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IMPROVEMENT OF AN OSTROWSKI TYPE INEQUALITY FOR MONOTONIC MAPPINGS AND ITS APPLICATION FOR SOME SPECIAL MEANS

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Abstract. We first improve two Ostrowski type inequalities for monotonic functions, then provide its application for special means.

Keywords – Ostrowski's Inequality, Trapezoid Inequality, Special Means.

1. Introduction.

In [1], Dragomir established the following Ostrowski's inequality for monotonic mappings.

Theorem 1. *Let $f : [a, b] \to R$ be a monotonic nondecreasing mapping on $[a, b]$. Then for all $x \in [a, b]$, we have the following inequality*

$$\begin{aligned}\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| &\le \frac{1}{b-a}\left\{ [2x-(a+b)]f(x) + \int_a^b sgn(t-x)f(t)dt \right\} \\ &\le \frac{1}{b-a}[(x-a)(f(x)-f(a)) + (b-x)(f(b)-f(x))] \\ &\le \left[\frac{1}{2} + \frac{|x-((a+b)/2)|}{b-a}\right](f(b)-f(a)). \qquad (1.1)\end{aligned}$$

And the constant $1/2$ *is the best possible one.*

In [2], Dragomir, Pečarić and Wang generalized Theorem 1 and proved

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Theorem 2. *Let $f : [a, b] \to R$ be a monotonic nondecreasing mapping on $[a, b]$ and t_1, t_2, $t_3 \in (a, b)$ be such that $t_1 \leq t_2 \leq t_3$. Then*

$$\begin{aligned}
&\left| \int_a^b f(x)dx - [(t_1 - a)f(a) + (t_3 - t_1)f(t_2) + (b - t_3)f(b)] \right| \\
\leq &(b - t_3)f(b) + (2t_2 - t_1 - t_3)f(t_2) - (t_1 - a)f(a) + \int_a^b T(x)f(x)dx \\
\leq &(b - t_3)(f(b) - f(t_3)) + (t_3 - t_2)(f(t_3) - f(t_2)) \\
&+(t_2 - t_1)(f(t_2) - f(t_1)) + (t_1 - a)(f(t_1) - f(a)) \\
\leq &\max\{t_1 - a, t_2 - t_1, t_3 - t_2, b - t_3\}(f(b) - f(a)), \qquad (1.2)
\end{aligned}$$

where $T(x) = sgn(t_1 - x)$, for $x \in [a, t_2]$, and $T(x) = sgn(t_3 - x)$, for $x \in [t_2, b]$.

In the present paper, we firstly improve the above results, and then provide its application for some special means.

2. Main Result.

We shall start with the following result.

Theorem 3. *Let $f : [a, b] \to R$ be a monotonic nondecreasing mapping on $[a, b]$ and let t_1, t_2, $t_3 \in [a, b]$ be such that $t_1 \leq t_2 \leq t_3$. Then*

$$\begin{aligned}
&\left| \int_a^b f(x)dx - [(t_1 - a)f(a) + (t_3 - t_1)f(t_2) + (b - t_3)f(b)] \right| \\
\leq &\max\{(b - t_3)(f(b) - f(t_3)) + (t_2 - t_1)(f(t_2) - f(t_1)), \\
&(t_3 - t_2)(f(t_3) - f(t_2)) + (t_1 - a)(f(t_1) - f(a))\} \qquad (2.1) \\
\leq &\max\{t_1 - a, t_2 - t_1, t_3 - t_2, b - t_3\}(f(b) - f(a)). \qquad (2.2)
\end{aligned}$$

Proof. Since $f(x)$ is a monotonic nondecreasing mapping on $[a, b]$, we have

$$\begin{aligned}
&\left| \int_a^b f(x)dx - [(t_1 - a)f(a) + (t_3 - t_1)f(t_2) + (b - t_3)f(b)] \right| \\
= &\left| \int_a^{t_1} (f(x) - f(a))dx + \int_{t_1}^{t_3} (f(x) - f(t_2))dx + \int_{t_3}^b (f(x) - f(b))dx \right| \\
= &\left| \left[\int_a^{t_1} (f(x) - f(a))dx + \int_{t_2}^{t_3} (f(x) - f(t_2))dx \right] \right. \\
&\left. - \left[\int_{t_1}^{t_2} (f(t_2) - f(x))dx + \int_{t_3}^b (f(b) - f(x))dx \right] \right| \\
\leq &\max\{(b - t_3)(f(b) - f(t_3)) + (t_2 - t_1)(f(t_2) - f(t_1)), \\
&(t_3 - t_2)(f(t_3) - f(t_2)) + (t_1 - a)(f(t_1) - f(a))\} \\
\leq &\max\{t_1 - a, t_2 - t_1, t_3 - t_2, b - t_3\}(f(b) - f(a)).
\end{aligned}$$

Thus (2.1) and (2.2) are proved.

