Some Bounds in terms of $\triangle$ - Seminorms for Ostrowski-Grüss Type Inequalities

SOME BOUNDS IN TERMS OF $\Delta-$SEMINORMS FOR OSTROWSKI-GRÜSS TYPE INEQUALITIES

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Abstract. In this paper we point out some bounds for the remainder of a generalised Ostrowski type formula by the use of $\Delta-$seminorms.

1. Introduction

As in [1], let $\{P_n\}_{n\in\mathbb{N}}$ and $\{Q_n\}_{n\in\mathbb{N}}$ be two sequences of harmonic polynomials, that is, polynomials satisfying

$$P_n'(t) = P_{n-1}(t), \quad P_0(t)=1, \quad t\in\mathbb{R}, \tag{1.1}$$

$$Q_n'(t) = Q_{n-1}(t), \quad Q_0(t)=1, \quad t\in\mathbb{R}. \tag{1.2}$$

In [1], the authors proved the following result.

Lemma 1. *Let $\{P_n\}_{n\in\mathbb{N}}$ and $\{Q_n\}_{n\in\mathbb{N}}$ be two harmonic polynomials. Set*

$$S_n(t,x) := \begin{cases} P_n(t), & t\in[a,x] \\ \\ Q_n(t), & t\in(x,b] \end{cases}, \quad (t,x)\in[a,b]^2. \tag{1.3}$$

Then we have the equality

$$\begin{aligned} &\int_a^b f(t)\,dt \\ &= \sum_{k=1}^{n}(-1)^{k+1}\Big[Q_k(b)f^{(k-1)}(b)+(P_k(x)-Q_k(x))f^{(k-1)}(x) \\ &\quad -P_k(a)f^{(k-1)}(a)\Big]+(-1)^n\int_a^b S_n(t,x)f^{(n)}(t)\,dt, \end{aligned} \tag{1.4}$$

provided that $f:[a,b]\to\mathbb{R}$ is such that $f^{(n-1)}$ is absolutely continuous on $[a,b]$.

Using the following "pre-Grüss" inequality

$$|T(f,g)| \le \frac{1}{2}\sqrt{T(f,f)}\,(\Gamma-\gamma), \tag{1.5}$$

where

$$T(f,g) := \frac{1}{b-a}\int_a^b f(x)g(x)\,dx - \frac{1}{(b-a)^2}\int_a^b f(x)\,dx\cdot\int_a^b g(x)\,dx$$

is the Chebychev functional and f,g are such that the previous integrals exist and $\gamma \le g(x) \le \Gamma$ for a.e. $x\in[a,b]$, the authors of [1] proved basically the

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following inequality for estimating the integral $\int_a^b f(t)\,dt$ in terms of the harmonic polynomials $\{P_n\}_{n\in\mathbb{N}}$ and $\{Q_n\}_{n\in\mathbb{N}}$.

Theorem 1. *Assume that $f:[a,b]\to\mathbb{R}$ is such that $f^{(n)}$ is integrable and $\gamma_n \leq f^{(n)} \leq \Gamma_n$ for all $t\in[a,b]$. Put*

$$U_n(x) := \frac{1}{b-a}\left[Q_{n+1}(b) - Q_{n+1}(x) + P_{n+1}(x) - P_{n+1}(a)\right]. \tag{1.6}$$

Then for all $x\in[a,b]$, we have the inequality

$$\begin{aligned} &\Bigg| \int_a^b f(t)\,dt - \sum_{k=1}^{n} (-1)^{k+1}\Big[Q_k(b) f^{(k-1)}(b) + (P_k(x) - Q_k(x)) f^{(k-1)}(x) \\ &\quad - P_k(a) f^{(k-1)}(a)\Big] - (-1)^n U_n(x)\Big[f^{(n-1)}(b) - f^{(n-1)}(a)\Big]\Bigg| \\ &\leq \frac{1}{2} K(x)(\Gamma_n - \gamma_n)(b-a), \end{aligned} \tag{1.7}$$

where

$$K(x) := \left\{\frac{1}{b-a}\int_a^x P_n^2(t)\,dt + \int_x^b Q_n^2(t)\,dt - [U_n(x)]^2\right\}^{\frac{1}{2}}. \tag{1.8}$$

A number of particular cases that were obtained by an appropriate choice of harmonic polynomials have also been presented in [1].

