A Weighted Interpolation for Jensen's Integral Inequality and Some Converse Results

A WEIGHTED INTERPOLATION FOR JENSEN'S INTEGRAL INEQUALITY AND SOME CONVERSE RESULTS

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Abstract. A weighted interpolation of Jensen's integral inequality and some applications for Hadamard's inequality are given.

1. Introduction

Let $\phi : \mathbb{R} \to \mathbb{R}$ be a convex mapping and $f, p : [a,b] \to \mathbb{R}$, $p(t) \geq 0$ on $[a,b]$ be measurable and such that p, fp and $\phi \circ f \cdot p$ are integrable on $[a,b]$. Then

$$\phi\left(\frac{\int_a^b f(t)\,p(t)\,dt}{\int_a^b p(t)\,dt}\right) \leq \frac{\int_a^b (\phi \circ f)(t)\,p(t)\,dt}{\int_a^b p(t)\,dt}, \tag{1.1}$$

provided that $\int_a^b p(t)\,dt > 0$.

The inequality (1.1) is well known in the literature as the *Jensen's weighted integral inequality.*

In 1996, Dragomir and Goh [21] proved a counterpart result for convex mappings of $n-$variables, in terms of the Theory of Probability. We mention here only the uni-dimensional version of their result:

$$\begin{aligned} 0 &\leq \frac{\int_a^b (\phi \circ f)(t)\,p(t)\,dt}{\int_a^b p(t)\,dt} - \phi\left(\frac{\int_a^b f(t)\,p(t)\,dt}{\int_a^b p(t)\,dt}\right) \\ &\leq \frac{1}{\int_a^b p(t)\,dt}\int_a^b (\phi' \circ f)(t) \cdot f(t)\,p(t)\,dt \\ &\quad - \frac{1}{\int_a^b p(t)\,dt}\int_a^b (\phi' \circ f)(t)\,p(t)\,dt \cdot \frac{1}{\int_a^b p(t)\,dt}\int_a^b f(t)\,p(t)\,dt, \end{aligned} \tag{1.2}$$

provided that all the integrals exist and ϕ is differentiable convex on $\mathbb{R}$.

In this paper we point out some refinements of (1.1) and their counterparts as suggested by (1.2). Some particular examples are also examined.

2. The Results

We will start with the following weighted refinement of Jensen's integral inequality (1.1).

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Theorem 1. *If ϕ, p and f are as above, $u_1, ..., u_k$ are nonnegative numbers and $U_k := \sum_{i=1}^{k} u_i > 0$, then we have the following inequality:*

$$\begin{aligned} &\phi\left(\frac{\int_a^b f(t)\,p(t)\,dt}{\int_a^b p(t)\,dt}\right) \\ \leq\ & \frac{\int_a^b ... \int_a^b \phi\left(\frac{f(t_1)+...+f(t_k)}{k}\right) p(t_1)...p(t_k)\,dt_1...dt_k}{\left(\int_a^b p(t)\,dt\right)^k} \\ \leq\ & \frac{\int_a^b ... \int_a^b \phi\left(\frac{u_1 f(t_1)+...+u_k f(t_k)}{U_k}\right) p(t_1)...p(t_k)\,dt_1...dt_k}{\left(\int_a^b p(t)\,dt\right)^k} \\ \leq\ & \frac{\int_a^b \phi(f(t))\,p(t)\,dt}{\int_a^b p(t)\,dt} \end{aligned} \tag{2.1}$$

holds.

Proof. By Jensen's inequality for integrals of the form $\underbrace{\int_a^b ... \int_a^b}_{k \text{ times}}$ we get that

$$\begin{aligned} &\frac{\int_a^b ... \int_a^b \phi\left(\frac{f(t_1)+...+f(t_k)}{k}\right) p(t_1)...p(t_k)\,dt_1...dt_k}{\int_a^b ... \int_a^b p(t_1)...p(t_k)\,dt_1...dt_k} \\ \geq\ & \phi\left(\frac{\int_a^b ... \int_a^b \left(\frac{f(t_1)+...+f(t_k)}{k}\right) p(t_1)...p(t_k)\,dt_1...dt_k}{\int_a^b ... \int_a^b p(t_1)...p(t_k)\,dt_1...dt_k}\right). \end{aligned} \tag{2.2}$$

As a simple calculation shows us that:

$$\int_a^b ... \int_a^b p(t_1)...p(t_k)\,dt_1...dt_k = \left(\int_a^b p(t)\,dt\right)^k$$

and:

$$\begin{aligned} &\int_a^b ... \int_a^b \left[\frac{f(t_1)+...+f(t_k)}{k}\right] p(t_1)...p(t_k)\,dt_1...dt_k \\ =\ & \left(\int_a^b p(t)\,dt\right)^{k-1} \int_a^b f(t)\,p(t)\,dt, \end{aligned}$$

then the inequality (2.2) gives us the first inequality in (2.1).
Define the functions

$$y_i : [a,b] \to I, y_i = y_i(t_1, ..., t_k), i = \overline{1,k}$$

given by

$$\begin{aligned} y_1 &:= \frac{u_1 f(t_1) + u_2 f(t_2) + ... + u_{k-1} f(t_{k-1}) + u_k f(t_k)}{U_k} \\ y_2 &:= \frac{u_k f(t_1) + u_1 f(t_2) + ... + u_{k-2} f(t_{k-1}) + u_{k-1} f(t_k)}{U_k} \end{aligned}$$

....

