On the Ky Fan Inequality

ON THE KY FAN INEQUALITY

S.S. DRAGOMIR AND F.P. SCARMOZZINO

Abstract. Some inequalities related to the Ky Fan and C.-L. Wang inequalities for weighted arithmetic and geometric means are given.

1. Introduction

In 1961, E.F. Beckenbach and R. Bellman published in their well known book "Inequalities" the following "unpublished result due to Ky Fan" [2, p. 5] (see also [1, p. 150]).

Theorem 1. *If* $0 < x_i \le \frac{1}{2}$, $(i = 1, \dots, n)$; *then:*

$$\left[\prod_{i=1}^{n} x_i \Big/ \prod_{i=1}^{n} (1 - x_i) \right]^{\frac{1}{n}} \le \sum_{i=1}^{n} x_i \Big/ \sum_{i=1}^{n} (1 - x_i) \tag{1.1}$$

with equality only if $x_1 = \dots = x_n$.

A generalisation of Ky Fan's inequality for weighted means was proved by C.-L. Wang in 1980, [9].

Theorem 2. *If* $0 < x_i \le \frac{1}{2}$, $(i = 1, \dots, n)$, *then*

$$\frac{A_n(\bar{p}, \bar{x})}{A_n(\bar{p}, 1 - \bar{x})} \ge \frac{G_n(\bar{p}, \bar{x})}{G_n(\bar{p}, 1 - \bar{x})}, \tag{1.2}$$

where $p_i > 0$ $(i = 1, \dots, n)$ *with* $\sum_{i=1}^{n} p_i = 1$ *and* $A_n(\bar{p}, \bar{x}) := \sum_{i=1}^{n} p_i x_i$ *is the weighted arithmetic mean,* $G_n(\bar{p}, \bar{x}) := \prod_{i=1}^{n} x_i^{p_i}$ *is the weighted geometric mean. The equality holds in (1.2) iff* $x_1 = \dots = x_n$.

For a survey on related results of Ky Fan's inequality, see [1] by H. Alzer.

For different refinements and generalisations, see [4] – [8].

2. The Results

The following result holds.

Theorem 3. *Assume that* $0 < m \le x_i \le M \le \frac{1}{2}$, $(i = 1, \dots, n)$, $p_i > 0$ $(i = 1, \dots, n)$, *with* $\sum_{i=1}^{n} p_i = 1$, *then we have the inequalities:*

$$\frac{A_n(\bar{p}, \bar{x})}{G_n(\bar{p}, \bar{x})} \ge \left[\frac{A_n(\bar{p}, \bar{x})}{G_n(\bar{p}, \bar{x})} \right]^{\frac{M^2}{(1-M)^2}} \ge \frac{A_n(\bar{p}, 1 - \bar{x})}{G_n(\bar{p}, 1 - \bar{x})} \ge \left[\frac{A_n(\bar{p}, \bar{x})}{G_n(\bar{p}, \bar{x})} \right]^{\frac{m^2}{(1-m)^2}} \ge 1. \tag{2.1}$$

The equality will hold in all inequalities iff $x_1 = \dots = x_n$.

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Proof. The first and the last inequality in (2.1) follow by the fact that $\frac{A_n(\bar{p},\bar{x})}{G_n(\bar{p},\bar{x})} \geq 1$ (by the weighted arithmetic mean - geometric mean inequality), $m \in \left(0, \frac{1}{2}\right]$ and $M \in \left(0, \frac{1}{2}\right]$.

We define the function $f:(0,1)\to\mathbb{R}$, $f(t)=\ln\left(\frac{1-t}{t}\right)+\alpha\ln t$ with $\alpha\in\mathbb{R}$. We have

$$f'(t) = -\frac{1}{t(1-t)} + \frac{\alpha}{t}, \quad t\in(0,1),$$

$$f''(t) = \frac{1-2t}{[t(1-t)]^2} - \frac{\alpha}{t^2} = \frac{1}{t^2}\left[\frac{1-2t}{(1-t)^2} - \alpha\right], \quad t\in(0,1).$$

If we consider the function $g:(0,1)\to\mathbb{R}$, $g(t)=\frac{1-2t}{(1-t)^2}$, then $g'(t)=\frac{2t(t-1)}{(t-1)^4}$, showing that the function g is monotonically strictly decreasing on $(0,1)$.

