (x-\gamma)^n f(x)\,dx \\ \text{and} \\ M_n(\gamma) = \int_a^b (x-\gamma)^n f(x)\,dx \end{cases}$$

with $[\alpha, \beta] \subseteq [a, b]$.

That is, m represents moments about γ of the subinterval $[\alpha, \beta]$ while M represents moments about γ of the interval $[a, b]$. It should be noted that if $\gamma = 0$, then (3.1) produces the moments about the origin, while taking $\gamma = m_1(0)$ (or $\gamma = M_1(0)$) gives the central moments.

The following theorem holds.

Theorem 5. *Let $f : [a, b] \to \mathbb{R}$ be an absolutely continuous mapping with $\gamma \in \mathbb{R}$ and $[\alpha, \beta] \subseteq [a, b]$. Then*

$$\text{(3.2)} \quad |m_n(\gamma) - \mathfrak{M}(a, b; f) A(\alpha, \beta; \gamma)|$$
$$\leq \begin{cases} \frac{\|f'\|_\infty}{2(b-a)} \Psi_1(a, \alpha, \beta, b; \gamma), & f' \in L_\infty[a, b]; \\ \frac{\|f'\|_p}{(q+1)^{\frac{1}{q}}(b-a)} \Psi_q(a, \alpha, \beta, b; \gamma), & f' \in L_p[a, b], \\ & p > 1,\ \frac{1}{p} + \frac{1}{q} = 1; \\ \|f'\|_1 \left[\frac{\Theta(\alpha,\beta;\gamma)}{2} + \frac{1}{b-a}\int_{\alpha-\gamma}^{\beta-\gamma} |v|^n \left|v - \left(\frac{a+b}{2} - \gamma\right)\right| dv\right], & f' \in L_1[a, b], \end{cases}$$

where $\mathfrak{M}(a, b; f)$ is as defined by (2.2),

$$\text{(3.3)} \quad (n+1) A(\alpha, \beta; \gamma) = (\beta - \gamma)^{n+1} - (\alpha - \gamma)^{n+1},$$

$$\text{(3.4)} \quad \Psi_r(a, \alpha, \beta, b; \gamma) = \int_{\alpha-\gamma}^{\beta-\gamma} |v|^n \left[(v + \gamma - a)^{r+1} + (b - \gamma - v)^{r+1}\right]^{\frac{1}{r}} dv$$

and

$$\text{(3.5)} \qquad (n+1)\Theta(\alpha, \beta; \gamma) = \begin{cases} (n+1) A(\alpha, \beta; \gamma), & \gamma \leq \alpha; \\ (\beta - \gamma)^{n+1} + (\alpha - \gamma)^{n+1}, & \alpha < \gamma \leq \beta; \\ (\gamma - \alpha)^{n+1} - (\gamma - \beta)^{n+1}, & \gamma > \beta. \end{cases}$$

Proof. Taking $u(x) = (x - \gamma)^n$ in (2.1) readily gives the left hand side of (3.2) after some minor algebra. Now, for the bounds. For $1 \leq r < \infty$ then the substitution $v = x - \gamma$ into

$$\int_\alpha^\beta |x - \gamma|^n \left[(x - a)^{r+1} + (b - x)^{r+1}\right]^{\frac{1}{r}} dx$$

produces $\Psi_r(a, \alpha, \beta, b; \gamma)$ as given by (3.4).

The last inequality is obtained on noting that

$$\int_{\alpha}^{\beta}|x-\gamma|^{n}dx=\int_{\alpha-\gamma}^{\beta-\gamma}|v|^{n}dv=\begin{cases}\int_{\alpha-\gamma}^{\beta-\gamma}v^{n}dv, & \gamma\leq\alpha;\\ \\ \int_{0}^{\gamma-\alpha}v^{n}dv+\int_{0}^{\beta-\gamma}v^{n}dv, & \alpha<\gamma\leq\beta;\\ \\ \int_{\gamma-\beta}^{\gamma-\alpha}v^{n}dv, & \gamma>\beta,\end{cases}$$

which, on evaluation, produces $\Theta(\alpha,\beta;\gamma)$ as given by (3.5).

