Monotonicity of Sequences Involving Convex Function and Sequence

MONOTONICITY OF SEQUENCES INVOLVING CONVEX FUNCTION AND SEQUENCE

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Abstract. Let f be an increasing, convex (concave, respectively) function defined on $[0,1]$, $\{a_i\}_{i\in\mathbb{N}}$ an increasing, positive sequence such that $\{i(a_i/a_{i+1}-1)\}_{i\in\mathbb{N}}$ decreases (the sequence $\{i(a_{i+1}/a_i-1)\}_{i\in\mathbb{N}}$ increses, respectively), then the sequence $\{(1/n)\sum_{i=1}^{n} f(a_i/a_n)\}_{n\in\mathbb{N}}$ is decreasing.

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1. Introduction

Let I be an interval in $\mathbb{R}$. Then $f : I \to \mathbb{R}$ is said to be convex if for all $x, y \in I$ and $\lambda \in [0,1]$,

$$f(\lambda x + (1-\lambda)y) \leqslant \lambda f(x) + (1-\lambda) f(y). \tag{1}$$

If (1) is strict for all $x \neq y$ and $\lambda \in (0,1)$, then f is said to be strictly convex.

If the inequality in (1) is reversed, then f is said to be concave. If inequality (1) is reversed and strict for all $x \neq y$ and $\lambda \in (0,1)$, then f is said to be strictly concave.

The finite difference of a sequence $\{a_i\}_{i\in\mathbb{N}}$ can be defined by

$$\Delta^0 a_i = a_i, \qquad \Delta a_i = a_{i+1} - a_i, \qquad \Delta^m a_i = \Delta(\Delta^{m-1} a_i). \tag{2}$$

We shall say that a sequence $\{a_i\}_{i\in\mathbb{N}}$ is convex of order m (m-convex) if $\Delta^m a_i \geqslant 0$ for $m \geqslant 0$, $i \in \mathbb{N}$. If $\{a_i\}_{i\in\mathbb{N}}$ is 2-convex, we have $a_{i+1} + a_{i-1} \geqslant 2a_i$ for $i \geqslant 2$, the sequence $\{a_i\}_{i\in\mathbb{N}}$ is called convex; if $a_{i+1} + a_{i-1} \leqslant 2a_i$ for $i \geqslant 2$,, we call $\{a_i\}_{i\in\mathbb{N}}$ being concave.

Let $\{a_i\}_{i\in\mathbb{N}}$ be a positive sequence. If $a_{i+1}a_{i-1} \geqslant a_i^2$ for $i \geqslant 2$, we call $\{a_i\}_{i\in\mathbb{N}}$ a logarithmically convex sequence; if $a_{i+1}a_{i-1} \leqslant a_i^2$ for $i \geqslant 2$,, we call $\{a_i\}_{i\in\mathbb{N}}$ a logarithmically concave sequence.

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Let f be a strictly increasing convex (or concave) function in $(0,1]$, Professor J.-C. Kuang in [4] verified that

$$\frac{1}{n}\sum_{k=1}^{n} f\Big(\frac{k}{n}\Big) > \frac{1}{n+1}\sum_{k=1}^{n+1} f\Big(\frac{k}{n+1}\Big) > \int_0^1 f(x)\,\mathrm{d}x. \tag{3}$$

In [11], the author generalized the results in [4] and obtained the following main result and some corollaries: Let f be a strictly increasing convex (or concave) function in $(0,1]$, then the sequence $\frac{1}{n}\sum_{i=k+1}^{n+k} f\big(\frac{i}{n+k}\big)$ is decreasing in n and k and has a lower bound $\int_0^1 f(t)\,\mathrm{d}t$, that is,

$$\frac{1}{n}\sum_{i=k+1}^{n+k} f\Big(\frac{i}{n+k}\Big) > \frac{1}{n+1}\sum_{i=k+1}^{n+k+1} f\Big(\frac{i}{n+k+1}\Big) > \int_0^1 f(t)\,\mathrm{d}t, \tag{4}$$

where k is a nonnegative integer, n a natural number.

