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A Trapezoid Type Inequality for Double Integrals

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A TRAPEZOID TYPE INEQUALITY FOR DOUBLE INTEGRALS

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ABSTRACT. In this paper, we point out a trapezoid like inequality for double integrals and apply it in connection with the Grüss inequality.

1. INTRODUCTION

In the recent papers [1] and [2], the authors proved the following inequality of the Ostrowski type for double integrals.

Theorem 1. *Let $f : [a, b] \times [c, d] \to \mathbb{R}$ be continuous on $[a, b] \times [c, d]$. Then we have the inequality:*

$$
(1.1) \quad \left| \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(t, s)\, ds dt \right.
$$

$$
\left. - \frac{1}{d-c} \int_c^d f(x, s)\, ds - \frac{1}{b-a} \int_a^b f(t, y)\, dt + f(x, y) \right|
$$

$$
\leq
\begin{cases}
\left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] \left[\frac{1}{4} + \left(\frac{y - \frac{c+d}{2}}{d-c} \right)^2 \right] (b-a)(d-c) \left\| \frac{\partial^2 f}{\partial t \partial s} \right\|_\infty \\[1em]
\quad if \ \frac{\partial^2 f}{\partial t \partial s} \in L_\infty([a, b] \times [c, d]); \\[1em]
\frac{1}{(q+1)^{\frac{2}{q}}} \left[\left(\frac{x-a}{b-a} \right)^{q+1} + \left(\frac{b-x}{b-a} \right)^{q+1} \right]^{\frac{1}{q}} \left[\left(\frac{y-c}{d-c} \right)^{q+1} + \left(\frac{d-y}{d-c} \right)^{q+1} \right]^{\frac{1}{q}} \\[1em]
\quad \times [(b-a)(d-c)]^{\frac{1}{q}} \left\| \frac{\partial^2 f}{\partial t \partial s} \right\|_p \\[1em]
\quad if \ \frac{\partial^2 f}{\partial t \partial s} \in L_p([a, b] \times [c, d]), \ p > 1, \ \frac{1}{p} + \frac{1}{q} = 1; \\[1em]
\left[\frac{1}{2} + \left| \frac{x - \frac{a+b}{2}}{b-a} \right| \right] \left[\frac{1}{2} + \left| \frac{y - \frac{c+d}{2}}{d-c} \right| \right] \left\| \frac{\partial^2 f}{\partial t \partial s} \right\|_1 \\[1em]
\quad if \ \frac{\partial^2 f}{\partial t \partial s} \in L_1([a, b] \times [c, d])
\end{cases}
$$

for all $(x, y) \in [a, b] \times [c, d]$, *where*

$$\left\| \frac{\partial^2 f}{\partial t \partial s} \right\|_\infty \quad : \quad = \quad \sup_{(t,s) \in [a,b] \times [c,d]} \left| \frac{\partial^2 f(t,s)}{\partial t \partial s} \right|,$$

$$\left\| \frac{\partial^2 f}{\partial t \partial s} \right\|_p \quad : \quad = \quad \left(\int_a^b \int_c^d \left| \frac{\partial^2 f(t,s)}{\partial t \partial s} \right|^p ds dt \right)^{\frac{1}{p}}$$

if $p \in [1, \infty)$.

The best inequality we can get from (1.1) is the one for which $x = \frac{a+b}{2}$ and $y = \frac{c+d}{2}$, obtaining:

Corollary 1. *With the assumptions in Theorem 1, we have the following mid-point type inequality:*

$$(1.2) \qquad \left| \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(t,s) \, ds dt - \frac{1}{d-c} \int_c^d f\left(\frac{a+b}{2}, s \right) ds \right.$$

$$\left. - \frac{1}{b-a} \int_a^b f\left(t, \frac{c+d}{2} \right) dt + f\left(\frac{a+b}{2}, \frac{c+d}{2} \right) \right|$$

$$\leq \begin{cases} \frac{1}{16} \left\| \frac{\partial^2 f}{\partial t \partial s} \right\|_\infty (b-a)(d-c) & \text{if } \frac{\partial^2 f}{\partial t \partial s} \in L_\infty \left([a,b] \times [c,d] \right); \\[2em] \frac{1}{4(q+1)^{\frac{2}{q}}} \left[(b-a)(d-c) \right]^{\frac{1}{q}} \left\| \frac{\partial^2 f}{\partial t \partial s} \right\|_p \\[1em] \quad \text{if } \frac{\partial^2 f}{\partial t \partial s} \in L_p \left([a,b] \times [c,d] \right), \quad \text{for } p > 1, \ \frac{1}{p} + \frac{1}{q} = 1; \\[2em] \frac{1}{4} \left\| \frac{\partial^2 f}{\partial t \partial s} \right\|_1 & \text{if } \frac{\partial^2 f}{\partial t \partial s} \in L_1 \left([a,b] \times [c,d] \right). \end{cases}$$

For some applications of the above results in Numerical Integration for cubature formulae see [1] and [2].

Another result of Ostrowski type was proved in [4].