$$y_k := \frac{u_2 f(t_1) + u_3 f(t_2) + \ldots + u_k f(t_{k-1}) + u_1 f(t_k)}{U_k}$$

where $t_1, \ldots, t_k \in [a,b]$.
Using Jensen's discrete inequality for the convex map ϕ and for the elements $y_1, \ldots, y_k \in I$, we have that:

$$\phi\left(\frac{y_1 + \ldots + y_k}{k}\right) \leq \frac{\phi(y_1) + \ldots + \phi(y_k)}{k},$$

and taking into account that:

$$\begin{aligned} \frac{y_1 + \ldots + y_k}{k} &= \frac{(u_1 + \ldots + u_k)(f(t_1) + \ldots + f(t_k))}{kU_k} \\ &= \frac{f(t_1) + \ldots + f(t_k)}{k} \end{aligned}$$

we get that

$$\begin{aligned} & \phi\left(\frac{f(t_1) + \ldots + f(t_k)}{k}\right) \\ \leq & \frac{1}{k}\left[\phi\left(\frac{u_1 f(t_1) + \ldots + u_k f(t_k)}{U_k}\right) + \phi\left(\frac{u_k f(t_1) + \ldots + u_{k-1} f(t_k)}{U_k}\right) + \ldots \right. \\ & \left. + \phi\left(\frac{u_2 f(t_1) + \ldots + u_1 f(t_k)}{U_k}\right)\right] \end{aligned}$$

for all $t_1, \ldots, t_k \in [a,b]$.
Integrating this inequality on $[a,b]^k$ we get that:

$$\begin{aligned} & \int_a^b \ldots \int_a^b \phi\left(\frac{f(t_1) + \ldots + f(t_k)}{k}\right) p(t_1) \ldots p(t_k)\, dt_1 \ldots dt_k \\ \leq & \frac{1}{k}\left[\int_a^b \ldots \int_a^b \phi\left(\frac{u_1 f(t_1) + \ldots + u_k f(t_k)}{U_k}\right) p(t_1) \ldots p(t_k)\, dt_1 \ldots dt_k + \ldots \right. \\ & \left. + \int_a^b \ldots \int_a^b \phi\left(\frac{u_2 f(t_1) + \ldots + u_1 f(t_k)}{U_k}\right) p(t_1) \ldots p(t_k)\, dt_1 \ldots dt_k\right]. \end{aligned} \tag{2.3}$$

However, it can be easily seen that:

$$\begin{aligned} & \int_a^b \ldots \int_a^b \phi\left(\frac{u_1 f(t_1) + \ldots + u_k f(t_k)}{U_k}\right) p(t_1) \ldots p(t_k)\, dt_1 \ldots dt_k \\ = & \ldots\ldots = \int_a^b \ldots \int_a^b \phi\left(\frac{u_2 f(t_1) + \ldots + u_1 f(t_k)}{U_k}\right) p(t_1) \ldots p(t_k)\, dt_1 \ldots dt_k \end{aligned}$$

and the inequality (2.3) gives us the second inequality in (2.3).
Finally, by Jensen's discrete inequality for the weights $u_1, \ldots, u_k$ we can write:

$$\phi\left(\frac{u_1 f(t_1) + \ldots + u_k f(t_k)}{U_k}\right) \leq \frac{u_1 (\phi \circ f)(t_1) + \ldots + u_k (\phi \circ f)(t_k)}{U_k}$$

for all $t_1, ..., t_k \in [a, b]$.
Integrating this inequality on $[a, b]^k$ we deduce that:

$$\begin{aligned}
& \int_a^b ... \int_a^b \phi\left(\frac{u_1 f(t_1) + ... + u_k f(t_k)}{U_k}\right) p(t_1) ... p(t_k)\, dt_1 ... dt_k \\
\leq\ & \frac{1}{U_k} \int_a^b ... \int_a^b \left[u_1 (\phi \circ f)(t_1) + ... + u_k (\phi \circ f)(t_k)\right] p(t_1) ... p(t_k)\, dt_1 ... dt_k \\
=\ & \left(\int_a^b p(t)\, dt\right)^{k-1} \int_a^b (\phi \circ f)(t)\, p(t)\, dt
\end{aligned}$$

and the last inequality in (2.1) is proved. □

Corollary 1. *Let ϕ and u_i $\left(i = \overline{1, k}\right)$ be defined as above. Then the following interpolation of Hadamard's integral inequality is valid:*

$$\begin{aligned}
\phi\left(\frac{a+b}{2}\right) & \leq \frac{1}{(b-a)^k} \int_a^b ... \int_a^b \phi\left(\frac{t_1 + ... + t_k}{k}\right) dt_1 ... dt_k \\
& \leq \frac{1}{(b-a)^k} \int_a^b ... \int_a^b \phi\left(\frac{u_1 t_1 + ... + u_k t_k}{U_k}\right) dt_1 ... dt_k \\
& \leq \frac{1}{b-a} \int_a^b \phi(t)\, dt.
\end{aligned} \tag{2.4}$$

Remark 1. *The first inequality in* (2.4) *was proved in* [4], *the second inequality in* (2.4) *was proved in* [19] *and the last inequality in* (2.4) *was proved in* [12].

3. Some Converses of the Above Results. The Unweighted Case

We shall start with the following result which gives a converse for the first inequality in (2.1).

