On Some Inequalities for the Moments of Guessing Mapping

ON SOME INEQUALITIES FOR THE MOMENTS OF GUESSING MAPPING

S. S. DRAGOMIR, J. PEČARIĆ, AND J. VAN DER HOEK

Abstract. Using some inequalities for real numbers and integrals we point out here some new inequalities for the moments of guessing mapping which generalize and improve the recent results of Arikan [2], Dragomir and van der Hoek [3]-[4] and Dragomir [10].

1. Introduction

J. L. Massey in [1], considered the problem of guessing the value of a realization of a discrete random variable X by asking questions of the form: "Is X equal to x?" until the answer is "Yes".

Let $G(X)$ denote the number of guesses required by a particular guessing strategy for $X = x$.

Massey observed that $E(G(X))$, the average number of guesses, is minimized by a guessing strategy that guesses the possible values of X in decreasing order of probability.

We begin by giving a formal and generalized statement of the above problem by following E. Arikan [2].

Let (X, Y) be a pair of random variables with X taking values in a finite set $\mathcal{X}$ of size n, Y taking values in the countable set $\mathcal{Y}$. Call a function $G(X)$ of the random variable X a *guessing function for* X if $G : \mathcal{X} \to \{1, ..., n\}$ is one-to-one. Call a function $G(X|Y)$ a *guessing function for* X *given* Y if, for any fixed value $Y = y, G(X|y)$ is a guessing function for X. $G(X|Y)$ will be thought of as the number of guesses required to determine X where the value of Y is given.

The following inequalities on the moments of $G(X)$ and $G(X|Y)$ were proved by E. Arikan in the recent paper [2].

Theorem 1. *For any arbitrary guessing function $G(X)$ and $G(X|Y)$ and any $p \geq 0$, we have:*

$$E(G(X)^p) \geq (1 + \ln n)^{-p} \left[\sum_{x \in \mathcal{X}} P_X(x)^{\frac{1}{1+p}} \right]^{1+p} \tag{1.1}$$

and

$$E[G(X|Y)^p] \geq (1 + \ln n)^{-p} \sum_{y \in \mathcal{Y}} \left[\sum_{x \in \mathcal{X}} P_{X,Y}(x, y)^{\frac{1}{1+p}} \right]^{1+p} \tag{1.2}$$

where $P_{X,Y}, P_X$ are the probability distributions of (X, Y) and X, respectively.

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To simplify the notation, we assume that the x_i $(x_i \in \mathcal{X})$ are numbered such that x_k is always the kth guess. That yields

$$E\left(G^p\right) = \sum_{k=1}^{n} k^p p_k,$$

where $p_k = \Pr\left(X = x_k\right)$, $k = 1, \ldots, n$.

The following estimation results for the $p-$moment of the guessing mappings was obtained by Dragomir and van der Hoek [3]:

Theorem 2. *Let X be a random variable having the probability distribution $p = (p_i)$, $i = \overline{1, n}$. Then we have the inequality:*

$$\left|E\left(G\left(X\right)^p\right) - \frac{1}{n}\sum_{i=1}^{n} i^p\right| \leq \frac{n\left(n^p - 1\right)}{4}\left(P_M - P_m\right) \tag{1.3}$$

where

$$P_M := \max\left\{p_i \left| i = \overline{1, n}\right.\right\} \text{ and } P_m := \min\left\{p_i \left| i = \overline{1, n}\right.\right\}$$

and $p > 0$.

Corollary 1. *If we assume that for a given $\varepsilon > 0$ and $n \geq 1$, we have*

$$0 \leq P_M - P_m < \frac{4\varepsilon}{n\left(n^p - 1\right)}, \tag{1.4}$$

then

$$\left|E\left(G\left(X\right)^p\right) - \frac{1}{n}\sum_{i=1}^{n} i^p\right| < \varepsilon. \tag{1.5}$$

Remark 1. *If we put in (1.3) $p = 1$, we get:*

$$\left|E\left(G\left(X\right)\right) - \frac{n+1}{2}\right| \leq \frac{n\left(n-1\right)}{4}\left(P_M - P_m\right). \tag{1.6}$$

If we choose in (1.3) $p = 2$, we get

$$\left|E\left(G\left(X\right)^2\right) - \frac{\left(n+1\right)\left(2n+1\right)}{6}\right| \leq \frac{n\left(n^2 - 1\right)}{4}\left(P_M - P_m\right) \tag{1.7}$$

and, finally, for $p = 3$, we obtain

$$\left|E\left(G\left(X\right)^3\right) - \frac{n\left(n+1\right)^2}{4}\right| \leq \frac{n\left(n^3 - 1\right)}{4}\left(P_M - P_m\right). \tag{1.8}$$

Theorem 3. *With the assumptions of Theorem 2, we have the inequality:*

$$\begin{aligned} &\left|\binom{p+1}{1}E\left(G\left(X\right)^p\right) - \binom{p+1}{2}E\left(G\left(X\right)^{p-1}\right) + \ldots\right. \\ &\quad \left. + \left(-1\right)^{p+1}\binom{p+1}{1}E\left(G\left(X\right)\right) + \left(-1\right)^{p+2} - n^p\right| \\ &\leq \frac{\left(p+1\right)n^{p+1}}{4}\left(P_M - P_m\right) \end{aligned} \tag{1.9}$$

provided that $p \in \mathbb{N}$, $p \geq 1$.