Theorem 2. *Let* $f : [a, b] \times [c, d] \to \mathbb{R}$ *be a mapping as in Theorem 1. Then we have the inequality:*

$$(1.3) \qquad \left| f(x, y) - \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(t,s) \, ds dt \right|$$

$$\leq \quad M_1(x) + M_2(y) + M_3(x, y),$$

where

$$M_1(x) = \begin{cases} \frac{\left[\frac{1}{4}(b-a)^2 + \left(x - \frac{a+b}{2} \right)^2 \right]}{b-a} \left\| \frac{\partial f}{\partial t} \right\|_\infty, & \text{if } \frac{\partial f}{\partial t} \in L_\infty \left([a,b] \times [c,d] \right); \\[2em] \frac{\left[\frac{(b-x)^{q_1+1} + (x-a)^{q_1+1}}{q_1+1} \right]^{\frac{1}{q_1}}}{(b-a)[(d-c)]^{\frac{1}{p_1}}} \left\| \frac{\partial f}{\partial t} \right\|_{p_1}, & \text{if } \frac{\partial f}{\partial t} \in L_{p_1} \left([a,b] \times [c,d] \right), \\[1em] & \quad p_1 > 1, \ \frac{1}{p_1} + \frac{1}{q_1} = 1; \\[1em] \frac{\left[\frac{1}{2}(b-a) + \left| x - \frac{a+b}{2} \right| \right]}{(b-a)(d-c)} \left\| \frac{\partial f}{\partial t} \right\|_1, & \text{if } \frac{\partial f}{\partial t} \in L_1 \left([a,b] \times [c,d] \right). \end{cases}$$

$$M_2(y) = \begin{cases} \dfrac{\left[\frac{1}{4}(d-c)^2+\left(y-\frac{c+d}{2}\right)^2\right]}{d-c}\left\|\dfrac{\partial f}{\partial s}\right\|_\infty, & \text{if} \quad \dfrac{\partial f}{\partial s} \in L_\infty\left([a,b]\times[c,d]\right); \\[4mm] \dfrac{\left[\frac{(d-y)^{q_2+1}+(y-c)^{q_2+1}}{q_2+1}\right]^{\frac{1}{q_2}}}{[(b-a)]^{\frac{1}{p_2}}(d-c)}\left\|\dfrac{\partial f}{\partial s}\right\|_{p_2}, & \text{if} \quad \dfrac{\partial f}{\partial s} \in L_{p_2}\left([a,b]\times[c,d]\right); \\ & p_2 > 1, \; \frac{1}{p_2}+\frac{1}{q_2}=1; \\[3mm] \dfrac{\left[\frac{1}{2}(d-c)+\left|y-\frac{c+d}{2}\right|\right]}{(b-a)(d-c)}\left\|\dfrac{\partial f}{\partial s}\right\|_1, & \text{if} \quad \dfrac{\partial f}{\partial s} \in L_1\left([a,b]\times[c,d]\right); \end{cases}$$

and

$$M_3(x,y) = \begin{cases} \dfrac{\left[\frac{1}{4}(b-a)^2+\left(x-\frac{a+b}{2}\right)^2\right]\left[\frac{1}{4}(d-c)^2+\left(y-\frac{c+d}{2}\right)^2\right]}{(b-a)(d-c)}\left\|\dfrac{\partial^2 f}{\partial t\partial s}\right\|_\infty, \\[3mm] \quad \text{if} \quad \dfrac{\partial^2 f}{\partial s\partial t} \in L_\infty\left([a,b]\times[c,d]\right); \\[5mm] \dfrac{\left[\frac{(b-x)^{q_3+1}+(x-a)^{q_3+1}}{q_3+1}\right]^{\frac{1}{q_3}}\left[\frac{(d-y)^{q_3+1}+(y-c)^{q_3+1}}{q_3+1}\right]^{\frac{1}{q_3}}}{(b-a)(d-c)}\left\|\dfrac{\partial^2 f}{\partial t\partial s}\right\|_{p_3}, \\[3mm] \quad \text{if} \quad \dfrac{\partial^2 f}{\partial s\partial t} \in L_{p_3}\left([a,b]\times[c,d]\right), \; p_3 > 1, \; \frac{1}{p_3}+\frac{1}{q_3}=1; \\[5mm] \dfrac{\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]\left[\frac{1}{2}(d-c)+\left|y-\frac{c+d}{2}\right|\right]}{(b-a)(d-c)}\left\|\dfrac{\partial^2 f}{\partial t\partial s}\right\|_1, \\[3mm] \quad \text{if} \quad \dfrac{\partial^2 f}{\partial s\partial t} \in L_1\left([a,b]\times[c,d]\right); \end{cases}$$

for all $(x,y) \in [a,b]\times[c,d]$, *where* $\|\cdot\|_p$ $(1 \le p \le \infty)$ *are the usual* $p-$*norms on* $[a,b]\times[c,d]$.