In the recent paper [2], Dragomir proved the following refinement of (1.7).

Theorem 2. *Assume that the mapping $f:[a,b]\to\mathbb{R}$ is such that $f^{(n-1)}$ is absolutely continuous on $[a,b]$ and $f^{(n)}\in L_2[a,b]$ $(n\geq 1)$. If we denote*

$$\Big[f^{(n-1)};a,b\Big] := \frac{f^{(n-1)}(b) - f^{(n-1)}(a)}{b-a},$$

then we have the inequality

$$\begin{aligned} &\Bigg| \int_a^b f(t)\,dt - \sum_{k=1}^{n} (-1)^{k+1}\Big[Q_k(b) f^{(k-1)}(b) + (P_k(x) - Q_k(x)) f^{(k-1)}(x) \\ &\quad - P_k(a) f^{(k-1)}(a)\Big] - (-1)^n \left[Q_{n+1}(b) - Q_{n+1}(x)\right. \\ &\quad \left. + P_{n+1}(x) - P_{n+1}(a)\right]\Big[f^{(n-1)};a,b\Big]\Bigg| \\ &\leq K(x)(b-a)\left[\frac{1}{b-a}\left\|f^{(n)}\right\|_2^2 - \left(\Big[f^{(n)};a,b\Big]\right)^2\right]^{\frac{1}{2}} \\ &\quad \left(\leq \frac{1}{2}K(x)(b-a)(\Gamma_n - \gamma_n) \quad \text{if} \quad f^{(n)}\in L_\infty(a,b)\right), \end{aligned} \tag{1.9}$$

for all $x\in[a,b]$ and $K(x)$ as is given in (1.8).

2. Some Preliminary Results Involving Lebesgue Norms on $[a,b]^2$

For $f\in L_p[a,b]$ $(p\in[1,\infty))$, we can define the functional (see also [3])

$$\|f\|_p^{\Delta} := \left(\int_a^b \int_a^b |f(t) - f(s)|^p\,dtds\right)^{\frac{1}{p}} \tag{2.1}$$

and for $f \in L_\infty [a,b]$, we can define

$$\|f\|_\infty^\Delta := ess \sup_{(t,s)\in[a,b]^2} |f(t) - f(s)| . \tag{2.2}$$

If we consider $f_\Delta : [a,b]^2 \to \mathbb{R}$,

$$f_\Delta (t,s) = f(t) - f(s) , \tag{2.3}$$

then, obviously

$$\|f\|_p^\Delta = \|f_\Delta\|_p , \ \ p \in [1,\infty] , \tag{2.4}$$

where $\|\cdot\|_p$ are the usual Lebesque p-norms on $[a,b]^2$.

Using the properties of the Lebesque $p-$norms, we may deduce the following semi-norm properties for $\|\cdot\|_p^\Delta$:

(i) $\|f\|_p^\Delta \geq 0$ for $f \in L_p [a,b]$ and $\|f\|_p^\Delta = 0$ implies that $f = c$ (c is a constant) a.e. in $[a,b]$;

(ii) $\|f+g\|_p^\Delta \leq \|f\|_p^\Delta + \|g\|_p^\Delta$ if $f,g \in L_p [a,b]$;

(iii) $\|\alpha f\|_p^\Delta = |\alpha| \, \|f\|_p^\Delta$.

We note that if $p = 2$, then,

$$\begin{aligned} \|f\|_2^\Delta &= \left(\int_a^b \int_a^b (f(t) - f(s))^2 \, dtds \right)^{\frac{1}{2}} \\ &= \sqrt{2} \left[(b-a) \|f\|_2^2 - \left(\int_a^b f(t) \, dt \right)^2 \right]^{\frac{1}{2}} . \end{aligned}$$

If $f : [a,b] \to \mathbb{R}$ is absolutely continuous on $[a,b]$, then we can point out the following bounds for $\|f\|_p^\Delta$ in terms of $\|f'\|_p$.