Further, $\int_{\alpha}^{\beta}|x-\gamma|^{n}\left|x-\frac{a+b}{2}\right|dx$ produces the integral term in the third inequality of (3.2) on making the substitution $v=x-\gamma$. ∎

Theorem 6. *Let $f:[a,b]\rightarrow\mathbb{R}$ be an absolutely continuous mapping $\gamma\in\mathbb{R}$ and $[\alpha,\beta]\subseteq[a,b]$. Then*

$$\begin{aligned}&|m_{n}(\gamma)-\mathfrak{M}(a,b;f)A(\alpha,\beta;\gamma)| \tag{3.6}\\ &\leq\begin{cases}\frac{\|f'\|_{\infty}}{b-a}\left\{\frac{|A(\alpha,\beta;\gamma)|}{2}\left[(\alpha-a)^{2}+(b-\beta)^{2}\right]\right.\\ \qquad\left.+\chi_{1}(a,\alpha,\beta,b;\gamma)\right\}, & f'\in L_{\infty}[a,b];\\ \\ \frac{\|f'\|_{p}}{b-a}\left\{\frac{|A(\alpha,\beta;\gamma)|^{q}}{q+1}\left[(\alpha-a)^{q+1}+(b-\beta)^{q+1}\right]\right.\\ \qquad\left.+\chi_{q}(a,\alpha,\beta,b;\gamma)\right\}^{\frac{1}{q}}, & f'\in L_{p}[a,b],\ p>1,\ \frac{1}{p}+\frac{1}{q}=1;\\ \\ \frac{\|f'\|_{1}}{b-a}\max\left\{|A(\alpha,\beta;\gamma)|\,\Theta,\sup_{t\in[\alpha,\beta]}|\phi(t)|\right\}, & f'\in L_{1}[a,b],\end{cases}\end{aligned}$$

where $\mathfrak{M}(a,b;f)$ is as given by (2.2), $(n+1)A(\alpha,\beta;\gamma)$ is as given by (3.3)

$$\begin{aligned}&\chi_{r}(a,\alpha,\beta,b;\gamma) \tag{3.7}\\ &=\frac{1}{(n+1)^{r}}\int_{\alpha-\gamma}^{\beta-\gamma}\Big|(b-a)v^{n+1}+v\left[(\alpha-\gamma)^{n+1}-(\beta-\gamma)^{n+1}\right]\\ &\qquad+(b-\gamma)(\alpha-\gamma)^{n+1}-(\gamma-a)(\beta-\gamma)^{n+1}\Big|^{r}dv\end{aligned}$$

and

$$\begin{aligned}&(n+1)|\phi(t)| \tag{3.8}\\ &=\Big|(b-a)(t-\gamma)^{n+1}+(t-\gamma)\left[(\alpha-\gamma)^{n+1}-(\beta-\gamma)^{n+1}\right]\\ &\qquad+(b-\gamma)(\alpha-\gamma)^{n+1}-(\gamma-a)(\beta-\gamma)^{n+1}\Big|.\end{aligned}$$

Proof. Taking $u(x)=(x-\gamma)^{n}$ in (2.9) gives the left hand side of (3.6). For the bounds.

From (2.10), with $u(x)=(x-\gamma)^{n}$, we obtain

$$\phi(t)=(t-a)A(t,\beta;\gamma)-(b-t)A(\alpha,t;\gamma),$$

where $A(\alpha,\beta;\gamma)$ is as given by (3.3) and thus

$$\begin{aligned}&(n+1)\left|\phi(t)\right|\\ = & \left|(t-a)\left[(\beta-\gamma)^{n+1}-(t-\gamma)^{n+1}\right]-(b-t)\left[(t-\gamma)^{n+1}-(\alpha-\gamma)^{n+1}\right]\right|\end{aligned}$$

and so

$$\begin{aligned}&(n+1)\left|\phi(t)\right|\\ = & \left|(b-a)(t-\gamma)^{n+1}-\left[(t-a)(\beta-\gamma)^{n+1}+(b-t)(\alpha-\gamma)^{n+1}\right]\right|,\end{aligned}\tag{3.9}$$

which produces (3.8) on expressing it as a polynomial in terms of $t-\gamma$.