Corollary 2. *With the assumptions in Theorem 1, we have the inequality*

$$(1.4) \qquad \left| f\left(\frac{a+b}{2},\frac{c+d}{2}\right) - \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)\,dsdt \right|$$
$$\le \quad \tilde{M}_1 + \tilde{M}_2 + \tilde{M}_3,$$

where

$$\tilde{M}_1 := \begin{cases} \frac{1}{4}(b-a)\left\|\dfrac{\partial f}{\partial t}\right\|_\infty, & \text{if} \quad \dfrac{\partial f}{\partial t} \in L_\infty\left([a,b]\times[c,d]\right) \\[3mm] \frac{1}{2}\left[\dfrac{(b-a)^{\frac{1}{q_1}}}{(q_1+1)^{\frac{1}{q_1}}(d-c)^{\frac{1}{p_1}}}\right]\left\|\dfrac{\partial f}{\partial t}\right\|_{p_1}, & \text{if} \quad \dfrac{\partial f}{\partial t} \in L_{p_1}\left([a,b]\times[c,d]\right) \\ & p_1 > 1, \; \frac{1}{p_1}+\frac{1}{q_1}=1; \\[3mm] \frac{1}{2(d-c)}\left\|\dfrac{\partial f}{\partial t}\right\|_1, & \text{if} \quad \dfrac{\partial f}{\partial t} \in L_1\left([a,b]\times[c,d]\right) \end{cases}$$

$$\tilde{M}_2 := \begin{cases} \frac{1}{4}(d-c)\left\|\dfrac{\partial f}{\partial s}\right\|_\infty, & \text{if} \quad \dfrac{\partial f}{\partial t} \in L_\infty\left([a,b]\times[c,d]\right) \\[3mm] \frac{1}{2}\left[\dfrac{(d-c)^{\frac{1}{q_2}}}{(q_2+1)^{\frac{1}{q_2}}(b-a)^{\frac{1}{p_2}}}\right]\left\|\dfrac{\partial f}{\partial s}\right\|_{p_2}, & \text{if} \quad \dfrac{\partial f}{\partial t} \in L_{p_2}\left([a,b]\times[c,d]\right) \\ & p_2 > 1, \; \frac{1}{p_2}+\frac{1}{q_2}=1; \\[3mm] \frac{1}{2(b-a)}\left\|\dfrac{\partial f}{\partial s}\right\|_1, & \text{if} \quad \dfrac{\partial f}{\partial t} \in L_1\left([a,b]\times[c,d]\right) \end{cases}$$

and

$$\tilde{M}_3 := \begin{cases} \frac{1}{16}\left(b-a\right)\left(d-c\right)\left\|\frac{\partial^2 f}{\partial t\partial s}\right\|_{\infty}, & if \quad \frac{\partial^2 f}{\partial t\partial s} \in L_{\infty}\left([a,b]\times[c,d]\right); \\[2em] \frac{1}{4}\cdot\frac{(b-a)^{\frac{1}{q_3}}(d-c)^{\frac{1}{q_3}}}{(q_3+1)^{\frac{2}{q_3}}}\left\|\frac{\partial^2 f}{\partial t\partial s}\right\|_{p_3}, & if \quad \frac{\partial^2 f}{\partial t\partial s} \in L_{p_3}\left([a,b]\times[c,d]\right); \\[1em] & \quad p_3 > 1,\ \frac{1}{p_3}+\frac{1}{q_3}=1, \\[1em] \frac{1}{4}\left\|\frac{\partial^2 f}{\partial t\partial s}\right\|_1, & if \quad \frac{\partial^2 f}{\partial t\partial s} \in L_1\left([a,b]\times[c,d]\right). \end{cases}$$

2. SOME INTEGRAL EQUALITIES

Let us start with the following integral identity.

Theorem 3. *Let* $f : [a,b]\times[c,d] \to \mathbb{R}$ *be a continuous mapping on* $[a,b]\times[c,d]$ *such that* $\frac{\partial f(a,\cdot)}{\partial y}$, $\frac{\partial f(b,\cdot)}{\partial y}$ *are continuous on* $[c,d]$, $\frac{\partial f(\cdot,c)}{\partial x}$, $\frac{\partial f(\cdot,d)}{\partial x}$ *are continuous on* $[a,b]$ *and* $\frac{\partial^2 f(\cdot,\cdot)}{\partial x\partial y}$ *is continuous on* $[a,b]\times[c,d]$. *Then we have the identity:*

$$(2.1)\qquad \int_a^b\int_c^d f\left(x,y\right)dydx + \left(b-a\right)\int_c^d\left(y-\frac{c+d}{2}\right)\left[\frac{\frac{\partial f(a,y)}{\partial y}+\frac{\partial f(b,y)}{\partial y}}{2}\right]dy$$

$$+ \left(d-c\right)\int_a^b\left(x-\frac{a+b}{2}\right)\left[\frac{\frac{\partial f(x,c)}{\partial x}+\frac{\partial f(x,d)}{\partial x}}{2}\right]dx$$

$$= \frac{f\left(a,c\right)+f\left(a,d\right)+f\left(b,c\right)+f\left(b,d\right)}{4}\left(b-a\right)\left(d-c\right)$$

$$+ \int_a^b\int_c^d\left(x-\frac{a+b}{2}\right)\left(y-\frac{c+d}{2}\right)\frac{\partial^2 f\left(x,y\right)}{\partial x\partial y}dydx.$$

Proof. A simple integration by parts gives

$$(2.2)\qquad \int_{\alpha}^{\beta} h\left(x\right)dx = \frac{h\left(\alpha\right)+h\left(\beta\right)}{2}\left(\beta-\alpha\right) - \int_{\alpha}^{\beta}\left(x-\frac{\alpha+\beta}{2}\right)h'\left(x\right)dx,$$

provided that $h : [\alpha,\beta] \to \mathbb{R}$ is absolutely continuous on $[\alpha,\beta]$.

