A Mapping Associated with Jensen's Inequality and Applications

A MAPPING ASSOCIATED WITH JENSEN'S INEQUALITY AND APPLICATIONS

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Abstract. In this paper we introduce a new mapping connected with the classical inequality due to Jensen and point out its main properties. Some applications related to well-known inequalities are also established.

1. Introduction

Let X be a real linear space and C a convex subset in X, where $f : C \to \mathbb{R}$ is a convex function. Suppose that $x_i \in C$ $(i \in I)$, $p_i \geq 0$, with $P_I := \sum_{i \in I} p_i > 0$, where $I \subset \mathbb{N}$ is a finite set of indices. The following inequality is well-known in literature as Jensen's discrete inequality:

$$f\left(\frac{1}{P_I}\sum_{i\in I} p_i x_i\right) \leq \frac{1}{P_I}\sum_{i\in I} p_i f(x_i). \tag{1.1}$$

Note that some of the classical inequalities (the arithmetic mean-geometric mean inequality, Levinson's inequality, Ky Fan's inequality, etc) are particular cases of this inequality (see also [6] and [8] where further references are given). For some results which give refinements, counterparts and inequalities related to Jensen's inequality (1.1), we refer the reader to the papers [1]–[5] and [7].

We introduce the following notations (see also [5]):

$$\mathfrak{P}_f(\mathbb{N}) := \{I \subset \mathbb{N} | I \text{ is finite}\}$$
$$\mathfrak{J}^+(\mathbb{R}) := \left\{p = (p_i)_{i\in\mathbb{N}} \,| p_i > 0 \text{ for all } i \in \mathbb{N}\right\}$$
$$\mathfrak{J}_*(C) := \left\{x = (x_i)_{i\in\mathbb{N}} \,| x_i \in C \text{ for all } i \in \mathbb{N}\right\}$$
$$Conv(C,\mathbb{R}) := \text{the cone of all convex mappings defined on } C.$$

In what follows, we consider the map $H : Conv(C,\mathbb{R}) \times \mathfrak{P}_f(\mathbb{N}) \times \mathfrak{J}^+(\mathbb{R}) \times \mathfrak{J}_*(C) \to \mathbb{R}_+$ and given by

$$H = H(f, I, p, x) := \left[\frac{1}{P_I}\sum_{i\in I} p_i f(x_i) - f\left(\frac{1}{P_I}\sum_{i\in I} p_i x_i\right)\right]^{P_I} \geq 0.$$

Further on, we shall point out some properties for the mapping H which improve inequality (1.1). Some particular examples are also examined.

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2. The Results

We will start with the following result.

Theorem 1. *Suppose that f,p and x are as above. Then for all I,J finite sets of indices with $I\cap J=\emptyset$ one has the inequality*

$$H(f,I\cup J,p,x)\geq H(f,I,p,x)\cdot H(f,J,p,x)\geq 0 \tag{2.1}$$

i.e., the mapping $H(f,\cdot,p,x)$ is supermultiplicative on $\mathfrak{P}_f(\mathbb{N})$.

Proof. Consider the mapping $L: Conv(C,\mathbb{R})\times\mathfrak{P}_f(\mathbb{N})\times\mathfrak{J}^+(\mathbb{R})\times\mathfrak{J}_*(C)\to\mathbb{R}_+$ given by

$$L(f,I,p,x):=\frac{1}{P_I}\sum_{i\in I}p_if(x_i)-f\left(\frac{1}{P_I}\sum_{i\in I}p_ix_i\right)\geq 0.$$

For all $I,J\in\mathfrak{P}_f(\mathbb{N})$ with $I\cap J=\emptyset$ one has:

$$\begin{aligned}L(f,I\cup J,p,x) &= \frac{1}{P_I+P_J}\left(\sum_{i\in I}p_if(x_i)+\sum_{j\in J}p_jf(x_j)\right)\\ &\quad -f\left(\frac{1}{P_I+P_J}\left(\sum_{i\in I}p_ix_i+\sum_{j\in J}p_jx_j\right)\right).\end{aligned}$$