Corollary 2. *If we assume that for a given $\varepsilon > 0$ and $n \geq 1$, we have:*

$$0 \leq P_M - P_m < \frac{4\varepsilon}{(p+1)\,n^{p+1}} \tag{1.10}$$

then

$$\begin{aligned} &\left| \binom{p+1}{1} E\left(G(X)^p\right) - \binom{p+1}{2} E\left(G(X)^{p-1}\right) + \ldots \right. \\ &\left. + (-1)^{p+1} \binom{p+1}{1} E\left(G(X)\right) + (-1)^{p+2} - n^p \right| \\ \leq\ & \varepsilon. \end{aligned} \tag{1.11}$$

Remark 2. *If in* (1.9) *we put $p = 1$, we get:*

$$\left| E\left(G(X)\right) - \frac{n+1}{2} \right| \leq \frac{n^2}{4}\left(P_M - P_m\right), \tag{1.12}$$

and if we choose $p = 2$, we get:

$$\left| E\left(G(X)^2\right) - E\left(G(X)\right) - \frac{n^2-1}{3} \right| \leq \frac{n^3}{4}\left(P_M - P_m\right). \tag{1.13}$$

Let us note that (1.6) is better than (1.12).

Theorem 4. *With the assumptions of Theorem 3, we have the inequality*

$$\begin{aligned} P_m \frac{p}{p+1} n^{p+1} \leq\ & n^p - \frac{1}{p+1}\left[\binom{p+1}{1} E\left(G(X)^p\right) + \ldots \right. \\ & \left. + (-1)^{p+1} \binom{p+1}{1} E\left(G(X)\right) + (-1)^{p+2} \right] \\ \leq\ & P_M \frac{p}{p+1} n^{p+1} \end{aligned} \tag{1.14}$$

where $p \in \mathbb{N}$ and $p \geq 1$.

Remark 3. *Let us note that in* [3], *Theorem 3 and its consequences are given with some misprints.*

In this paper we shall give some improvements and generalizations of previous results.

2. On Arikan's Inequalities

First, let us prove the following general result:

Theorem 5. *For an arbitrary guessing function $G(X)$ and $G(X|Y)$ and any $p \in \mathbb{R}$ we have*

$$E\left(G(X)^p\right) \geq S_{\frac{p}{\alpha}}(n)^{\alpha} \left[\sum_{x \in \mathcal{X}} P_X(x)^{\frac{1}{1-\alpha}} \right]^{1-\alpha} \tag{2.1}$$

and

$$E\left(G(X|Y)^p\right) \geq S_{\frac{p}{\alpha}}(n)^{\alpha} \sum_{y \in \mathcal{Y}} \left[\sum_{x \in \mathcal{X}} P_{X,Y}(x,y)^{\frac{1}{1-\alpha}} \right]^{1-\alpha} \tag{2.2}$$

where either $\alpha > 1$ or $\alpha < 0$ and

$$S_m(n) = \sum_{i=1}^{n} i^m. \tag{2.3}$$

If $\alpha \in (0,1)$ we have reverse inequalities in (2.1) *and* (2.2).

Proof. Inequality (2.1) is a simple consequence of a reverse Hölder inequality

$$\sum_{i=1}^{n} a_i b_i \geq \left(\sum_{i=1}^{n} a_i^{\frac{1}{\alpha}} \right)^{\alpha} \left(\sum_{i=1}^{n} b_i^{\frac{1}{1-\alpha}} \right)^{1-\alpha} \tag{2.4}$$

where α is either greater than 1 or less than 0. Set $a_i = i^p$, $b_i = p_i$ and we shall get (2.1). If $\alpha \in (0,1)$ we have reverse inequality in (2.4), that is Hölder inequality is valid, so for $a_i = i^p$ and $b_i = p_i$ we get reverse inequality in (2.1).
Since

$$\begin{aligned} E\left(G\left(X\,|Y\right)^p\right) &= \sum_y P(y)\, E\left[G\left(X\,|Y = y\right)^p\right] \\ &\geq \sum_y P(y)\, S_{\frac{p}{\alpha}}(n)^{\alpha} \left[\sum_x P(x\,|y)^{\frac{1}{1-\alpha}} \right]^{1-\alpha} \\ &= S_{\frac{p}{\alpha}}(n)^{\alpha} \sum_y \left[\sum_x P(x,y)^{\frac{1}{1-\alpha}} \right]^{1-\alpha}, \end{aligned}$$

we have (2.2). Similarly we can prove the corresponding reverse result. ∎

The following generalization of Theorem 1 is valid:

