Some Inequalities Involving the Geometric Mean of Natural Numbers and the Ratio of Gamma Functions

SOME INEQUALITIES INVOLVING THE GEOMETRIC MEAN OF NATURAL NUMBERS AND THE RATIO OF GAMMA FUNCTIONS

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Abstract. In this article, using Stirling's formula, the series-expansion of digamma functions and other techniques, two inequalities involving the geometric mean of natural numbers and the ratio of gamma functions are obtained.

1. Introduction

In [1], Dr. H. Alzer proved that the inequalities

$$\frac{n+2\sqrt{2}-1}{n+1} \le \frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}} < \frac{n+2}{n+1} \tag{1}$$

hold for all integers $n \ge 1$. The lower and upper bounds in (1) are the best possible.

He also verified in [2] that the inequality

$$\frac{[\Gamma(x+2)]^{1/(x+1)}}{[\Gamma(x+1)]^{1/x}} < \frac{x+2}{x+1} \tag{2}$$

holds for $x \ge 2$.

Since $\Gamma(n+1) = n!$, the right hand side in (1) can be deduced from inequality (2) only if we let $x = n \ge 2$. Moreover, the right hand side in (1) refines the inequality

$$\frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}} < \frac{n+1}{n}, \tag{3}$$

which was obtained in [12] by H. Minc and L. Sathre.

Recently, in [18] and [22], the first author obtained the following

$$\frac{n+k}{n+m+k} < \frac{\sqrt[n]{(n+k)!/k!}}{\sqrt[n+m]{(n+m+k)!/k!}} < \sqrt{\frac{n+k}{n+m+k}} \tag{4}$$

for positive integers n and m and nonnegative integer k.

The inequality (3) was refined by Dr. H. Alzer in [3]: Let $n \in \mathbb{N}$, then, for any $r > 0$, we have

$$\frac{n}{n+1} \le \left(\frac{1}{n}\sum_{i=1}^{n} i^r \Bigg/ \frac{1}{n+1}\sum_{i=1}^{n+1} i^r\right)^{1/r} < \frac{\sqrt[n]{n!}}{\sqrt[n+1]{(n+1)!}}. \tag{5}$$

The lower and upper bounds are the best possible.

Many new and simple proofs of the inequalities in (5) and some generalizations were given in [5, 6, 10, 11, 14, 17, 21, 23, 24, 27].

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The left hand side of inequality (5) was generalized in [16]: Let n and m be natural numbers, k a nonnegative integer. Then

$$\frac{n+k}{n+m+k} < \left(\frac{1}{n}\sum_{i=k+1}^{n+k} i^r \Bigg/ \frac{1}{n+m}\sum_{i=k+1}^{n+m+k} i^r\right)^{1/r}, \tag{6}$$

where r is any given positive real number. The lower bound is the best possible.

The integral analogue of (6) was presented in [8] and [15]: Let $b > a > 0$ and $\delta > 0$ be real numbers, then, for any given positive $r \in \mathbb{R}$, we have

$$\begin{aligned}\frac{b}{b+\delta} &< \left(\frac{b+\delta-a}{b-a}\cdot\frac{b^{r+1}-a^{r+1}}{(b+\delta)^{r+1}-a^{r+1}}\right)^{1/r}\\ &= \left(\frac{1}{b-a}\int_a^b x^r\,\mathrm{d}x \Bigg/ \frac{1}{b+\delta-a}\int_a^{b+\delta} x^r\,\mathrm{d}x\right)^{1/r}\\ &< \frac{[b^b/a^a]^{1/(b-a)}}{[(b+\delta)^{b+\delta}/a^a]^{1/(b+\delta-a)}}.\end{aligned} \tag{7}$$

The lower and upper bounds in (7) are the best possible.

The inequality (7) was generalized to an inequality for linear positive functionals in [7].

Recently, results related to those above were obtained in [19]. These results were generalisations for monotonic sequences involving convex functions as follows:

- For $a > 1$, let $n \in \mathbb{N}$ and $r > 0$, then
$$\left(\frac{1}{n}\sum_{i=1}^{n} a^{ir} \Bigg/ \frac{1}{n+1}\sum_{i=1}^{n+1} a^{ir}\right)^{1/r} > \frac{1}{a}. \tag{8}$$
- For $n, m \in \mathbb{N}$, $k \in \mathbb{N}\cup\{0\}$ and $r > 0$, we have
$$\frac{1}{a^m} < \left\{\frac{1}{a^n}\sum_{i=k+1}^{n+k} a^{ir} \Bigg/ \frac{1}{a^{n+m}}\sum_{i=k+1}^{n+m+k} a^{ir}\right\}^{1/r}, \tag{9}$$
that is,
$$\frac{1}{a^{m(r+1)}} \le \sum_{i=k+1}^{n+k} a^{ir} \Bigg/ \sum_{i=k+1}^{n+m+k} a^{ir}, \tag{10}$$
where $a > 1$ is a positive real number.
- If $\{a_i\}_{i\in\mathbb{N}}$ is an increasing, positive sequence such that $\left\{i\left(\frac{a_{i+1}}{a_i}-1\right)\right\}_{i\in\mathbb{N}}$ increases, then we have
$$\frac{a_n}{a_{n+1}} \le \sqrt[n]{\prod_{i=1}^{n}(a_i+a_n)} \Bigg/ \sqrt[n+1]{\prod_{i=1}^{n+1}(a_i+a_{n+1})} \le \sqrt[n]{\prod_{i=1}^{n} a_i} \Bigg/ \sqrt[n+1]{\prod_{i=1}^{n+1} a_i}. \tag{11}$$
- If φ is an increasing, convex, positive function defined on $(0,\infty)$ such that $\left\{\varphi(i)\left[\frac{\varphi(i)}{\varphi(i+1)}-1\right]\right\}_{i\in\mathbb{N}}$ decreases, then
$$\frac{[\varphi(n)]^{n/\varphi(n)}}{[\varphi(n+1)]^{(n+1)/\varphi(n+1)}} \le \sqrt[\varphi(n)]{\prod_{i=1}^{n}[\varphi(i)+\varphi(n)]} \Bigg/ \sqrt[\varphi(n+1)]{\prod_{i=1}^{n+1}[\varphi(i)+\varphi(n+1)]}. \tag{12}$$

These inequalities generalize those obtained in [10], [17], and [22].

In this article, we will prove the following inequalities

Theorem 1. *For $m, n \in \mathbb{N}$ and nonnegative integer k, we have*

$$\frac{\sqrt[n]{(n+k)!/k!}}{\sqrt[n+m]{(n+m+k)!/k!}} > \frac{n+k+1}{n+m+k+1}. \tag{13}$$

Theorem 2. *The function*

$$\frac{[\Gamma(x+y+1)/\Gamma(y+1)]^{1/x}}{x+y+1} \tag{14}$$

is decreasing in $x \geq 1$ for fixed $y \geq 0$.

If the above hold, then, for positive real numbers x and y, we have

$$\frac{x+y+1}{x+y+2} \leq \frac{[\Gamma(x+y+1)/\Gamma(y+1)]^{1/x}}{[\Gamma(x+y+2)/\Gamma(y+1)]^{1/(x+1)}}. \tag{15}$$

Remark 1. If we take $x, y \in \mathbb{N}$, then the right hand side of (4) and inequality (13) follow from (15).

2. Proofs of Theorems

Proof of Theorem 1. Inequality (13) can be rearranged so that we have

$$\frac{n+k+1}{\sqrt[n]{(n+k)!/k!}} < \frac{n+m+k+1}{\sqrt[n+m]{(n+m+k)!/k!}},$$

which is equivalent to

$$\frac{n+k+1}{\sqrt[n]{(n+k)!/k!}} < \frac{n+k+2}{\sqrt[n+1]{(n+k+1)!/k!}}. \tag{16}$$

When $k = 0$, inequality (16) follows from the right inequality in (1).

When $k \geq 1$, the inequality (16) can be rewritten as

$$\left[\frac{(n+k)!}{k!}\right]^{1/n} > \frac{(n+k+1)^{n+2}}{(n+k+2)^{n+1}}. \tag{17}$$

In [10] and [13, p. 184], the following inequalities were given for $n \in \mathbb{N}$

$$\sqrt{2\pi n}\left(\frac{n}{e}\right)^n < n! < \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \exp\frac{1}{12n}. \tag{18}$$

Inequality (18) is related to the Stirling's formula. In [25], Professor J. Sándor and L. Debnath proved a new form of the Stirling's formula: For all positive integers $n \geq 2$, we have the double inequality

$$\sqrt{2\pi}\,\mathrm{e}^{-n}\, n^{n+1/2} < n! < \left(\frac{n}{n-1}\right)^{1/2}\sqrt{2\pi}\,\mathrm{e}^{-n}\, n^{n+1/2}. \tag{19}$$

By substituting the inequalities in (18) into the left term of inequality (17), we see that it is sufficient to prove the following

$$\left[\sqrt{2\pi(n+k)}\left(\frac{n+k}{e}\right)^{n+k}\right]^{1/n} > \frac{(n+k+1)^{n+2}}{(n+k+2)^{n+1}}\left[\sqrt{2\pi k}\left(\frac{k}{e}\right)^k \exp\frac{1}{12k}\right]^{1/n}. \tag{20}$$

