A Generalisation Of Chebyshev's Inequality For Functions Of Several Variables

A GENERALISATION OF CHEBYSHEV'S INEQUALITY FOR FUNCTIONS OF SEVERAL VARIABLES

P. CERONE, S.S. DRAGOMIR, G. HANNA, AND J.E. PEČARIĆ

ABSTRACT. The current note serves to develop generalisations of Chebyshev's inequality for Hölder functions of several variables.

1. INTRODUCTION

For two measurable functions $f, g : [a, b] \to \mathbb{R}$, define the functional, which is known in the literature as Chebychev's functional

$$T(f, g; a, b) := \frac{1}{b-a}\int_a^b f(x) g(x) \, dx - \frac{1}{(b-a)^2}\int_a^b f(x) \, dx \cdot \int_a^b g(x) \, dx, \tag{1.1}$$

provided that the involved integrals exist. Results involving (1.1) abound in the literature, see for example [1] – [13].

The following inequality is well known as the Grüss inequality [10]

$$|T(f, g; a, b)| \le \frac{1}{4}(M - m)(N - n), \tag{1.2}$$

provided that $m \le f \le M$ and $n \le g \le N$ a.e. on $[a, b]$, where m, M, n, N are real numbers. The constant $\frac{1}{4}$ in (1.2) is the best possible.

Another inequality of this type is due to Chebychev (see for example [13, p. 207]). Namely, if f, g are absolutely continuous on $[a, b]$ and $f', g' \in L_\infty[a, b]$ and $\|f'\|_\infty := ess \sup_{t \in [a,b]} |f'(t)|$, then

$$|T(f, g; a, b)| \le \frac{1}{12}\|f'\|_\infty \|g'\|_\infty (b - a)^2 \tag{1.3}$$

and the constant $\frac{1}{12}$ is the best possible.

Cerone and Dragomir [2] have pointed out generalizations of the above results for integrals defined on two different intervals $[a, b]$ and $[c, d]$.

They defined the functional (generalised Chebychev functional)

$$\begin{aligned} T(f, g; a, b, c, d) \ :&= M(fg; a, b) + M(fg; c, d) \\ &\quad - M(f; a, b) M(g; c, d) - M(f; c, d) M(g; a, b), \end{aligned} \tag{1.4}$$

where the integral mean is defined by

$$M(f; a, b) := \frac{1}{b-a}\int_a^b f(x) \, dx \tag{1.5}$$

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and obtained a variety of bounds using a generalisation of Korkine's identity. Budimir, Cerone and Pečarić [1] obtained the bounds for (1.4) for f and g of Hölder type and also, for a weighted version of (1.4). In particular, they obtained the following result.

Theorem 1. *Let $f, g : I \subseteq \mathbb{R} \to \mathbb{R}$ be measurable on I and the intervals $[a,b]$, $[c,d] \subset I$. Further, suppose that f and g are of Hölder type so that for $x \in [a,b]$, $y \in [c,d]$*

$$|f(x) - f(y)| \leq H_1 |x-y|^r \quad \text{and} \quad |g(x) - g(y)| \leq H_2 |x-y|^s, \tag{1.6}$$

where H_1, $H_2 > 0$ and $r, s \in (0,1]$ are fixed. The following inequality then holds,

$$\begin{aligned} &(\theta+1)(\theta+2)|T(f,g;a,b,c,d)| \\ \leq\ & \frac{H_1 H_2}{(b-a)(d-c)} \left[|b-c|^{\theta+2} - |b-d|^{\theta+2} + |d-a|^{\theta+2} - |c-a|^{\theta+2} \right], \end{aligned} \tag{1.7}$$

where $\theta = r + s$ and $T(f,g;a,b,c,d)$ is as defined by (1.4) and (1.5).

Hanna, Dragomir and Cerone [11] obtained bounds for a two dimensional Chebychev functional for functions of Hölder type and applied them for perturbed Taylor-like formulae in $\mathbb{R}^2$.

It is the express aim of this note to obtain bounds for a Chebychev functional defined on an $n-$dimensional hypercube where the functions are of Hölder type.