Using (2.2), we can write:

$$(2.3)\qquad \int_a^b f\left(x,y\right)dx = \left(b-a\right)\frac{f\left(a,y\right)+f\left(b,y\right)}{2} - \int_a^b\left(x-\frac{a+b}{2}\right)\frac{\partial f\left(x,y\right)}{\partial x}dx$$

for all $y \in [c,d]$.

Integrating (2.3) on the interval $[c,d]$, we obtain

$$\int_c^d\left(\int_a^b f\left(x,y\right)dx\right)dy = \frac{1}{2}\left(b-a\right)\left[\int_c^d f\left(a,y\right)dy + \int_c^d f\left(b,y\right)dy\right]$$

$$- \int_c^d\left(\int_a^b\left(x-\frac{a+b}{2}\right)\frac{\partial f\left(x,y\right)}{\partial x}dx\right)dy.$$

Using Fubini's theorem, we can state:

$$(2.4) \quad \int_a^b \int_c^d f(x,y)\, dy dx = \frac{1}{2}(b-a)\left[\int_c^d f(a,y)\, dy + \int_c^d f(b,y)\, dy\right]$$
$$-\int_a^b \left(x - \frac{a+b}{2}\right)\left(\int_c^d \frac{\partial f(x,y)}{\partial x}\, dy\right) dx.$$

By the identity (2.2), we can also state:

$$(2.5) \quad \int_c^d f(a,y)\, dy = \frac{f(a,c)+f(a,d)}{2}(d-c) - \int_c^d \left(y - \frac{c+d}{2}\right)\frac{\partial f(a,y)}{\partial y}\, dy,$$

$$(2.6) \quad \int_c^d f(b,y)\, dy = \frac{f(b,c)+f(b,d)}{2}(d-c) - \int_c^d \left(y - \frac{c+d}{2}\right)\frac{\partial f(b,y)}{\partial y}\, dy,$$

and

$$(2.7) \quad \int_c^d \frac{\partial f(x,y)}{\partial x}\, dy = \frac{1}{2}\left[\frac{\partial f(x,c)}{\partial x} + \frac{\partial f(x,d)}{\partial x}\right](d-c)$$
$$-\int_c^d \left(y - \frac{c+d}{2}\right)\frac{\partial^2 f(x,y)}{\partial x \partial y}\, dy.$$

Now, using (2.4) and (2.5)-(2.7), we have successively

$$\int_a^b \int_c^d f(x,y)\, dy dx$$
$$= \frac{1}{2}(b-a)\left[\frac{f(a,c)+f(a,d)}{2}(d-c) - \int_c^d \left(y - \frac{c+d}{2}\right)\frac{\partial f(a,y)}{\partial y}\, dy\right.$$
$$\left. + \frac{f(b,c)+f(b,d)}{2}(d-c) - \int_c^d \left(y - \frac{c+d}{2}\right)\frac{\partial f(b,y)}{\partial y}\, dy\right]$$
$$- \int_a^b \left(x - \frac{a+b}{2}\right)\left[\frac{1}{2}\left[\frac{\partial f(x,c)}{\partial x} + \frac{\partial f(x,d)}{\partial x}\right](d-c)\right.$$
$$\left. - \int_c^d \left(y - \frac{c+d}{2}\right)\frac{\partial^2 f(x,y)}{\partial x \partial y}\, dy\right] dx$$

$$= \frac{f(a,c) + f(a,d) + f(b,c) + f(b,d)}{4}(b-a)(d-c)$$

$$-\frac{1}{2}(b-a)\int_c^d \left(y - \frac{c+d}{2}\right)\frac{\partial f(a,y)}{\partial y}dy$$

$$-\frac{1}{2}(b-a)\int_c^d \left(y - \frac{c+d}{2}\right)\frac{\partial f(b,y)}{\partial y}dy$$

$$-\frac{1}{2}(d-c)\int_a^b \left(x - \frac{a+b}{2}\right)\frac{\partial f(x,c)}{\partial x}dx$$

$$-\frac{1}{2}(d-c)\int_a^b \left(x - \frac{a+b}{2}\right)\frac{\partial f(x,d)}{\partial x}dx$$

$$+\int_a^b \int_c^d \left(x - \frac{a+b}{2}\right)\left(y - \frac{c+d}{2}\right)\frac{\partial^2 f(x,y)}{\partial x \partial y}dydx$$

and the identity (2.1) is proved.   □

The following corollary holds:

Corollary 3. *Let* $f, g : [a,b] \to \mathbb{R}$ *be differentiable on* (a,b) *and such that* f', $g' \in L_1(a,b)$. *Then we have the equality:*

$$(2.8) \qquad (b-a)\int_a^b f(x)g(x)\,dx - \int_a^b f(x)\,dx \int_a^b g(x)\,dx$$

$$= \frac{[f(a) - f(b)][g(a) - g(b)]}{4}(b-a)^2$$

$$+\int_a^b \left(x - \frac{a+b}{2}\right)\left(\frac{g(a) + g(b)}{2} - g(x)\right)f'(x)\,dx$$

$$+\int_a^b \left(x - \frac{a+b}{2}\right)\left(\frac{f(a) + f(b)}{2} - f(x)\right)g'(x)\,dx$$

$$-\int_a^b \left(x - \frac{a+b}{2}\right)f'(x)\,dx \int_a^b \left(x - \frac{a+b}{2}\right)g'(x)\,dx.$$

Proof. The above identity can be proved by direct computation. We give here a proof based on the previous identity (2.1).