With the help of thses conclusions, we can deduce the Alzer's inequality, the Minc-Sathre's inequality, and more other inequalities involving the sum of powers of positive numbers. These inequalities have been investigated by many mathematicians. For more information, please refer to the references in this paper.

In this article, by similar method as in [4, 11], considering the convexity of a given function or sequence, using the Hermite-Hadamard inequality in [3, 7], we obtain

Theorem 1. *Let f be an increasing, convex (concave, respectively) function defined on $[0,1]$, $\{a_i\}_{i\in\mathbb{N}}$ an increasing, positive sequence such that $\{i(a_i/a_{i+1}-1)\}_{i\in\mathbb{N}}$ decreases (the sequence $\{i(a_{i+1}/a_i-1)\}_{i\in\mathbb{N}}$ increses, respectively), then the sequence $\{(1/n)\sum_{i=1}^n f(a_i/a_n)\}_{n\in\mathbb{N}}$ is decreasing. That is*

$$\frac{1}{n}\sum_{i=1}^{n} f\Big(\frac{a_i}{a_n}\Big) \geqslant \frac{1}{n+1}\sum_{i=1}^{n+1} f\Big(\frac{a_i}{a_{n+1}}\Big) \geqslant \int_0^1 f(t)\,\mathrm{d}t. \tag{5}$$

Theorem 2. *Let f be an increasing, convex (or concave), positive function defined on $[0,1]$, φ an increasing, convex, positive function defined on $(0,\infty)$ such that $\{\varphi(i)\,[\varphi(i)/\varphi(i+1)-1]\}_{i\in\mathbb{N}}$ decreases, then $\{(1/\varphi(n))\sum_{i=1}^n f(\varphi(i)/\varphi(n))\}_{n\in\mathbb{N}}$ is decreasing. That is*

$$\frac{1}{\varphi(n)}\sum_{i=1}^{n} f\left(\frac{\varphi(i)}{\varphi(n)}\right) \geqslant \frac{1}{\varphi(n+1)}\sum_{i=1}^{n+1} f\left(\frac{\varphi(i)}{\varphi(n+1)}\right). \tag{6}$$

2. Proofs of Theorems

Proof of Theorem 1. The left inequality in (5) is equivalent to

$$n\sum_{i=1}^{n+1} f\Big(\frac{a_i}{a_{n+1}}\Big) \leqslant (n+1)\sum_{i=1}^{n} f\Big(\frac{a_i}{a_n}\Big),$$

$$n\sum_{i=1}^{n} f\Big(\frac{a_i}{a_{n+1}}\Big) + nf(1) \leqslant (n+1)\sum_{i=1}^{n} f\Big(\frac{a_i}{a_n}\Big),$$

$$\begin{aligned}
n\sum_{i=1}^{n} f\Big(\frac{a_i}{a_{n+1}}\Big) &\leqslant \sum_{i=1}^{n}\Big[(i-1)f\Big(\frac{a_{i-1}}{a_n}\Big)+(n-i+1)f\Big(\frac{a_i}{a_n}\Big)\Big],\\
\sum_{i=1}^{n} f\Big(\frac{a_i}{a_{n+1}}\Big) &\leqslant \sum_{i=1}^{n}\Big[\frac{i-1}{n}f\Big(\frac{a_{i-1}}{a_n}\Big)+\Big(1-\frac{i-1}{n}\Big)f\Big(\frac{a_i}{a_n}\Big)\Big], \tag{7}
\end{aligned}$$

where we let $a_0=0$.

Since the sequence $\left\{i\left(\frac{a_i}{a_{i+1}}-1\right)\right\}_{i\in\mathbb{N}}$ decreases and $\left\{i\left(\frac{a_{i+1}}{a_i}-1\right)\right\}_{i\in\mathbb{N}}$ increases, then we have

$$n\Big(\frac{a_n}{a_{n+1}}-1\Big)\leqslant (i-1)\Big(\frac{a_{i-1}}{a_i}-1\Big), \tag{8}$$

$$n\Big(\frac{a_{n+1}}{a_n}-1\Big)\geqslant i\Big(\frac{a_{i+1}}{a_i}-1\Big). \tag{9}$$