Theorem 3. *Assume that $f : [a,b] \to \mathbb{R}$ is absolutely continuous on $[a,b]$.*

(i) *If $p \in [1,\infty)$, then we have the inequality*

$$\|f\|_p^\Delta \leq \begin{cases} \frac{2^{\frac{1}{p}} (b-a)^{1+\frac{2}{p}}}{[(p+1)(p+2)]^{\frac{1}{p}}} \|f'\|_\infty & \text{if} \quad f' \in L_\infty [a,b] ; \\ \\ \frac{\left(2\beta^2\right)^{\frac{1}{p}} (b-a)^{\frac{1}{\beta}+\frac{2}{p}}}{[(p+\beta)(p+2\beta)]^{\frac{1}{p}}} \|f'\|_\alpha & \text{if} \quad f' \in L_\alpha [a,b] , \\ & \qquad \alpha > 1, \ \frac{1}{\alpha} + \frac{1}{\beta} = 1; \\ \\ (b-a)^{\frac{2}{p}} \|f'\|_1 & \text{if} \quad f' \in L_1 [a,b] , \end{cases} \tag{2.5}$$

(ii) *If $p = \infty$, then we have the inequality*

$$\|f\|_\infty^\Delta \leq \begin{cases} (b-a) \|f'\|_\infty & \text{if} \quad f' \in L_\infty [a,b] ; \\ \\ (b-a)^{\frac{1}{\beta}} \|f'\|_\alpha & \text{if} \quad f' \in L_\alpha [a,b] , \ \alpha > 1, \ \frac{1}{\alpha} + \frac{1}{\beta} = 1; \\ \\ \|f'\|_1 . \end{cases} \tag{2.6}$$

Proof. As $f:[a,b]\to\mathbb{R}$ is absolutely continuous, then $f(t)-f(s)=\int_s^t f'(u)\,du$ for all $t,s\in[a,b]$, and then

$$\begin{aligned}(2.7)\qquad &|f(t)-f(s)|\\ &=\left|\int_s^t f'(u)\,du\right|\le\begin{cases}|t-s|\,\|f'\|_\infty & \text{if}\quad f'\in L_\infty[a,b]\,;\\[2mm] |t-s|^{\frac{1}{\beta}}\,\|f'\|_\alpha & \text{if}\quad f'\in L_\alpha[a,b]\,,\ \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1;\\[2mm] \|f'\|_1 & \text{if}\quad f'\in L_1[a,b]\end{cases}\end{aligned}$$

and so for $p\in[1,\infty)$, we may write

$$\begin{aligned}&|f(t)-f(s)|^p\\ &\le\begin{cases}|t-s|^p\,\|f'\|_\infty^p & \text{if}\quad f'\in L_\infty[a,b]\,;\\[2mm] |t-s|^{\frac{p}{\beta}}\,\|f'\|_\alpha^p & \text{if}\quad f'\in L_\alpha[a,b]\,,\ \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1;\\[2mm] \|f'\|_1^p & \text{if}\quad f'\in L_1[a,b]\,,\end{cases}\end{aligned}$$

and then from (2.3), (2.4)

$$(2.8)\qquad \|f\|_p^\Delta\le\begin{cases}\|f'\|_\infty\left(\int_a^b\int_a^b|t-s|^p\,dtds\right)^{\frac{1}{p}} & \text{if}\quad f'\in L_\infty[a,b]\,;\\[2mm] \|f'\|_\alpha\left(\int_a^b\int_a^b|t-s|^{\frac{p}{\beta}}\,dtds\right)^{\frac{1}{p}} & \text{if}\quad f'\in L_\alpha[a,b]\,,\\ & \quad \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1;\\[2mm] \|f'\|_1\left(\int_a^b\int_a^b dtds\right)^{\frac{1}{p}} & \text{if}\quad f'\in L_1[a,b]\,.\end{cases}$$

Further, since

$$\begin{aligned}\left(\int_a^b\int_a^b|t-s|^p\,dtds\right)^{\frac{1}{p}} &= \left[\int_a^b\left(\int_a^t(t-s)^p\,ds+\int_t^b(s-t)^p\,ds\right)dt\right]^{\frac{1}{p}}\\ &= \left(\int_a^b\left[\frac{(t-a)^{p+1}+(b-t)^{p+1}}{p+1}\right]dt\right)^{\frac{1}{p}}\\ &= \frac{2^{\frac{1}{p}}(b-a)^{1+\frac{2}{p}}}{[(p+1)(p+2)]^{\frac{1}{p}}},\end{aligned}$$

giving

$$\left(\int_a^b\int_a^b|t-s|^{\frac{p}{\beta}}\,dtds\right)^{\frac{1}{p}}=\frac{\left(2\beta^2\right)^{\frac{1}{p}}(b-a)^{\frac{1}{\beta}+\frac{2}{p}}}{[(p+\beta)(p+2\beta)]^{\frac{1}{p}}},$$

and

$$\left(\int_a^b\int_a^b dtds\right)^{\frac{1}{p}}=(b-a)^{\frac{2}{p}}\,,$$

we obtain, from (2.8), the stated result (2.5).

