On the Convergence of Generalized Singular Integrals

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by

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Abstract

In this article we study the degree of L^p-approximation ($1 \leq p \leq +\infty$) to the unit, by univariate and multivariate variants of the Jackson-type generalizations of Picard, Gauss–Weierstrass and Poisson–Cauchy singular integrals.

Part A: Univariate Results

1. Introduction

Let f be a function from $\mathbf{R}$ into itself. For $r \in \mathbf{N}$, the rth L_p-modulus of smoothness over $\mathbf{R}$ ($1 \leq p \leq +\infty$) is defined by

$$\omega_r(f;\delta)_X = \sup_{|h| \leq \delta} \|\Delta_h^r f\|_X,$$

where

$$\Delta_h^r f(x) = \sum_{i=0}^{r} (-1)^i \binom{r}{i} f(x+ih), \quad r \in \mathbf{N},$$

$$X = L^p(\mathbf{R}) \quad \text{or} \quad X = L^p_{2\pi}(\mathbf{R}),$$
$$\|f\|_{L^p(\mathbf{R})} = \left(\int_{-\infty}^{+\infty} |f(x)|^p \, dx\right)^{1/p},$$
$$\|f\|_{L^p_{2\pi}(\mathbf{R})} = \left(\int_{-\pi}^{\pi} |f(x)|^p \, dx\right)^{1/p}.$$

Next, for $\xi > 0$ we consider the Jackson-type generalizations of Picard, Poisson–Cauchy and Gauss–Weierstrass singular integrals introduced in [3] by

$$P_{n,\xi}(f;x) = -\frac{1}{2\xi}\sum_{k=1}^{n+1}(-1)^k\binom{n+1}{k}\int_{-\infty}^{+\infty} f(x+kt)e^{-|t|/\xi}\, dt,$$
$$Q_{n,\xi}(f;x) = \frac{1}{-\left(\frac{2}{\xi}\right)\tan^{-1}\left(\frac{\pi}{\xi}\right)}\sum_{k=1}^{n+1}(-1)^k\binom{n+1}{k}\int_{-\pi}^{\pi} \frac{f(x+kt)}{t^2+\xi^2}\, dt.$$

and

$$W_{n,\xi}(f;x) = -\frac{1}{2C(\xi)}\sum_{k=1}^{n+1}(-1)^k\binom{n+1}{k}\int_{-\pi}^{\pi} f(x+kt)e^{-t^2/\xi^2}\, dt,$$
$$C(\xi) = \int_0^{\pi} e^{-t^2/\xi^2}\, dt,$$

respectively (the above operators are introduced by generalizing the usual Picard, Poisson–Cauchy and Gauss–Weierstrass singular integrals, by following the same idea which is used to define the Jackson's generalized operator in classical approximation theory).

Here we consider only f such that $P_{n,\xi}(f;x)$, $Q_{n,\xi}(f;x)$, $W_{n,\xi}(f;x) \in \mathbf{R}$, for all $x \in \mathbf{R}$.

2. L^p-approximation, $1 \le p < +\infty$

The first main result of this section is

Theorem 2.1. *Here take $X = L^1(\mathbf{R})$ (for $P_{n,\xi}$), $X = L^1_{2\pi}(\mathbf{R})$, (for $W_{n,\xi}$, $Q_{n,\xi}$), $\xi > 0$, $n \in \mathbf{N}$, $f \in X$. Then*

$$\|f - P_{n,\xi}\|_X \le \left[\sum_{k=0}^{n+1}\binom{n+1}{k}k!\right]\omega_{n+1}(f;\xi)_X, \quad \xi > 0; \tag{1}$$
$$\|f - W_{n,\xi}(f)\|_X \le \left[1/\int_0^{\pi} e^{-u^2}\, du\right]\left[\int_0^{+\infty}(u+1)^{n+1}e^{-u^2}\, du\right]$$

$$\cdot\, \omega_{n+1}(f;\xi)_{L^1_{2\pi}(\mathbf{R})}, \quad 0<\xi\le 1; \tag{2}$$

$$\|f-Q_{n,\xi}(f)\|_X \le K(n,\xi)\omega_{n+1}(f;\xi)_{L^1_{2\pi}(\mathbf{R})}, \quad \xi>0, \tag{3}$$

where $K(n,\xi)=\left[1/\tan^{-1}\frac{\pi}{\xi}\right]\int_0^{\pi/\xi}\frac{(u+1)^{n+1}}{u^2+1}\,du$.