Theorem 2. *Suppose that ϕ, p and f are as above. Then one has the inequality:*

$$\begin{aligned}
0 & \leq \frac{\int_a^b ... \int_a^b \phi\left(\frac{f(t_1) + ... + f(t_k)}{k}\right) p(t_1) ... p(t_k)\, dt_1 ... dt_k}{\left(\int_a^b p(t)\, dt\right)^k} - \phi\left(\frac{\int_a^b f(t)\, p(t)\, dt}{\int_a^b p(t)\, dt}\right) \\
& \leq \frac{\int_a^b ... \int_a^b \phi'_+\left(\frac{f(t_1) + ... + f(t_k)}{k}\right) f(t_1)\, p(t_1) ... p(t_k)\, dt_1 ... dt_k}{\left(\int_a^b p(t)\, dt\right)^k} \\
& \quad - \frac{\int_a^b f(t)\, p(t)\, dt}{\int_a^b p(t)\, dt} \cdot \frac{\int_a^b ... \int_a^b \phi'_+\left(\frac{f(t_1) + ... + f(t_k)}{k}\right) p(t_1) ... p(t_k)\, dt_1 ... dt_k}{\left(\int_a^b p(t)\, dt\right)^k},
\end{aligned} \tag{3.1}$$

where ϕ'_+ is the right derivative of ϕ on $\mathring{I}$.

Proof. As ϕ is convex on I, we can write that

$$\phi(x) - \phi(y) \geq \phi'_+(y)(x - y) \text{ for all } x, y \in \mathring{I}. \tag{3.2}$$

where $\phi'_+(\cdot)$ is the right derivative of ϕ which is a nonnegative mapping on $\mathring{I}$. Choosing

$$x=\frac{\int_a^b f(t)\,p(t)\,dt}{\int_a^b p(t)\,dt} \text{ and } y=\frac{f(t_1)+...+f(t_k)}{k},$$

we deduce the inequality:

$$\begin{aligned}&\phi\left(\frac{\int_a^b f(t)\,p(t)\,dt}{\int_a^b p(t)\,dt}\right)-\phi\left(\frac{f(t_1)+...+f(t_k)}{k}\right)\\ \geq\;& \phi'_+\left(\frac{f(t_1)+...+f(t_k)}{k}\right)\left(\frac{\int_a^b f(t)\,p(t)\,dt}{\int_a^b p(t)\,dt}-\frac{f(t_1)+...+f(t_k)}{k}\right)\end{aligned}$$

for all $t_1,...,t_k$ in $[a,b]$.
Integrating this inequality on $[a,b]^n$ we easily deduce the inequality:

$$\begin{aligned}&\frac{\int_a^b...\int_a^b \phi\left(\frac{f(t_1)+...+f(t_k)}{k}\right)p(t_1)...p(t_k)\,dt_1...dt_k}{\left(\int_a^b p(t)\,dt\right)^k}-\phi\left(\frac{\int_a^b f(t)\,p(t)\,dt}{\int_a^b p(t)\,dt}\right)\\
&\leq\frac{1}{\left(\int_a^b p(t)\,dt\right)^k}\\
&\times\int_a^b...\int_a^b\phi'_+\left(\frac{1}{k}\sum_{i=1}^k f(t_i)\right)\left(\frac{1}{k}\sum_{i=1}^k f(t_i)\right)p(t_1)...p(t_k)\,dt_1...dt_k\\
&-\frac{\int_a^b f(t)\,p(t)\,dt}{\int_a^b p(t)\,dt}\cdot\frac{\int_a^b...\int_a^b\phi'_+\left(\frac{1}{k}\sum_{i=1}^k f(t_i)\right)p(t_1)...p(t_k)\,dt_1...dt_k}{\left(\int_a^b p(t)\,dt\right)^k}\\
&=\frac{1}{\left(\int_a^b p(t)\,dt\right)^k}\int_a^b...\int_a^b\phi'_+\left(\frac{1}{k}\sum_{i=1}^k f(t_i)\right)f(t_1)\,p(t_1)...p(t_k)\,dt_1...dt_k\\
&-\frac{\int_a^b f(t)\,p(t)\,dt}{\int_a^b p(t)\,dt}\cdot\frac{\int_a^b...\int_a^b\phi'_+\left(\frac{1}{k}\sum_{i=1}^k f(t_i)\right)p(t_1)...p(t_k)\,dt_1...dt_k}{\left(\int_a^b p(t)\,dt\right)^k},\end{aligned}\tag{3.3}$$

and the theorem is proved. □

The result embodied in (3.1) can be completed by the following one:

Theorem 3. *With the above assumptions, one has the inequality:*

$$\begin{aligned}&\frac{\int_a^b...\int_a^b\phi'_+\left(\frac{f(t_1)+...+f(t_k)}{k}\right)f(t_1)\,p(t_1)...p(t_k)\,dt_1...dt_k}{\left(\int_a^b p(t)\,dt\right)^k}\\
&-\frac{\int_a^b f(t)\,p(t)\,dt}{\int_a^b p(t)\,dt}\cdot\frac{\int_a^b...\int_a^b\phi'_+\left(\frac{f(t_1)+...+f(t_k)}{k}\right)p(t_1)...p(t_k)\,dt_1...dt_k}{\left(\int_a^b p(t)\,dt\right)^k}\end{aligned}\tag{3.4}$$

$$\leq \frac{1}{k^{\frac{1}{2}}}\left[\frac{1}{\left(\int_a^b p(t)\,dt\right)^k}\int_a^b\cdots\int_a^b\left[\phi'_+\left(\frac{f(t_1)+\dots+f(t_k)}{k}\right)\right]^2 p(t_1)\dots p(t_k)\,dt_1\dots dt_k\right]^{\frac{1}{2}}$$
$$\times\left[\frac{\int_a^b f^2(t)\,p(t)\,dt}{\int_a^b p(t)\,dt}-\left(\frac{\int_a^b f(t)\,p(t)\,dt}{\int_a^b p(t)\,dt}\right)^2\right]^{\frac{1}{2}}$$

for all $k \geq 1$.