Using (2.7) we have (for $p=\infty$) that

$$\|f\|_{\infty}^{\Delta} \leq \begin{cases} \|f'\|_{\infty} \, ess \sup\limits_{(t,s)\in[a,b]^2} |t-s| \\ \\ \|f'\|_{\alpha} \, ess \sup\limits_{(t,s)\in[a,b]} |t-s|^{\frac{1}{\beta}} \\ \\ \|f'\|_{1} \end{cases} = \begin{cases} (b-a)\|f'\|_{\infty} \\ \\ (b-a)^{\frac{1}{\beta}} \|f'\|_{\alpha} \\ \\ \|f'\|_{1} \end{cases}$$

and the inequality (2.6) is also proved. ∎

3. Some Bounds in Terms of $\Delta-$Seminorms

We start with the following result which obtains bounds for the left hand side of (1.9) (or equivalently (1.7)) in terms of the $\Delta-$seminorms of the previous section.

Theorem 4. *Let $\{P_n\}_{n\in\mathbb{N}}$ and $\{Q_n\}_{n\in\mathbb{N}}$ be two harmonic polynomials. Set $S_n(\cdot,\cdot)$ as in Lemma 1 and assume that $f:[a,b]\to\mathbb{R}$ is such that $f^{(n-1)}$ is absolutely continuous on $[a,b]$. Then we have the inequality:*

$$\begin{aligned} (3.1)\quad & \Bigg| \int_a^b f(t)\,dt - \sum_{k=1}^{n} (-1)^{k+1} \Big[Q_k(b) f^{(k-1)}(b) + (P_k(x) - Q_k(x)) f^{(k-1)}(x) \\ & -P_k(a) f^{(k-1)}(a) \Big] - (-1)^n \left[Q_{n+1}(b) - Q_{n+1}(x) \right. \\ & \left. + P_{n+1}(x) - P_{n+1}(a) \right] \left[f^{(n-1)}; a, b \right] \Bigg| \\ \leq \quad & \frac{1}{2(b-a)} \times \begin{cases} \|S_n(\cdot,x)\|_1^{\Delta} \left\| f^{(n)} \right\|_{\infty}^{\Delta} & \text{if } \; f^{(n)} \in L_{\infty}[a,b]; \\ \\ \|S_n(\cdot,x)\|_q^{\Delta} \left\| f^{(n)} \right\|_{p}^{\Delta} & \text{if } \; \frac{1}{p}+\frac{1}{q}=1,\ p>1, \\ & \quad\; f^{(n)} \in L_p[a,b]; \\ \|S_n(\cdot,x)\|_{\infty}^{\Delta} \left\| f^{(n)} \right\|_{1}^{\Delta} \end{cases} \end{aligned}$$

and the $\Delta-$seminorms $\|\cdot\|_p^{\Delta}$ $(p\in[1,\infty])$ are defined as in Section 2.

Proof. Recall Korkine's identity

$$(3.2)\qquad T(h,g) = \frac{1}{2(b-a)^2} \int_a^b \int_a^b (h(t)-h(s))(g(t)-g(s))\,dtds,$$

where $T(\cdot,\cdot)$ is the Chebychev functional. That is, we recall that

$$T(h,g) = \frac{1}{b-a}\int_a^b h(x)g(x)\,dx - \frac{1}{(b-a)^2}\int_a^b h(x)\,dx \cdot \int_a^b g(x)\,dx,$$

provided that all the involved integrals exist.