Consider the mapping $h : [a,b]^2 \to \mathbb{R}$ given by $h(x,y) := (f(x) - f(y))(g(x) - g(y))$ and write the equality (2.1) for h and the interval $[a,b]^2$ to get

$$(2.9) \quad \int_a^b \int_a^b h(x,y)\,dxdy + \frac{1}{2}(b-a)\int_a^b \left(y - \frac{a+b}{2}\right)\left[\frac{\partial h(a,y)}{\partial y} + \frac{\partial h(b,y)}{\partial y}\right]dy$$

$$+\frac{1}{2}(b-a)\int_a^b \left(x - \frac{a+b}{2}\right)\left[\frac{\partial h(x,a)}{\partial x} + \frac{\partial h(x,b)}{\partial x}\right]dx$$

$$= \frac{h(a,a) + h(a,b) + h(b,a) + h(b,b)}{4}(b-a)^2$$

$$+\int_a^b \int_a^b \left(x - \frac{a+b}{2}\right)\left(y - \frac{a+b}{2}\right)\frac{\partial^2 h(x,y)}{\partial x \partial y}dxdy.$$

Note that

$$\frac{\partial h\left(x,y\right)}{\partial x} = f'\left(x\right)\left(g\left(x\right)-g\left(y\right)\right)+g'\left(x\right)\left(f\left(x\right)-f\left(y\right)\right)$$
$$\frac{\partial h\left(x,y\right)}{\partial y} = f'\left(y\right)\left(g\left(y\right)-g\left(x\right)\right)+g'\left(y\right)\left(f\left(y\right)-f\left(x\right)\right)$$

and then

$$\frac{1}{2}\left[\frac{\partial h\left(a,y\right)}{\partial y}+\frac{\partial h\left(b,y\right)}{\partial y}\right]$$
$$= f'\left(y\right)\left(g\left(y\right)-\frac{g\left(a\right)+g\left(b\right)}{2}\right)+g'\left(y\right)\left(f\left(y\right)-\frac{f\left(a\right)+f\left(b\right)}{2}\right)$$

and

$$\frac{1}{2}\left[\frac{\partial h\left(x,a\right)}{\partial x}+\frac{\partial h\left(x,b\right)}{\partial x}\right]$$
$$= f'\left(x\right)\left(g\left(x\right)-\frac{g\left(a\right)+g\left(b\right)}{2}\right)+g'\left(x\right)\left(f\left(x\right)-\frac{f\left(a\right)+f\left(b\right)}{2}\right).$$

In addition, we note that

$$\frac{\partial^2 h\left(x,y\right)}{\partial x\partial y} = -f'\left(x\right)g'\left(y\right)-f'\left(y\right)g'\left(x\right).$$

We have

$$\begin{aligned}
I \quad : \quad &= \frac{1}{2}\int_a^b\left(y-\frac{a+b}{2}\right)\left[\frac{\partial h\left(a,y\right)}{\partial y}+\frac{\partial h\left(b,y\right)}{\partial y}\right]dy \\
&= \int_a^b\left(y-\frac{a+b}{2}\right)\left[f'\left(y\right)\left(g\left(y\right)-\frac{g\left(a\right)+g\left(b\right)}{2}\right)\right. \\
&\quad\left. +g'\left(y\right)\left(f\left(y\right)-\frac{f\left(a\right)+f\left(b\right)}{2}\right)\right]dy \\
&= \int_a^b\left(x-\frac{a+b}{2}\right)\left(g\left(x\right)-\frac{g\left(a\right)+g\left(b\right)}{2}\right)f'\left(x\right)dx \\
&\quad +\int_a^b\left(x-\frac{a+b}{2}\right)\left(f\left(x\right)-\frac{f\left(a\right)+f\left(b\right)}{2}\right)g'\left(x\right)dx
\end{aligned}$$

and, similarly,

$$\frac{1}{2}\int_a^b\left(x-\frac{a+b}{2}\right)\left[\frac{\partial h\left(x,a\right)}{\partial x}+\frac{\partial h\left(x,b\right)}{\partial x}\right]dx = I.$$

On the other hand,

$$\frac{h\left(a,a\right)+h\left(a,b\right)+h\left(b,a\right)+h\left(b,b\right)}{4} = \frac{\left(f\left(a\right)-f\left(b\right)\right)\left(g\left(a\right)-g\left(b\right)\right)}{2}$$

and

$$\int_a^b \int_a^b \left(x - \frac{a+b}{2} \right) \left(y - \frac{a+b}{2} \right) \frac{\partial^2 h\,(x,y)}{\partial x \partial y} dx dy$$

$$= -\int_a^b \int_a^b \left(x - \frac{a+b}{2} \right) \left(y - \frac{a+b}{2} \right) [f'\,(x)\,g'\,(y) + f'\,(y)\,g'\,(x)]\,dx dy$$

$$= -2 \int_a^b \int_a^b \left(x - \frac{a+b}{2} \right) \left(y - \frac{a+b}{2} \right) f'\,(x)\,g'\,(y)\,dx dy$$

$$= -2 \int_a^b \left(x - \frac{a+b}{2} \right) f'\,(x)\,dx \int_a^b \left(x - \frac{a+b}{2} \right) g'\,(x)\,dx.$$