Hence,

$$\begin{aligned}&\int_\alpha^\beta \left|\phi(t)\right|^r dt\\ = & \frac{1}{(n+1)^r}\int_\alpha^\beta \left|(b-a)(t-\gamma)^{n+1}+(t-\gamma)\left[(\alpha-\gamma)^{n+1}-(\beta-\gamma)^{n+1}\right]\right.\\ & \left.+(b-\gamma)(\alpha-\gamma)^{n+1}-(\gamma-a)(\beta-\gamma)^{n+1}\right|^r dt,\end{aligned}$$

which, on substitution of $v=t-\gamma$ produces $\chi_r(a,\alpha,\beta,b;\gamma)$ as defined in (3.7).

The last inequality in (3.6) is obtained from the third inequality in (2.9) and from (2.10) on taking $u(x)=(x-\gamma)^n$. ∎

Remark 8. *The above two theorems provide quite general results, producing bounds for the moments over a subinterval in terms of the $L_p[a,b]$ norms of the derivative of the function. Taking $\alpha=a$ and $\beta=b$ gives results involving $M_n(\gamma)$ rather than $m_n(\gamma)$ as defined by (3.1). As stated previously at the start of this section, taking $\gamma=0$ and $\gamma=m_1(0)$ (or $M_1(0)$) produces the moments about the origin and the central moments respectively. Taking $n=0$ reproduces the corollaries of the previous section.*

The following corollaries investigate in some detail, but just a few specialisations. We will restrict the examples to taking $\alpha=a$ and $\beta=b$.

Corollary 7. *Let $f:[a,b]\to\mathbb{R}$ be absolutely continuous and $\gamma\in\mathbb{R}$. Then*

$$\begin{aligned}&\left|M_n(\gamma)-\mathfrak{M}(a,b;f)\frac{(b-\gamma)^{n+1}-(a-\gamma)^{n+1}}{n+1}\right|\\ \le & \begin{cases}\dfrac{\|f'\|_\infty\tilde{\chi}_1(\gamma)}{n+1}, & f'\in L_\infty[a,b];\\[2ex] \dfrac{\|f'\|_p\tilde{\chi}_q^{\frac{1}{q}}(\gamma)}{n+1}, & f'\in L_p[a,b],\ p>1,\ \frac{1}{p}+\frac{1}{q}=1;\\[2ex] \dfrac{\|f'\|_1\sup_{t\in[a,b]}\left|\tilde{\phi}(t)\right|}{n+1}, & f'\in L_1[a,b],\end{cases}\end{aligned}\tag{3.10}$$

where

$$\tilde{\phi}(t)=(t-\gamma)^{n+1}-\left[\left(\frac{t-a}{b-a}\right)(b-\gamma)^{n+1}+\left(\frac{b-t}{b-a}\right)(a-\gamma)^{n+1}\right]\tag{3.11}$$

and

$$\tilde{\chi}_r(\gamma) = \int_a^b \left|\tilde{\phi}(t)\right|^r dt. \tag{3.12}$$

Proof. Taking $\alpha = a$ and $\beta = b$ in Theorem 6, we obtain that $\tilde{\phi}(t) = \frac{(n+1)\phi(t)}{b-a}$ from (3.9) as given in (3.11) and $\tilde{\chi}_r(\gamma) = \frac{(n+1)^r}{b-a}\chi_r(a,a,b,b;\gamma)$ as shown by (3.12). ∎

The results of Corollary 7 may be simplified if the nature of $\tilde{\phi}(t)$ as given by (3.11) were known. The following lemma examines the behaviour of $\tilde{\phi}(t)$.

Lemma 2. *For $\tilde{\phi}(t)$ given by (3.11) we have*

$$\tilde{\phi}(t)\begin{cases} <0 & \begin{cases} n \text{ odd}, & \text{any } \gamma \text{ and } t\in(a,b) \\ n \text{ even} & \begin{cases} \gamma<a, & t\in(a,b) \\ a<\gamma<b, & t\in[c,b) \end{cases} \end{cases} \\ >0, & n \text{ even} \begin{cases} \gamma>b, & t\in(a,b) \\ a<\gamma<b, & t\in(a,c) \end{cases} \end{cases}. \tag{3.13}$$

where $\tilde{\phi}(c) = 0$, $a < c < b$ and

$$c\begin{cases} >\gamma, & \gamma<\frac{a+b}{2} \\ =\gamma, & \gamma=\frac{a+b}{2} \\ <\gamma, & \gamma>\frac{a+b}{2}. \end{cases}$$