Consequently for $t\in(m,M)$, we have

$$\frac{1-2M}{(1-M)^2} = g(M) \leq g(t) \leq g(m) = \frac{1-2m}{(1-m)^2}. \tag{2.2}$$

Using (2.2) we observe that the function f is strictly convex on (m,M) if $\alpha \leq \frac{1-2M}{(1-M)^2}$.

Applying Jensen's discrete inequality for the function $f:(m,M)\to\mathbb{R}$, $f(t)=\ln\left(\frac{1-t}{t}\right)+\alpha\ln t$, with $\alpha\leq\frac{1-2M}{(1-M)^2}$, we deduce

$$\begin{aligned}\sum_{i=1}^n p_i\left[\ln\left(\frac{1-x_i}{x_i}\right)+\alpha\ln x_i\right] &= \sum_{i=1}^n p_i f(x_i) \geq f\left(\sum_{i=1}^n p_i x_i\right) \\ &= \ln\left(\frac{1-\sum_{i=1}^n p_i x_i}{\sum_{i=1}^n p_i x_i}\right) + \alpha\ln\left(\sum_{i=1}^n p_i x_i\right),\end{aligned}$$

which is equivalent to

$$\ln\left[\frac{G_n(\bar{p},1-\bar{x})}{G_n(\bar{p},\bar{x})}\right] + \alpha\ln G_n(\bar{p},\bar{x}) \geq \ln\left[\frac{A_n(\bar{p},1-\bar{x})}{A_n(\bar{p},\bar{x})}\right] + \alpha\ln A_n(\bar{p},\bar{x})$$

or, moreover, to

$$\ln\left[\frac{G_n(\bar{p},\bar{x})}{A_n(\bar{p},\bar{x})}\right]^\alpha \geq \ln\left[\frac{A_n(\bar{p},1-\bar{x})}{A_n(\bar{p},\bar{x})} \Big/ \frac{G_n(\bar{p},1-\bar{x})}{G_n(\bar{p},\bar{x})}\right],$$

that is,

$$\left[\frac{G_n(\bar{p},\bar{x})}{A_n(\bar{p},\bar{x})}\right]^{\alpha-1} \geq \frac{A_n(\bar{p},1-\bar{x})}{G_n(\bar{p},1-\bar{x})}. \tag{2.3}$$

Now, we observe that the inequality (2.3) is the best possible if α is maximal, i.e., $\alpha=\frac{1-2M}{(1-M)^2}$, getting

$$\left[\frac{G_n(\bar{p},\bar{x})}{A_n(\bar{p},\bar{x})}\right]^{\frac{1-2M}{(1-M)^2}-1} \geq \frac{A_n(\bar{p},1-\bar{x})}{G_n(\bar{p},1-\bar{x})},$$

which is clearly equivalent to the second inequality in (2.1).

The third inequality is produced in a similar fashion, using the function $h(t)=\beta\ln t-\ln\left(\frac{1-t}{t}\right)$ which is strictly convex on (m,M) if $\beta\geq\frac{1-2m}{(1-m)^2}$.

The case of equality follows by the fact that in Jensen's inequality for strictly convex functions, the equality holds iff $x_1=\cdots=x_n$.

We omit the details. ∎

Remark 1. *Since Wang's inequality (1.2) is equivalent to:*

$$\frac{A_n(\bar{p},\bar{x})}{G_n(\bar{p},\bar{x})} \geq \frac{A_n(\bar{p},1-\bar{x})}{G_n(\bar{p},1-\bar{x})}, \tag{2.4}$$

then the first part of (2.1) may be seen as a refinement of Wang's result while the second part

$$\frac{A_n(\bar{p},1-\bar{x})}{G_n(\bar{p},1-\bar{x})} \geq \left[\frac{A_n(\bar{p},\bar{x})}{G_n(\bar{p},\bar{x})}\right]^{\frac{m^2}{(1-m)^2}} \tag{2.5}$$

can be considered a counterpart of (1.2).

Now, let us recall the Lah-Ribarić inequality for convex functions (see for example [3, p. 140]).