Now, by (2.9) (dividing by 2), we get the identity:

$$\frac{1}{2} \int_a^b \int_a^b (f\,(x) - f\,(y))\,(g\,(x) - g\,(y))\,dx dy$$

$$+ \int_a^b \left(x - \frac{a+b}{2} \right) \left(g\,(x) - \frac{g\,(a) + g\,(b)}{2} \right) f'\,(x)\,dx$$

$$+ \int_a^b \left(x - \frac{a+b}{2} \right) \left(f\,(x) - \frac{f\,(a) + f\,(b)}{2} \right) g'\,(x)\,dx$$

$$= \frac{(f\,(a) - f\,(b))\,(g\,(a) - g\,(b))}{4}\,(b-a)^2$$

$$- \int_a^b \left(x - \frac{a+b}{2} \right) f'\,(x)\,dx \int_a^b \left(x - \frac{a+b}{2} \right) g'\,(x)\,dx.$$

As it is well known that

$$\frac{1}{2} \int_a^b \int_a^b (f\,(x) - f\,(y))\,(g\,(x) - g\,(y))\,dx dy$$

$$= (b-a) \int_a^b f\,(x)\,g\,(x)\,dx - \int_a^b f\,(x)\,dx \int_a^b g\,(x)\,dx,$$

the inequality (2.8) is completely proved.   □

3. Some Integral Inequalities for $\|\cdot\|_\infty$ −Norm

The following inequality holds.

Theorem 4. *Let f be as in Theorem 3 and assume that*

$$\left\| \frac{\partial^2 f}{\partial x \partial y} \right\|_\infty := \sup_{(x,y) \in (a,b) \times (c,d)} \left| \frac{\partial^2 f\,(x,y)}{\partial x \partial y} \right| < \infty.$$

Then we have the estimation

(3.1)
$$\left| \int_a^b \int_c^d f(x,y)\, dy dx + (b-a) \int_c^d \left(y - \frac{c+d}{2} \right) f_2(y)\, dy \right.$$

$$+ (d-c) \int_a^b \left(x - \frac{a+b}{2} \right) f_1(x)\, dx$$

$$\left. - \frac{f(a,c) + f(a,d) + f(b,c) + f(b,d)}{4} (b-a)(d-c) \right|$$

$$\leq \frac{1}{16} (b-a)^2 (d-c)^2 \left\| \frac{\partial^2 f}{\partial x \partial y} \right\|_\infty,$$

where

$$f_1(x) := \frac{1}{2} \left[\frac{\partial f(x,c)}{\partial x} + \frac{\partial f(x,d)}{\partial x} \right], \ x \in [a,b]$$

and

$$f_2(y) := \frac{1}{2} \left[\frac{\partial f(a,y)}{\partial y} + \frac{\partial f(b,y)}{\partial y} \right], \ y \in [c,d].$$

Proof. Using the identity (2.1) and the properties of the integral, we can state:

$$\left| \int_a^b \int_c^d f(x,y)\, dy dx + (b-a) \int_c^d \left(y - \frac{c+d}{2} \right) f_2(y)\, dy \right.$$

$$+ (d-c) \int_a^b \left(x - \frac{a+b}{2} \right) f_1(x)\, dx$$

$$\left. - \frac{f(a,c) + f(a,d) + f(b,c) + f(b,d)}{4} (b-a)(d-c) \right|$$

$$\leq \int_a^b \int_c^d \left| x - \frac{a+b}{2} \right| \left| y - \frac{c+d}{2} \right| \left| \frac{\partial^2 f(x,y)}{\partial x \partial y} \right| dy dx$$

$$\leq \left\| \frac{\partial^2 f}{\partial x \partial y} \right\|_\infty \int_a^b \left| x - \frac{a+b}{2} \right| dx \int_c^d \left| y - \frac{c+d}{2} \right| dy$$

$$= \left\| \frac{\partial^2 f}{\partial x \partial y} \right\|_\infty \frac{(b-a)^2}{4} \cdot \frac{(d-c)^2}{4}$$

and the inequality (3.1) is proved. $\square$

Another inequality which employs the $\|\cdot\|_\infty$ −norm of f_1 and f_2 which can be useful in practice is embodied in the following theorem:

Theorem 5. *Let f be as in Theorem 4 and assume that*

$$\|f_1\|_\infty := \sup_{x \in [a,b]} |f_1(x)| < \infty, \ \|f_2\|_\infty := \sup_{x \in [a,b]} |f_2(x)| < \infty.$$

Then

$$(3.2) \quad \left| \int_a^b \int_c^d f(x,y)\, dy dx - \frac{f(a,c) + f(a,d) + f(b,c) + f(b,d)}{4} (b-a)(d-c) \right|$$

$$\leq \quad \frac{1}{4}(b-a)(d-c)$$

$$\times \left[(b-a)\|f_1\|_\infty + (d-c)\|f_2\|_\infty + \frac{(b-a)(d-c)}{4} \left\| \frac{\partial^2 f}{\partial x \partial y} \right\|_\infty \right].$$

