Some New Inequalities for Jeffreys Divergence Measure in Information Theory

SOME NEW INEQUALITIES FOR JEFFREYS DIVERGENCE MEASURE IN INFORMATION THEORY

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Abstract. Some new inequalities for the well-known Jeffreys divergence measure in Information Theory are given.

1. Introduction

One of the important issues in many applications of Probability Theory is finding an appropriate measure of *distance* (or *difference* or *discrimination*) between two probability distributions. A number of divergence measures for this purpose have been proposed and extensively studied by Jeffreys [22], Kullback and Leibler [31], Rényi [42], Havrda and Charvat [20], Kapur [25], Sharma and Mittal [44], Burbea and Rao [5], Rao [41], Lin [34], Csiszár [10], Ali and Silvey [1], Vajda [52], Shioya and Da-te [45] and others (see for example [25] and the references therein).

These measures have been applied in a variety of fields such as: anthropology [41], genetics [37], finance, economics, and political science [43], [47], [48], biology [39], the analysis of contingency tables [19], approximation of probability distributions [9], [26], signal processing [23], [24] and pattern recognition [3], [8].

Assume that a set χ and the $\sigma-$finite measure μ are given. Consider the set of all probability densities on μ to be $\Omega := \left\{p|p : \chi \to \mathbb{R},\, p(x) \geq 0,\ \int_\chi p(x)\, d\mu(x) = 1\right\}$. The Kullback-Leibler divergence [31] is well known among the information divergences. It is defined as:

$$D_{KL}(p,q) := \int_\chi p(x) \log\left[\frac{p(x)}{q(x)}\right] d\mu(x)\,, \quad p,q \in \Omega, \tag{1.1}$$

where log is to base 2.

In Information Theory and Statistics, various divergences are applied in addition to the Kullback-Leibler divergence. These are the: *variation distance* D_v, *Hellinger distance* D_H [21], χ^2-*divergence* D_{χ^2}, $\alpha-$*divergence* D_α, *Bhattacharyya distance* D_B [4] , *Harmonic distance* D_{Ha}, *Jeffreys distance* D_J [22], *triangular discrimination* D_Δ [49], etc... They are defined as follows:

$$D_v(p,q) := \int_\chi |p(x) - q(x)|\, d\mu(x)\,, \quad p,q \in \Omega; \tag{1.2}$$

$$D_H(p,q) := \int_\chi \left[\sqrt{p(x)} - \sqrt{q(x)}\right]^2 d\mu(x)\,, \quad p,q \in \Omega; \tag{1.3}$$

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$$D_{\chi^2}(p,q) := \int_{\chi} p(x)\left[\left(\frac{q(x)}{p(x)}\right)^2 - 1\right] d\mu(x), \quad p,q \in \Omega; \tag{1.4}$$

$$D_{\alpha}(p,q) := \frac{4}{1-\alpha^2}\left[1 - \int_{\chi} [p(x)]^{\frac{1-\alpha}{2}} [q(x)]^{\frac{1+\alpha}{2}} d\mu(x)\right], \quad p,q \in \Omega; \tag{1.5}$$

$$D_B(p,q) := \int_{\chi} \sqrt{p(x)\,q(x)}\,d\mu(x), \quad p,q \in \Omega; \tag{1.6}$$

$$D_{Ha}(p,q) := \int_{\chi} \frac{2p(x)\,q(x)}{p(x)+q(x)}\,d\mu(x), \quad p,q \in \Omega; \tag{1.7}$$

$$D_J(p,q) := \int_{\chi} [p(x) - q(x)] \ln\left[\frac{p(x)}{q(x)}\right] d\mu(x), \quad p,q \in \Omega; \tag{1.8}$$

$$D_{\Delta}(p,q) := \int_{\chi} \frac{[p(x)-q(x)]^2}{p(x)+q(x)}\,d\mu(x), \quad p,q \in \Omega. \tag{1.9}$$

For other divergence measures, see the paper [25] by Kapur or the book on line [46] by Taneja. For a comprehensive collection of preprints available on line, see the RGMIA web site `http://rgmia.vu.edu.au/papersinfth.html`