Using (3.2) and the identity (1.4), gives (see also [1]):

$$\begin{aligned}
(3.3)\qquad & \int_a^b f(t)\,dt-\sum_{k=1}^{n}(-1)^{k+1}\Big[Q_k(b)\,f^{(k-1)}(b)+(P_k(x)-Q_k(x))\,f^{(k-1)}(x) \\
& -P_k(a)\,f^{(k-1)}(a)\Big]-(-1)^n\left[Q_{n+1}(b)-Q_{n+1}(x)\right. \\
& \left.+P_{n+1}(x)-P_{n+1}(a)\right]\left[f^{(n-1)};a,b\right] \\
= \quad & \frac{1}{2(b-a)^2}\int_a^b\int_a^b (S_n(t,x)-S_n(s,x))\,(f^{(n)}(t)-f^{(n)}(s))dtds.
\end{aligned}$$

Using Hölder's integral inequality for double integrals, we may write

$$\begin{aligned}
B &:= \left|\int_a^b\int_a^b (S_n(t,x)-S_n(s,x))\,(f^{(n)}(t)-f^{(n)}(s))dtds\right| \\
&\leq \left(\int_a^b\int_a^b |S_n(t,x)-S_n(s,x)|^q\,dtds\right)^{\frac{1}{q}}\left(\int_a^b\int_a^b \left|f^{(n)}(t)-f^{(n)}(s)\right|^p dtds\right)^{\frac{1}{p}} \\
&= \|S_n(\cdot,x)\|_q^{\Delta}\left\|f^{(n)}\right\|_p^{\Delta},
\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$, $p>1$.

If $q=1$, we have

$$B\leq \|S_n(\cdot,x)\|_1^{\Delta}\left\|f^{(n)}\right\|_{\infty}^{\Delta}$$

and if $p=1$, then

$$B\leq \|S_n(\cdot,x)\|_{\infty}^{\Delta}\left\|f^{(n)}\right\|_1^{\Delta}.$$

Further, using the identity (3.3) and the properties of modulus, we obtain (3.1). ∎

Remark 1. *For $p=q=2$, we recapture Theorem 2 and so Theorem 4 represents an Ostrowski-Grüss result whose bounds are given in terms of the $\Delta-$seminorms whose properties are given in Section 2.*

The following corollary holds.

Corollary 1. *Let $\{P_n\}_{n\in\mathbb{N}}$,$\{Q_n\}_{n\in\mathbb{N}}$ and $\{S_n\}_{n\in\mathbb{N}}$ be as in Theorem 4. If $f:[a,b]\to\mathbb{R}$ is such that $f^{(n)}$ is absolutely continuous, then we have the inequality:*

$$\begin{aligned}
(3.4)\qquad & \left|\int_a^b f(t)\,dt-\sum_{k=1}^{n}(-1)^{k+1}\Big[Q_k(b)\,f^{(k-1)}(b)+(P_k(x)-Q_k(x))\,f^{(k-1)}(x)\right. \\
& -P_k(a)\,f^{(k-1)}(a)\Big]-(-1)^n\left[Q_{n+1}(b)-Q_{n+1}(x)\right. \\
& \left.\left.+P_{n+1}(x)-P_{n+1}(a)\right]\left[f^{(n-1)};a,b\right]\right|
\end{aligned}$$

$$
\leq \begin{cases}
\frac{1}{2}\left\|S_n(\cdot,x)\right\|_1^{\Delta}\left\|f^{(n+1)}\right\|_\infty & \text{if } f^{(n+1)}\in L_\infty[a,b]; \\
\frac{1}{2}(b-a)^{\frac{1}{\beta}-1}\left\|S_n(\cdot,x)\right\|_1^{\Delta}\left\|f^{(n+1)}\right\|_\alpha & \text{if } f^{(n+1)}\in L_\alpha[a,b], \\
 & \quad \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1; \\
\frac{1}{2(b-a)}\left\|S_n(\cdot,x)\right\|_1^{\Delta}\left\|f^{(n+1)}\right\|_1 & \\
\frac{2^{\frac{1}{p}-1}(b-a)^{\frac{2}{p}}}{[(p+1)(p+2)]^{\frac{1}{p}}}\left\|S_n(\cdot,x)\right\|_q^{\Delta}\left\|f^{(n+1)}\right\|_\infty & \text{if } \frac{1}{p}+\frac{1}{q}=1,\ p>1, \\
 & \quad f^{(n+1)}\in L_\alpha[a,b]; \\
\frac{2^{\frac{1}{p}-1}\left(\beta^2\right)^{\frac{1}{p}}(b-a)^{\frac{1}{\beta}+\frac{2}{p}-1}}{[(p+\beta)(p+2\beta)]^{\frac{1}{p}}}\left\|S_n(\cdot,x)\right\|_q^{\Delta}\left\|f^{(n+1)}\right\|_\alpha & \text{if } f^{(n+1)}\in L_\alpha[a,b], \\
 & \quad \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1, \\
 & \quad p>1,\ \frac{1}{p}+\frac{1}{q}=1, \\
\frac{1}{2}(b-a)^{\frac{2}{p}-1}\left\|S_n(\cdot,x)\right\|_q^{\Delta}\left\|f^{(n+1)}\right\|_1 & \text{if } \frac{1}{p}+\frac{1}{q}=1,\ p>1; \\
\frac{(b-a)^2}{6}\left\|S_n(\cdot,x)\right\|_\infty^{\Delta}\left\|f^{(n+1)}\right\|_\infty & \text{if } f^{(n+1)}\in L_\infty[a,b]; \\
\frac{\beta^2(b-a)^{\frac{1}{\beta}+1}}{2(\beta+1)(\beta+2)}\left\|S_n(\cdot,x)\right\|_\infty^{\Delta}\left\|f^{(n+1)}\right\|_\alpha & \text{if } \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1, \\
 & \quad f^{(n+1)}\in L_\alpha[a,b]; \\
\frac{1}{2}(b-a)\left\|S_n(\cdot,x)\right\|_\infty^{\Delta}\left\|f^{(n+1)}\right\|_1 . &
\end{cases}
$$