Remark. For fixed $n\in\mathbf{N}$, by (1) and (2) it follows that

$$\|f-P_{n,\xi}(f)\|_X\to 0, \quad \|f-W_{n,\xi}(f)\|_X\to 0 \quad \text{as } \xi\to 0.$$

On the other hand, because $K(n,\xi)\to+\infty$, as $\xi\to 0$, by (3) we do not obtain, in general, the convergence $\|f-Q_{n,\xi}(f)\|_X\to 0$ as $\xi\to 0$. However, in some particular cases the convergence holds, as can be seen by the following.

Corollary 2.1. *If* $f^{(n+1)}\in L^1_{2\pi}(\mathbf{R})$ *and* $f^{(n)}$ *is absolutely continuous on* $\mathbf{R}$*, then*

$$\|f-Q_{n,\xi}(f)\|_{L^1_{2\pi}(\mathbf{R})}\le C_n\xi, \quad 0<\xi\le 1$$

where $C_n>0$ *is a constant independent of* f *and* ξ.

The second main result of the section follows.

Theorem 2.2. *Let us consider* $X=L^p(\mathbf{R})$ *(for* $P_{n,\xi}$*),* $X=L^p_{2\pi}(\mathbf{R})$ *(for* $W_{n,\xi}$*,* $Q_{n,\xi}$*),* $0<\xi\le 1$, $n\in\mathbf{N}$, $1<p<+\infty$, $\frac{1}{p}+\frac{1}{q}=1$, $f\in X$. *Then*

$$\|f-P_{n,\xi}(f)\|_X\le (2/q)^{1/q}\|g\|_{L^p(\mathbf{R}_+)}\omega_{n+1}(f;\xi)_X,$$

where $g(u)=(u+1)^{n+1}e^{-u/2}$;

$$\|f-W_{n,\xi}(f)\|_X\le\left(\sqrt{\frac{\pi}{2q}}\right)^{1/q}\frac{1}{\int_0^{\pi}e^{-u^2}\,du}\|h\|_{L^p(\mathbf{R}_+)}\omega_{n+1}(f;\xi)_X,$$

where $h(u)=(u+1)^{n+1}e^{-u^2/2}$;

$$\|f-Q_{n,\xi}(f)\|_X\le K_p(n,\xi)\omega_{n+1}(f;\xi)_{L^p_{2\pi}(\mathbf{R})},$$

where $K_p(n,\xi)=\left[\frac{1}{\tan^{-1}\frac{\pi}{\xi}}\int_0^{\pi/\xi}(u+1)^{(n+1)p}\frac{1}{u^2+1}\,du\right]^{1/p}$.

Remark. Theorem 2.2 shows us that

$$\|f - P_{n,\xi}(f)\|_X \le C_1\omega_{n+1}(f;\xi)_X, \quad \|f - W_{n,\xi}(f)\|_X \le C_2\omega_{n+1}(f;\xi)_X$$

where $C_1, C_2 > 0$ are independent of f, n and ξ, while $K_p(n,\xi)$ in the third estimation (in Theorem 2.2) tends to $+\infty$ with $\xi \to 0$. In this case, as in Corollary 2.1 we can improve the estimation of $\|f - Q_{n,\xi}(f)\|_X$.