In [35], Lin and Wong (see also [34]) introduced the following divergence

$$D_{LW}(p,q) := \int_{\chi} p(x) \log\left[\frac{p(x)}{\frac{1}{2}p(x) + \frac{1}{2}q(x)}\right] d\mu(x), \quad p,q \in \Omega. \tag{1.10}$$

In other words, Lin-Wong divergence is represented as follows, using the Kullback-Leibler divergence:

$$D_{LW}(p,q) = D_{KL}\left(p, \frac{1}{2}p + \frac{1}{2}q\right). \tag{1.11}$$

Lin and Wong have shown various inequalities as follows

$$D_{LW}(p,q) \le \frac{1}{2} D_{KL}(p,q); \tag{1.12}$$

$$D_{LW}(p,q) + D_{LW}(q,p) \le D_v(p,q) \le 2; \tag{1.13}$$

$$D_{LW}(p,q) \le 1. \tag{1.14}$$

In [45], Shioya and Da-te improved $(1.13) - (1.14)$ by showing that

$$D_{LW}(p,q) \le \frac{1}{2} D_v(p,q) \le 1. \tag{1.15}$$

For classical and new results in comparing different kinds of divergence measures, see the papers [22]-[45] where further references are given.

In [18], Dragomir and Wang proved, amongst others, the following midpoint inequality

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \le \frac{1}{4}(b-a)(\Gamma - \gamma), \tag{1.16}$$

provided that f is absolutely continuous and the derivative $f' : [a, b] \to \mathbb{R}$ satisfies the condition

$$\gamma \leq f'(t) \leq \Gamma \text{ for a.e. } t \in [a, b]. \tag{1.17}$$

With the same assumptions for the mapping f, but using a finer argument based in a "pre-Grüss" inequality, the authors of [38] improved (1.1) as follows

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(t)\, dt \right| \leq \frac{1}{4\sqrt{3}} (b-a)(\Gamma - \gamma). \tag{1.18}$$

For other results concerning the midpoint and trapezoid inequality, see the recent papers [6]-[7] and the website `http://rgmia.vu.edu.au/`.

The main aim of this paper is to point out some new midpoint and trapezoid type inequalities and apply them for the Jeffreys divergence measure D_J.

2. Some Analytic Inequalities

The following result holds.

Lemma 1. *Let $f : [a, b] \to \mathbb{R}$ be an absolutely continuous mapping on $[a, b]$ with $f' \in L_2[a, b]$. Then we have the inequality:*

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(t)\, dt \right| \leq \frac{b-a}{2\sqrt{3}} \left[\frac{1}{b-a} \|f'\|_2^2 - ([f; a, b])^2 \right]^{\frac{1}{2}}, \tag{2.1}$$

where

$$[f; a, b] := \frac{f(b) - f(a)}{b-a}.$$

Proof. Start with the following identity which can be easily proved by the integration by parts formula

$$f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(t)\, dt = \frac{1}{b-a} \int_a^b m(t) f'(t)\, dt, \tag{2.2}$$

where

$$m(t) := \begin{cases} t-a & \text{if} \quad t \in \left[a, \frac{a+b}{2}\right] \\ \\ t-b & \text{if} \quad t \in \left(\frac{a+b}{2}, b\right] \end{cases}.$$

Using *Korkine's identity*, i.e., we recall it

$$\begin{aligned} &\frac{1}{b-a} \int_a^b u(t) v(t)\, dt - \frac{1}{b-a} \int_a^b u(t)\, dt \cdot \frac{1}{b-a} \int_a^b v(t)\, dt \\ &= \frac{1}{2(b-a)^2} \int_a^b \int_a^b (u(t) - u(s))(v(t) - v(s))\, dt ds, \end{aligned} \tag{2.3}$$

and this identity can be proved by direct computation, we may write that

$$\begin{aligned} &\frac{1}{b-a} \int_a^b m(t) f'(t)\, dt - \frac{1}{b-a} \int_a^b m(t)\, dt \cdot \frac{1}{b-a} \int_a^b f'(t)\, dt \\ &= \frac{1}{2(b-a)^2} \int_a^b \int_a^b (m(t) - m(s))(f'(t) - f'(s))\, dt ds. \end{aligned} \tag{2.4}$$