By the convexity of f one has

$$\begin{aligned}& f\left(\frac{P_I}{P_I+P_J}\left(\frac{1}{P_I}\sum_{i\in I}p_ix_i\right)+\frac{P_J}{P_I+P_J}\left(\frac{1}{P_J}\sum_{j\in J}p_jx_j\right)\right)\\ &\leq \frac{P_I}{P_I+P_J}f\left(\frac{1}{P_I}\sum_{i\in I}p_ix_i\right)+\frac{P_J}{P_I+P_J}f\left(\frac{1}{P_J}\sum_{j\in J}p_jx_j\right)\end{aligned}$$

and thus

$$\begin{aligned}L(f,I\cup J,p,x) &\geq \frac{1}{P_I+P_J}\left[\sum_{i\in I}p_if(x_i)+\sum_{j\in J}p_jf(x_j)\right]\\ &\quad -\frac{P_If\left(\frac{1}{P_I}\sum_{i\in I}p_ix_i\right)+P_Jf\left(\frac{1}{P_J}\sum_{j\in J}p_jx_j\right)}{P_I+P_J}\\ &= \frac{P_IL(f,I,p,x)+P_JL(f,J,p,x)}{P_I+P_J}.\end{aligned}$$

Using the elementary arithmetic mean-geometric mean inequality:

$$\frac{\alpha a+\beta b}{\alpha+\beta}\geq a^{\frac{\alpha}{\alpha+\beta}}b^{\frac{\beta}{\alpha+\beta}}\quad\text{with } a,b\geq 0,\ \alpha,\beta\geq 0 \text{ with } \alpha+\beta>0,$$

we obtain:

$$L(f,I\cup J,p,x)\geq[L(f,I,p,x)]^{\frac{P_I}{P_{I\cup J}}}[L(f,J,p,x)]^{\frac{P_J}{P_{I\cup J}}}$$

that is,

$$[L(f,I\cup J,p,x)]^{P_{I\cup J}} \geq [L(f,I,p,x)]^{P_I}[L(f,J,p,x)]^{P_J} \tag{2.2}$$

and the inequality (2.1) is obtained. ∎

The following corollary also holds.

Corollary 1. *Let H_0 be a fixed set in $\mathfrak{P}_f(\mathbb{N})$ with $L(f,H_0,p,x)>0$. Then for all $I,J\in\mathfrak{P}_f(\mathbb{N})$ with $I\supset J$ and $I\backslash J=H_0$, one has the inequality*

$$\left[\frac{L(f,I,p,x)}{L(f,H_0,p,x)}\right]^{P_I} \geq \left[\frac{L(f,J,p,x)}{L(f,H_0,p,x)}\right]^{P_J}. \tag{2.3}$$

Proof. Using the inequality (2.2), we can write

$$\begin{aligned}
[L(f,I,p,x)]^{P_I} &= [L(f,H_0\cup J,p,x)]^{P_{H_0\cup J}} \\
&\geq [L(f,H_0,p,x)]^{P_{H_0}}[L(f,J,p,x)]^{P_J} \\
&= [L(f,H_0,p,x)]^{P_I-P_J}[L(f,J,p,x)]^{P_J},
\end{aligned}$$

whence we obtain the inequality (2.3). ∎

Another result for the mapping H defined above is given in the following theorem.

Theorem 2. *Suppose that $f:C\subseteq X\to\mathbb{R}$ is a convex function on the convex set C, $I\in\mathfrak{P}_f(\mathbb{N})$ and $x=(x_i)_{i\in\mathbb{N}}\in\mathfrak{J}_*(C)$. Then for all $p,q\in\mathfrak{J}^+(\mathbb{R})$, one has the inequality*

$$H(f,I,p+q,x)\geq H(f,I,p,x)\,H(f,I,q,x)\geq 0 \tag{2.4}$$

that is, the mapping $H(f,I,\cdot,x)$ is supermultiplicative on $\mathfrak{J}^+(\mathbb{R})$.