Theorem 6. *For an arbitrary guessing function $G(X)$ and $G(X\,|Y)$ and all $p \in \mathbb{R}$, we have*

$$E\left(G(X)^p\right) \geq K(\alpha,p) \left[\sum_{x\in\mathcal{X}} P_X(x)^{\frac{1}{1-\alpha}} \right]^{1-\alpha}; \tag{2.5}$$

and

$$E\left(G\left(X\,|Y\right)^p\right) \geq K(\alpha,p)^{\alpha} \sum_{y\in\mathcal{Y}} \left[\sum_{x\in\mathcal{X}} P_{X,Y}(x,y)^{\frac{1}{1-\alpha}} \right]^{1-\alpha} \tag{2.6}$$

where $P_{X,Y}, P_X$ are probability distributions of (X,Y) and X respectively and where

$$K(\alpha,p) = \begin{cases} \left(\frac{p}{p+\alpha}\right)^{\alpha}, & \text{for } p > -\alpha, \alpha < 0 \\ (1+\ln n)^{-p}, & \text{for } p = -\alpha, \alpha < 0 \\ \left(\frac{\alpha n^{\frac{p}{\alpha}+1}}{p+\alpha}\right)^{\alpha}, & \text{for } 0 < p < -\alpha, \alpha < 0 \\ \left(\frac{\alpha n^{\frac{p}{\alpha}+1}}{p+\alpha} + n^{\frac{p}{\alpha}}\right)^{\alpha}, & \text{for } p \leq 0, \alpha < 0 \\ 1, & \text{for } p < -\alpha, \alpha > 0 \\ (\ln n)^{-p}, & \text{for } p = -\alpha, \alpha > 0 \\ \left(\frac{\alpha n^{\frac{p}{\alpha}+1}}{p+\alpha} - \frac{\alpha}{p+\alpha}\right)^{\alpha}, & \text{for } -\alpha < p < 0, \alpha > 0 \\ \left(\frac{\alpha n^{\frac{p}{\alpha}+1}}{p+\alpha}\right)^{\alpha}, & \text{for } p > 0, \alpha > 0 \end{cases}. \tag{2.7}$$

We also have

$$E\left(G\left(X\right)^{p}\right)\leq\tilde{K}\left(\alpha,p\right)\left[\sum_{x\in\mathcal{X}}P_{X}\left(x\right)^{\frac{1}{1-\alpha}}\right]^{1-\alpha}\tag{2.8}$$

and

$$E\left(G\left(X\left|Y\right.\right)^{p}\right)\leq\tilde{K}\left(\alpha,p\right)^{\alpha}\sum_{y\in\mathcal{Y}}\left[\sum_{x\in\mathcal{X}}P_{X,Y}\left(x,y\right)^{\frac{1}{1-\alpha}}\right]^{1-\alpha}\tag{2.9}$$

where $\alpha\in(0,1)$ *and*

$$\tilde{K}\left(\alpha,p\right)=\left\{\begin{array}{c}\left(\frac{p}{\alpha+p}\right)^{\alpha},\ \text{for } p<-\alpha\\ \left(1+\ln n\right)^{-p},\ \text{for } p=-\alpha\\ \left(\frac{\alpha n^{\frac{p}{\alpha}+1}}{p+\alpha}\right)^{\alpha},\ \text{for } -\alpha<p<0\\ \left(\frac{\alpha n^{\frac{p}{\alpha}+1}}{p+\alpha}+n^{\frac{p}{\alpha}}\right)^{\alpha},\ \text{for } p\geq 0\end{array}\right..$$

Proof. It follows from Theorem 5 and the following result obtained in [5], (see also [6, p. 118]):

$$\left.\begin{array}{c}1\\ \ln n\\ \frac{n^{r+1}}{r+1}-\frac{1}{r+1}\\ \frac{n^{r+1}}{r+1}\end{array}\right\}\leq\sum_{k=1}^{n}k^{r}\leq\left\{\begin{array}{c}\frac{r}{r+1},\ \text{for } r<-1,\\ 1+\ln n,\ \text{for } r=-1,\\ \frac{n^{r+1}}{r+1},\ \text{for } -1<r<0,\\ \frac{n^{r+1}}{r+1}+n^{r},\ \text{for } r\geq 0.\end{array}\right.\tag{2.10}$$

∎

Corollary 3. *Let the conditions of Theorem 1 be satisfied. Then* (2.5) , (2.6) , (2.8) *and* (2.9) *are valid with*

$$K\left(\alpha,p\right)=\left\{\begin{array}{c}\left(\frac{p}{p+\alpha}\right)^{\alpha},\ \text{for } p>-\alpha,\\ \left(1+\ln n\right)^{-p},\ \text{for } r=-\alpha,\\ \left(\frac{\alpha n^{\frac{p}{\alpha}+1}}{p+\alpha}\right)^{\alpha},\ \text{for } \alpha>0 \text{ and } \alpha<-p<0;\end{array}\right.\tag{2.11}$$

and

$$\tilde{K}\left(\alpha,p\right)=\left(\frac{\alpha n^{\frac{p}{\alpha}+1}}{p+\alpha}+n^{\frac{p}{\alpha}}\right)^{\alpha}.\tag{2.12}$$

Moreover, we can give some improvements of Arikan inequalities (1.1) and (1.2). Namely, the following results are valid.