Inequality (8) can be rewritten as

$$\frac{(i-1)a_{i-1}+(n-i+1)a_i}{na_n}\geqslant \frac{a_i}{a_{n+1}}, \tag{10}$$

and inequality (9) yields

$$\begin{aligned}
(n+1)\Big(\frac{a_{n+1}}{a_n}-1\Big)&\geqslant i\Big(\frac{a_{i+1}}{a_i}-1\Big),\\
\frac{ia_{i+1}+(n-i+1)a_i}{(n+1)a_{n+1}}&\leqslant \frac{a_i}{a_n}. \tag{11}
\end{aligned}$$

Since f is increasing, from (10) and (11), we have

$$f\left(\frac{(i-1)a_{i-1}+(n-i+1)a_i}{na_n}\right)\geqslant f\Big(\frac{a_i}{a_{n+1}}\Big), \tag{12}$$

$$f\left(\frac{ia_{i+1}+(n-i+1)a_i}{(n+1)a_{n+1}}\right)\leqslant f\Big(\frac{a_i}{a_n}\Big). \tag{13}$$

If f is convex, then

$$\frac{i-1}{n}f\Big(\frac{a_{i-1}}{a_n}\Big)+\Big(1-\frac{i-1}{n}\Big)f\Big(\frac{a_i}{a_n}\Big)\geqslant f\left(\frac{(i-1)a_{i-1}+(n-i+1)a_i}{na_n}\right). \tag{14}$$

Combination of (14) with (12) leads to

$$f\Big(\frac{a_i}{a_{n+1}}\Big)\leqslant \Big[\frac{i-1}{n}f\Big(\frac{a_{i-1}}{a_n}\Big)+\Big(1-\frac{i-1}{n}\Big)f\Big(\frac{a_i}{a_n}\Big)\Big], \tag{15}$$

inequality (7) follows.

If f is concave, then

$$\begin{aligned}
&\frac{i}{n+1}f\Big(\frac{a_{i+1}}{a_{n+1}}\Big)+\Big(1-\frac{i}{n+1}\Big)f\Big(\frac{a_i}{a_{n+1}}\Big)\\
&\leqslant f\Big(\frac{i}{n+1}\cdot\frac{a_{i+1}}{a_{n+1}}+\frac{n-i+1}{n+1}\cdot\frac{a_i}{a_{n+1}}\Big)\\
&= f\Big(\frac{ia_{i+1}+(n-i+1)a_i}{(n+1)a_{n+1}}\Big).
\end{aligned} \tag{16}$$

¿From (13) and (16), we obtain

$$\sum_{i=1}^{n}\Big[\frac{i}{n+1}f\Big(\frac{a_{i+1}}{a_{n+1}}\Big)+\Big(1-\frac{i}{n+1}\Big)f\Big(\frac{a_i}{a_{n+1}}\Big)\Big]\leqslant\sum_{i=1}^{n}f\Big(\frac{a_i}{a_n}\Big),$$

that is

$$\begin{aligned}&\sum_{i=1}^{n}\frac{i}{n+1}f\Big(\frac{a_{i+1}}{a_{n+1}}\Big)+\sum_{i=1}^{n}\Big(\frac{n}{n+1}-\frac{i-1}{n+1}\Big)f\Big(\frac{a_i}{a_{n+1}}\Big)\\&=\frac{n}{n+1}f(1)+\frac{n}{n+1}\sum_{i=1}^{n}f\Big(\frac{a_i}{a_{n+1}}\Big)\\&=\frac{n}{n+1}\sum_{i=1}^{n+1}f\Big(\frac{a_i}{a_{n+1}}\Big)\leqslant\sum_{i=1}^{n}f\Big(\frac{a_i}{a_n}\Big).\end{aligned}\tag{17}$$

The final line in (17) implies the left inequality in (5).