If $f:[a,b]\subset\mathbb{R}\to\mathbb{R}$ is convex on $[a,b]$, $x_i\in[a,b]$, $p_i\geq 0$ $(i=1,\dots,n)$ and $\sum_{i=1}^n p_i=1$, then

$$\sum_{i=1}^n p_i f(x_i) \leq \frac{b-\sum_{i=1}^n p_i x_i}{b-a}\cdot f(a) + \frac{\sum_{i=1}^n p_i x_i - a}{b-a}\cdot f(b). \tag{2.6}$$

Now, we can state and prove the following inequality related to the Ky Fan result.

Theorem 4. *Assume that $0<m\leq x_i\leq M\leq\frac{1}{2}$, $p_i>0$ $\quad(i=1,\dots,n)$ with $\sum_{i=1}^n p_i=1$, then we have the inequalities:*

$$\begin{aligned}
&\left(\frac{1-m}{m^{\left(\frac{m}{1-m}\right)^2}}\right)^{\frac{M-A_n(\bar{p},\bar{x})}{M-m}}\left(\frac{1-M}{M^{\left(\frac{m}{1-m}\right)^2}}\right)^{\frac{A_n(\bar{p},\bar{x})-m}{M-m}}\cdot G_n(\bar{p},\bar{x})^{\left(\frac{m}{1-m}\right)^2} \\
\leq\ & G_n(\bar{p},1-\bar{x}) \\
\leq\ & \left(\frac{1-m}{m^{\left(\frac{M}{1-M}\right)^2}}\right)^{\frac{M-A_n(\bar{p},\bar{x})}{M-m}}\left(\frac{1-M}{M^{\left(\frac{M}{1-M}\right)^2}}\right)^{\frac{A_n(\bar{p},\bar{x})-m}{M-m}} G_n(\bar{p},\bar{x})^{\left(\frac{M}{1-M}\right)^2}.
\end{aligned} \tag{2.7}$$

Proof. From the proof of Theorem 3, we know that the function $f:(m,M)\subset\left(0,\frac{1}{2}\right]\to\mathbb{R}$, $f(t)=\ln\left(\frac{1-t}{t}\right)+\frac{1-2M}{(1-M)^2}\ln t$ is strictly convex on (m,M). Now, if we apply the Lah-Ribarić inequality for f as above, $a=m$ and $b=M$, we get:

$$\begin{aligned}
&\sum_{i=1}^n p_i\left[\ln\left(\frac{1-x_i}{x_i}\right)+\frac{1-2M}{(1-M)^2}\ln x_i\right] \\
=\ & \sum_{i=1}^n p_i f(x_i) \leq \frac{M-\sum_{i=1}^n p_i x_i}{M-m} f(m) + \frac{\sum_{i=1}^n p_i x_i - m}{M-m} f(M) \\
=\ & \frac{M-\sum_{i=1}^n p_i x_i}{M-m}\left[\ln\left(\frac{1-m}{m}\right)+\frac{1-2M}{(1-M)^2}\ln m\right] \\
& + \frac{\sum_{i=1}^n p_i x_i - m}{M-m}\left[\ln\left(\frac{1-M}{M}\right)+\frac{1-2M}{(1-M)^2}\ln M\right],
\end{aligned}$$

which is equivalent to

$$\begin{aligned}
&\ln\left[\frac{G_n(\bar{p},1-\bar{x})}{G_n(\bar{p},\bar{x})}\right]+\frac{1-2M}{(1-M)^2}\ln G_n(\bar{p},\bar{x}) \\
\leq\quad & \frac{M-A_n(\bar{p},\bar{x})}{M-m}\left[\ln\left(\frac{1-m}{m}\right)+\ln(m)^{\frac{1-2M}{(1-M)^2}}\right] \\
&+\frac{A_n(\bar{p},\bar{x})-m}{M-m}\left[\ln\left(\frac{1-M}{M}\right)+\ln(M)^{\frac{1-2M}{(1-M)^2}}\right],
\end{aligned}$$

that is,

$$\begin{aligned}
&\frac{G_n(\bar{p},1-\bar{x})}{G_n(\bar{p},\bar{x})}\cdot\left[G_n(\bar{p},\bar{x})\right]^{\frac{1-2M}{(1-M)^2}} \\
\leq\quad & \left((1-m)\,m^{\left\{\frac{1-2M}{(1-M)^2}-1\right\}}\right)^{\frac{M-A_n(\bar{p},\bar{x})}{M-m}}\cdot\left((1-M)\,M^{\left\{\frac{1-2M}{(1-M)^2}-1\right\}}\right)^{\frac{A_n(\bar{p},\bar{x})-m}{M-m}}
\end{aligned}$$

from which we obtain the second inequality in (2.7).