Proof. By Schwartz's inequality for multiple integrals we can write:

$$\frac{1}{\left(\int_a^b p(t)\,dt\right)^k}\int_a^b\cdots\int_a^b \phi'_+\left(\frac{f(t_1)+\dots+f(t_k)}{k}\right) \tag{3.5}$$
$$\times\left(\frac{f(t_1)+\dots+f(t_k)}{k}-\frac{\int_a^b f(t)\,p(t)\,dt}{\int_a^b p(t)\,dt}\right)p(t_1)\dots p(t_k)\,dt_1\dots dt_k$$

$$\leq \left\{\frac{1}{\left(\int_a^b p(t)\,dt\right)^k}\int_a^b\cdots\int_a^b\left[\phi'_+\left(\frac{f(t_1)+\dots+f(t_k)}{k}\right)\right]^2 p(t_1)\dots p(t_k)\,dt_1\dots dt_k\right\}^{\frac{1}{2}}$$
$$\times\left\{\frac{1}{\left(\int_a^b p(t)\,dt\right)^k}\int_a^b\cdots\int_a^b\left[\frac{f(t_1)+\dots+f(t_k)}{k}\right.\right.$$
$$\left.\left.-\frac{\int_a^b f(t)\,p(t)\,dt}{\int_a^b p(t)\,dt}\right]^2 p(t_1)\dots p(t_k)\,dt_1\dots dt_k\right\}^{\frac{1}{2}}$$

Now, let us compute the integral

$$I = \int_a^b\cdots\int_a^b\left[\frac{f(t_1)+\dots+f(t_k)}{k}-\frac{\int_a^b f(t)\,p(t)\,dt}{\int_a^b p(t)\,dt}\right]^2 p(t_1)\dots p(t_k)\,dt_1\dots dt_k$$
$$= \int_a^b\cdots\int_a^b\left[\left(\frac{f(t_1)+\dots+f(t_k)}{k}\right)^2-2\left(\frac{f(t_1)+\dots+f(t_k)}{k}\right)\frac{\int_a^b f(t)\,p(t)\,dt}{\int_a^b p(t)\,dt}\right.$$
$$\left.+\left(\frac{\int_a^b f(t)\,p(t)\,dt}{\int_a^b p(t)\,dt}\right)^2\right]p(t_1)\dots p(t_k)\,dt_1\dots dt_k$$
$$= \frac{1}{k^2}\int_a^b\cdots\int_a^b\sum_{i,j=1}^k f(t_i)\,f(t_j)\,p(t_1)\dots p(t_k)\,dt_1\dots dt_k$$
$$-\frac{2k\left(\int_a^b p(t)\,dt\right)\int_a^b f(t)\,p(t)\,dt}{k}\cdot\frac{\int_a^b f(t)\,p(t)\,dt}{\int_a^b p(t)\,dt}$$
$$+\left(\int_a^b p(t)\,dt\right)^k\left(\frac{\int_a^b f(t)\,p(t)\,dt}{\int_a^b p(t)\,dt}\right)^2$$

$$
\begin{aligned}
&= \frac{k\int_a^b f^2(t)\,p(t)\,dt\left(\int_a^b p(t)\,dt\right)^{k-1}}{k^2} \\
&\quad+\frac{2\sum_{1\le i<j\le n}\int_a^b\ldots\int_a^b f(t_i)\,f(t_j)\,p(t_1)\ldots p(t_k)\,dt_1\ldots dt_k}{k^2} \\
&\quad-\left(\int_a^b p(t)\,dt\right)^{k-2}\left(\int_a^b f(t)\,p(t)\,dt\right)^2 \\
&= \frac{1}{k^2}\left[k\int_a^b f^2(t)\,p(t)\,dt\left(\int_a^b p(t)\,dt\right)^{k-1}\right. \\
&\quad+k(k-1)\left(\int_a^b p(t)\,dt\right)^{k-2}\left(\int_a^b f(t)\,p(t)\,dt\right)^2 \\
&\quad\left.-k^2\left(\int_a^b p(t)\,dt\right)^{k-2}\left(\int_a^b f(t)\,p(t)\,dt\right)^2\right] \\
&= \frac{1}{k}\left[\int_a^b f^2(t)\,p(t)\,dt\left(\int_a^b p(t)\,dt\right)^{k-1}-\left(\int_a^b p(t)\,dt\right)^{k-2}\left(\int_a^b f(t)\,p(t)\,dt\right)^2\right].
\end{aligned}
$$

Now, using the inequality (3.5) for the above representation of f we deduce the inequality (3.4). □

The following corollary is important:

Corollary 2. *Assume that $\phi : I \to \mathbb{R}$ is convex on the interval I and $\sup_{x\in I}\left|\phi'_+(x)\right| = M < \infty$. Thus, with the above assumptions for f and p we have that:*

$$
\begin{aligned}
&\lim_{k\to\infty}\frac{1}{\left(\int_a^b p(t)\,dt\right)^k}\int_a^b\ldots\int_a^b\phi\left(\frac{f(t_1)+\ldots+f(t_k)}{k}\right)p(t_1)\ldots p(t_k)\,dt_1\ldots dt_k \\
&= \inf_{\substack{k\in\mathbb{N}\\ k\ge 1}}\frac{1}{\left(\int_a^b p(t)\,dt\right)^k}\int_a^b\ldots\int_a^b\phi\left(\frac{f(t_1)+\ldots+f(t_k)}{k}\right)p(t_1)\ldots p(t_k)\,dt_1\ldots dt_k \\
&= \phi\left(\frac{\int_a^b f(t)\,p(t)\,dt}{\int_a^b p(t)\,dt}\right).
\end{aligned}
$$