However,

$$\int_a^b m(t)\,dt = 0$$

and then, by (2.2) and (2.4), we have the representation:

$$\begin{aligned} (2.5)\qquad & f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)\,dt \\ = \; & \frac{1}{2(b-a)^2}\int_a^b\int_a^b (m(t)-m(s))(f'(t)-f'(s))\,dtds. \end{aligned}$$

Using the Cauchy-Buniakowski-Schwartz integral inequality for double integrals, we have

$$\begin{aligned} (2.6)\qquad & \frac{1}{2(b-a)^2}\int_a^b\int_a^b |(m(t)-m(s))(f'(t)-f'(s))|\,dtds \\ \le \; & \left[\frac{1}{2(b-a)^2}\int_a^b\int_a^b (m(t)-m(s))^2\,dtds\right]^{\frac{1}{2}} \\ & \times\left[\frac{1}{2(b-a)^2}\int_a^b\int_a^b (f'(t)-f'(s))^2\,dtds\right]^{\frac{1}{2}} \end{aligned}$$

and as

$$\begin{aligned} & \frac{1}{2(b-a)^2}\int_a^b\int_a^b (m(t)-m(s))^2\,dtds \\ = \; & \frac{1}{b-a}\int_a^b m^2(t)\,dt - \left(\frac{1}{b-a}\int_a^b m(t)\,dt\right)^2 = \frac{(b-a)^2}{12} \end{aligned}$$

and

$$\begin{aligned} & \frac{1}{2(b-a)^2}\int_a^b\int_a^b (f'(t)-f'(s))^2\,dtds \\ = \; & \frac{1}{b-a}\int_a^b (f'(t))^2\,dt - \left(\frac{1}{b-a}\int_a^b f'(t)\,dt\right)^2, \end{aligned}$$

then, by (2.5) and (2.6), we deduce (2.1). ∎

Remark 1. *For another proof of this inequality, see* [2].

Remark 2. *Taking into account, by the Grüss inequality, we have that*

$$(2.7)\qquad 0 \le \frac{1}{b-a}\|f'\|_2^2 - ([f;a,b])^2 \le \frac{1}{4}(\Gamma-\gamma),$$

then (2.1) is an improvement of (1.18) in the case when $f' \in L_\infty[a,b]$ and satisfies (1.17).

Corollary 1. *For any $a,b>0$, we have the inequality*

$$(2.8)\qquad 0 \le (b-a)(\ln b - \ln a) - 2\cdot\frac{(b-a)^2}{a+b} \le \frac{(b-a)^4}{6\sqrt{a^3b^3}}.$$

Proof. Choose $f:(0,\infty)\to\mathbb{R}$, $f(x)=\frac{1}{x}$. Then

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &= \frac{2}{a+b}, \\ \frac{1}{b-a}\int_a^b f(t)\,dt &= \frac{\ln b-\ln a}{b-a}, \\ \frac{1}{b-a}\left\|f'\right\|_2^2-\left([f;a,b]\right)^2 &= \frac{(b-a)^2}{3a^3b^3}, \end{aligned}$$

and then, by (2.1), we get (by the convexity of f) that

$$0\le\frac{\ln b-\ln a}{b-a}-\frac{2}{a+b}\le\frac{(b-a)^2}{6\sqrt{a^3b^3}},$$

which is clearly equivalent to (2.8). ∎

The following lemma also holds.