Proof. As above, we have

$$\begin{aligned}
&L(f,I,p+q,x) \\
= {}& \frac{1}{P_I+Q_I}\left[\sum_{i\in I}p_if(x_i)+\sum_{i\in I}q_if(x_i)\right]-f\left(\frac{1}{P_I+Q_I}\left(\sum_{i\in I}p_ix_i+\sum_{i\in I}q_ix_i\right)\right) \\
\geq {}& \frac{1}{P_I+Q_I}\left[\sum_{i\in I}p_if(x_i)+\sum_{i\in I}q_if(x_i)\right] \\
&-\frac{P_If\left(\sum_{i\in I}p_ix_i/P_I\right)+Q_If\left(\sum_{i\in I}q_ix_i/Q_I\right)}{P_I+Q_I} \\
= {}& \frac{P_IL(f,I,p,x)+Q_IL(f,I,q,x)}{P_I+Q_I} \\
\geq {}& [L(f,I,p,x)]^{\frac{P_I}{P_I+Q_I}}[L(f,I,q,x)]^{\frac{Q_I}{P_I+Q_I}}
\end{aligned}$$

whence we obtain

$$[L(f,I,p+q,x)]^{P_I+Q_I}\geq [L(f,I,p,x)]^{P_I}[L(f,I,q,x)]^{Q_I}$$

and the inequality (2.4) is proved. ∎

The following corollary also holds.

Corollary 2. *Suppose that f, I, x are as above and $p, q \in \mathfrak{J}^{+}(\mathbb{R})$ such that $p - q = e \in \mathfrak{J}^{+}(\mathbb{R})$ with $L(f, I, e, x) > 0$. Then we have the inequality*

$$\left[\frac{L(f, I, p, x)}{L(f, I, e, x)}\right]^{P_I} \geq \left[\frac{L(f, I, q, x)}{L(f, I, e, x)}\right]^{Q_I}. \tag{2.5}$$

Proof. One has

$$\begin{aligned}
[L(f, I, p, x)]^{P_I} &= [L(f, I, e + q, x)]^{E_I + Q_I} \\
&\geq [L(f, I, e, x)]^{E_I} [L(f, I, q, x)]^{Q_I} \\
&= [L(f, I, e, x)]^{P_I - Q_I} [L(f, I, q, x)]^{Q_I},
\end{aligned}$$

which proves the inequality (2.5). ∎

Now, suppose that $q, e \in \mathfrak{J}^{+}(\mathbb{R})$ and $L(f, I, e, x) > 0$. We shall consider the mapping $Q : [0, \infty) \to [0, \infty)$ given by

$$Q(t) := \left[\frac{L(f, I, te + q, x)}{L(f, I, e, x)}\right]^{(tE_I + Q_I)}.$$

The main properties of this mapping are given in the following theorem.

Theorem 3. *With the above assumptions, one has*

(*i*) *The mapping Q is logarithmically concave on $[0, \infty)$;*
(*ii*) *The mapping Q is monotonic increasing on $[0, \infty)$;*
(*iii*) *We have the bound*

$$\inf_{t \in [0, \infty)} Q(t) = Q(0) = \left[\frac{L(f, I, q, x)}{L(f, I, e, x)}\right]^{Q_I}. \tag{2.6}$$

Proof. The proof is as follows.