Proof. As

$$
\phi'_+\left(\frac{f(t_1)+\ldots+f(t_k)}{k}\right)\le M,
$$

we get that (see (3.1) and (3.3)):

$$\begin{aligned}
0 &\leq \frac{1}{\left(\int_a^b p(t)\,dt\right)^k}\int_a^b \cdots \int_a^b \phi\left(\frac{f(t_1)+\ldots+f(t_k)}{k}\right) p(t_1)\ldots p(t_k)\,dt_1\ldots dt_k \\
&\quad -\phi\left(\frac{\int_a^b f(t)p(t)\,dt}{\int_a^b p(t)\,dt}\right) \\
&\leq \frac{M}{k^{\frac{1}{2}}}\left[\frac{\int_a^b f^2(t)p(t)\,dt}{\int_a^b p(t)\,dt}-\left(\frac{\int_a^b f(t)p(t)\,dt}{\int_a^b p(t)\,dt}\right)^2\right]^{\frac{1}{2}} \to 0
\end{aligned}$$

as $k \to \infty$. □

4. Some Converses of the Above Results. The Weighted Case

We shall start with the following converse inequality:

Theorem 4. *Suppose that ϕ, p and f are as above and $(U_k)_{k\in\mathbb{N}}$ are strictly positive. Then we have the inequality:*

$$\begin{aligned}
(4.1)\qquad 0 &\leq \frac{\int_a^b \cdots \int_a^b \phi\left(\frac{u_1 f(t_1)+\ldots+u_k f(t_k)}{U_k}\right) p(t_1)\ldots p(t_k)\,dt_1\ldots dt_k}{\left(\int_a^b p(t)\,dt\right)^k} \\
&\quad -\phi\left(\frac{\int_a^b f(t)p(t)\,dt}{\int_a^b p(t)\,dt}\right) \\
&\leq \frac{\int_a^b \cdots \int_a^b \phi'_+\left(\frac{u_1 f(t_1)+\ldots+u_k f(t_k)}{U_k}\right) f(t_1)\,p(t_1)\ldots p(t_k)\,dt_1\ldots dt_k}{\left(\int_a^b p(t)\,dt\right)^k} \\
&\quad -\frac{\int_a^b f(t)p(t)\,dt}{\int_a^b p(t)\,dt}\cdot\frac{\int_a^b \cdots \int_a^b \phi'_+\left(\frac{u_1 f(t_1)+\ldots+u_k f(t_k)}{U_k}\right) p(t_1)\ldots p(t_k)\,dt_1\ldots dt_k}{\left(\int_a^b p(t)\,dt\right)^k},
\end{aligned}$$

where $\phi'_+(\cdot)$ is the right derivative of ϕ on $\mathring{I}$.

Proof. As ϕ is convex on I, we can write that:

$$\phi(x)-\phi(y) \geq \phi'_+(y)(x-y) \text{ for all } x, y \in \mathring{I}.$$

Choosing

$$x = \frac{\int_a^b f(t)p(t)\,dt}{\int_a^b p(t)\,dt}$$

and

$$y = \frac{u_1 f(t_1)+\ldots+u_k f(t_k)}{U_k}$$

we deduce the inequality:

$$\begin{aligned} & \phi\left(\frac{\int_a^b f(t)\,p(t)\,dt}{\int_a^b p(t)\,dt}\right)-\phi\left(\frac{u_1 f(t_1)+\ldots+u_k f(t_k)}{U_k}\right) \\ \geq\ & \phi'_+\left(\frac{u_1 f(t_1)+\ldots+u_k f(t_k)}{U_k}\right)\left(\frac{\int_a^b f(t)\,p(t)\,dt}{\int_a^b p(t)\,dt}-\frac{u_1 f(t_1)+\ldots+u_k f(t_k)}{U_k}\right) \end{aligned}$$

for all $t_1,\ldots,t_k$ in $[a,b]$.
Integrating this inequality on $[a,b]^n$ we can easily deduce the inequality (4.1). We shall omit the details. □

The result embodied in (4.1) can be completed by the following one:

Theorem 5. *With the above assumptions, one has the inequality:*

$$\begin{aligned} 0 \leq\ & \frac{\int_a^b\cdots\int_a^b \phi'_+\left(\frac{u_1 f(t_1)+\ldots+u_k f(t_k)}{U_k}\right) f(t_1)\,p(t_1)\ldots p(t_k)\,dt_1\ldots dt_k}{\left(\int_a^b p(t)\,dt\right)^k} \\ & -\frac{\int_a^b f(t)\,p(t)\,dt}{\int_a^b p(t)\,dt}\,\frac{\int_a^b\cdots\int_a^b \phi'_+\left(\frac{u_1 f(t_1)+\ldots+u_k f(t_k)}{U_k}\right) p(t_1)\ldots p(t_k)\,dt_1\ldots dt_k}{\left(\int_a^b p(t)\,dt\right)^k} \\ \leq\ & \sqrt{\frac{\sum_{i=1}^k u_i^2}{U_k^2}}\left[\frac{1}{\left(\int_a^b p(t)\,dt\right)^k}\right. \\ & \left.\times\int_a^b\cdots\int_a^b\left[\phi'_+\left(\frac{u_1 f(t_1)+\ldots+u_k f(t_k)}{U_k}\right)\right]^2 p(t_1)\ldots p(t_k)\,dt_1\ldots dt_k\right]^{\frac{1}{2}} \\ & \times\left[\frac{\int_a^b f^2(t)\,p(t)\,dt}{\int_a^b p(t)\,dt}-\left(\frac{\int_a^b f(t)\,p(t)\,dt}{\int_a^b p(t)\,dt}\right)^2\right]^{\frac{1}{2}}. \end{aligned} \tag{4.2}$$