Lemma 2. *Let $f:[a,b]\to\mathbb{R}$ be an absolutely continuous mapping on $[a,b]$ with $f'\in L_2[a,b]$. Then we have the inequality:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\le\frac{b-a}{2\sqrt{3}}\left[\frac{1}{b-a}\left\|f'\right\|_2^2-\left([f;a,b]\right)^2\right]^{\frac{1}{2}}. \tag{2.9}$$

Proof. In the recent paper [17], Dragomir and Mabizela proved the following identity which can be easily verified by direct computation:

$$\begin{aligned} &\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt \\ &= \frac{1}{2(b-a)^2}\int_a^b\int_a^b\left(f'(t)-f'(s)\right)(t-s)\,dtds. \end{aligned} \tag{2.10}$$

Using (2.10) and the Cauchy-Buniakowski-Schwartz integral inequality for double integrals, we have

$$\begin{aligned} &\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \\ &\le \frac{1}{2(b-a)^2}\int_a^b\int_a^b\left|\left(f'(t)-f'(s)\right)(t-s)\right|dtds \\ &\le \left[\frac{1}{2(b-a)^2}\int_a^b\int_a^b\left(f'(t)-f'(s)\right)^2dtds\right]^{\frac{1}{2}} \\ &\quad\times\left[\frac{1}{2(b-a)^2}\int_a^b\int_a^b(t-s)^2\,dtds\right]^{\frac{1}{2}} \end{aligned} \tag{2.11}$$

and as

$$\begin{aligned} &\frac{1}{2(b-a)^2}\int_a^b\int_a^b\left(f'(t)-f'(s)\right)^2dtds \\ &= \frac{1}{b-a}\int_a^b\left(f'(t)\right)^2dt-\left(\frac{1}{b-a}\int_a^b f'(t)\,dt\right)^2, \end{aligned}$$

and

$$\frac{1}{2(b-a)^2}\int_a^b\int_a^b (t-s)^2\,dtds = \frac{1}{b-a}\int_a^b t^2dt - \left(\frac{1}{b-a}\int_a^b tdt\right)^2 = \frac{(b-a)^2}{12},$$

then, from (2.11), we deduce the desired inequality (2.9). ∎

Remark 3. *If we assume that f' satisfies (1.17), then by (2.7), we can deduce the inequality*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \le \frac{1}{4\sqrt{3}}(b-a)(\Gamma-\gamma), \tag{2.12}$$

which improves a similar result from [38] *with the constant $\frac{1}{4}$.*

The following corollary also holds.

Corollary 2. *For any $a,b>0$, we have the inequality*

$$0 \le \frac{a+b}{2ab}(b-a)^2 - (\ln b - \ln a)(b-a) \le \frac{(b-a)^4}{6\sqrt{a^3b^3}}. \tag{2.13}$$

3. Some New Inequalities for Jeffreys Divergence

The following inequalities involving the Jeffreys divergence are known (see for example the book on line by Taneja [46])

$$D_{Ha}(p,q) \ge \exp\left[-\frac{1}{2}D_J(p,q)\right], \quad p,q\in\Omega, \tag{3.1}$$

$$D_{Ha}(p,q) \ge 1-\frac{1}{4}D_J(p,q), \quad p,q\in\Omega \tag{3.2}$$

and

$$D_J(p,q) \ge 4\left[1-D_B(p,q)\right], \quad p,q\in\Omega, \tag{3.3}$$

where $D_{Ha}(\cdot,\cdot)$ is the Harmonic distance and $D_B(\cdot,\cdot)$ is the Bhattacharyya distance.

The following result holds.

Theorem 1. *We have the inequality*

$$2D_\Delta(p,q) \le D_J(p,q) \le \frac{1}{2}\left[D_{\chi^2}(p,q)+D_{\chi^2}(q,p)\right], \quad p,q\in\Omega, \tag{3.4}$$

where D_{χ^2} is the chi-square distance and D_Δ is the triangular discrimination.