2. Bounds for the Chebychev Functional

If we consider the Chebyshev functional:

$$\begin{aligned} D_n(f,g) \quad : &= \frac{1}{\prod_{k=1}^{n}(b_k - a_k)} \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} f(x_1,\ldots,x_n)\, g(x_1,\ldots,x_n)\, dx_1 \ldots dx_n \\ & - \frac{1}{\prod_{k=1}^{n}(b_k - a_k)^2} \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} f(x_1,\ldots,x_n)\, dx_1 \ldots dx_n \\ & \times \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} g(x_1,\ldots,x_n)\, dx_1 \ldots dx_n \\ &= \frac{1}{\nu([\bar{a},\bar{b}])} \int_{\bar{a}}^{\bar{b}} f(\bar{x})\, g(\bar{x})\, d\bar{x} - \frac{1}{\nu([\bar{a},\bar{b}])^2} \int_{\bar{a}}^{\bar{b}} f(\bar{x})\, d\bar{x} \cdot \int_{\bar{a}}^{\bar{b}} g(\bar{x})\, d\bar{x}, \end{aligned}$$

where $\nu([\bar{a},\bar{b}]) := \prod_{k=1}^{n}(b_k - a_k)$, then we can state the following generalisation of Chebyshev's inequality.

Theorem 2. *Let $f, g : [\bar{a},\bar{b}] \to \mathbb{R}$ be of Hölder type. That is,*

$$|f(\bar{x}) - f(\bar{y})| \leq \sum_{i=1}^{n} L_i |x_i - y_i|^{p_i}, \quad \bar{x}, \bar{y} \in [\bar{a},\bar{b}], \tag{2.1}$$

$$|g(\bar{x}) - g(\bar{y})| \leq \sum_{i=1}^{n} H_i |x_i - y_i|^{q_i}, \quad \bar{x}, \bar{y} \in [\bar{a},\bar{b}], \tag{2.2}$$

where L_i, $H_i > 0$ and p_i, $q_i \in (0,1]$ are fixed for $i = 1, 2, \ldots, n$.
Then we have the inequality:

$$|D_n(f,g)| \leq \sum_{i=1}^{n} L_i H_i \frac{(b_i - a_i)^{p_i+q_i}}{(p_i+q_i+1)(p_i+q_i+2)} + 2\sum_{\substack{i\neq j\\ i,j=1}}^{n} L_i H_j \frac{(b_i-a_i)^{p_i}(b_j-a_j)^{q_j}}{(p_i+1)(p_i+2)(q_j+1)(q_j+2)} \tag{2.3}$$

and the inequality is sharp.

Proof. We have

$$|f(\bar{x}) - f(\bar{y})| \leq \sum_{i=1}^{n} L_i |x_i - y_i|^{p_i} \tag{2.4}$$

and

$$|g(\bar{x}) - g(\bar{y})| \leq \sum_{i=1}^{n} H_i |x_i - y_i|^{q_i}. \tag{2.5}$$

If we multiply (2.4) and (2.5), we may get

$$\begin{aligned} & |(f(\bar{x}) - f(\bar{y}))(g(\bar{x}) - g(\bar{y}))| \\ \leq & \sum_{i,j=1}^{n} L_i H_j |x_i - y_i|^{p_i} |x_j - y_j|^{q_j} \\ = & \sum_{i=1}^{n} L_i H_i |x_i - y_i|^{p_i+q_i} + \sum_{\substack{i\neq j\\ i,j=1}}^{n} L_i H_j |x_i - y_i|^{p_i} |x_j - y_j|^{q_j}. \end{aligned}$$

If we integrate over $\bar{x}, \bar{y} \in [\bar{a}, \bar{b}] = \prod_{i=1}^{n} [a_i, b_i] := [a_1, b_1] \times [a_n, b_n]$ we get from Korkine's identity

$$\begin{aligned} & |D_n(f,g)| \\ \leq & \frac{1}{2\left[\prod_{i=1}^{n}(b_i - a_i)\right]^2} \int_{\bar{a}}^{\bar{b}} \int_{\bar{a}}^{\bar{b}} |(f(\bar{x}) - f(\bar{y}))(g(\bar{x}) - g(\bar{y}))| \, d\bar{x} d\bar{y} \\ \leq & \frac{1}{2\left[\prod_{i=1}^{n}(b_i - a_i)\right]^2} \left[\sum_{i=1}^{n} L_i H_i \int_{\bar{a}}^{\bar{b}} \int_{\bar{a}}^{\bar{b}} |x_i - y_i|^{p_i+q_i} \, d\bar{x} d\bar{y} \right. \\ & \left. + \sum_{\substack{i\neq j\\ i,j=1}}^{n} L_i H_j \int_{\bar{a}}^{\bar{b}} \int_{\bar{a}}^{\bar{b}} |x_i - y_i|^{p_i} |x_j - y_j|^{q_j} \, d\bar{x} d\bar{y} \right]. \end{aligned} \tag{2.6}$$