Proof. By Schwartz's inequality for multiple integrals, we can write :

$$\begin{aligned} & \frac{1}{\left(\int_a^b p(t)\,dt\right)^k}\int_a^b\cdots\int_a^b \phi'_+\left(\frac{u_1 f(t_1)+\ldots+u_k f(t_k)}{U_k}\right) \\ & \times\left(\frac{u_1 f(t_1)+\ldots+u_k f(t_k)}{U_k}-\frac{\int_a^b f(t)\,p(t)\,dt}{\int_a^b p(t)\,dt}\right) p(t_1)\ldots p(t_k)\,dt_1\ldots dt_k \end{aligned} \tag{4.3}$$

$$\leq \left\{ \frac{1}{\left(\int_a^b p(t)\,dt\right)^k} \times \int_a^b \cdots \int_a^b \left[\phi'_+\left(\frac{u_1 f(t_1) + \dots + u_k f(t_k)}{U_k}\right)\right]^2 p(t_1)\dots p(t_k)\,dt_1 \dots dt_k \right\}^{\frac{1}{2}}$$

$$\times \left\{ \frac{1}{\left(\int_a^b p(t)\,dt\right)^k} \times \int_a^b \cdots \int_a^b \left[\frac{u_1 f(t_1) + \dots + u_k f(t_k)}{U_k} - \frac{\int_a^b f(t)\,p(t)\,dt}{\int_a^b p(t)\,dt}\right]^2 p(t_1)\dots p(t_k)\,dt_1 \dots dt_k \right\}^{\frac{1}{2}}.$$

Now, let us compute the integral

$$\begin{aligned}
& I(u) \\
: &= \int_a^b \cdots \int_a^b \left[\frac{u_1 f(t_1) + \dots + u_k f(t_k)}{U_k} - \frac{\int_a^b f(t)\,p(t)\,dt}{\int_a^b p(t)\,dt}\right]^2 p(t_1)\dots p(t_k)\,dt_1 \dots dt_k \\
&= \int_a^b \cdots \int_a^b \frac{\left(u_1 f(t_1) + \dots + u_k f(t_k)\right)^2}{U_k^2} p(t_1)\dots p(t_k)\,dt_1 \dots dt_k \\
& \quad -2\frac{\int_a^b f(t)\,p(t)\,dt}{\int_a^b p(t)\,dt} \int_a^b \cdots \int_a^b \left(\frac{u_1 f(t_1) + \dots + u_k f(t_k)}{U_k}\right) p(t_1)\dots p(t_k)\,dt_1 \dots dt_k \\
& \quad + \left(\frac{\int_a^b f(t)\,p(t)\,dt}{\int_a^b p(t)\,dt}\right)^2 \left(\int_a^b p(t)\,dt\right) \\
&= \frac{1}{U_k^2}\left[\int_a^b \cdots \int_a^b \sum_{i=1}^k u_i^2 f^2(t_i)\,p(t_1)\dots p(t_k)\,dt_1 \dots dt_k \right. \\
& \quad \left. + 2\sum_{1\leq i<j\leq k} u_i u_j \int_a^b \cdots \int_a^b f(t_i)\,f(t_j)\,p(t_1)\dots p(t_k)\,dt_1 \dots dt_k \right] \\
& \quad -2\frac{\int_a^b f(t)\,p(t)\,dt}{\int_a^b p(t)\,dt} \cdot \left(\int_a^b p(t)\,dt\right)^{k-1} \cdot \frac{U_k \int_a^b f(t)\,p(t)\,dt}{U_k} \\
& \quad + \left(\frac{\int_a^b f(t)\,p(t)\,dt}{\int_a^b p(t)\,dt}\right)^2 \left(\int_a^b p(t)\,dt\right)^k \\
&= \left[\frac{1}{U_k^2}\sum_{i=1}^k u_i^2 \left(\int_a^b p(t)\,dt\right)^{k-1} \int_a^b f^2(t)\,p(t)\,dt \right. \\
& \quad \left. + 2\sum_{1\leq i<j\leq k} u_i u_j \left(\int_a^b f(t)\,p(t)\,dt\right)^2 \left(\int_a^b p(t)\,dt\right)^{k-2}\right]
\end{aligned}$$

$$-2\left(\int_a^b f(t)\,p(t)\,dt\right)^2\left(\int_a^b p(t)\,dt\right)^{k-2}+\left(\int_a^b f(t)\,p(t)\,dt\right)^2\left(\int_a^b p(t)\,dt\right)^{k-2}$$

$$=\frac{1}{U_k^2}\left[\left(\sum_{i=1}^k u_i^2\right)\int_a^b f^2(t)\,p(t)\,dt\int_a^b p(t)\,dt\right.$$

$$\left.+\,2\left(\sum_{1\le i<j\le k}u_iu_j\right)\left(\int_a^b f(t)\,p(t)\,dt\right)^2\right]\left(\int_a^b p(t)\,dt\right)^{k-2}$$

$$-\left(\int_a^b f(t)\,p(t)\,dt\right)^2\left(\int_a^b p(t)\,dt\right)^{k-2}$$

$$=\frac{\left(\int_a^b p(t)\,dt\right)^{k-2}}{U_k^2}\times\left[\left(\sum_{i=1}^k u_i^2\right)\int_a^b f^2(t)\,p(t)\,dt\int_a^b p(t)\,dt\right.$$

$$\left.+\,2\left(\sum_{1\le i<j\le k}u_iu_j\right)\left(\int_a^b f(t)\,p(t)\,dt\right)^2-U_k^2\left(\int_a^b f(t)\,p(t)\,dt\right)^2\right]$$

$$=\frac{\sum_{i=1}^k u_i^2}{U_k^2}\left(\int_a^b p(t)\,dt\right)^{k-2}\left[\int_a^b f^2(t)\,p(t)\,dt\int_a^b p(t)\,dt-\left(\int_a^b f(t)\,p(t)\,dt\right)^2\right].$$

