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AN ALGEBRAIC INEQUALITY, II

BAI-NI GUO AND FENG QI

ABSTRACT. In this article, using inequality between logarithmic mean and one-parameter mean, which can be deduced from monotonicity of the extended mean values, an integral analogue of J. S. Martins' inequality is proved. An open problem is proposed.

1. Introduction

In [1, 12, 35, 37], double inequalities were proved using mathematical induction and other techniques, which can be expressed as

$$\frac{n}{n+1} \leqslant \left(\frac{1}{n}\sum_{i=1}^{n} i^r \bigg/ \frac{1}{n+1}\sum_{i=1}^{n+1} i^r\right)^{1/r} \leqslant \frac{\sqrt[n]{n!}}{\sqrt[n+1]{(n+1)!}}, \tag{1}$$

where $r > 0$ and $n \in \mathbb{N}$.

We call the left-hand side of inequality (1) H. Alzer's inequality [1], and the right-hand side of inequality (1) J. S. Martins' inequality [10]. The inequality between the two ends of (1) is called Minc-Sathre's inequality [11].

In [16], the second author generalised Alzer's inequality and obtained

$$\frac{n+k}{n+m+k} < \left(\frac{1}{n}\sum_{i=k+1}^{n+k} i^r \bigg/ \frac{1}{n+m}\sum_{i=k+1}^{n+m+k} i^r\right)^{1/r}, \tag{2}$$

where r is a given positive real number, n and m are natural numbers, and k is a nonnegative integer. The lower bound in (2) is the best possible.

In [3], Martins' inequality was generalised: Let $\{a_i\}_{i=1}^{\infty}$ be an increasing sequence of positive real numbers and

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(1) for any positive integer $\ell > 1$,

$$\frac{a_{\ell+1}}{a_\ell} \le \frac{a_\ell}{a_{\ell-1}}; \tag{3}$$

(2) for any positive integer $\ell > 1$,

$$\left(\frac{a_{\ell+1}}{a_\ell}\right)^\ell \ge \left(\frac{a_\ell}{a_{\ell-1}}\right)^{\ell-1}. \tag{4}$$

Then, for any natural numbers n and m, we have

$$\left(\frac{1}{n}\sum_{i=1}^{n} a_i^r \Big/ \frac{1}{n+m}\sum_{i=1}^{n+m} a_i^r\right)^{1/r} < \frac{\sqrt[n]{a_n!}}{\sqrt[n+m]{a_{n+m}!}}, \tag{5}$$

where $n, m \in \mathbb{N}$ and r is a positive number, $a_i!$ denotes $\prod_{i=1}^{n} a_i$. The upper bound is best possible.

As a corollary of inequality (5), we have: Let a and b be positive real numbers, k a nonnegative integer, and $m, n \in \mathbb{N}$. Then, for any real number $r > 0$, we have

$$\left(\frac{1}{n}\sum_{i=k+1}^{n+k} (ai+b)^r \Big/ \frac{1}{n+m}\sum_{i=k+1}^{n+m+k} (ai+b)^r\right)^{1/r} < \frac{\sqrt[n]{\prod_{i=k+1}^{n+k}(ai+b)}}{\sqrt[n+m]{\prod_{i=k+1}^{n+m+k}(ai+b)}}. \tag{6}$$

Inequalities (5) and (6) answer an open problem proposed in [15, 16].

The Alzer's inequality and inequality (2) have been generalised and extended by many mathematicians. For more information, please refer to [2, 4, 5, 8, 15, 20, 23, 26, 34]. The Minc-Sathre's inequality was generalised in [8, 15, 18, 20, 24, 26]. In [18, 24, 26], the following inequalities were obtained:

$$\frac{n+k+1}{n+m+k+1} < \left(\prod_{i=k+1}^{n+k} i\right)^{1/n} \Big/ \left(\prod_{i=k+1}^{n+m+k} i\right)^{1/(n+m)} \leqslant \sqrt{\frac{n+k}{n+m+k}}, \tag{7}$$

where $n, m \in \mathbb{N}$ and k is a nonnegative integer.

In [13], the second author proved the integral analogue of inequality (2): Let $b > a > 0$ and $\delta > 0$ be real numbers. Then, for any given positive $r \in \mathbb{R}$, we have

$$\begin{aligned} &\left(\frac{1}{b-a}\int_a^b x^r \,\mathrm{d}x \Big/ \frac{1}{b+\delta-a}\int_a^{b+\delta} x^r \,\mathrm{d}x\right)^{1/r} \\ &= \left(\frac{b+\delta-a}{b-a}\cdot\frac{b^{r+1}-a^{r+1}}{(b+\delta)^{r+1}-a^{r+1}}\right)^{1/r} \\ &> \frac{b}{b+\delta}. \end{aligned} \tag{8}$$

The lower bound in (8) is the best possible.

Inequality (8) has been generalised to inequalities for a positive functional in [6].

In this paper, by monotonicity of the extended mean values, from which an inequality between the logarithmic mean and one-parameter mean is deduced, we will prove an open problem proposed in [13] as follows:

Theorem 1. *Let $b > a > 0$ and $\delta > 0$ be real numbers. Then, for any positive $r \in \mathbb{R}$, we have*

$$\left(\frac{b+\delta-a}{b-a}\cdot\frac{b^{r+1}-a^{r+1}}{(b+\delta)^{r+1}-a^{r+1}}\right)^{1/r} < \frac{[b^b/a^a]^{1/(b-a)}}{[(b+\delta)^{b+\delta}/a^a]^{1/(b+\delta-a)}}. \tag{9}$$

The upper bound in (9) *is the best possible.*

At last, we propose an open problem.