Finally, by definition of definite integral, the right inequality in (5) follows. □

Proof of Theorem 2. Since

$$\varphi(i)\left(\frac{\varphi(i)}{\varphi(i+1)}-1\right)\leqslant\varphi(i-1)\left(\frac{\varphi(i-1)}{\varphi(i)}-1\right),\tag{18}$$

therefore we obtain

$$\varphi(n)\left(\frac{\varphi(n)}{\varphi(n+1)}-1\right)\leqslant\varphi(i-1)\left(\frac{\varphi(i-1)}{\varphi(i)}-1\right),\tag{19}$$

that is

$$\frac{\varphi(i)}{\varphi(n+1)}\leqslant\frac{\varphi^2(i-1)+[\varphi(n)-\varphi(i-1)]\varphi(i)}{\varphi^2(n)}.\tag{20}$$

¿From monotonicity of f, letting $\varphi(0)=0$, we have

$$f\left(\frac{\varphi(i)}{\varphi(n+1)}\right)\leqslant f\left(\frac{\varphi^2(i-1)+[\varphi(n)-\varphi(i-1)]\varphi(i)}{\varphi^2(n)}\right),\tag{21}$$

$$\sum_{i=1}^{n}f\left(\frac{\varphi(i)}{\varphi(n+1)}\right)\leqslant\sum_{i=1}^{n}f\left(\frac{\varphi^2(i-1)+[\varphi(n)-\varphi(i-1)]\varphi(i)}{\varphi^2(n)}\right).\tag{22}$$

Since φ is convex and f is positive, if f is convex, then

$$\begin{aligned}&\sum_{i=1}^{n}f\left(\frac{\varphi^2(i-1)+[\varphi(n)-\varphi(i-1)]\varphi(i)}{\varphi^2(n)}\right)\\&\leqslant\sum_{i=1}^{n}\left\{\frac{\varphi(i-1)}{\varphi(n)}f\left(\frac{\varphi(i-1)}{\varphi(n)}\right)+\frac{\varphi(n)-\varphi(i-1)}{\varphi(n)}f\left(\frac{\varphi(i)}{\varphi(n)}\right)\right\}\\&\leqslant\sum_{i=1}^{n}\left\{\frac{\varphi(i-1)}{\varphi(n)}f\left(\frac{\varphi(i-1)}{\varphi(n)}\right)+\frac{\varphi(n+1)-\varphi(i)}{\varphi(n)}f\left(\frac{\varphi(i)}{\varphi(n)}\right)\right\}.\end{aligned}\tag{23}$$

¿From (22) and (23), we get

$$\sum_{i=1}^{n} f\left(\frac{\varphi(i)}{\varphi(n+1)}\right) \leqslant \sum_{i=1}^{n}\left\{\frac{\varphi(i-1)}{\varphi(n)} f\left(\frac{\varphi(i-1)}{\varphi(n)}\right)+\frac{\varphi(n+1)-\varphi(i)}{\varphi(n)} f\left(\frac{\varphi(i)}{\varphi(n)}\right)\right\}, \tag{24}$$

that is

$$\begin{aligned} &\varphi(n) \sum_{i=1}^{n} f\left(\frac{\varphi(i)}{\varphi(n+1)}\right) \\ &\leqslant \sum_{i=1}^{n}\left\{\varphi(i-1) f\left(\frac{\varphi(i-1)}{\varphi(n)}\right)+[\varphi(n+1)-\varphi(i)] f\left(\frac{\varphi(i)}{\varphi(n)}\right)\right\} \\ &=\varphi(n+1) \sum_{i=1}^{n} f\left(\frac{\varphi(i)}{\varphi(n)}\right)-\varphi(n) f(1). \end{aligned} \tag{25}$$

Inequality (25) is equivalent to

$$\varphi(n+1) \sum_{i=1}^{n} f\left(\frac{\varphi(i)}{\varphi(n)}\right) \geqslant \varphi(n) \sum_{i=1}^{n+1} f\left(\frac{\varphi(i)}{\varphi(n+1)}\right), \tag{26}$$

$$\frac{1}{\varphi(n)} \sum_{i=1}^{n} f\left(\frac{\varphi(i)}{\varphi(n)}\right) \geqslant \frac{1}{\varphi(n+1)} \sum_{i=1}^{n+1} f\left(\frac{\varphi(i)}{\varphi(n+1)}\right). \tag{27}$$