Corollary 1. *Let f be defined as in Theorem 3. Then*

$$
\begin{aligned}
&\left| \int_a^b f(x)dx - [(x-a)f(a) + (b-x)f(b)] \right| \\
&\leq \max\{(b-x)(f(b)-f(x)), (x-a)(f(x)-f(a))\} \\
&\leq \max\{x-a, b-x\} \max\{(f(x)-f(a)), (f(b)-f(x))\} \\
&\leq \left[\frac{1}{2}(b-a) + \left| x - \frac{a+b}{2} \right| \right] (f(b)-f(a)).
\end{aligned}
$$

For $x = (a+b)/2$, we get trapezoid inequality.

Corollary 2. *Let f be defined as in Theorem 3. Then*

$$
\begin{aligned}
&\left| \int_a^b f(x)dx - \frac{f(a)+f(b)}{2}(b-a) \right| \\
\leq& \frac{b-a}{2} \max \left\{ \left(f\left(\frac{a+b}{2}\right) - f(a) \right), \left(f(b) - f\left(\frac{a+b}{2}\right) \right) \right\} \qquad (2.3) \\
\leq& \frac{1}{2}(b-a)(f(b)-f(a)).
\end{aligned}
$$

For $t_1 = a$, $t_2 = x$, $t_3 = b$, we get Theorem 1.

3. Application for Special Means.

In this section, we shall give application of Corollary 2. Let us recall the following means.

1. The arithmetic mean:

$$A = A(a,b) := \frac{a+b}{2}, \quad a, b \geq 0.$$

2. The geometric mean:

$$G = G(a,b) := \sqrt{ab}, \quad a, b \geq 0.$$

3. The harmonic mean:

$$H = H(a,b) := \frac{2}{1/a + 1/b}, \quad a, b \geq 0.$$

4. The logarthmic mean:

$$L = L(a,b) := \frac{b-a}{\ln b - \ln a}, \quad a, b \geq 0, a \neq b; \ If \ a = b, \ then \ L(a,b) = a.$$

5. The identric mean:

$$I = I(a,b) := \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{1/(b-a)}, \quad a, b \geq 0, a \neq b; \ If \ a = b, \ then \ I(a,b) = a.$$

6. The p-logarthmic mean:

$$L_p = L_p(a,b) := \left[\frac{b^{p+1} - a^{p+1}}{(p+1)(b-a)}\right]^{1/p}, \quad a \neq b;\ If\ a = b,\ then\ L_p(a,b) = a,$$

where $p \neq -1, 0$ and $a, b > 0$.

The following simple relationships are known in the literature

$$H \leq G \leq L \leq I \leq A.$$

We are going to use inequality (2.3) in the following equivalent version:

$$\begin{aligned} &\left|\frac{1}{b-a}\int_a^b f(t)dt - \frac{f(a)+f(b)}{2}\right| \\ \leq &\frac{1}{2}\max\left\{\left(f\left(\frac{a+b}{2}\right) - f(a)\right), \left(f(b) - f\left(\frac{a+b}{2})\right)\right\} \\ \leq &\frac{1}{2}(f(b) - f(a)), \end{aligned} \tag{3.1}$$

where $f : [a,b] \to R$ is monotonic nondecreasing on $[a,b]$.

5.1. Mapping $f(x) = x^p$

Consider the mapping $f : [a,b] \subset (0,\infty) \to R, f(x) = x^p, p > 0$. Then

$$\frac{1}{b-a}\int_a^b f(t)dt = L_p^p(a,b),$$

$$\frac{f(a)+f(b)}{2} = A(a^p, b^p),$$

$$f(b) - f(a) = p(b-a)L_{p-1}^{p-1}.$$

Then by (3.1), we get

$$\begin{aligned} \left|L_p^p(a,b) - A(a^p,b^p)\right| \leq &\frac{1}{2}\max\left\{\left(\frac{a+b}{2}\right)^p - a^p, b^p - \left(\frac{a+b}{2}\right)^p\right\} \\ = &\frac{1}{2}\left[b^p - \left(\frac{a+b}{2}\right)^p\right] = \frac{1}{2}\left(b^p - a^p\right) - \frac{1}{2}\left(\left(\frac{a+b}{2}\right)^p - a^p\right) \\ \leq &\frac{1}{2}p(b-a)L_{p-1}^{p-1} - \frac{p(b-a)a^{p-1}}{4}. \end{aligned} \tag{3.2}$$

Remark 1. The following result was proved in [2].