Now, using the inequality (4.3) we deduce (4.2). □

The above theorem has the following important corollary:

Corollary 3. *Assume that $\phi : I \to \mathbb{R}$ is convex on the interval I and $\sup_{x\in I}\left|\phi'_+(x)\right| = M < \infty$. Then with the above assumptions for f and p and if:*

$$\lim_{k\to\infty}\frac{u_1^2+...+u_k^2}{(u_1+...+u_2)^2}=0,\ u_i\ge 0\,(i\in\mathbb{N})$$

then we have:

$$\lim_{k\to\infty}\frac{\int_a^b...\int_a^b\phi\left(\frac{u_1f(t_1)+...+u_kf(t_k)}{U_k}\right)p(t_1)...p(t_k)\,dt_1...dt_k}{\left(\int_a^b p(t)\,dt\right)^k}=\phi\left(\frac{\int_a^b f(t)\,p(t)\,dt}{\int_a^b p(t)\,dt}\right).$$

5. A Bound for the Difference $\varphi_k - \varphi_{k+1}$

In this section, we investigate the difference

$$\varphi_k-\varphi_{k+1},\ k\ge 1$$

where:

$$\varphi_k:=\frac{\int_a^b...\int_a^b\phi\left(\frac{f(t_1)+...+f(t_k)}{k}\right)p(t_1)...p(t_k)\,dt_1...dt_k}{\left(\int_a^b p(t)\,dt\right)^k},k\ge 1.$$

We know (see the inequality (1.1)) that:

$$\phi\left(\frac{\int_a^b f(t)\,p(t)\,dt}{\int_a^b p(t)\,dt}\right)\le\varphi_{k+1}\le\varphi_k\le...\le\varphi_1=\frac{\int_a^b f(t)\,p(t)\,dt}{\int_a^b p(t)\,dt}\tag{5.1}$$

for all $k \in \mathbb{N}$, $k \geq 1$.

The following theorem holds:

Theorem 6. *Suppose that* $\phi : I \subseteq \mathbb{R} \to \mathbb{R}$ *is a convex mapping on the interval* I, p *is nonnegative on* $[a, b]$, $\int_a^b p(t)\,dt > 0$, *and* $f : [a, b] \to I$ *is continuous on* $[a, b]$. *Then we have the inequality:*

$$
\begin{aligned}
(5.2)\qquad & 0 \leq \varphi_k - \varphi_{k+1} \\
& \leq \frac{1}{k+1} \frac{1}{\left[\int_a^b p(t)\,dt\right]^{k+1}} \\
& \times \left[\int_a^b \dots \int_a^b \phi'_+ \left(\frac{f(t_1) + \dots + f(t_k)}{k}\right) f(t_1)\, p(t_1) \dots p(t_k)\, dt_1 \dots dt_k\right. \\
& \times \int_a^b p(t)\,dt \\
& - \int_a^b \dots \int_a^b \phi'_+ \left(\frac{f(t_1) + \dots + f(t_k)}{k}\right) p(t_1) \dots p(t_k)\, dt_1 \dots dt_k \\
& \left.\times \int_a^b f(t)\, p(t)\,dt\right]
\end{aligned}
$$

for all $k \geq 1$.

Proof. As ϕ is convex, we can write:

$$
\begin{aligned}
& \phi\left(\frac{f(t_1) + \dots + f(t_{k+1})}{k+1}\right) - \phi\left(\frac{f(t_1) + \dots + f(t_k)}{k}\right) \\
\geq\ & \phi'_+ \left(\frac{f(t_1) + \dots + f(t_k)}{k}\right) \left(\frac{f(t_1) + \dots + f(t_{k+1})}{k+1} - \frac{f(t_1) + \dots + f(t_k)}{k}\right) \\
=\ & \phi'_+ \left(\frac{f(t_1) + \dots + f(t_k)}{k}\right) \left[\frac{k f(t_{k+1}) - (f(t_1) + \dots + f(t_k))}{k(k+1)}\right]
\end{aligned}
$$

for all $t_1, \dots, t_{k+1} \in [a, b]$ and $k \geq 1$.

Integrating this inequality on $[a, b]^{k+1}$ we can write:

$$
\begin{aligned}
& \frac{1}{\left(\int_a^b p(t)\,dt\right)^{k+1}} \int_a^b \dots \int_a^b \phi\left(\frac{f(t_1) + \dots + f(t_{k+1})}{k+1}\right) p(t_1) \dots p(t_k)\, dt_1 \dots dt_k \\
& - \frac{1}{\left(\int_a^b p(t)\,dt\right)^{k}} \int_a^b \dots \int_a^b \phi\left(\frac{f(t_1) + \dots + f(t_k)}{k}\right) p(t_1) \dots p(t_k)\, dt_1 \dots dt_k
\end{aligned}
$$

$$\geq \frac{1}{k+1}\left[\frac{1}{\left(\int_a^b p(t)\,dt\right)^k}\right.$$
$$\times \int_a^b \cdots \int_a^b \phi'_+\left(\frac{f(t_1)+\dots+f(t_k)}{k}\right) p(t_1)\dots p(t_k)\,dt_1\dots dt_k \frac{\int_a^b f(t)\,p(t)\,dt}{\int_a^b p(t)\,dt}$$
$$-\frac{1}{\left(\int_a^b p(t)\,dt\right)^{k+1}} \int_a^b \cdots \int_a^b \phi'_+\left(\frac{f(t_1)+\dots+f(t_k)}{k}\right)$$
$$\left.\times \left(\frac{f(t_1)+\dots+f(t_k)}{k}\right) p(t_1)\dots p(t_{k+1})\,dt_1\dots dt_{k+1}\right]$$