To prove the first inequality, we apply the Lah-Ribarić inequality for the function $h:(m,M)\to\mathbb{R}$, $h(t)=\frac{1-2m}{(1-m)^2}\ln t-\ln\left(\frac{1-t}{t}\right)$ which is strictly convex on (m,M).

We omit the details. ∎

Finally, let us recall Dragomir-Ionescu's inequality for differentiable convex functions (see [7])

$$\begin{aligned}
0\quad &\leq\quad \sum_{i=1}^{n}p_i f(x_i)-f\left(\sum_{i=1}^{n}p_i x_i\right) \\
&\leq\quad \sum_{i=1}^{n}p_i x_i f'(x_i)-\sum_{i=1}^{n}p_i x_i\sum_{i=1}^{n}p_i f'(x_i)
\end{aligned}\tag{2.8}$$

provided $f:(a,b)\subseteq\mathbb{R}\to\mathbb{R}$ is differentiable convex on (a,b), $x_i\in(a,b)$ and $p_i>0$ $(i=1,\dots,n)$ with $\sum_{i=1}^{n}p_i=1$.

If f is strictly convex on (a,b), then the equality holds in (2.8) iff $x_1=\cdots=x_n$, we may state the following result.

Theorem 5. *With the assumptions of Theorem 4, we have*

$$\begin{aligned}
&\exp\left[A_n(\bar{p},\bar{x})\,A_n\left(\bar{p},\frac{1}{\bar{x}(1-\bar{x})}\right)-A_n\left(\bar{p},\frac{1}{1-\bar{x}}\right)\right] \\
&\times\left[\frac{1-2M}{(1-M)^2}\left\{1-A_n(\bar{p},\bar{x})\,A_n\left(\bar{p},\frac{1}{\bar{x}}\right)\right\}\right]\times\left[\frac{A_n(\bar{p},\bar{x})}{G_n(\bar{p},\bar{x})}\right]^{\frac{1-2M}{(1-M)^2}} \\
\geq\quad & \left[\frac{G_n(\bar{p},1-\bar{x})}{G_n(\bar{p},\bar{x})}\right]\Big/\left[\frac{A_n(\bar{p},1-\bar{x})}{A_n(\bar{p},\bar{x})}\right] \\
\geq\quad & \exp\left[A_n(\bar{p},\bar{x})\,A_n\left(\bar{p},\frac{1}{\bar{x}(1-\bar{x})}\right)-A_n\left(\bar{p},\frac{1}{1-\bar{x}}\right)\right] \\
&\times\left[\frac{1-2m}{(1-m)^2}\left\{1-A_n(\bar{p},\bar{x})\,A_n\left(\bar{p},\frac{1}{\bar{x}}\right)\right\}\right]\times\left[\frac{A_n(\bar{p},\bar{x})}{G_n(\bar{p},\bar{x})}\right]^{\frac{1-2m}{(1-m)^2}},
\end{aligned}\tag{2.9}$$

where $\frac{1}{\bar{x}}$ *denotes the vector* $\left(\frac{1}{x_1},\dots,\frac{1}{x_n}\right)$, $\bar{y}\cdot\bar{z}:=(y_1z_1,\dots,z_ny_n)$, *and* $\bar{x}\in\mathbb{R}^n$, $\bar{x}>\bar{0}$ *(i.e.,* $x_i>0$ *for any* $i\in\{1,\dots,n\}$*),* $\bar{y},\bar{z}\in\mathbb{R}^n$.