Proof. From (3.11), $\tilde{\phi}(a) = \tilde{\phi}(b) = 0$.
Further,

$$\tilde{\phi}'(t) = (n+1)(t-\gamma)^n - \frac{(b-\gamma)^{n+1} - (a-\gamma)^{n+1}}{b-a} \tag{3.14}$$

and

$$\tilde{\phi}''(t) = (n+1)\,n\,(t-\gamma)^{n-1}. \tag{3.15}$$

Thus, for n odd, $\tilde{\phi}''(t) > 0$, $t \in [a,b]$ and so $\tilde{\phi}(t) < 0$ for $t \in (a,b)$.

For n even, the behaviour depends also on γ and t. $\tilde{\phi}''(t) > 0$ for any $t \in [a,b]$ if $\gamma < a$ and for $t \in (c,b)$ if $a < \gamma < b$, where $\tilde{\phi}(c) = 0$. Thus, $\tilde{\phi}(t) < 0$ over these regions.

Now $\tilde{\phi}''(t) < 0$ for $t - \gamma < 0$. That is, for $t \in [a,b]$ if $\gamma > b$ and for $t \in (a,c)$ if $a < \gamma < b$, where $\tilde{\phi}(c) = 0$. Hence $\tilde{\phi}(t) > 0$ for these cases and the lemma is proved. It is straightforward to see that as $\tilde{\phi}(a) = \tilde{\phi}(b) = 0$ and ϕ is concave, then c relative to γ is as stated in the lemma. ∎

Lemma 3. *For $\tilde{\chi}_1(t)$ as given by (3.12) and (3.11), we have from*

$$\tilde{\chi}_1(t) \tag{3.16}$$

$$= \begin{cases} \frac{B-A}{2}\left[B^{n+1}-A^{n+1}\right]-\frac{B^{n+2}-A^{n+2}}{n+2}, & \left\{\begin{array}{l} n \text{ odd and any } \gamma \\ n \text{ even and } \gamma < a \end{array}\right. ; \\ \frac{2C^{n+2}-B^{n+2}-A^{n+2}}{n+2}+\frac{1}{2(b-a)}\left\{\left[(b-a)^2-2(c-a)^2\right]B^{n+1}\right. & \\ \qquad \left.+\left[2(b-c)^2-(b-a)^2\right]\right\}A^{n+1}, & n \text{ even and } a<\gamma<b; \\ \frac{B^{n+2}-A^{n+2}}{n+2}-\frac{B-A}{2}\left[B^{n+1}-A^{n+1}\right], & n \text{ even and } \gamma > b, \end{cases}$$

where

$$\begin{cases} B=b-\gamma,\ A=a-\gamma,\ C=c-\gamma, \\ C_1=\int_a^c C(t)\,dt,\ \ C_2=\int_c^b C(t)\,dt, \\ \text{with} \quad C(t)=\left(\frac{t-a}{b-a}\right)B^{n+1}+\left(\frac{b-t}{b-a}\right)A^{n+1} \end{cases} \tag{3.17}$$

and $\tilde{\phi}(c)=0$ with $a<c<b$.

Proof. From (3.11)

$$\tilde{\phi}(t)=(t-\gamma)^{n+1}-C(t), \tag{3.18}$$

where $C(t)$ is as given in (3.17).