Proof. We use the celebrated Hermite-Hadamard inequality for convex functions

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(t)\,dt \le \frac{f(a)+f(b)}{2} \tag{3.5}$$

and choose $f(t)=\frac{1}{t}$ to get

$$\frac{2}{a+b} \le \frac{\ln b-\ln a}{b-a} \le \frac{a+b}{2ab},$$

which is equivalent to

$$\frac{2(b-a)^2}{a+b} \le (b-a)(\ln b-\ln a) \le \frac{a+b}{2ab}(b-a)^2. \tag{3.6}$$

If we choose in (3.6) $b = q(x)$, $a = q(x)$, $x \in \chi$, then we get

$$\begin{aligned} \frac{2\left(q\left(x\right)-p\left(x\right)\right)^{2}}{p\left(x\right)+q\left(x\right)} &\leq \left(q\left(x\right)-p\left(x\right)\right)\left(\ln q\left(x\right)-\ln p\left(x\right)\right) \\ &\leq \frac{p\left(x\right)+q\left(x\right)}{2p\left(x\right)q\left(x\right)}\left(q\left(x\right)-p\left(x\right)\right)^{2} \end{aligned}$$

and integrating over x on χ, we deduce

$$\begin{aligned} 2D_{\Delta}\left(p,q\right) &\leq D_{J}\left(p,q\right) \\ &\leq \frac{1}{2}\left[\int_{\chi}\frac{\left(q\left(x\right)-p\left(x\right)\right)^{2}}{q\left(x\right)}d\mu\left(x\right)+\int_{\chi}\frac{\left(q\left(x\right)-p\left(x\right)\right)^{2}}{p\left(x\right)}d\mu\left(x\right)\right] \\ &= \frac{1}{2}\left[\int_{\chi}\frac{q^{2}\left(x\right)-2p\left(x\right)q\left(x\right)+p^{2}\left(x\right)}{q\left(x\right)}d\mu\left(x\right)\right. \\ &\quad \left.+\int_{\chi}\frac{q^{2}\left(x\right)-2p\left(x\right)q\left(x\right)+p^{2}\left(x\right)}{p\left(x\right)}d\mu\left(x\right)\right] \\ &= \frac{1}{2}\left[\int_{\chi}\frac{p^{2}\left(x\right)}{q\left(x\right)}d\mu\left(x\right)-1+\int_{\chi}\frac{q^{2}\left(x\right)}{p\left(x\right)}d\mu\left(x\right)-1\right] \\ &= \frac{1}{2}\left[D_{\chi^{2}}\left(q,p\right)+D_{\chi^{2}}\left(p,q\right)\right] \end{aligned}$$

and the inequality (3.4) is deduced. ∎

Using the analytic inequalities established in Section 2, we can prove the following counterpart results as well.

Theorem 2. *For all $p, q \in \Omega$, we have*

$$0 \leq D_{J}\left(p,q\right)-2D_{\Delta}\left(p,q\right) \leq \frac{1}{6}D_{*}\left(p,q\right), \tag{3.7}$$

where

$$D_{*}\left(p,q\right) := \int_{\chi}\frac{\left(p\left(x\right)-q\left(x\right)\right)^{4}}{\sqrt{p^{3}\left(x\right)q^{3}\left(x\right)}}d\mu\left(x\right).$$

The proof follows by the inequality (2.8) by a similar procedure as in the proof of Theorem 1 and we omit the details.

By the use of the analytic inequality (2.13), we may state the following theorem.

Theorem 3. *For each $p, q \in \Omega$, we have*

$$0 \leq \frac{1}{2}\left[D_{\chi^{2}}\left(p,q\right)+D_{\chi^{2}}\left(q,p\right)\right]-D_{J}\left(p,q\right) \leq \frac{1}{6}D_{*}\left(p,q\right). \tag{3.8}$$

References

[1] S.M. ALI and S.D. SILVEY, A general class of coefficients of divergence of one distribution from another, *J. Roy. Statist. Soc. Sec B,* **28** (1966), 131-142.

[2] N.S. BARNETT, S.S. DRAGOMIR and A. SOFO, Better bounds for an inequality of the Ostrowski type with applications, *RGMIA Research Report Collection,* **3** (1) (2000), Article 1.

[3] M. BETH BASSAT, f-entropies, probability of error and feature selection, *Inform. Control*, **39** (1978), 227-242.

[4] A. BHATTACHARYYA, On a measure of divergence between two statistical populations defined by their probability distributions, *Bull. Calcutta Math. Soc.,* **35** (1943), 99-109.