Theorem 7. *Let the conditions of Theorem 1 be fulfilled. Then*

$$E\left(G\left(X\right)^{p}\right)>C^{-p}\left[\sum_{x\in\mathcal{X}}P_{X}\left(x\right)^{\frac{1}{1+p}}\right]^{1+p}\tag{2.13}$$

and

$$E\left(G\left(X\left|Y\right.\right)^{p}\right)>C^{-p}\sum_{y\in\mathcal{Y}}\left[\sum_{x\in\mathcal{X}}P_{X,Y}\left(x,y\right)^{\frac{1}{1+p}}\right]^{1+p},\tag{2.14}$$

where C is either

$$C=\gamma+\ln n+\frac{1}{2n}+\frac{1}{12n^{2}}+\frac{1}{n^{3}} \tag{2.15}$$

or

$$C=\gamma+\ln\left(n+\frac{1}{2}\right)+\frac{1}{24n^{2}} \tag{2.16}$$

or

$$C=\gamma+\ln\left(n+\frac{1}{2}\right)+\frac{1}{24\left(n+\frac{1}{2}\right)^{2}}-\frac{7}{960}\frac{1}{\left(n+1\right)^{3}}, \tag{2.17}$$

where γ is Euler's constant.

Proof. It follows from Theorem 5 for $\alpha=-p$ that

$$E\left(G\left(X\right)^{p}\right)\geq S_{-1}\left(n\right)^{-p}\left[\sum_{x\in\mathcal{X}}P_{X}\left(x\right)^{\frac{1}{1+p}}\right]^{1+p} \tag{2.18}$$

and

$$E\left(G\left(X\left|Y\right.\right)^{p}\right)\geq S_{-1}\left(n\right)^{-p}\sum_{y\in\mathcal{Y}}\left[\sum_{x\in\mathcal{X}}P_{X,Y}\left(x,y\right)^{\frac{1}{1+p}}\right]^{1+p}. \tag{2.19}$$

On the other hand, the following results are known:

$$S_{-1}\left(n\right)<\gamma+\ln n+\frac{1}{2n}+\frac{1}{12n^{2}}+\frac{1}{n^{3}}$$

([7], [6, p. 120])

$$S_{-1}\left(n\right)<\gamma+\ln\left(n+\frac{1}{2}\right)+\frac{1}{24n^{2}} \tag{2.20}$$

([8], [6, p. 126])

$$S_{-1}\left(n\right)<\gamma+\ln\left(n+\frac{1}{2}\right)+\frac{1}{24\left(n+\frac{1}{2}\right)^{2}}-\frac{7}{960}\frac{1}{\left(n+1\right)^{3}}$$

([8], [6, p. 120]).
Now, (2.18), (2.19), and (2.20) give Theorem 7. ∎

The following result is also valid.

Theorem 8. *Let the assumptions of Theorem 1 be fulfilled. Then*

$$E\left(G\left(X\right)^{p}\right)>\left(\frac{\pi^{3}}{6}-\frac{n+\frac{1}{2}}{n^{2}+n+\frac{1}{3}}\right)^{-\frac{p}{2}}\left[\sum_{x\in\mathcal{X}}P_{X}\left(x\right)^{\frac{2}{2+p}}\right]^{\frac{2+p}{2}} \tag{2.21}$$

and

$$E\left(G\left(X\left|Y\right.\right)^{p}\right)\geq\left(\frac{\pi^{3}}{6}-\frac{n+\frac{1}{2}}{n^{2}+n+\frac{1}{3}}\right)^{-\frac{p}{2}}\sum_{y\in\mathcal{Y}}\left[\sum_{x\in\mathcal{X}}P_{X,Y}\left(x,y\right)^{\frac{2}{2+p}}\right]^{\frac{2+p}{2}}. \tag{2.22}$$

Proof. For $\alpha=-\frac{p}{2}$, (2.1) becomes

$$E\left(G\left(X\right)^{p}\right)>S_{-2}\left(n\right)^{-\frac{p}{2}}\left[\sum_{x\in\mathcal{X}}P_{X}\left(x\right)^{\frac{2}{2+p}}\right]^{\frac{p+2}{2}}. \tag{2.23}$$

Moreover, the following inequality is well-known [9]:

$$\sum_{k=1}^{n}k^{-2}<\frac{\pi^{2}}{6}-\frac{n+\frac{1}{2}}{n^{2}+n+\frac{1}{3}}. \tag{2.24}$$

Now, (2.23) and (2.24) give (2.21). Similarly we can prove (2.22). ∎

3. On Inequalities of Dragomir and van der Hoek

Firstly, we shall prove the following improvement of Theorem 2.