3. Uniform Approximation by $Q_{n,\xi}$ Operator

We present

Theorem 3.1. *For $0 < \xi \le 1$, $n \in \mathbf{N}$, $f \in X = C_{2\pi}(\mathbf{R})$, we get the estimation*

$$\|f - Q_{n,\xi}(f)\|_X \le K(n,\xi)\omega_{n+1}(f;\xi)_X,$$

where $K(n,\xi)$ is given by Theorem 2.1.

Corollary 3.2. *If $f^{(n+1)} \in C_{2\pi}(\mathbf{R}) = X$, then*

$$\|f - Q_{n,\xi}(f)\|_X \le C_n\xi, \quad 0 < \xi \le 1,$$

where $C_n > 0$ is independent of f and ξ.

Part B: Multivariate Results

4. Introduction

Let f be a function defined on $\mathbf{R}^m$ with values in $\mathbf{R}$.

Let $x = (x_1, \dots, x_m)$, $h = (h_1, \dots, h_m) \in \mathbf{R}^m$. Let us denote

$$\Delta_h^r f(x) = \sum_{i=0}^{r} (-1)^{r-i} \binom{r}{i} f(x + ih), \quad r \in \mathbf{N}.$$

We define the rth L^p-modulus of smoothness over $\mathbf{R}^m$, $1 \le p \le +\infty$, by

$$\omega_r(f;\delta)_p = \sup\{\|\Delta_h^r f(\cdot)\|_{L^p(\mathbf{R}^m)}; |h| \le \delta\},$$

where $|h| = (|h_1|, |h_2|, \ldots, |h_m|)$, $\delta = (\delta_1, \ldots, \delta_m)$, $|h| \le \delta$ means $|h_i| \le \delta_i$, $i = \overline{i,m}$ and

$$\|f\|_{L^p(\mathbf{R}^m)} = \left\{\int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} |f(x_1, \ldots, x_m)|^p dx_1 \ldots dx_m\right\}^{1/p}, \quad \text{if } 1 \le p < +\infty,$$

$$\|f\|_{L^\infty(\mathbf{R}^m)} = \sup\{|f(x_1, \ldots, x_m)|; x_i \in \mathbf{R}, i = \overline{i,m}\}, \quad \text{if } p = +\infty.$$

When $f \in L^p_{2\pi}(\mathbf{R}^m) = \{f : \mathbf{R}^m \to \mathbf{R};\ f$ is 2π-periodic in each variable and $\|f\|_{L^p_{2\pi}(\mathbf{R}^m)} < +\infty\}$, the rth L^p-modulus of smoothness is defined as above, where

$$\|f\|_{L^p_{2\pi}(\mathbf{R}^m)} = \left\{\int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} |f(x_1, \ldots, x_m)|^p \, dx_1 \ldots dx_m\right\}^{1/p}, \quad \text{if } 1 \le p < +\infty,$$

$$\|f\|_{L^p_{2\pi}(\mathbf{R}^m)} = \sup\{|f(x_1, \ldots, x_m)|; x_i \in [-\pi, \pi], i = \overline{1,m}\}, \quad \text{if } p = +\infty.$$

Next, for $\xi = (\xi_1, \ldots, \xi_m) > 0$ (i.e., $\xi_i > 0$, $i = \overline{1,m}$), we consider the multivariate variants of the Jackson-type generalizations of Picard, Poisson–Cauchy and Gauss–Weierstrass singular integrals introduced in [2] by

$$P_{n,\xi}(f;x) = -\frac{1}{\prod_{i=1}^{m}(2\xi_i)} \sum_{k=1}^{n+1} (-1)^k \binom{n+1}{k} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} f(x_1 + kt_1, \ldots, x_m + kt_m) \prod_{i=1}^{m} e^{-|t_i|/\xi_i} \, dt_1 \ldots dt_m,$$