2. Lemmas

In [36], Stolarsky defined the extended mean values $E(r,s;x,y)$ and proved that it is continuous on the domain $\{(r,s;x,y) : r,s \in \mathbb{R}, x,y > 0\}$ as follows

$$E(r,s;x,y) = \left(\frac{r}{s}\cdot\frac{y^s-x^s}{y^r-x^r}\right)^{1/(s-r)}, \qquad rs(r-s)(x-y) \neq 0; \tag{10}$$

$$E(r,0;x,y) = \left(\frac{1}{r}\cdot\frac{y^r-x^r}{\ln y-\ln x}\right)^{1/r}, \qquad r(x-y) \neq 0; \tag{11}$$

$$E(r,r;x,y) = e^{-1/r}\left(\frac{x^{x^r}}{y^{y^r}}\right)^{1/(x^r-y^r)}, \qquad r(x-y) \neq 0; \tag{12}$$

$$E(0,0;x,y) = \sqrt{xy}, \qquad x \neq y; \tag{13}$$

$$E(r,s;x,x) = x, \qquad x = y.$$

Leach and Sholander [9] showed that $E(r,s;x,y)$ are increasing with both r and s, or with both x and y. The monotonicities of E have also been researched by the authors and others in [7, 25] and [27]–[32] using different ideas and simpler methods. In [19], the logarithmic convexity of E was proved.

Most two variable means are special cases of E, for example

$$E(0,1;x,y) = L(x,y), \tag{14}$$

$$E(r,r+1;x,y) = J_r(x,y), \quad r > 0. \tag{15}$$

They are called the logarithmic mean and the one-parameter mean, respectively, and, by the monotonicity of E, we have

$$J_r(x,y) > L(x,y), \quad r > 0. \tag{16}$$

Recently, the concepts of mean values has been generalised in [14, 17, 21, 22], [27]–[31] and [33].

3. Proof of Theorem 1

Inequality (9) can be rewritten as follows

$$\frac{b^{r+1}-a^{r+1}}{(b-a)(b^b/a^a)^{r/(b-a)}}<\frac{(b+\delta)^{r+1}-a^{r+1}}{(b+\delta-a)((b+\delta)^{b+\delta}/a^a)^{r/(b+\delta-a)}}. \tag{17}$$

Therefore, it suffices to prove the function

$$\phi(x)=\frac{x^{r+1}-a^{r+1}}{(x-a)(x^x/a^a)^{r/(x-a)}} \tag{18}$$

is decreasing in $x>a>0$.

Taking the logarithm and differentiating yields

$$\begin{aligned}\frac{\mathrm{d}\ln\phi(x)}{\mathrm{d}x}&=\frac{a^{2+r}+a^{2+r}r-a^{1+r}x-a^{1+r}rx-a^2x^r-a^2rx^r+ax^{1+r}+arx^{1+r}}{(a-x)^2(a^{1+r}-x^{1+r})}\\&\quad+\frac{rx^{2+r}\ln x-a^{1+r}rx\ln x-arx^{1+r}\ln x+a^{1+r}r\ln(a^{-a}x^x)-rx^{1+r}\ln(a^{-a}x^x)}{(a-x)^2(a^{1+r}-x^{1+r})}\\&=\frac{a[(1+r)(x-a)(x^r-a^r)+r(x^{1+r}-a^{1+r})(\ln a-\ln x)]}{(a-x)^2(a^{1+r}-x^{1+r})}\\&=\frac{a(1+r)(x^r-a^r)(\ln a-\ln x)\left(\frac{r(x^{1+r}-a^{1+r})}{(r+1)(x^r-a^r)}-\frac{x-a}{\ln x-\ln a}\right)}{(a-x)^2(a^{1+r}-x^{1+r})}\\&=\frac{a(1+r)(x^r-a^r)(\ln a-\ln x)\left(J_r(a,x)-L(a,x)\right)}{(a-x)^2(a^{1+r}-x^{1+r})}\\&<0.\end{aligned} \tag{19}$$

Thus, the function $\ln\phi(x)$, and $\phi(x)$, is decreasing.

Here L'Hospital's rule yields

$$\lim_{r\to0^+}\left(\frac{b+\delta-a}{b-a}\cdot\frac{b^{r+1}-a^{r+1}}{(b+\delta)^{r+1}-a^{r+1}}\right)^{1/r}=\frac{[b^b/a^a]^{1/(b-a)}}{[(b+\delta)^{b+\delta}/a^a]^{1/(b+\delta-a)}}. \tag{20}$$

Hence, the upper bound in inequality (9) is the best possible. The proof is complete.

4. Open Problem

Lastly, we propose the following

Open Problem. *Let $b>a>0$ and $\delta>0$ be real numbers, $f(x)$ a positive integrable function. Then, for any given positive $r\in\mathbb{R}$, we have*

$$\begin{aligned}\frac{\sup_{x\in[a,b]}f(x)}{\sup_{x\in[a,b+\delta]}f(x)}&<\left(\frac{1}{b-a}\int_a^b f^r(x)\,\mathrm{d}x\Big/\frac{1}{b+\delta-a}\int_a^{b+\delta}f^r(x)\,\mathrm{d}x\right)^{1/r}\\&<\exp\left(\frac{\int_a^b\ln f(x)\,\mathrm{d}x}{b-a}-\frac{\int_a^{b+\delta}\ln f(x)\,\mathrm{d}x}{b+\delta-a}\right).\end{aligned} \tag{21}$$

The lower and upper bounds in (21) are the best possible.

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