Theorem 9. *Let assumptions of Theorem 2 be fulfilled. Then*

$$\begin{aligned} &\left|E\left(G\left(X\right)^{p}\right)-\frac{1}{n}S_{p}\left(n\right)\right| \\ \leq&\left(\frac{1}{n}\left[\frac{n^{2}}{4}\right]\right)^{\frac{1}{2}}\left(P_{M}-P_{m}\right)\left[S_{2p}\left(n\right)-\frac{1}{n}S_{p}\left(n\right)^{2}\right]^{\frac{1}{2}} \\ \leq&\frac{1}{n}\left[\frac{n^{2}}{4}\right]\left(n^{p}-1\right)\left(P_{M}-P_{m}\right) \\ \leq&\frac{n\left(n^{p}-1\right)}{4}\left(P_{M}-P_{m}\right). \end{aligned} \tag{3.1}$$

Proof. Let us note that the following inequality of Biernacki, Pidek and Ryll-Nardzewski is well-known (see for example [6, p. 30]):
Let a and b be two real n-tuples such that

$$u\leq a_{i}\leq U\text{ and }v\leq b_{i}\leq V,\ \ (i=1,...,n).$$

Then

$$\left|D\left(a,b\right)\right|\leq\frac{1}{n}\left[\frac{n^{2}}{4}\right]\left(V-v\right)\left(U-u\right) \tag{3.2}$$

where

$$D\left(a,b\right)=\sum_{i=1}^{n}a_{i}b_{i}-\frac{1}{n}\sum_{i=1}^{n}a_{i}\sum_{i=1}^{n}b_{i}.$$

This result is stronger than that of Theorem 2.4 in [3].
Denote by

$$A=\frac{1}{n}\sum_{i=1}^{n}a_{i},\ \ B=\frac{1}{n}\sum_{i=1}^{n}b_{i},$$

then

$$\begin{aligned} D\left(a,b\right)&=\sum_{i=1}^{n}\left(a_{i}-A\right)\left(b_{i}-B\right) \\ &\leq\left(\sum_{i=1}^{n}\left(a_{i}-A\right)^{2}\right)^{\frac{1}{2}}\left(\sum_{i=1}^{n}\left(b_{i}-B\right)^{2}\right)^{\frac{1}{2}} \\ &=\left(D\left(a,a\right)\right)^{\frac{1}{2}}\left(D\left(b,b\right)\right)^{\frac{1}{2}}. \end{aligned} \tag{3.3}$$

By (3.2) we have

$$D(a,a) \le \frac{1}{n}\left[\frac{n^2}{4}\right](U-u)^2,$$

so (3.3) becomes

$$|D(a,b)| \le \sqrt{\frac{1}{n}\left[\frac{n^2}{4}\right]}(U-u)\sqrt{D(b,b)}. \tag{3.4}$$

For $b_i = i^p, a_i = p_i, U = P_M, u = P_m$, we get the first inequality in (3.1) since $\sum_{i=1}^n p_i = 1$.
The second inequality follows from (3.2), since from (3.2) we have for $a_i = i^p, b_i = i^p$:

$$D(\{i^p\},\{i^p\}) \le \frac{1}{n}\left[\frac{n^2}{4}\right](n^p-1)^2.$$

That is,

$$\left|S_{2p}(n) - \frac{1}{n}S_p(n)^2\right| \le \frac{1}{n}\left[\frac{n^2}{4}\right](n^p-1)^2,$$

while the last inequality is obvious. ∎

Remark 4. *If we put in* (3.1) $p=1$, *we get the following improvement of* (1.6).

$$\begin{aligned}\left|E(G(X)) - \frac{n+1}{2}\right| &\le \left\{\frac{n^2-1}{12}\left[\frac{n^2}{4}\right]\right\}^{\frac{1}{2}}(P_M-P_m)\\ &\le \frac{1}{n}\left[\frac{n^2}{4}\right](n-1)(P_M-P_m)\\ &\le \frac{n(n-1)}{4}(P_M-P_m).\end{aligned}$$

If we choose in (3.1) $p=2$, *we get the following improvement of* (1.7)

$$\begin{aligned}&\left|E\left(G(X)^2\right) - \frac{(n+1)(2n+1)}{6}\right|\\ \le\ & \left[\frac{(n-1)(n+1)(2n+1)(8n+1)}{180}\left[\frac{n^2}{4}\right]\right]^{\frac{1}{2}}(P_M-P_m)\\ \le\ & \frac{1}{n}\left[\frac{n^2}{4}\right]\left(n^2-1\right)(P_M-P_m)\\ \le\ & \frac{n\left(n^2-1\right)}{4}(P_M-P_m).\end{aligned}$$

And finally, for $p=3$, *we obtain the improvement of* (1.2).

Corollary 4. *If we assume that for a given* $\varepsilon > 0$ *and* $n \ge 1$ *we have*

$$0 \le P_M - P_m < \varepsilon\left\{\frac{1}{n}\left[\frac{n^2}{4}\right]\left[S_{2p}(n) - \frac{1}{n}S_p(n)^2\right]\right\}^{\frac{1}{2}}, \tag{3.5}$$

then (1.5) *is valid.*

Theorem 10. *With assumptions of Theorem 2, we have the inequality*

$$\begin{aligned} &\left|\binom{p+1}{1} E\left(G(X)^{p}\right)-\binom{p+1}{2} E\left(G(X)^{p-1}\right)+\ldots\right. \\ &\left.+(-1)^{p+1}\binom{p+1}{1} E(G(X))+(-1)^{p+2}-n^{p}\right| \\ \leq\; & \frac{p n^{p+1}\left(P_{M}-P_{m}\right)}{2 \sqrt{2 p+1}} \leq \frac{(p+1) n^{p+1}}{4}\left(P_{M}-P_{m}\right) \end{aligned} \tag{3.6}$$

provided that $p \in \mathbb{N}$, $p \geq 1$.