$$Q_{n,\xi}(f;x) = -\frac{1}{\prod_{i=1}^{m}\left[\frac{2}{\xi_i}\tan^{-1}\left(\frac{\pi}{\xi_i}\right)\right]} \sum_{k=1}^{n+1} (-1)^k \binom{n+1}{k} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \frac{f(x_1 + kt_1, \ldots, x_m + kt_m)}{\prod_{i=1}^{m}(t_i^2 + \xi_i^2)} \, dt_1 \ldots dt_m,$$

and

$$W_{n,\xi}(f;x) = -\frac{1}{\prod_{i=1}^{m}(2C(\xi_i))}\sum_{k=1}^{n+1}(-1)^k\binom{n+1}{k}\int_{-\pi}^{\pi}$$

$$\cdots\int_{-\pi}^{\pi} f(x_1+kt_1,\ldots,x_m+kt_m)\prod_{i=1}^{m} e^{-t_i^2/\xi_i^2}\,dt_1\ldots,dt_m,$$

respectively, where $C(\xi_i) = \int_0^\pi e^{-t_i^2/\xi_i^2}\,dt_i$, $i=\overline{1,m}$, $x=(x_1,\ldots,x_m)\in\mathbf{R}^m$.

We denote

$$Erf(x_i) = \frac{2}{\sqrt{\pi}}\int_0^{x_i} e^{-t_i^2}\,dt_i, \quad x_i\in\mathbf{R}.$$

Here we obtain analogous results as in Part A.

5. L^p-approximation $1\le p\le +\infty$

The first main result here is

Theorem 5.1. *Let* $X = L^1(\mathbf{R}^m)$ *(for* $P_{n,\xi}$*),* $X = L^1_{2\pi}(\mathbf{R}^m)$ *(for* $W_{n,\xi}$*,* $Q_{n,\xi}$*),* $\xi\in\mathbf{R}^m$*,* $\xi>0$*,* $n\in\mathbf{N}$*,* $f\in X$*. Then*

$$\|f-P_{n,\xi}(f)\|_X \le \left[\sum_{k=0}^{n+1}\binom{n+1}{k}k!\right]^m \omega_{n+1}(f;\xi)_X, \quad \xi>0;$$

$$\|f-W_{n,\xi}(f)\|_X \le \left[\frac{\int_0^{+\infty}(u+1)^{n+1}e^{-u^2}\,du}{\int_0^\pi e^{-u^2}\,du}\right]^m \omega_{n+1}(f;\xi)_{L^1_{2\pi}(\mathbf{R}^m)},$$

$$0\le\xi\le 1;$$

and

$$\|f-Q_{n,\xi}(f)\|_X \le \left[\prod_{i=1}^{m}K(n,\xi_i)\right]\omega_{n+1}(f;\xi)_X, \quad \xi>0,$$

where

$$K(n,\xi_i) = \left[\int_0^{\pi/\xi_i}\frac{(u+1)^{n+1}}{u^2+1}\,du\right]/\tan^{-1}\frac{\pi}{\xi_i}, \quad i=\overline{i,m}.$$

The next main result has as follows:

Theorem 5.2. *Let $X = L^p(\mathbf{R}^m)$ (for $P_{n,\xi}$), $X = L^p_{2\pi}(\mathbf{R}^m)$ (for $W_{n,\xi}$, $Q_{n,\xi}$), $\xi \in \mathbf{R}^m$, $0 < \xi_i \le 1$, $i = \overline{1,m}$, $n \in \mathbf{N}$, $1 < p < +\infty$, $\frac{1}{p} + \frac{1}{q} = 1$, $f \in X$. Then*

$$\|f - P_{n,\xi}(f)\|_X \le \left(\frac{2}{q}\right)^{m/q} \|g\|^m_{L^p(\mathbf{R}_+)} \omega_{n+1}(f;\xi)_X,$$

where $g(u) = (u+1)^{n+1} e^{-u/2}$, $u \in \mathbf{R}_+$;