(*i*) Let $t_1, t_2 \in [0, \infty)$ and $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$. We have

$$\begin{aligned}
& Q(\alpha t_1 + \beta t_2) \\
= & \left[\frac{L(f, I, (\alpha t_1 + \beta t_2) e + (\alpha + \beta) q, x)}{L(f, I, e, x)}\right]^{(\alpha t_1 + \beta t_2) E_I + (\alpha + \beta) Q_I} \\
= & \frac{H(f, I, \alpha(t_1 e + q) + \beta(t_2 e + q), x)}{[L(f, I, e, x)]^{\alpha(t_1 E_I + Q_I) + \beta(t_2 E_I + Q_I)}} \\
\geq & \frac{H(f, I, \alpha(t_1 e + q), x) H(f, I, \beta(t_2 e + q), x)}{[L(f, I, e, x)]^{\alpha(t_1 E_I + Q_I)} [L(f, I, e, x)]^{\beta(t_2 E_I + Q_I)}} \\
= & \left[\frac{L(f, I, \alpha(t_1 e + q), x)}{[L(f, I, e, x)]}\right]^{\alpha(t_1 E_I + Q_I)} \left[\frac{L(f, I, \beta(t_2 e + q), x)}{[L(f, I, e, x)]}\right]^{\beta(t_2 E_I + Q_I)} \\
= & \left\{\left[\frac{L(f, I, t_1 e + q, x)}{[L(f, I, e, x)]}\right]^{(t_1 E_I + Q_I)}\right\}^{\alpha} \left\{\left[\frac{L(f, I, t_2 e + q, x)}{[L(f, I, e, x)]}\right]^{(t_2 E_I + Q_I)}\right\}^{\beta} \\
= & [Q(t_1)]^{\alpha} [Q(t_2)]^{\beta}
\end{aligned}$$

as for all $\delta > 0$, $s \in \mathfrak{J}^{+}(\mathbb{R})$, $L(f, I, \delta s, x) = L(f, I, s, x)$, and the logarithmic concavity of Q is proved.

(*ii*) Let $0 \le t_1 < t_2 < \infty$. Then

$$\begin{aligned}
Q(t_2) &= \left[\frac{L(f,I,(t_2-t_1)e+t_1e+q,x)}{L(f,I,e,x)}\right]^{(t_2-t_1)E_I+t_1E_I+Q_I} \\
&= \frac{H(f,I,(t_2-t_1)e+t_1e+q,x)}{[L(f,I,e,x)]^{((t_2-t_1)E_I+t_1E_I+Q_I)}} \\
&\ge \frac{H(f,I,(t_2-t_1)e,x)\,H(f,I,t_1e+q,x)}{[L(f,I,e,x)]^{(t_2-t_1)E_I}\,[L(f,I,e,x)]^{t_1E_I+Q_I}} \\
&= \left[\frac{L(f,I,(t_2-t_1)e,x)}{L(f,I,e,x)}\right]^{(t_2-t_1)E_I}\left[\frac{L(f,I,t_1e+q,x)}{L(f,I,e,x)}\right]^{t_1E_I+Q_I} \\
&= \left[\frac{L(f,I,t_1e+q,x)}{L(f,I,e,x)}\right]^{t_1E_I+Q_I} = Q(t_1)
\end{aligned}$$

as $L(f,I,(t_1-t_2)e,x) = L(f,I,e,x)$; and the monotonicity of Q is proved.

(*iii*) Now, since $Q(t) \ge Q(0)$ for all $t \in [0,\infty)$, the bound (2.6) is obtained.

∎

3. Applications

1. Let $f : C \subseteq X \to \mathbb{R}$ be a convex mapping on the convex set C and $p_i \ge 0$, $i = \overline{1,2n}$ with $\sum_{i=1}^{2n} p_i > 0$, $\sum_{i=1}^{n} p_{2i} > 0$, $\sum_{i=1}^{n} p_{2i-1} > 0$ and $x_i \in C$ $\left(i = \overline{1,2n}\right)$. Then we have the inequality:

$$\begin{aligned}
&\frac{1}{P_{2n}}\sum_{i=1}^{2n} p_i f(x_i) - f\left(\frac{1}{P_{2n}}\sum_{i=1}^{2n} p_i x_i\right) \\
\ge\ &\left[\frac{1}{\sum_{i=1}^{n} p_{2i}}\sum_{i=1}^{n} p_{2i} f(x_{2i}) - f\left(\frac{1}{\sum_{i=1}^{n} p_{2i}}\sum_{i=1}^{n} p_{2i}x_{2i}\right)\right]^{\sum_{i=1}^{n} p_{2i} / P_{2n}} \\
&\times\left[\frac{1}{\sum_{i=1}^{n} p_{2i-1}}\sum_{i=1}^{n} p_{2i-1} f(x_{2i-1}) - f\left(\frac{1}{\sum_{i=1}^{n} p_{2i-1}}\sum_{i=1}^{n} p_{2i-1}x_{2i-1}\right)\right]^{\sum_{i=1}^{n} p_{2i-1} / P_{2n}} \\
\ge\ & 0.
\end{aligned}$$

2. With the above assumptions and assuming that $p_i \ge 0$ $\left(i = \overline{1,2n-1}\right)$, with $\sum_{i=1}^{2n} p_i > 0$, $\sum_{i=1}^{n} p_{2i} > 0$, $\sum_{i=1}^{n} p_{2i-1} > 0$ and $x_i \in C$ $\left(i = \overline{1,2n-1}\right)$, one has the

inequality

$$\begin{aligned}
&\frac{1}{P_{2n-1}}\sum_{i=1}^{2n-1}p_i f(x_i) - f\left(\frac{1}{P_{2n-1}}\sum_{i=1}^{2n-1}p_i x_i\right)\\
\geq\ &\left[\frac{1}{\sum_{i=1}^{n}p_{2i-1}}\sum_{i=1}^{n}p_{2i-1}f(x_{2i-1}) - f\left(\frac{1}{\sum_{i=1}^{n}p_{2i-1}}\sum_{i=1}^{n}p_{2i-1}x_{2i-1}\right)\right]^{\sum_{i=1}^{n}p_{2i-1}/P_{2n-1}}\\
&\times\left[\frac{1}{\sum_{i=1}^{n}p_{2i}}\sum_{i=1}^{n}p_{2i}f(x_{2i}) - f\left(\frac{1}{\sum_{i=1}^{n}p_{2i}}\sum_{i=1}^{n}p_{2i}x_{2i}\right)\right]^{\sum_{i=1}^{n}p_{2i}/P_{2n-2}}.
\end{aligned}$$

3. Let $f : C \subseteq X \to \mathbb{R}$ be a convex mapping on C, $x_i \in C$ $\left(i = \overline{1,n}\right)$ and $\alpha_i \in \left(0, \frac{\pi}{2}\right)$, $i = \overline{1,n}$. Then we have the inequality

$$\begin{aligned}
&\frac{1}{n}\sum_{i=1}^{n}f(x_i) - f\left(\frac{1}{n}\sum_{i=1}^{n}x_i\right)\\
\geq\ &\left[\frac{1}{\sum_{i=1}^{n}\sin^2\alpha_i}\sum_{i=1}^{n}\sin^2\alpha_i f(x_i) - f\left(\frac{\sum_{i=1}^{n}\sin^2\alpha_i\cdot x_i}{\sum_{i=1}^{n}\sin^2\alpha_i}\right)\right]^{\sum_{i=1}^{n}\sin^2\alpha_i/n}\\
&\times\left[\frac{1}{\sum_{i=1}^{n}\cos^2\alpha_i}\sum_{i=1}^{n}\cos^2\alpha_i f(x_i) - f\left(\frac{\sum_{i=1}^{n}\cos^2\alpha_i\cdot x_i}{\sum_{i=1}^{n}\cos^2\alpha_i}\right)\right]^{\sum_{i=1}^{n}\cos^2\alpha_i/n}\\
\geq\ &0.
\end{aligned}$$