[5] I. BURBEA and C.R. RAO, On the convexity of some divergence measures based on entropy function, *IEEE Trans. Inf. Th.,* **28** (3) (1982), 489-495.

[6] P. CERONE and S.S. DRAGOMIR, Midpoint type rules from an inequalities point of view, *RGMIA Research Report Collect.,* **2** (1999), No. 7, Article 1 (`http://rgmia.vu.edu.au/v2n7.html`)

[7] P. CERONE and S.S. DRAGOMIR, Trapezoidal type rules from an inequalities point of view, *RGMIA Research Report Collect.,* **2** (1999), No. 6, Article 8 (`http://rgmia.vu.edu.au/v2n6.html`)

[8] C.H. CHEN, *Statistical Pattern Recognition,* Rocelle Park, New York, Hoyderc Book Co., 1973.

[9] C.K. CHOW and C.N. LIN, Approximating discrete probability distributions with dependence trees, *IEEE Trans. Inf. Th.,* **14** (3) (1968), 462-467.

[10] I. CSISZÁR, Information-type measures of difference of probability distributions and indirect observations, *Studia Math. Hungarica,* **2** (1967), 299-318.

[11] I. CSISZÁR, On topological properties of $f-$divergences, *Studia Math. Hungarica,* **2** (1967), 329-339.

[12] I. CSISZÁR, A note on Jensen's inequality, *Studia Sci. Math. Hung.,* **1** (1966), 185-188.

[13] I. CSISZÁR, On topological properties of $f-$divergences, *Studia Math. Hungarica,* **2** (1967), 329-339.

[14] I. CSISZÁR, A note on Jensen's inequality, *Studia Sci. Math. Hung.,* **1** (1966), 185-188.

[15] I. CSISZÁR and J. KÖRNER, *Information Theory: Coding Theorem for Discrete Memoryless Systems,* Academic Press, New York, 1981.

[16] D. DACUNHA-CASTELLE, *Ecole d'ete de Probability de Saint-Flour*, III-1977, Berlin, Heidelberg: Springer 1978.

[17] S.S. DRAGOMIR and S. MABIZELA, Some error estimates in the trapezoidal quadrature rule, *RGMIA Research Report Collect.,* **2** (1999), No. 5, Article 6 (`http://rgmia.vu.edu.au/v2n5.html`)

[18] S.S. DRAGOMIR and S. WANG, A new inequality of Ostrowski-Grüss type and its applications to the estimation of error bounds for special means and for some numerical quadrature rules, *Comp. Math. Appl.,* **33** (1997), No. 11, 15-20.

[19] D.V. GOKHALE and S. KULLBACK, *Information in Contingency Tables*, New York, Marcel Dekker, 1978.

[20] J.H. HAVRDA and F. CHARVAT, Quantification method classification process: concept of structural α-entropy, *Kybernetika*, **3** (1967), 30-35.

[21] E. HELLINGER, Neue Bergrüirdung du Theorie quadratisher Formerus von uneudlichvieleu Veränderlicher, *J. für reine and Augeur. Math.,* **36** (1909), 210-271.

[22] H. JEFFREYS, An invariant form for the prior probability in estimating problems, *Proc. Roy. Soc. London*, **186** A (1946), 453-461.

[23] T.T. KADOTA and L.A. SHEPP, On the best finite set of linear observables for discriminating two Gaussian signals, *IEEE Trans. Inf. Th.,* **13** (1967), 288-294.

[24] T. KAILATH, The divergence and Bhattacharyya distance measures in signal selection, *IEEE Trans. Comm. Technology.,* Vol COM-15 (1967), 52-60.

[25] J.N. KAPUR, A comparative assessment of various measures of directed divergence, *Advances in Management Studies*, **3** (1984), 1-16.

[26] D. KAZAKOS and T. COTSIDAS, A decision theory approach to the approximation of discrete probability densities, *IEEE Trans. Perform. Anal. Machine Intell.,* **1** (1980), 61-67.

[27] J.H.B. KEMPERMAN, On the optimum note of transmitting information, *Ann. Math. Statist.,* **40** (1969), 2158-2177.