$$\|f - W_{n,\xi}(f)\|_X \le \left[\left(\sqrt{\frac{\pi}{2q}}\right)^{1/q} \|h\|_{L^p(\mathbf{R}_+)} \Big/ \int_0^{\pi} e^{-u^2}\, du\right]^m \omega_{n+1}(f;\xi)_X,$$
$$0 < \xi \le 1,$$

where $h(u) = (u+1)^{n+1} e^{-u^2/2}$; and

$$\|f - Q_{n,\xi}(f)\|_X \le \left[\prod_{i=1}^m K_p(n,\xi_i)\right] \omega_{n+1}(f;\xi)_X, \quad 0 \le \xi \le 1,$$

where $K_p(n,\xi_i) = \left[\int_0^{\pi/\xi_i} [(u+1)^{(n+1)p}/(u^2+1)]\, du / \tan^{-1}\frac{\pi}{\xi_i}\right]^{1/p}$.

The last result has to do with the uniform convergence.

Theorem 5.3. *Let $X = L^\infty(\mathbf{R}^m)$ (for $P_{n,\xi}$), $X = L^\infty_{2\pi}(\mathbf{R}^m)$ (for $W_{n,\xi}$, $Q_{n,\xi}$), $\xi \in \mathbf{R}^m$, $0 < \xi \le 1$, $n \in \mathbf{N}$, $f \in X$. Then*

$$\|f - P_{n,\xi}(f)\|_X \le \left[\sum_{k=0}^m \binom{n+1}{k} k!\right]^m \omega_{n+1}(f;\xi)_X, \quad \xi > 0;$$
$$\|f - W_{n,\xi}(f)\|_X \le \left[\frac{\int_0^{+\infty} (u+1)^{n+1} e^{-u^2}\, du}{\int_0^{\pi} e^{-u^2}\, du}\right]^m \omega_{n+1}(f;\xi)_X, \quad 0 < \xi \le 1;$$

and

$$\|f - Q_{n,\xi}(f)\|_X \le \left[\prod_{i=1}^m K(n,\xi_i)\right] \omega_{n+1}(f;\xi)_X, \quad \xi > 0,$$

where $K(n,\xi_i)$ is given in Theorem 5.1.

Remark. Theorems 5.1–5.3 show us that while the generalized operators $P_{n,\xi}$ and $W_{n,\xi}$ give very good estimates (such that if $\xi \to 0$, i.e., $\xi_i \to 0$, $i = \overline{1,m}$, then $P_{n,\xi}(f) \to f$, $W_{n,\xi}(f) \to f$), for the generalized operator $Q_{n,\xi}(f)$ in general this does not happen, because if $\xi_i \to 0$, we have $K_p(n,\xi_i) \to +\infty$, for all $1 \le p < +\infty$.

However, under some smoothness conditions for f, (as for example if $\left|\frac{\partial^{|k|} f}{\partial x_1^{k_1} \cdots \partial x_m^{k_m}}\right| \leq M$ on $\mathbf{R}^m$, for all $|k| \in \{0, 1, \ldots, n+1\}$, where $|k| = k_1 + \cdots + k_m$, $k_i \in \mathbf{N} \cup \{0\}$, $i = \overline{1, m}$) and reasoning as in the univariate case, Part A, we easily get that $Q_{n,\xi}(f) \to f$, as $\xi \to 0$.

We intend to publish the above results with full proofs and more discussion elsewhere.

References

[1] Anastassiou, G. A. and S. G. Gal, General theory of global smoothness preservation by singular integrals, univariate case, *J. Comp. Anal. Appl.* **1**, No. 3 (1999), 289–317.

[2] Anastassiou, G. A. and S. G. Gal, Global smoothness preservation by multivariate singular integrals, *Bull. Austal. Math. Soc.* **61** (2000), 489–506.

[3] Gal, S. G., Degree of approximation of continuous functions by some singular integrals, *Rev. Anal. Numér. Théor. Approx.* (Cluj), Tome XXVII, No. 2 (1998), 251–261.