Now, we have that

$$A_i := \int_{\bar{a}}^{\bar{b}} \int_{\bar{a}}^{\bar{b}} |x_i - y_i|^{p_i+q_i} \, d\bar{x} d\bar{y} = \prod_{\substack{k\neq i\\ k=1}}^{n} (b_k - a_k)^2 \int_{a_i}^{b_i} \int_{a_i}^{b_i} |x_i - y_i|^{p_i+q_i} \, dx_i dy_i$$

and as

$$\int_c^d \int_c^d |x - y|^r \, dx dy = 2 \frac{(d-c)^{r+2}}{(r+1)(r+2)}, \tag{2.7}$$

then we get

$$\begin{aligned} A_i &= \prod_{\substack{k\neq i\\k=1}}^{n}(b_k-a_k)^2\cdot\frac{2(b_i-a_i)^{p_i+q_i+2}}{(p_i+q_i+1)(p_i+q_i+2)} \\ &= 2\prod_{k=1}^{n}(b_k-a_k)^2\frac{(b_i-a_i)^{p_i+q_i}}{(p_i+q_i+1)(p_i+q_i+2)}. \end{aligned}$$

Also,

$$\begin{aligned} A_{ij} &:= \int_{\bar a}^{\bar b}\int_{\bar a}^{\bar b}|x_i-y_i|^{p_i}|x_j-y_j|^{q_j}\,d\bar x d\bar y \\ &= \prod_{\substack{k\neq i,j\\k=1}}^{n}(b_k-a_k)^2\int_{a_i}^{b_i}\int_{a_i}^{b_i}|x_i-y_i|^{p_i}\,dx_idy_i\cdot\int_{a_j}^{b_j}\int_{a_j}^{b_j}|x_j-y_j|^{q_j}\,dx_jdy_j \\ &= \prod_{\substack{k\neq i,j\\k=1}}^{n}(b_k-a_k)^2\cdot\frac{2(b_i-a_i)^{p_i+2}}{(p_i+1)(p_i+2)}\cdot\frac{2(b_j-a_j)^{q_j+2}}{(q_j+1)(q_j+2)} \\ &= 4\prod_{k=1}^{n}(b_k-a_k)^2\frac{(b_i-a_i)^{p_i}(b_j-a_j)^{q_j}}{(p_i+1)(p_i+2)(q_j+1)(q_j+2)} \end{aligned}$$

where we have utilised (2.7).

Further, by (2.6), we have:

$$\begin{aligned} &|D_n(f,g)| \\ \leq\;& \frac{1}{2\left[\prod_{i=1}^{n}(b_i-a_i)\right]^2}\left[\sum_{i=1}^{n}L_iH_i-2\prod_{k=1}^{n}(b_k-a_k)^2\frac{(b_i-a_i)^{p_i+q_i}}{(p_i+q_i+1)(p_i+q_i+2)}\right. \\ &\left.+4\sum_{\substack{i\neq j\\i,j=1}}^{n}L_iH_j\prod_{k=1}^{n}(b_k-a_k)^2\frac{(b_i-a_i)^{p_i}(b_j-a_j)^{q_j}}{(p_i+1)(p_i+2)(q_j+1)(q_j+2)}\right] \\ =\;& \sum_{i=1}^{n}L_iH_i\frac{(b_i-a_i)^{p_i+q_i}}{(p_i+q_i+1)(p_i+q_i+2)} \\ &+2\sum_{\substack{i\neq j\\i,j=1}}^{n}L_iH_j\frac{(b_i-a_i)^{p_i}(b_j-a_j)^{q_j}}{(p_i+1)(p_i+2)(q_j+1)(q_j+2)} \end{aligned}$$

and the result (2.6) is thus verified. The sharpness follows from the sharpness of the Chebychev functional for $n=1$ (see for example [12]). ∎

Corollary 1. *Let $f,g:[\bar a,\bar b]\to\mathbb{R}$ be Lipschitzian with constants L_i, $H_i>0$. That is,*

$$|f(\bar x)-f(\bar y)|\leq\sum_{i=1}^{n}L_i|x_i-y_i|,\quad \bar x,\bar y\in[\bar a,\bar b]$$

and

$$|g(\bar x)-g(\bar y)|\leq\sum_{i=1}^{n}H_i|x_i-y_i|,\quad \bar x,\bar y\in[\bar a,\bar b].$$

Then the inequality

$$|D_n(f,g)| \le \frac{1}{12}\sum_{i=1}^{n} L_i H_i (b_i - a_i)^2 + \frac{1}{18}\sum_{\substack{i\neq j\\ i,j=1}}^{n} L_i H_i (b_i - a_i)(b_j - a_j) \tag{2.8}$$

holds and is sharp.