Taking the logarithm on both sides of inequality (20), simplifying directly and using standard arguments, we obtain

$$\frac{2k+1}{2n}\ln\left(1+\frac{n}{k}\right)+(n+1)\ln\left(1+\frac{1}{n+k+1}\right)-\ln\left(1+\frac{1}{n+k}\right)-\frac{1}{12kn}-1 > 0. \tag{21}$$

In [9, pp. 367–368], [13, pp. 273–274] and [20], we have for $t > 0$

$$\ln\left(1+\frac{1}{t}\right) > \frac{2}{2t+1}, \tag{22}$$

$$\ln(1+t) < \frac{t(2+t)}{2(1+t)}. \tag{23}$$

Thus, to get inequality (21), it suffices to show that

$$\frac{2(n+1)}{2(n+k+1)+1} + \frac{2k+1}{2k+n} - \frac{2(n+k)+1}{2(n+k)(n+k+1)} - \frac{1}{12kn} - 1 > 0,$$

which can be deduced from the following

$$\begin{aligned} &12kn^4(k-1) + 2n^3(n+5k)(k^2-1) + 5n^3(k^3-1) + 6k^2n^2(kn-1) \\ &+ 3n^2(k^3n-1) + 2k(n^2+3k)(k^2n-1) + k^2n(kn^2-1) \\ &+ 9kn(k^2n^2-1) + 10k^3(n^3-1) + 2k^2n(k+12)(n^2-1) \\ &+ 4k^4(6n^2-1) + 6k^3n(3k+10n) + 10k^2n^4 > 0. \end{aligned} \tag{24}$$

The proof is complete. □

Proof of Theorem 2. For a fixed real number $y \geq 0$, define

$$w(x) = \frac{\ln\Gamma(x+y+1) - \ln\Gamma(y+1)}{x} - \ln(x+y+1), \quad x \in [1,\infty). \tag{25}$$

A simple calculation reveals that

$$w'(x) = \frac{\ln\Gamma(y+1) - \ln\Gamma(x+y+1)}{x^2} - \frac{1}{x+y+1} + \frac{\psi(x+y+1)}{x}, \tag{26}$$

where $\psi = \Gamma'/\Gamma$ denotes the logarithmic derivatives of the gamma function. It is also called a digamma function in [4, p. 71].

It is well-known that

$$\Gamma(z+1) = z\Gamma(z), \quad \operatorname{Re}(z) > 0; \tag{27}$$

$$\psi(x) < \ln x - \frac{1}{2x}, \quad x > 1; \tag{28}$$

$$\psi'(z) = \sum_{i=0}^{\infty} \frac{1}{(i+z)^2};. \tag{29}$$

The inequality (28) can be found in [9, 12, 13] respectively. For more on formula (29), please refer to formula (8.12) in Theorem 8.3, page 93 in [26].

Using the formulae (27) and (29) and inequalities (23) and (28) and direct computation, we have

$$\begin{aligned} \frac{[x^2w'(x)]'}{x} &= \psi'(x+y+1) - \frac{x+2y+2}{(x+y+1)^2} \\ &= \sum_{i=1}^{\infty} \frac{1}{(x+y+i)^2} - \frac{x+2y+2}{(x+y+1)^2} \\ &< \frac{1}{(x+y+1)^2} + \int_1^{\infty} \frac{\mathrm{d}t}{(x+y+t)^2} - \frac{x+2y+2}{(x+y+1)^2} \\ &= -\frac{y}{(x+y+1)^2} \\ &< 0, \end{aligned} \tag{30}$$

and

$$
\begin{aligned}
w'(1) &= \ln\Gamma(1+y) - \ln\Gamma(2+y) + \psi(2+y) - \frac{1}{2+y}\\
&= \psi(2+y) - \ln(1+y) - \frac{1}{2+y}\\
&< \ln(2+y) - \ln(1+y) - \frac{1}{2(2+y)} - \frac{1}{2+y}\\
&= \ln\left(1+\frac{1}{1+y}\right) - \frac{3}{2(2+y)}\\
&< \frac{2y+3}{2(1+y)(2+y)} - \frac{3}{2(2+y)}\\
&= -\frac{y}{2(1+y)(2+y)}\\
&< 0.
\end{aligned}
\tag{31}
$$

Thus, the function $x^2w'(x)$ is decreasing, $x^2w'(x) < w'(1) < 0$, and the function $w(x)$ is decreasing with $x > 1$. That is, the function $\left[\Gamma(x+y+1)/\Gamma(y+1)\right]^{1/x}/(x+y+1)$ is decreasing with $x > 1$ for fixed $y \geq 0$. This completes the proof. □

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