Proof. Since the function $f:(m,M)\subset\left(0,\frac{1}{2}\right]\to\mathbb{R}$, $f(t)=\ln\left(\frac{1-t}{t}\right)+\frac{1-2M}{(1-M)^2}\ln t$ is strictly convex on (m,M), by (2.8) we may state that

$$\begin{aligned}
&\sum_{i=1}^{n}p_i\left[\ln\left(\frac{1-x_i}{x_i}\right)+\frac{1-2M}{(1-M)^2}\ln x_i\right]-\ln\left(\frac{1-\sum_{i=1}^{n}p_ix_i}{\sum_{i=1}^{n}p_ix_i}\right)\\
&-\frac{1-2M}{(1-M)^2}\ln\left(\sum_{i=1}^{n}p_ix_i\right)\\
=\;&\sum_{i=1}^{n}p_if(x_i)-f\left(\sum_{i=1}^{n}p_ix_i\right)\le\sum_{i=1}^{n}p_ix_if'(x_i)-\sum_{i=1}^{n}p_ix_i\sum_{i=1}^{n}p_if'(x_i)\\
=\;&\sum_{i=1}^{n}p_ix_i\left[\frac{1-2M}{(1-M)^2}\cdot\frac{1}{x_i}-\frac{1}{x_i(1-x_i)}\right]\\
&-\sum_{i=1}^{n}p_ix_i\sum_{i=1}^{n}p_i\left[\frac{1-2M}{(1-M)^2}\cdot\frac{1}{x_i}-\frac{1}{x_i(1-x_i)}\right],
\end{aligned}$$

which is equivalent to

$$\begin{aligned}
&\ln\left[\frac{G_n(\bar{p},1-\bar{x})}{G_n(\bar{p},\bar{x})}\right]+\frac{1-2M}{(1-M)^2}\ln G_n(\bar{p},\bar{x})-\ln\left[\frac{A_n(\bar{p},1-\bar{x})}{A_n(\bar{p},\bar{x})}\right]\\
&-\frac{1-2M}{(1-M)^2}\ln A_n(\bar{p},\bar{x})\\
\le\;&\frac{1-2M}{(1-M)^2}-A_n\left(\bar{p},\frac{1}{1-\bar{x}}\right)\\
&-A_n(\bar{p},\bar{x})\times\left[\frac{1-2M}{(1-M)^2}A_n\left(\bar{p},\frac{1}{\bar{x}}\right)-A_n\left(\bar{p},\frac{1}{\bar{x}(1-\bar{x})}\right)\right],
\end{aligned}$$

which is equivalent to

$$\begin{aligned}
&\ln\left[\left[\frac{G_n(\bar{p},1-\bar{x})}{G_n(\bar{p},\bar{x})}\right]\Big/\left[\frac{A_n(\bar{p},1-\bar{x})}{A_n(\bar{p},\bar{x})}\right]\right]\\
\le\;&\ln\left[\frac{A_n(\bar{p},\bar{x})}{G_n(\bar{p},\bar{x})}\right]^{\frac{1-2M}{(1-M)^2}}+\frac{1-2M}{(1-M)^2}\left[1-A_n(\bar{p},\bar{x})A_n\left(\bar{p},\frac{1}{\bar{x}}\right)\right]\\
&+A_n(\bar{p},\bar{x})A_n\left(\bar{p},\frac{1}{\bar{x}(1-\bar{x})}\right)-A_n\left(\bar{p},\frac{1}{1-\bar{x}}\right)\\
=\;&\ln\left\{\left[\frac{A_n(\bar{p},\bar{x})}{G_n(\bar{p},\bar{x})}\right]^{\frac{1-2M}{(1-M)^2}}\cdot\exp\left[\frac{1-2M}{(1-M)^2}\left\{1-A_n(\bar{p},\bar{x})A_n\left(\bar{p},\frac{1}{\bar{x}}\right)\right\}\right]\right.\\
&\left.\times\exp\left[A_n(\bar{p},\bar{x})A_n\left(\bar{p},\frac{1}{\bar{x}(1-\bar{x})}\right)-A_n\left(\bar{p},\frac{1}{1-\bar{x}}\right)\right]\right\},
\end{aligned}$$

hence the first inequality in (2.9).

The second inequality follows by (2.8) applied for the strictly convex function $h(t)=\frac{1-2m}{(1-m)^2}\ln t-\ln\left(\frac{1-t}{t}\right)$, $t\in(m,M)$.

We omit the details. ∎

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School of Communications and Informatics, Victoria University of Technology, PO Box 14428, Melbourne City MC 8001, Victoria, Australia.

E-mail address: sever@matilda.vu.edu.au

URL: http://rgmia.vu.edu.au/SSDragomirWeb.html

E-mail address: fps@matilda.vu.edu.au

URL: http://cams.vu.edu.au/staff/fernandos.html