Proof. As in Theorem 4, we have, by the identity (2.1) that

$$\left| \int_a^b \int_c^d f(x,y)\, dy dx - \frac{f(a,c) + f(a,d) + f(b,c) + f(b,d)}{4} (b-a)(d-c) \right|$$

$$\leq \quad (b-a)\int_c^d \left| y - \frac{c+d}{2} \right| |f_2(y)|\, dy + (d-c)\int_a^b \left| x - \frac{a+b}{2} \right| |f_1(x)|\, dx$$

$$+ \int_a^b \int_c^d \left| x - \frac{a+b}{2} \right| \left| y - \frac{c+d}{2} \right| \left| \frac{\partial^2 f(x,y)}{\partial x \partial y} \right| dy dx$$

$$\leq \quad (b-a)\|f_2\|_\infty \frac{(d-c)^2}{4} + (d-c)\|f_1\|_\infty \frac{(b-a)^2}{4}$$

$$+ \frac{(b-a)^2}{4} \cdot \frac{(d-c)^2}{4} \left\| \frac{\partial^2 f}{\partial x \partial y} \right\|_\infty$$

$$= \quad \frac{1}{4}(b-a)(d-c)$$

$$\times \left[(d-c)\|f_2\|_\infty + (b-a)\|f_1\|_\infty + \frac{(b-a)(d-c)}{4} \left\| \frac{\partial^2 f}{\partial x \partial y} \right\|_\infty \right].$$

Hence, the proof is completed.   □

Remark 1. *If we know that*

$$\left\| \frac{\partial f}{\partial x} \right\|_\infty := \sup_{(x,y) \in (a,b) \times (c,d)} \left| \frac{\partial f(x,y)}{\partial x} \right| < \infty$$

and

$$\left\| \frac{\partial f}{\partial y} \right\|_\infty := \sup_{(x,y) \in (a,b) \times (c,d)} \left| \frac{\partial f(x,y)}{\partial y} \right| < \infty,$$

then, obviously

$$\|f_1\|_\infty \leq \left\| \frac{\partial f}{\partial x} \right\|_\infty, \quad \|f_2\|_\infty \leq \left\| \frac{\partial f}{\partial y} \right\|_\infty$$

and by (3.2) we deduce

$$(3.3) \quad \left| \int_a^b \int_c^d f(x,y)\, dy dx - \frac{f(a,c) + f(a,d) + f(b,c) + f(b,d)}{4} (b-a)(d-c) \right|$$

$$\leq \quad \frac{1}{4}(b-a)(d-c)$$

$$\times \left[(b-a)\left\| \frac{\partial f}{\partial x} \right\|_\infty + (d-c)\left\| \frac{\partial f}{\partial y} \right\|_\infty + \frac{(b-a)(d-c)}{4} \left\| \frac{\partial^2 f}{\partial x \partial y} \right\|_\infty \right].$$

Now, let us recall Grüss' inequality:

If $f, g : [a, b] \to \mathbb{R}$ are integrable on $[a, b]$ and satisfy the condition

$$(3.4) \qquad m_1 \le f(x) \le M_1, \;\; m_2 \le g(x) \le M_2 \;\; \text{for all } x \in [a, b],$$

then we have the inequality:

$$(3.5) \qquad \left| (b-a) \int_a^b f(x) g(x) \, dx - \int_a^b f(x) \, dx \int_a^b g(x) \, dx \right|$$

$$\le \;\; \frac{1}{4} (M_2 - m_2)(M_1 - m_1)(b-a)^2$$

and the constant $\frac{1}{4}$ is the best possible one.

Using Corollary 3, we can state a similar result.

Theorem 6. *Let f, g be as in Corollary 3 and assume that $f', g', f, g \in L_\infty [a, b]$. Then we have the inequality:*

$$(3.6) \qquad \left| (b-a) \int_a^b f(x) g(x) \, dx - \int_a^b f(x) \, dx \cdot \int_a^b g(x) \, dx \right|$$

$$\le \;\; \frac{1}{4}(b-a)^2 \left[|(f(a) - f(b))(g(a) - g(b))| + \left\| \frac{g(a) + g(b)}{2} - g \right\| \|f'\|_\infty \right.$$

$$\left. + \left\| \frac{f(a) + f(b)}{2} - f \right\|_\infty \|g'\|_\infty + \frac{(b-a)^2}{4} \|f'\|_\infty \|g'\|_\infty \right],$$

where, by $\left\| \frac{g(a) + g(b)}{2} - g \right\|_\infty$, we understand

$$\left\| \frac{g(a) + g(b)}{2} - g \right\|_\infty = \sup_{x \in [a, b]} \left| \frac{g(a) + g(b)}{2} - g(x) \right|.$$

Proof. Using the identity (2.8) and the properties of modulus, we have

$$\left| (b-a) \int_a^b f(x) g(x) \, dx - \int_a^b f(x) \, dx \int_a^b g(x) \, dx \right|$$

$$\le \;\; \frac{(b-a)^2}{4} |(f(a) - f(b))(g(a) - g(b))|$$

$$+ \int_a^b \left| x - \frac{a+b}{2} \right| \left| \frac{g(a) + g(b)}{2} - g(x) \right| |f'(x)| \, dx$$

$$+ \int_a^b \left| x - \frac{a+b}{2} \right| \left| \frac{f(a) + f(b)}{2} - f(x) \right| |g'(x)| \, dx$$

$$+ \int_a^b \left| x - \frac{a+b}{2} \right| |f'(x)| \, dx \int_a^b \left| x - \frac{a+b}{2} \right| |g'(x)| \, dx$$