Now assume f is concave. Then

$$\begin{aligned} \frac{\varphi(i)}{\varphi(n+1)} f\left(\frac{\varphi(i+1)}{\varphi(n+1)}\right)+\left(1-\frac{\varphi(i)}{\varphi(n+1)}\right) f\left(\frac{\varphi(i)}{\varphi(n+1)}\right) \\ \leqslant f\left(\frac{\varphi(i) \varphi(i+1)+\varphi(i) \varphi(n+1)-\varphi^{2}(i)}{\varphi^{2}(n+1)}\right). \end{aligned} \tag{28}$$

Since φ is increasing and convex, then easy computation gives us

$$\frac{\varphi(i) \varphi(i+1)+\varphi(i) \varphi(n+1)-\varphi^{2}(i)}{\varphi^{2}(n+1)} \leqslant \frac{\varphi(i)}{\varphi(n)}. \tag{29}$$

Therefore, from convexity of φ, we have

$$\begin{aligned} \sum_{i=1}^{n} f\left(\frac{\varphi(i)}{\varphi(n)}\right) &\geqslant \sum_{i=1}^{n}\left\{\frac{\varphi(i)}{\varphi(n+1)} f\left(\frac{\varphi(i+1)}{\varphi(n+1)}\right)+\left(1-\frac{\varphi(i)}{\varphi(n+1)}\right) f\left(\frac{\varphi(i)}{\varphi(n+1)}\right)\right\} \\ &=\sum_{i=1}^{n+1} \frac{\varphi(n+1)+\varphi(i-1)-\varphi(i)}{\varphi(n+1)} f\left(\frac{\varphi(i)}{\varphi(n+1)}\right) \\ &\geqslant \frac{\varphi(n)}{\varphi(n+1)} \sum_{i=1}^{n} f\left(\frac{\varphi(i)}{\varphi(n+1)}\right). \end{aligned} \tag{30}$$

The proof is complete. □

3. Corollaries

As special cases of Theorem 1 and 2, we will obtain many inequalities for the sum of powers of positive numbers.

Corollary 1. *Let f be an increasing, convex (or concave) function defined on $[0,1]$, $\{a_i\}_{i\in\mathbb{N}}$ a logarithmically convex (or a logarithmically concave), increasing, positive sequence, then the sequence $\{(1/n)\sum_{i=1}^{n} f(a_i/a_n)\}_{n\in\mathbb{N}}$ is decreasing.*

Corollary 2. *Let f be an increasing, convex (or concave), positive function defined on $[0,1]$, φ an increasing, convex, logarithmically convex, positive function defined on $(0,\infty)$, then the sequence $\{(1/\varphi(n))\sum_{i=1}^{n} f(\varphi(i)/\varphi(n))\}_{n\in\mathbb{N}}$ is decreasing.*

It is clear that the function $f(x)=x^r$ is strictly increasing in $[0,1]$ for $r>0$ and is convex for $r\geqslant 1$, and is concave for $0<r<1$, take $a_i=i$ in Theorem 1, then we have

Corollary 3 ([1])**.** *Let $n\in\mathbb{N}$, then for any $r>0$, we have*

$$\frac{n}{n+1}\leqslant\left(\frac{1}{n}\sum_{i=1}^{n} i^r\Bigg/\frac{1}{n+1}\sum_{i=1}^{n+1} i^r\right)^{1/r}. \tag{31}$$

The lower bound is best possible.

If let $f(x)=x^r$, $r>0$, $x\in[0,1]$, and $a_i=i+k$, k is a given natural number, in Theorem 1, then we obtain

Corollary 4 ([10])**.** *Let n and m be natural numbers, k a nonnegative integer. Then*

$$\frac{n+k}{n+m+k}<\left(\frac{1}{n}\sum_{i=k+1}^{n+k} i^r\Bigg/\frac{1}{n+m}\sum_{i=k+1}^{n+m+k} i^r\right)^{1/r}, \tag{32}$$

where r is any given positive real number. The lower bound is best possible.