Proof. Let us note that the following inequality is well-known as Grüss' inequality ([3, Lemma 2.3]).
Let $h, g : [a, b] \to \mathbb{R}$ be two integrable functions such that

$$m_1 \leq g(x) \leq M_1,\ m_2 \leq h(x) \leq M_2 \text{ for all } x \in (a, b).$$

Then we have the estimation:

$$|D(g, h)| \leq \frac{1}{4}(M_1 - m_1)(M_2 - m_2), \tag{3.7}$$

where

$$D(a, b) = \frac{1}{b-a}\int_a^b g(x) h(x)\,dx - \frac{1}{b-a}\int_a^b g(x)\,dx \cdot \frac{1}{b-a}\int_a^b h(x)\,dx.$$

On the other hand, it is also well-known that

$$|D(g, h)| \leq (D(g, g))^{\frac{1}{2}} (D(h, h))^{\frac{1}{2}}. \tag{3.8}$$

Since by (3.7) we have

$$|D(g, g)| \leq \frac{1}{4}(M_1 - m_1)^2, \tag{3.9}$$

we get from (3.8)

$$|D(g, h)| \leq \frac{1}{2}(M_1 - m_1)(D(h, h))^{\frac{1}{2}}. \tag{3.10}$$

For $g = f$ and $h(x) = (x-a)^p$ (3.6) becomes

$$\begin{aligned} &\left|\frac{1}{b-a}\int_a^b (x-a)^p f(x)\,dx - \frac{(b-a)^{p-1}}{p+1}\int_a^b f(x)\,dx\right| \\ \leq\; & \frac{1}{2}(M-m)(b-a)^p \frac{p}{(p+1)\sqrt{2p+1}} \end{aligned} \tag{3.11}$$

where

$$M := \sup_{x\in[a,b]} f(x) < \infty \text{ and } m := \inf_{x\in[a,b]} f(x) > 0 \tag{3.12}$$

and $p \geq 0$. Since $\frac{p}{(p+1)\sqrt{2p+1}} \leq \frac{1}{2}$ we have the following improvement of Lemma 2.5 from [3]:

Let $f:[a,b]\to\mathbb{R}$ be an integrable mapping. Then we have the inequality

$$\left|\int_a^b (x-a)^p f(x)\,dx-\frac{(b-a)^p}{p+1}\int_a^b f(x)\,dx\right| \leq \frac{p(b-a)^{p+1}(M-m)}{2(p+1)\sqrt{2p+1}} \leq \frac{(b-a)^{p+1}}{4}(M-m), \tag{3.13}$$

where M and m are as defined by (3.12) and $p\geq 0$.
Now, set in the previous result, $a=0, b=n, f(x)=p_{i+1}, x\in[i,i+1), i=0,...,n-1$. Then we have

$$\int_0^n f(x)\,dx=\sum_{i=1}^n p_i=1,$$

$$\begin{aligned}\int_0^n x^p f(x)\,dx &= \frac{1}{p+1}\left[\binom{p+1}{1}\sum_{i=1}^n i^p p_i-\binom{p+2}{2}\sum_{i=1}^n i^{p-1}p_i+...\right.\\ &\quad\left.+(-1)^{p+1}\binom{p+1}{1}\sum_{i=1}^n ip_i+(-1)^{p+2}\right]\end{aligned}$$

and from (3.13) we get (3.6). ∎

Corollary 5. *If we assume that for a given $\varepsilon>0$ and $n\geq 1$, we have:*

$$0\leq P_M-P_m<\frac{2\varepsilon\sqrt{2p+1}}{pn^{p+1}},$$

then

$$\begin{aligned}&\left|\binom{p+1}{1}E(G(X)^p)-\binom{p+1}{2}E\left(G(X)^{p-1}\right)+...\right.\\ &\left.+(-1)^{p+1}\binom{p+1}{1}E(G(X))+(-1)^{p+2}-n^p\right|\\ \leq\ &\varepsilon.\end{aligned} \tag{3.14}$$

Remark 5. *If in* (3.6) *we put $p=1$, we get the following improvement of* (1.13) :

$$\left|E(G(X))-\frac{n+1}{2}\right|\leq\frac{n^2}{4\sqrt{3}}(P_M-P_m). \tag{3.15}$$

In addition if we choose $p=2$ we have an improvement of (1.13) :

$$\left|E\left(G(X)^2\right)-E(G(X))-\frac{n^2-1}{3}\right|\leq\frac{n^2}{3\sqrt{5}}(P_M-P_m). \tag{3.16}$$