4. Let X be a normed linear space and $p \geq 1$. Then for all $I, J \in \mathfrak{P}_f(\mathbb{N})$, where $I \cap J = \emptyset$, and $p_i \geq 0$ with P_I, $P_J > 0$, one has the inequality:

$$\begin{aligned}
&P_{I\cup J}^{p-1}\sum_{i\in I\cup J}p_i\|x_i\|^p - \left\|\sum_{i\in I\cup J}p_i x_i\right\|^p\\
\geq\ &\frac{P_{I\cup J}^{p}}{P_I^{p\cdot\frac{P_I}{P_{I\cup J}}}P_J^{p\cdot\frac{P_J}{P_{I\cup J}}}}\left(P_I^{p-1}\sum_{i\in I}p_i\|x_i\|^p - \left\|\sum_{i\in I}p_i x_i\right\|^p\right)^{\frac{P_I}{P_{I\cup J}}}\\
&\times\left(P_J^{p-1}\sum_{j\in J}p_j\|x_j\|^p - \left\|\sum_{j\in J}p_j x_j\right\|^p\right)^{\frac{P_J}{P_{I\cup J}}}\\
\geq\ &0
\end{aligned}$$

for all $x_i \in X$, $i \in I \cup J$.

If we assume that p_i, $q_i \geq 0$ so that P_I, $Q_I > 0$, $I \in \mathfrak{P}_f(\mathbb{N})$, then one has the inequality:

$$
\begin{aligned}
& (P_I+Q_I)^{p-1}\sum_{i\in I}(p_i+q_i)\|x_i\|^p-\left\|\sum_{i\in I}(p_i+q_i)\,x_i\right\|^p \\
\geq\ & \frac{(P_I+Q_I)^p}{P_I^{p\cdot\frac{P_I}{P_I+Q_I}}\cdot Q_I^{q\cdot\frac{Q_I}{P_I+Q_I}}}\left(P_I^{p-1}\sum_{i\in I}p_i\|x_i\|^p-\left\|\sum_{i\in I}p_ix_i\right\|^p\right)^{\frac{P_I}{P_I+Q_I}} \\
& \times\left(Q_I^{p-1}\sum_{i\in I}q_i\|x_i\|^p-\left\|\sum_{i\in I}q_ix_i\right\|^p\right)^{\frac{Q_I}{P_I+Q_I}}
\end{aligned}
$$

for all $x_i\in X$ $(i\in I)$.

5. Now, let $x_i>0$ and $p_i\geq 0$ $(i\in\mathbb{N})$ so that $P_I,\ P_J>0,\ I,J\in\mathfrak{P}_f(\mathbb{N})$, $I\cap J=\emptyset$. Denote

$$A(I,p,x):=\frac{1}{P_I}\sum_{i\in I}p_ix_i$$

and

$$G(I,p,x):=\left(\prod_{i\in I}x_i^{p_i}\right)^{1/P_I}.$$

The following inequality is well-known in the literature as the arithmetic mean-geometric mean inequality:

$$A(I,p,x)\geq G(I,p,x). \tag{3.1}$$

By Theorem 1 applied for the mapping $f:(0,\infty)\to\mathbb{R}$, $f(x)=-\ln x$, we have:

$$\frac{A(I\cup J,p,x)}{G(I\cup J,p,x)}\geq\exp\left\{\left[\ln\left(\frac{A(I,p,x)}{G(I,p,x)}\right)\right]^{\frac{P_I}{P_{I\cup J}}}\left[\ln\left(\frac{A(J,p,x)}{G(J,p,x)}\right)\right]^{\frac{P_J}{P_{I\cup J}}}\right\}\geq 1,$$

which gives a refinement of the well-known inequality (3.1).

If $p_i,\ q_i\geq 0$ such that $P_I,\ Q_I>0$ and $x_i>0$ $(i\in I)$, then by Theorem 2 we have:

$$\frac{A(I,p+q,x)}{G(I,p+q,x)}\geq\exp\left\{\left[\ln\left(\frac{A(I,p,x)}{G(I,p,x)}\right)\right]^{\frac{P_I}{P_I+Q_I}}\left[\ln\left(\frac{A(I,q,x)}{G(I,q,x)}\right)\right]^{\frac{Q_I}{P_I+Q_I}}\right\}\geq 1,$$

which also gives a refinement of (3.1).

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