$$\leq \frac{(b-a)^2}{4} \left| (f(a) - f(b))(g(a) - g(b)) \right|$$

$$+ \left\| \frac{g(a) + g(b)}{2} - g \right\| \|f'\|_\infty \int_a^b \left| x - \frac{a+b}{2} \right| dx$$

$$+ \left\| \frac{f(a) + f(b)}{2} - f \right\|_\infty \|g'\|_\infty \int_a^b \left| x - \frac{a+b}{2} \right| dx$$

$$+ \|f'\|_\infty \|g'\|_\infty \left(\int_a^b \left| x - \frac{a+b}{2} \right| dx \right)^2$$

and the inequality (3.6) is proved. $\square$

4. Some Integral Inequalities for $\|\cdot\|_p$ −Norm, $p \in [1, \infty)$.

The following result also holds.

Theorem 7. *Let $f : [a, b] \times [c, d] \to \mathbb{R}$ be as in Theorem 3 and assume that*

$$(4.1) \qquad \left\| \frac{\partial^2 f}{\partial x \partial y} \right\|_p := \left(\int_a^b \int_c^d \left| \frac{\partial^2 f(x, y)}{\partial x \partial y} \right|^p dy dx \right)^{\frac{1}{p}} < \infty, \quad p \in [1, \infty).$$

Then we have the estimate

$$(4.2) \qquad \left| \int_a^b \int_c^d f(x, y) \, dy dx + (b - a) \int_c^d \left(y - \frac{c+d}{2} \right) f_2(y) \, dy \right.$$

$$+ (d - c) \int_a^b \left(x - \frac{a+b}{2} \right) f_1(x) \, dx$$

$$\left. - \frac{f(a, c) + f(a, d) + f(b, c) + f(b, d)}{4} (b - a)(d - c) \right|$$

$$\leq \begin{cases} \frac{(b-a)(d-c)}{4} \left\| \frac{\partial^2 f}{\partial x \partial y} \right\|_1 & \text{if} \quad \frac{\partial^2 f}{\partial x \partial y} \in L_1[a, b]; \\[3mm] \left[\frac{(b-a)^{1+\frac{1}{q}}(d-c)^{1+\frac{1}{q}}}{4(q+1)^{\frac{2}{q}}} \right] \left\| \frac{\partial^2 f}{\partial x \partial y} \right\|_p, & \text{if} \quad \frac{\partial^2 f}{\partial x \partial y} \in L_p[a, b] \ \text{and} \\[1mm] & \qquad\qquad p > 1, \ \frac{1}{p} + \frac{1}{q} = 1, \end{cases}$$

where

$$f_1(x) = \frac{1}{2} \left[\frac{\partial f(x, c)}{\partial x} + \frac{\partial f(x, d)}{\partial x} \right], \quad x \in [a, b]$$

and

$$f_2(y) = \frac{1}{2} \left[\frac{\partial f(a, y)}{\partial y} + \frac{\partial f(b, y)}{\partial y} \right], \quad y \in [c, d].$$

Proof. It is easy to see that

$$\int_a^b \int_c^d \left| x - \frac{a+b}{2} \right| \left| y - \frac{c+d}{2} \right| \left| \frac{\partial^2 f(x,y)}{\partial x \partial y} \right| dy dx$$

$$\leq \sup_{(x,y)\in[a,b]\times[c,d]} \left| x - \frac{a+b}{2} \right| \left| y - \frac{c+d}{2} \right| \int_a^b \int_c^d \left| \frac{\partial^2 f(x,y)}{\partial x \partial y} \right| dy dx$$

$$= \frac{(b-a)(d-c)}{4} \left\| \frac{\partial^2 f}{\partial x \partial y} \right\|_1$$

and, by Hölder's inequality for double integrals, $(p > 1, \frac{1}{p} + \frac{1}{q} = 1)$, we have

$$\int_a^b \int_c^d \left| x - \frac{a+b}{2} \right| \left| y - \frac{c+d}{2} \right| \left| \frac{\partial^2 f(x,y)}{\partial x \partial y} \right| dy dx$$

$$\leq \left(\int_a^b \int_c^d \left| x - \frac{a+b}{2} \right|^q \left| y - \frac{c+d}{2} \right|^q dy dx \right)^{\frac{1}{q}} \left(\int_a^b \int_c^d \left| \frac{\partial^2 f(x,y)}{\partial x \partial y} \right|^p dy dx \right)^{\frac{1}{p}}$$

$$= \left[\frac{(b-a)^{1+\frac{1}{q}}}{2^q(q+1)} \right]^{\frac{1}{q}} \left[\frac{(d-c)^{1+\frac{1}{q}}}{2^q(q+1)} \right]^{\frac{1}{q}} \left\| \frac{\partial^2 f}{\partial x \partial y} \right\|_p$$

$$= \frac{(b-a)^{1+\frac{1}{q}}(d-c)^{1+\frac{1}{q}}}{4(q+1)^{\frac{2}{q}}} \left\| \frac{\partial^2 f}{\partial x \partial y} \right\|_p.$$

Using the representation (2.1) we get the desired result. $\qquad\square$

Remark 2. *Similar results to those in Theorems 5 and 6 may be stated, but we omit the details.*

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