Let us note that in the proof of Theorem 3, Grüss' integral inequality was used in [3]. *Of course, a discrete Grüss inequality, that is a stronger version of this inequality,* *such as* (3.2), *can be similarly used. Namely, we have*

$$\begin{aligned}(3.17)\qquad & \left|\binom{p+1}{1}E\left(G\left(X\right)^{p}\right)-\binom{p+1}{2}E\left(G\left(X\right)^{p-1}\right)+...\right.\\ & \left.+\left(-1\right)^{p+1}\binom{p+1}{1}E\left(G\left(X\right)\right)+\left(-1\right)^{p+2}-n^{p}\right|\\ \leq\ & \left|\sum_{i=1}^{n}\left[i^{p+1}-\left(i-1\right)^{p+1}\right]p_{i}-\frac{1}{n}\sum_{i=1}^{n}\left[i^{p+1}-\left(i-1\right)^{p+1}\right]\sum_{i=1}^{n}p_{i}\right|\\ \leq\ & \frac{1}{n}\left[\frac{n^{2}}{4}\right]\left(n^{p+1}-\left(n-1\right)^{p+1}-1\right)\left(P_{M}-P_{m}\right)\end{aligned}$$

since

$$\max_{i=1,..,n}\left[i^{p+1}-\left(i-1\right)^{p+1}\right]=n^{p+1}-\left(n-1\right)^{p+1}$$

and

$$\min_{i=1,..,n}\left[i^{p+1}-\left(i-1\right)^{p+1}\right]=1.$$

For $p=1$ *we have*

$$(3.18)\qquad \left|E\left(G\left(X\right)\right)-\frac{n+1}{2}\right|\leq\frac{n-1}{n}\left[\frac{n^{2}}{4}\right]\left(P_{M}-P_{m}\right).$$

As

$$\frac{n-1}{n}\left[\frac{n^{2}}{4}\right]\leq\frac{\left(n-1\right)n}{4},$$

it is clear that (3.18) *is better than* (1.6) *and* (1.12).
For $p=2$ *we have*

$$(3.19)\qquad \left|E\left(G\left(X\right)^{2}\right)-E\left(G\left(X\right)\right)-\frac{n^{2}-1}{3}\right|\leq\left(n-1\right)\left[\frac{n^{2}}{4}\right]\left(P_{M}-P_{m}\right)$$

which is better than (1.13).

Finally, we shall give a companion inequality to that given in (1.14).

Theorem 11. *With the assumptions of Theorem 3, we have the inequality*

$$\begin{aligned}(3.20)\qquad & P_{m}n^{p+1}\\ \leq\ & \binom{p+1}{1}E\left(G\left(X\right)^{p}\right)+...+\left(-1\right)^{p+1}\binom{p+1}{1}E\left(G\left(X\right)\right)+\left(-1\right)^{p+1}\\ \leq\ & P_{M}n^{p+1}.\end{aligned}$$

Proof. Let $m\leq f\left(u\right)\leq M$ for any $u\in\left[a,b\right]$. Then

$$mu^{n}\leq u^{n}f\left(u\right)\leq Mu^{n}.$$

That is,

$$m\int_{a}^{b}u^{n}du\leq\int_{a}^{b}u^{n}f\left(u\right)du\leq M\int_{a}^{b}u^{n}du$$

$$(3.21)\qquad m\frac{\left(b-a\right)^{p+1}}{p+1}\leq\int_{a}^{b}u^{n}f\left(u\right)du\leq M\frac{\left(b-a\right)^{p+1}}{p+1}.$$

Set $a=0, b=n, f(u)=p_i, u\in[i-1,i), i=1,...,n, m=P_M, M=P_m$. Then (3.21) becomes

$$\frac{P_m n^{p+1}}{p+1}\leq\sum_{i=1}^{n}\frac{i^{p+1}-(i-1)^{p+1}}{p+1}a_i\leq\frac{P_M n^{p+1}}{p+1},$$

which is, in fact (3.20). ∎

4. On Some Inequalities of Dragomir

Recently S.S. Dragomir [10] obtained some related results for guessing mappings. Let us note that some of his results can be improved if we use Biernacki, Pideck and Ryll-Nardzewski's inequality (3.2) instead of his Lemma 2.7. The rest of the proofs are the same, so here we shall give only improvements of results without proofs.

Let us consider the arithmetic means

$$A_n(p,x):=\sum_{i=1}^{n}p_ix_i \text{ where } p_i\geq 0 \text{ and } \sum_{i=1}^{n}p_i\geq 1,$$

$$A_n(q,x):=\sum_{i=1}^{n}q_ix_i \text{ where } q_i\geq 0 \text{ and } \sum_{i=1}^{n}q_i\geq 1$$

and

$$x=(x_i)_{i=\overline{1,n}}\in\mathbb{R}\ \ (i=1,...,n).$$

Theorem 12. *With the above assumptions for p,q and x, we have*

$$\begin{aligned}&\left|A_n(p,x)-A_n(q,x)-\frac{1}{n-1}(x_n-x_1)\sum_{i=1}^{n-1}(n-i)(p_i-q_i)\right|\\ \leq\ & g(n)(\Gamma-\gamma)(\Delta-\delta),\end{aligned}$$

where $g(n)=\frac{1}{n-1}\left[\frac{(n-1)^2}{4}\right]$ and $[\cdot]$ denotes the integer part, provided that

$$\delta\leq\Delta x_i\leq\Delta \text{ for all } i=1,...,n-1$$

and

$$\gamma\leq P_i-Q_i\leq\Gamma \text{ for all } i=1,...,n-1$$

where

$$\Delta x_i:=x_{i+1}-x_i\ \ (i=1,...,n-1), P_i:=\sum_{k=1}^{i}p_k,\ Q_i:=\sum_{k=1}^{i}q_k\ \ (\text{for } i=1,...,n).$$