The proof follows by Theorem 4 and by Theorem 3 (Section 2) applied for the $\Delta-$seminorm of the mapping $f^{(n)}$. We omit the details.

Remark 2. *If we choose*

$$
S_n(t,x)=\begin{cases} P_n(t), & t\in[a,x], \\ kH_n(t-x)+P_n(t), & t\in(x,b] \end{cases} \tag{3.5}
$$

with $P_n(\cdot)$ and $H_n(\cdot-x)$ harmonic polynomials satifying (1.1) and $H_n(0)=0$ for all $n\in\mathbb{N}$ then, $S_n(\cdot,x)$ is absolutely continuous on $[a,b]$. Hence, the bounds obtained in Theorem 3 will hold for $\left\|S_n(\cdot,x)\right\|_q$, $q\geq 1$ in terms of $\left\|S_n'(\cdot,x)\right\|_\gamma$, $\gamma\geq 1$. Here,

$$
S_n'(t,x)=\begin{cases} P_{n-1}(t), & t\in[a,x], \\ kH_{n-1}(t-x)+P_{n-1}(t), & t\in(x,b], \end{cases} \tag{3.6}
$$

where the differentiation is with respect to t within each of the two subintervals. With the above development, we have from Theorem 3:

(*i*) *For $q\in[1,\infty)$,*

$$
\left\|S_n(\cdot,x)\right\|_q^{\Delta}\leq\begin{cases}
2^{\frac{1}{q}}(b-a)^{1+\frac{2}{q}}\left\|S_n'(\cdot,x)\right\|_\infty, & S_n'\in L_\infty[a,b]; \\
\frac{\left(2\delta^2\right)^{\frac{1}{q}}(b-a)^{\frac{1}{\delta}+\frac{2}{q}}}{(q+\delta)(q+2\delta)^{\frac{1}{q}}}\left\|S_n'(\cdot,x)\right\|_\gamma, & S_n'\in L_\gamma[a,b], \\
 & \gamma>1,\ \frac{1}{\gamma}+\frac{1}{\delta}=1; \\
(b-a)^{\frac{2}{q}}\left\|S_n'(\cdot,x)\right\|_1, & S_n'\in L_1[a,b],
\end{cases} \tag{3.7}
$$

(ii)

$$\|S_n(\cdot,x)\|_{\infty}^{\Delta} \leq \begin{cases} (b-a)\|S_n'(\cdot,x)\|_{\infty}, & S_n' \in L_\infty[a,b]; \\ (b-a)^{\frac{1}{\delta}}\|S_n'(\cdot,x)\|_{\gamma}, & S_n' \in L_\gamma[a,b], \\ & \gamma>1,\ \frac{1}{\gamma}+\frac{1}{\delta}=1; \\ \|S_n'(\cdot,x)\|_1, & S_n' \in L_1[a,b], \end{cases} \tag{3.8}$$

and so substitution into the right hand side of Corollary 1 would give bounds involving 27 branches.