Proof. Taking $p_i = q_i = 1$ for $i = 1, 2, \ldots, n$ in (2.6) readily produces (2.8). ∎

Remark 1. *Result (2.8) is presented in* [12, p. 305], *however, the coefficients of the sums are interchanged. Further, it is apparent that for $x, y \in [a, b]$ and f absolutely continuous, then*

$$\left|\frac{f(x) - f(y)}{x - y}\right| \le \sup_{z\in[a,b]} |f'(z)| = L,$$

demonstrating that a function satisfying a Lipschitzian condition is a weaker condition than one whose derivative belongs to L_∞.

Remark 2. *If $n = 1$, then for $f, g \in [a_1, b_1] \to \mathbb{R}$*

$$|D_1(f,g)| = |T(f,g,a,b)| \le L_1 H_1 \frac{(b_1 - a_1)^{p_1+q_1+1}}{(p_1+q_1+1)(p_1+q_1+2)} \tag{2.9}$$

with

$$|f(x_1) - f(y_1)| \le L_1 |x_1 - y_1|^{p_1}, \ \ |g(x_1) - g(y_1)| \le H_1 |x_1 - y_1|^{q_1}$$

$x_1, y_1 \in [a_1, b_1]$ and $p_1, q_1 \in (0, 1]$.

If $n = 2$ then for $f, g \in [a_1, b_1] \times [a_2, b_2] \to \mathbb{R}$

$$\begin{aligned}
&|D_2(f,g)| \\
&= \left| \frac{1}{(b_1-a_1)(b_2-a_2)} \int_{a_1}^{b_1}\int_{a_2}^{b_2} f(x_1,x_2)\, g(x_1,x_2)\, dx_1 dx_2 \right. \\
&\quad - \frac{1}{(b_1-a_1)^2 (b_2-a_2)^2} \int_{a_1}^{b_1}\int_{a_2}^{b_2} f(x_1,x_2)\, dx_1 dx_2 \\
&\quad \left. \times \int_{a_1}^{b_1}\int_{a_2}^{b_2} g(x_1,x_2)\, dx_1 dx_2 \right| \\
&\le L_1 H_1 \frac{(b_1-a_1)^{p_1+q_1}}{(p_1+q_1+1)(p_1+q_1+2)} + L_2 H_2 \frac{(b_2-a_2)^{p_2+q_2}}{(p_2+q_2+1)(p_2+q_2+2)} \\
&\quad + 2L_1 H_2 \frac{(b_1-a_1)^{p_1}(b_2-a_2)^{q_2}}{(p_1+1)(p_1+2)(q_2+1)(q_2+2)} \\
&\quad + 2L_2 H_1 \frac{(b_2-a_2)^{p_2}(b_1-a_1)^{q_1}}{(p_2+1)(p_2+2)(q_1+1)(q_1+2)},
\end{aligned} \tag{2.10}$$

with

$$|f(x_i) - f(y_i)| \le L_i |x_i - y_i|^{p_i}, \ \ x_i, y_i \in [a_i, b_i], \ i = 1, 2,$$

and

$$|g(x_i) - g(y_i)| \le H_i |x_i - y_i|^{q_i}, \ \ x_i, y_i \in [a_i, b_i], \ i = 1, 2.$$

Thus (2.10) recaptures the result of Hanna, Dragomir and Cerone [11].

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School of Communications and Informatics, Victoria University of Technology, PO Box 14428, Melbourne City MC, Victoria 8001, Australia

E-mail address: pc@matilda.vu.edu.au
URL: http://sci.vu.edu.au/staff/peterc.html

E-mail address: sever@matilda.vu.edu.au
URL: http://rgmia.vu.edu.au/SSDragomirWeb.html

E-mail address: georgey@matilda.vu.edu.au

Faculty of Textile Technology, University of Zagreb, Pierottijeva 6, 41000 Zagreb, Croatia.

E-mail address: pecaric@mahazu.hazu.hr
URL: http://mahazu.hazu.hr/DepMPCS/indexJP.html