Corollary 6. *With the above assumptions for x and if*

$$\tilde{\gamma}\leq P_i-\frac{i}{n}\leq\tilde{\Gamma},$$

then we have the estimation:

$$\begin{aligned}&\left|A_n(p,x)-A_n(q,x)-\frac{1}{n-1}(x_n-x_1)\sum_{i=1}^{n-1}(n-i)\left(p_i-\frac{1}{n}\right)\right|\\ \leq\ & g(n)\left(\tilde{\Gamma}-\tilde{\gamma}\right)(\Delta-\delta).\end{aligned}$$

Theorem 13. *Suppose that p, q, x satisfy the condition*

$$\alpha \leq X_i \leq X, \ i = 1, ..., n-1$$

where

$$X_i := \sum_{k=1}^{i} x_n \ \ (i = 1, ..., n)$$

and

$$\varphi \leq \Delta (p_i - q_i) \leq \phi, \ i = 1, ..., n-1.$$

Then we have

$$\begin{aligned} & \left| A_n (p, x) - A_n (q, x) - X_n (p_n - q_n) \right. \\ & \left. + \frac{1}{n-1} [p_n - p_1 - (q_n - q_1)] \sum_{i=1}^{n-1} (n-i) x_i \right| \\ \leq \ & g(n) (X - x) (\phi - \varphi). \end{aligned}$$

Corollary 7. *If x and p satisfy the conditions*

$$x \leq X_i \leq X, \ i = 1, ..., n-1$$

and

$$\tilde{\varphi} \leq \Delta p_i \leq \tilde{\phi}, \ i = 1, ..., n-1$$

then we have the bound

$$\begin{aligned} & \left| A_n (p, x) - A_n (x) - X_n \left(p_n - \frac{1}{n} \right) + \frac{1}{n-1} (p_n - p_1) \sum_{i=1}^{n-1} (n-i) x_i \right| \\ \leq \ & g(n) (X - x) \left(\tilde{\phi} - \tilde{\varphi} \right). \end{aligned}$$

To simplify the notation for the next result, we assume that the x_i are numbered so that x_k is always the k^{th} guess. This yields

$$E (G^n) = \sum_{i=1}^{n} i^p p_i \quad (p \geq 0).$$

Now, if we consider another guessing mapping L, we can write

$$E (L^p) = \sum_{i=1}^{n} i^p p_{\sigma(i)}$$

where σ is a permutation of the indices $\{1, ..., n\}$. We also use the notation $P_i = \sum_{k=1}^{i} p_k$, $P_{\sigma(i)} = \sum_{k=1}^{i} P_{\sigma(n)}$, and $S_p (k) = \sum_{i=1}^{k} i^n, k = 1, ..., n$.

Theorem 14. *Let $G(X)$ and $L(X)$ be the guessing mappings associated with random variable X and $E (G (X)^p), E (L (X)^p)$ $(p \geq 1)$ their $p-$moments. Then we have*

$$\begin{aligned} & \left| E (G (X)^p) - E (L (X)^p) - \frac{n^p - 1}{n-1} \sum_{i=1}^{n-1} (n-i) \left(P_i - P_{\sigma(n)} \right) \right| \\ \leq \ & g(n) (n^p - 1) (M_{p,\sigma} - m_{p,\sigma}), \ p \geq 1 \end{aligned}$$

where

$$M_{p,\sigma} := \max_{i=\overline{1,n-1}} \left(P_i - P_{\sigma(i)} \right)$$

and

$$m_{p,\sigma} := \min_{i=\overline{1,n-1}} \left(P_i - P_{\sigma(i)}\right).$$

Theorem 15. *With the above assumptions, we have*

$$\begin{aligned} &\Big| E\left(G\left(X\right)^p\right) - E\left(L\left(X\right)^p\right) - \frac{S_p\left(n\right) - S_{p+1}\left(n\right)}{n-1}\left[p_n - p_1 - p_{\sigma(n)} + p_{\sigma(1)}\right] \\ &- S\left(n\right)\left(p_n - p_{\sigma(n)}\right)\Big| \\ \leq\ & g\left(n\right)\left(S_p\left(n\right) - 1\right)\left(\phi_{p,\sigma} - \varphi_{p,\sigma}\right) \end{aligned}$$

where

$$\phi_{p,\sigma} = \max_{i=\overline{1,n-1}} \left(\Delta p_i - \Delta p_{\sigma(i)}\right)$$

and

$$\varphi_{p,\sigma} = \min_{i=\overline{1,n-1}} \left(\Delta p_i - \Delta p_{\sigma(i)}\right).$$

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