3.2. Mapping $f(x) = -1/x$

Consider the mapping $f : [a, b] \subset (0, \infty) \to R, f(x) = -1/x$. Then

$$\frac{1}{b-a}\int_a^b f(t)dt = -L^{-1}(a,b),$$
$$\frac{f(a)+f(b)}{2} = -\frac{A(a,b)}{G^2(a,b)},$$
$$f(b) - f(a) = \frac{b-a}{G^2(a,b)}.$$

Then by (3.1), we get

$$\begin{aligned}\left|\frac{A(a,b)}{G^2(a,b)} - L^{-1}(a,b)\right| \leq & \frac{1}{2}\max\left\{\frac{1}{a} - \frac{2}{a+b}, \frac{2}{a+b} - \frac{1}{b}\right\}\\ =&\frac{1}{2}\frac{b-a}{a(a+b)} = \frac{1}{2}\frac{b-a}{ab} - \frac{1}{2}\frac{b-a}{b(a+b)}\\ \leq&\frac{1}{2}\frac{b-a}{G^2(a,b)} - \frac{1}{2}\frac{b-a}{b(a+b)}.\end{aligned}$$

Thus we get

$$0 \leq AL - G^2 \leq \frac{1}{2}\frac{b}{a+b}(b-a)L. \tag{3.3}$$

Remark 2. The following result was proved in [2].

$$0 \leq AG - G^2 \leq \frac{1}{2}(b-a)L.$$

3.3. Mapping $f(x) = \ln x$

Consider the mapping $f : [a, b] \subset (0, \infty) \to R, f(x) = \ln x$. Then

$$\frac{1}{b-a}\int_a^b f(t)dt = \ln I(a,b),$$
$$\frac{f(a)+f(b)}{2} = \ln G(a,b),$$
$$f(b) - f(a) = \frac{b-a}{L(a,b)}.$$

Then by (3.1), we get

$$\begin{aligned}|\ln I(a,b) - \ln G(a,b)| \leq&\frac{1}{2}\max\left\{\ln\frac{a+b}{2} - \ln a, \ln b - \ln\frac{a+b}{2}\right\}\\ =&\frac{1}{2}\ln\frac{a+b}{2a} = \frac{1}{2}\frac{b-a}{L(a,b)} - \frac{1}{2}\ln\frac{2b}{a+b}.\end{aligned}$$

Thus we get

$$1 \leq \frac{I}{G} \leq \sqrt{\frac{a+b}{2b}}e^{\frac{1}{2}\frac{b-a}{L(a,b)}}. \tag{3.4}$$

Remark 3. The following result was proved in [2].

$$1 \leq \frac{I}{G} \leq e^{\frac{1}{2}\frac{b-a}{L(a,b)}}.$$

References

[1] S. S. Dragomir, *Ostrowski's inequality for monotonic mapping and applications*, J. KSIAM (to appear).

[2] S. S. Dragomir J. Pečarić and S. Wang, *The Unified Treatment of Trapezoid, Simpson, and Ostrowski Type Inequality for Monotonic Mappings and Applications*, Mathematical and Computer Modelling, **31** (2000), 61-70.

[3] S. S. Dragomir and S. Wang, *An Inequality of Ostrowski-Grüss' Type and Its Applications to the Estimation of Error Bounds for Some Special Means and for Some Numerical Quadrature Rules*, Computers Math. Applic., **33 (11)** (1997), 15-20.

[4] S. S. Dragomir and S. Wang, *Applications of Ostrowski inequality to the estimation of error bounds for some special means and some numerical quadrature rules*, Appl. Math. Lett., **11 (1)** (1998), 105-109.

[5] M. Matić J. Pečarić and N. Ujević, *Improvement and Further Generalization of Inequalities of Ostrowski-Grüss Type*, Computers Math. Applic., **39 (3/4)** (2000), 161-175.

[6] D. S. Mitrinović, J. Pečarić and A. M. Fink, *Classical and New Inequalities in Analysis*, Kluwer Academic, Dordrecht, 1993.

[7] D. S. Mitrinović, J. Pečarić and A. M. Fink, *Inequalities Involving Functions and Their Integrals and Derivatives*, Kluwer Academic, Dordrecht, 1991.

[8] G. S. Yang and K. L. Tseng, *On Certain Integral Inequalities Related to Hermite-Hadamard Inequalities*, J. Math. Anal. Appl., **239** (1999), 180-187.