Corollary 2. *Let $\{P_n\}_{n\in\mathbb{N}}$ be as in Theorem 4. Then for $f:[a,b]\to\mathbb{R}$ and $f^{(n)}$ absolutely continuous, we have*

$$\begin{aligned} &\left|\int_a^b f(t)\,dt - \sum_{k=1}^{n}(-1)^{k+1}\left[P_k(b)f^{(k-1)}(b)-P_k(a)f^{(k-1)}(a)\right]\right. \\ &\left. -(-1)^n\left[P_{n+1}(b)-P_{n+1}(a)\right]\left[f^{(n-1)};a,b\right]\right| \end{aligned} \tag{3.9}$$

$$\leq \begin{cases} \frac{1}{2}\|P_n\|_1^{\Delta}\left\|f^{(n+1)}\right\|_\infty & \text{if } f^{(n+1)}\in L_\infty[a,b]; \\ \frac{1}{2}(b-a)^{\frac{1}{\beta}-1}\|P_n\|_1^{\Delta}\left\|f^{(n+1)}\right\|_\alpha & \text{if } f^{(n+1)}\in L_\alpha[a,b], \\ & \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1; \\ \frac{1}{2(b-a)}\|P_n\|_1^{\Delta}\left\|f^{(n+1)}\right\|_1 & \\ \frac{2^{\frac{1}{p}-1}(b-a)^{\frac{2}{p}}}{[(p+1)(p+2)]^{\frac{1}{p}}}\|P_n\|_q^{\Delta}\left\|f^{(n+1)}\right\|_\infty & \text{if } \frac{1}{p}+\frac{1}{q}=1,\ p>1, \\ & f^{(n+1)}\in L_\alpha[a,b]; \\ \frac{2^{\frac{1}{p}-1}\left(\beta^2\right)^{\frac{1}{p}}(b-a)^{\frac{1}{\beta}+\frac{2}{p}-1}}{[(p+\beta)(p+2\beta)]^{\frac{1}{p}}}\|P_n\|_q^{\Delta}\left\|f^{(n+1)}\right\|_\alpha & \text{if } f^{(n+1)}\in L_\alpha[a,b], \\ & \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1, \\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1, \\ \frac{1}{2}(b-a)^{\frac{2}{p}-1}\|P_n\|_q^{\Delta}\left\|f^{(n+1)}\right\|_1 & \text{if } \frac{1}{p}+\frac{1}{q}=1,\ p>1; \\ \frac{(b-a)^2}{6}\|P_n\|_\infty^{\Delta}\left\|f^{(n+1)}\right\|_\infty & \text{if } f^{(n+1)}\in L_\infty[a,b]; \\ \frac{\beta^2(b-a)^{\frac{1}{\beta}+1}}{2(\beta+1)(\beta+2)}\|P_n\|_\infty^{\Delta}\left\|f^{(n+1)}\right\|_\alpha & \text{if } \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1, \\ & f^{(n+1)}\in L_\alpha[a,b]; \\ \frac{1}{2}(b-a)\|P_n\|_\infty^{\Delta}\left\|f^{(n+1)}\right\|_1. & \end{cases}$$

where for $q\in[1,\infty)$

$$\|P_n\|_q^{\Delta} \leq \begin{cases} 2^{\frac{1}{q}}(b-a)^{1+\frac{2}{q}}\|P_{n-1}\|_\infty, & P_n\in L_\infty[a,b]; \\ \left(2\delta^2\right)^{\frac{1}{q}}(b-a)^{\frac{1}{\delta}+\frac{2}{q}}\|P_{n-1}\|_\gamma, & P_n\in L_\gamma[a,b], \\ & \gamma>1,\ \frac{1}{\gamma}+\frac{1}{\delta}=1; \\ (b-a)^{\frac{2}{q}}\|P_{n-1}\|_1, & P_n\in L_1[a,b], \end{cases} \tag{3.10}$$

and

$$\|P_n\|_\infty^\Delta \leq \begin{cases} (b-a)\|P_{n-1}\|_\infty, & P_n \in L_\infty[a,b]; \\ (b-a)^{\frac{1}{\delta}}\|P_{n-1}\|_\gamma, & P_n \in L_\gamma[a,b], \\ & \gamma > 1,\ \frac{1}{\gamma}+\frac{1}{\delta}=1; \\ \|P_{n-1}\|_1, & P_n \in L_1[a,b]. \end{cases} \tag{3.11}$$

Proof. Taking $k=0$ in (3.5) or equivalently $Q_n(t) \equiv P_n(t)$ in (1.3) gives from Corollary 1 the stated results where we have used (3.6) – (3.8). ∎