[28] C. KRAFT, Some conditions for consistency and uniform consistency of statistical procedures, *Univ. of California Pub. in Statistics*, **1** (1955), 125-142.

[29] S. KULLBACK, A lower bound for discrimination information in terms of variation, *IEEE Trans. Inf. Th.,* **13** (1967), 126-127.

[30] S. KULLBACK, Correction to a lower bound for discrimination information in terms of variation, *IEEE Trans. Inf. Th.,* **16** (1970), 771-773.

[31] S. KULLBACK and R.A. LEIBLER, On information and sufficiency, *Ann. Math. Stat.*, **22** (1951), 79-86.

[32] S. KULLBACK and R.A. LEIBLER, On information and sufficiency, *Annals Math. Statist.,* **22** (1951), 79-86.

[33] L. LECAM, *Asymptotic Methods in Statistical Decision Theory,* New York: Springer, 1986.
[34] J. LIN, Divergence measures based on the Shannon entropy, *IEEE Trans. Inf. Th.,* **37** (1) (1991), 145-151.
[35] J. LIN and S.K.M. WONG, A new directed divergence measure and its characterization, *Int. J. General Systems,* **17** (1990), 73-81.
[36] H.P. McKEAN, JR., Speed of approach to equilibrium for Koc's caricature of a Maximilian gas, *Arch. Ration. Mech. Anal.,* **21** (1966), 343-367.
[37] M. MEI, The theory of genetic distance and evaluation of human races, *Japan J. Human Genetics*, **23** (1978), 341-369.
[38] C.E.M. PEARCE, J. PEČARIĆ, N. UJEVIĆ and S. VAROŠANEC, Generaliasations of some inequalities of Ostrowski-Grüss type, *Math. Ineq. & Appl.,* **3** (2000), No. 1, 25-34.
[39] E.C. PIELOU, *Ecological Diversity,* Wiley, New York, 1975.
[40] M.S. PINSKER, Information and Information Stability of Random variables and processes, (in Russian), Moscow: Izv. Akad. Nouk, 1960.
[41] C.R. RAO, Diversity and dissimilarity coefficients: a unified approach, *Theoretic Population Biology*, **21** (1982), 24-43.
[42] A. RÉNYI, On measures of entropy and information, *Proc. Fourth Berkeley Symp. Math. Stat. and Prob., University of California Press*, **1** (1961), 547-561.
[43] A. SEN, *On Economic Inequality,* Oxford University Press, London 1973.
[44] B.D. SHARMA and D.P. MITTAL, New non-additive measures of relative information, *Journ. Comb. Inf. Sys. Sci.,* **2** (4)(1977), 122-132.
[45] H. SHIOYA and T. DA-TE, A generalisation of Lin divergence and the derivative of a new information divergence, *Elec. and Comm. in Japan*, **78** (7) (1995), 37-40.
[46] I. J. TANEJA, *Generalised Information Measures and their Applications* (http://www.mtm.ufsc.br/~taneja/bhtml/bhtml.html).
[47] H. THEIL, *Economics and Information Theory*, North-Holland, Amsterdam, 1967.
[48] H. THEIL, *Statistical Decomposition Analysis*, North-Holland, Amsterdam, 1972.
[49] F. TOPSOE, Some inequalities for information divergence and related measures of discrimination, *Res. Rep. Coll., RGMIA,* **2** (1) (1999), 85-98.
[50] G.T. TOUSSAINT, Sharper lower bounds for discrimination in terms of variation, *IEEE Trans. Inf. Th.,* **21** (1975), 99-100.
[51] I. VAJDA, Note on discrimination information and variation, *IEEE Trans. Inf. Th.,* **16** (1970), 771-773.
[52] I. VAJDA, *Theory of Statistical Inference and Information,* Dordrecht-Boston, Kluwer Academic Publishers, 1989.
[53] V.A. VOLKONSKI and J. A. ROZANOV, Some limit theorems for random function -I, (English Trans.), *Theory Prob. Appl., (*USSR), **4** (1959), 178-197.

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