Now,

$$\int_a^c (t-\gamma)^{n+1}\,dt=\frac{C^{n+2}-A^{n+2}}{n+2},\quad \int_c^b (t-\gamma)^{n+1}\,dt=\frac{B^{n+2}-C^{n+2}}{n+2}$$

and so

$$\int_a^b (t-\gamma)^{n+1}\,dt=\frac{B^{n+2}-A^{n+2}}{n+2}.$$

In addition,

$$\begin{aligned} C_1 &= \int_a^c C(t)\,dt=\frac{1}{2(b-a)}\left\{(c-a)^2B^{n+1}+\left[(b-a)^2-(b-c)^2\right]A^{n+1}\right\}, \\ C_2 &= \int_\gamma^b C(t)\,dt=\frac{1}{2(b-a)}\left\{\left[(b-a)^2-(c-a)^2\right]B^{n+1}+(b-c)^2A^{n+1}\right\}, \end{aligned}$$

and

$$C_1+C_2=\int_a^b C(t)\,dt=\frac{B-A}{2}\left(B^{n+1}+A^{n+1}\right).$$

Thus, using Lemma 2, (3.11), (3.12) and (3.18), then gives the results as stated in the lemma, after some algebraic manipulation. ∎

Lemma 4. *For $\tilde{\phi}(t)$ as defined by (3.11), then*

$$\sup_{t\in[a,b]}\left|\tilde{\phi}(t)\right| = \begin{cases} C(t^*) - \frac{B^{n+1}-A^{n+1}}{(n+1)(B-A)}, & n \text{ odd, } n \text{ even and } \gamma < a; \\ \\ \frac{B^{n+1}-A^{n+1}}{(n+1)(B-A)} - C(t^*) & n \text{ even and } \gamma > b; \\ \\ \frac{m_1+m_2}{2} + \left|\frac{m_1-m_2}{2}\right| & n \text{ even and } a < \gamma < b, \end{cases} \tag{3.19}$$

where $(t^* - \gamma)^n = \frac{B^{n+1}-A^{n+1}}{(n+1)(B-A)}$, $C(t)$ is as defined in (3.17), $m_1 = \tilde{\phi}(t_1^*)$, $m_2 = -\tilde{\phi}(t_2^*)$ and t^*, t_1^*, t_2^* are given by (3.20) and (3.21).

Proof. From Lemma 2, we know that $\tilde{\phi}(a) = \tilde{\phi}(b) = 0$ and so the maximum occurs at t^* where $\tilde{\phi}'(t^*) = 0$, that is, from (3.14)

$$(n+1)(t^* - \gamma)^n - \frac{B^{n+1}-A^{n+1}}{B-A} = 0. \tag{3.20}$$

For n even and $a < \gamma < b$, then there are two solutions to (3.20). Let these be t_1^* and t_2^* with $t_1^* < t_2^*$.

That is,

$$\begin{aligned} t_1^* &= \gamma - \left(\frac{B^{n+1}-A^{n+1}}{(n+1)(B-A)}\right)^{\frac{1}{n}}, \\ t_2^* &= \gamma + \left(\frac{B^{n+1}-A^{n+1}}{(n+1)(B-A)}\right)^{\frac{1}{n}}. \end{aligned} \tag{3.21}$$

Now, using the fact that

$$\max\{m_1, m_2\} = \frac{m_1+m_2}{2} + \left|\frac{m_1-m_2}{2}\right|,$$

the proof of the lemma is thus completed. ∎

Corollary 8. *Let $f : [a,b] \to \mathbb{R}_+$ be an absolutely continuous p.d.f. associated with a random variable X, then, the expectation $E[X]$ satisfies the inequalities*

$$\left|E(X) - \frac{a+b}{2}\right| \tag{3.22}$$

$$\leq \begin{cases} \frac{(b-a)^3}{6}\|f'\|_\infty, & f' \in L_\infty[a,b]; \\ \\ \left(\frac{b-a}{2}\right)^{2+\frac{1}{q}}\left[\int_0^{\frac{\pi}{4}}\sec^{2(q+1)}(\theta)\,d\theta\right]^{\frac{1}{q}}\|f'\|_p, & f' \in L_p[a,b],\ p>1, \\ & \frac{1}{p}+\frac{1}{q}=1; \\ \frac{(b-a)^2}{8}\|f'\|_1, & f' \in L_1[a,b], \end{cases}$$

where $E[X] = \int_a^b xf(x)\,dx$.