Let $a_i=a^{i+k}$, $a>1$, k is a nonnegative integer, and let $f(x)=x^r$, $r>0$, $x\in[0,1]$, in Theorem 1, then

Corollary 5. *For $a>1$, let $n\in\mathbb{N}$ and $r>0$, then*

$$\left(\frac{1}{n}\sum_{i=1}^{n} a^{ir}\Bigg/\frac{1}{n+1}\sum_{i=1}^{n+1} a^{ir}\right)^{1/r}>\frac{1}{a}. \tag{33}$$

Let $\varphi(x)=a^{x+k}$, $x>0$, $a>1$, k is a nonnegative integer, and let $f(x)=x^r$, $r>0$, $x\in[0,1]$ in Theorem 2, then

Corollary 6. *For $n,m\in\mathbb{N}$, $k\in\mathbb{N}\cup\{0\}$ and $r>0$, we have*

$$\frac{1}{a^m}<\left\{\frac{1}{a^n}\sum_{i=k+1}^{n+k} a^{ir}\Bigg/\frac{1}{a^{n+m}}\sum_{i=k+1}^{n+m+k} a^{ir}\right\}^{1/r}, \tag{34}$$

that is,

$$\frac{1}{a^{m(r+1)}} \leqslant \sum_{i=k+1}^{n+k} a^{ir} \Bigg/ \sum_{i=k+1}^{n+m+k} a^{ir} \,, \tag{35}$$

where $a > 1$ is a positive real number.

Let $f(x) = x^r$, $r > 0$, $x \in [0,1]$, $\varphi = x + k$, k is a given natural number in Theorem 2, then we have

Corollary 7. *Let n and m be natural numbers, k a nonnegative integer, then*

$$\left(\frac{1}{n+k}\sum_{i=k+1}^{n+k} i^r \Bigg/ \frac{1}{n+k+m}\sum_{i=k+1}^{n+k+m} i^r\right)^{1/r} > \frac{n+k}{n+k+m}, \tag{36}$$

where r is any given positive real number.

Since $\ln(1+x)$ and $\ln\frac{x}{1+x}$ are strictly increasing concave function in $(0,1]$, let $f(x) = \ln(1+x)$ or $f(x) = \ln\frac{x}{1+x}$ in (5) respectively, by direct calculation, we have

Corollary 8. *If $\{a_i\}_{i\in\mathbb{N}}$ is an increasing, positive sequence such that $\left\{i\left(\frac{a_{i+1}}{a_i} - 1\right)\right\}_{i\in\mathbb{N}}$ increases, then we have*

$$\frac{a_n}{a_{n+1}} \leqslant \sqrt[n]{\prod_{i=1}^{n}(a_i + a_n)} \Bigg/ \sqrt[n+1]{\prod_{i=1}^{n+1}(a_i + a_{n+1})} \leqslant \sqrt[n]{\prod_{i=1}^{n} a_i} \Bigg/ \sqrt[n+1]{\prod_{i=1}^{n+1} a_i}\,. \tag{37}$$

Let $f(x) = \ln(1+x)$ in (6), by direct computation, we obtain

Corollary 9. *If the function φ is an increasing, convex, positive function defined on $(0,\infty)$ such that $\{\varphi(i)\,[\varphi(i)/\varphi(i+1) - 1]\}_{i\in\mathbb{N}}$ decreases, then*

$$\frac{[\varphi(n)]^{n/\varphi(n)}}{[\varphi(n+1)]^{(n+1)/\varphi(n+1)}} \leqslant \sqrt[\varphi(n)]{\prod_{i=1}^{n}[\varphi(i) + \varphi(n)]} \Bigg/ \sqrt[\varphi(n+1)]{\prod_{i=1}^{n+1}[\varphi(i) + \varphi(n+1)]}\,. \tag{38}$$

Remark 1. The inequalities (37) and (38) generalize those obtained in [4], [11], and [14].

Remark 2. If taking more special functions f, φ, and $\{a_i\}_{i\in\mathbb{N}}$ in Theorem 1 and 2, we can obtain more new concrete inequalities involving sums or products of positive sequences.

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