Remark 3. *As an example, take*

$$\tilde{P}_n(t) = \frac{(t-\theta)^n}{n!} \tag{3.12}$$

with

$$\theta = (1-\lambda)a + \lambda b, \quad \lambda \in [0,1],$$

then,

$$\begin{aligned} \left\|\tilde{P}_n\right\|_\infty &= \sup_{t\in[a,b]}\left|\tilde{P}_n(t)\right| = \frac{(b-a)^n}{n!}\max\{\lambda^n,(1-\lambda)^n\} \\ &= \frac{(b-a)^n}{n!}\left[\frac{1}{2}|\left|\lambda-\frac{1}{2}\right|\right]^n. \end{aligned} \tag{3.13}$$

Further,

$$\begin{aligned} \left\|\tilde{P}_n\right\|_\gamma &= \left(\int_a^b \left|\tilde{P}_n(t)\right|^\gamma dt\right)^{\frac{1}{\gamma}} \qquad (\gamma\in[1,\infty)) \\ &= \frac{1}{n!}\left[\int_a^\theta (\theta-t)^{n\gamma}\,dt + \int_\theta^b (t-\theta)^{n\gamma}\,dt\right]^{\frac{1}{\gamma}} \\ &= \frac{1}{n!}\left[\frac{(\theta-a)^{n\gamma+1}+(b-\theta)^{n\gamma+1}}{n\gamma+1}\right]^{\frac{1}{\gamma}}, \end{aligned}$$

giving

$$\left\|\tilde{P}_n\right\|_\gamma = \frac{(b-a)^{n+\frac{1}{\gamma}}}{n!}\left[\frac{\lambda^{n\gamma+1}+(1-\lambda)^{n\gamma+1}}{n\gamma+1}\right]^{\frac{1}{\gamma}}. \tag{3.14}$$

Thus, from (3.9) with $P_n(t)$ as given by (3.12) gives

$$\begin{aligned} \tau_\lambda \quad : &= \left|\int_a^b f(t)\,dt - \sum_{k=1}^n (-1)^{k+1}\frac{(b-a)^k}{k!}\right. \\ &\quad \times\left[(1-\lambda)^k f^{(k-1)}(b) - \lambda^k f^{(k-1)}(a)\right] - (-1)^n\frac{(b-a)^{n+1}}{(n+1)!} \\ &\quad \left.\times\left[(1-\lambda)^{n+1} f^{(n)}(b) - \lambda^{n+1} f^{(n)}(a)\right]\left[f^{(n-1)};a,b\right]\right| \\ &\leq B\left(\left\|\tilde{P}_n\right\|^\Delta\right), \end{aligned} \tag{3.15}$$

where $B\left(\|P_n\|^{\Delta}\right)$ *is the right hand side of (3.9) with* $\left\|\tilde{P}_n\right\|_q^{\Delta}$, $\left\|\tilde{P}_n\right\|_\infty^{\Delta}$ *given by (3.10), (3.11) on using (3.13) and (3.14).*
For $\lambda = \frac{1}{2}$ *the left hand side of (3.15) simplifies to*

$$\begin{aligned} \frac{1}{b-a}\tau_{\frac{1}{2}} &= \left| \frac{1}{b-a}\int_a^b f(t)\,dt - \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k!}\left(\frac{b-a}{2}\right)^k \left[f^{(k-1)};a,b\right] \right. \\ &\quad \left. - \frac{(-1)^n}{(n+1)!}\left(\frac{b-a}{2}\right)^{n+1}\left[f^{(n)};a,b\right] \times \left[f^{(n-1)};a,b\right] \right|, \end{aligned}$$

where $[g;a,b] = \frac{g(b)-g(a)}{b-a}$.

References

[1] C.E.M. PEARCE, J. PEČARIĆ, N. UJEVIĆ and S. VAROŠANEC, Generalisations of some inequalities of Ostrowski-Grüss type, *Math. Ineq. & Appl.,* **3** (1), (2000), 25-34.

[2] S.S. DRAGOMIR, Better bounds in some Ostrowski-Grüss type inequalities, *RGMIA Res. Rep. Coll.,* **3** (1), (2000), Article 3.

[3] P. CERONE and S.S. DRAGOMIR, New bounds for a perturbed generalised Taylor's formula, *RGMIA Res. Rep. Coll.,* **3** (2), (2000), Article 8.

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