Proof. Taking $n = 1$ in Corollary 7 and Using Lemmae 2 - 4, gives the above results after some elementary algebra. In particular,

$$\tilde{\phi}(t) = t^2 - (a+b)t + ab = \left(t - \frac{a+b}{2}\right)^2 + \left(\frac{b-a}{2}\right)^2.$$

and t^*, the one solution to $\tilde{\phi}'(t) = 0$, is $t^* = \frac{a+b}{2}$. ∎

Corollary 9. *Let $f:[a,b]\to\mathbb{R}_+$ be an absolutely continuous p.d.f. associated with a random variable X. Then the variance, $\sigma^2(X)$ is such that*

$$\left|\sigma^2(X)-S\right| \tag{3.23}$$
$$\leq\begin{cases}\left\{\frac{C^4}{2}-\frac{1}{b-a}\left[(c-a)^3B^3-(b-c)^2A^3\right]\right.\\ \quad\left.+\left(B^2+A^2\right)\left(\frac{b-a}{2}\right)^2-\frac{(AB)^2}{2}\right\}\frac{\|f'\|_\infty}{3}, & f'\in L_\infty[a,b];\\ \left[\int_a^b\left|\hat{\phi}(t)\right|^q dt\right]^{\frac{1}{q}}\frac{\|f'\|_p}{3}, & f'\in L_p[a,b],\\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1;\\ \left[m_1+m_2+|m_2-m_1|\right]\frac{\|f'\|_1}{6}, & f'\in L_1[a,b],\end{cases}$$

where

$$S=\frac{B^3-A^3}{3(b-a)},$$
$$\begin{cases} & A=a-\gamma,\ B=b-\gamma,\ C=c-\gamma,\ \gamma=E[X],\\ & c\text{ satisfies }\hat{\phi}(c)=0,\ a<c<b,\\ \text{with} & m_1=\hat{\phi}\left[E[X]-S^{\frac{1}{2}}\right],\ m_2=\hat{\phi}\left[E[X]+S^{\frac{1}{2}}\right]\\ \text{and} & \hat{\phi}(\cdot)\ \text{as given by (2.23).}\end{cases} \tag{3.24}$$

$a<\gamma=E(X)<b.$

Proof. Taking $n=2$ in Corollary 7 gives from (3.11)

$$\hat{\phi}(t)=(t-\gamma)^3+\left(\frac{b-t}{b-a}\right)(\gamma-a)^3-\left(\frac{t-a}{b-a}\right)(b-\gamma)^3, \tag{3.25}$$

where $a<\gamma=E(X)<b$.

Now, from Lemmas 2 and 3 with $n=2$ and $a<\gamma<b$ gives

$$\begin{aligned}\tilde{\chi}_1(t) &= \frac{2C^4-B^4-A^4}{4}+\frac{1}{2(b-a)}\left\{\left[(b-a)^2-2(c-a)^2\right]B^3\right.\\ &\quad\left.+\left[2(b-c)^2-(b-a)^2\right]A^3\right\}\\ &= \frac{C^4}{2}-\frac{1}{b-a}\left[(c-a)^2B^3-(b-c)^2A^3\right]\\ &\quad+\frac{B-A}{2}\left[B^3-A^3\right]-\frac{B^4+A^4}{4}.\end{aligned} \tag{3.26}$$

Now,

$$\begin{aligned}&\frac{B-A}{2}\left[B^3-A^3\right]-\frac{B^4+A^4}{4}\\ &= \frac{1}{4}\left\{2(B-A)\left(B^3-A^3\right)-\left(B^4+A^4\right)\right\}\\ &= \frac{1}{4}\left[B^4+A^4-2AB\left(B^2+A^2\right)\right]\\ &= \frac{1}{4}\left[\left(B^2+A^2\right)^2-2(AB)^2-2AB\left(B^2+A^2\right)\right]\\ &= \frac{1}{4}\left[\left(B^2+A^2\right)(B-A)^2-2(AB)^2\right]\end{aligned}$$

and so substitution into (3.26) gives the first inequality in (3.23) for $f' \in L_\infty[a,b]$ on using (3.16) and the fact that $B - A = b - a$.

For $f' \in L_p[a,b]$, the bound is not given explicitly but is as presented in (3.10) with $\tilde{\phi}(t)$ replaced by $\hat{\phi}(t)$ from (3.25).

Now, for $f' \in L_1[a,b]$, using Lemma 3 with $n = 2$ and in particular (3.19) and (3.21) gives the stated result. The corollary is thus, now, completely proved. ∎

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