$$= \frac{1}{k+1}\left[\frac{1}{\left(\int_a^b p(t)\,dt\right)^k}\right.$$
$$\times \int_a^b \cdots \int_a^b \phi'_+\left(\frac{f(t_1)+\dots+f(t_k)}{k}\right) p(t_1)\dots p(t_k)\,dt_1\dots dt_k \frac{\int_a^b f(t)\,p(t)\,dt}{\int_a^b p(t)\,dt}$$
$$\left.-\frac{1}{\left(\int_a^b p(t)\,dt\right)^{k}} \int_a^b \cdots \int_a^b \phi'_+\left(\frac{f(t_1)+\dots+f(t_k)}{k}\right) p(t_1)\dots p(t_k)\,dt_1\dots dt_k\right].$$

Thus, the inequality (5.1) is proved. □

The above result can be completed by the following one:

Theorem 7. *With the above assumptions one has*

$$(5.3)\quad \frac{1}{k+1}\frac{1}{\left[\int_a^b p(t)\,dt\right]^{k+1}}$$
$$\times\left[\int_a^b \cdots \int_a^b \phi'_+\left(\frac{f(t_1)+\dots+f(t_k)}{k}\right) f(t_1)\,p(t_1)\dots p(t_k)\,dt_1\dots dt_k \int_a^b p(t)\,dt\right.$$
$$\left.-\int_a^b \cdots \int_a^b \phi'_+\left(\frac{f(t_1)+\dots+f(t_k)}{k}\right) p(t_1)\dots p(t_k)\,dt_1\dots dt_k \int_a^b f(t)\,p(t)\,dt\right]$$

$$\leq c_k\left[\frac{1}{\left(\int_a^b p(t)\,dt\right)^k}\right.$$
$$\left.\times \int_a^b \cdots \int_a^b \left[\phi'_+\left(\frac{f(t_1)+\dots+f(t_k)}{k}\right)\right]^2 p(t_1)\dots p(t_k)\,dt_1\dots dt_k \int_a^b p(t)\,dt\right]^{\frac{1}{2}}$$
$$\times\left[\frac{\int_a^b f^2(t)\,p(t)\,dt}{\int_a^b p(t)\,dt} - \left(\frac{\int_a^b f(t)\,p(t)\,dt}{\int_a^b p(t)\,dt}\right)^2\right]^{\frac{1}{2}},$$

where

$$c_k = \frac{1}{\sqrt{k(k+1)}}.$$

Proof. It can easily be seen that the first membership of the inequality (5.3) is in fact:

$$\begin{aligned} &\frac{1}{k+1} \frac{1}{\left(\int_a^b p(t)\,dt\right)^{k+1}} \int_a^b \cdots \int_a^b \phi'_+ \left(\frac{f(t_1) + \ldots + f(t_k)}{k} \right) \\ &\times \left[f(t_{k+1}) - \frac{f(t_1) + \ldots + f(t_k)}{k} \right] p(t_1) \ldots p(t_{k+1})\,dt_1 \ldots dt_{k+1}. \end{aligned}$$

Using the Cauchy-Buniakowsky-Schwartz inequality for multiple integrals we get that

$$\begin{aligned} &\frac{1}{\left(\int_a^b p(t)\,dt\right)^{k+1}} \int_a^b \cdots \int_a^b \phi'_+ \left(\frac{f(t_1) + \ldots + f(t_k)}{k} \right) \\ &\times \left[f(t_{k+1}) - \frac{f(t_1) + \ldots + f(t_k)}{k} \right] p(t_1) \ldots p(t_{k+1})\,dt_1 \ldots dt_{k+1} \int_a^b p(t)\,dt \\ \leq\ & \left(\frac{1}{\left(\int_a^b p(t)\,dt\right)^{k+1}} \right. \\ &\left. \times \int_a^b \cdots \int_a^b \left[\phi'_+ \left(\frac{f(t_1) + \ldots + f(t_k)}{k} \right) \right]^2 p(t_1) \ldots p(t_{k+1})\,dt_1 \ldots dt_{k+1} \right)^{\frac{1}{2}} \\ &\times \left(\frac{1}{\left(\int_a^b p(t)\,dt\right)^{k+1}} \right. \\ &\left. \times \int_a^b \cdots \int_a^b \left[f(t_{k+1}) - \frac{f(t_1) + \ldots + f(t_k)}{k} \right]^2 p(t_1) \ldots p(t_{k+1})\,dt_1 \ldots dt_{k+1} \right)^{\frac{1}{2}}. \end{aligned}$$

Let us compute the integral:

$$\begin{aligned} I \ :&= \int_a^b \cdots \int_a^b \left[f(t_{k+1}) - \frac{f(t_1) + \ldots + f(t_k)}{k} \right]^2 p(t_1) \ldots p(t_{k+1})\,dt_1 \ldots dt_{k+1} \\ &= \int_a^b \cdots \int_a^b \left(f^2(t_{k+1}) - 2f(t_{k+1}) \cdot \frac{f(t_1) + \ldots + f(t_k)}{k} \right. \\ &\quad \left. + \left(\frac{f(t_1) + \ldots + f(t_k)}{k} \right)^2 \right) p(t_1) \ldots p(t_{k+1})\,dt_1 \ldots dt_{k+1} \end{aligned}$$

$$= \left(\int_a^b p(t)\,dt\right)^k \int_a^b f^2(t)\,p(t)\,dt - 2\left(\int_a^b f(t)\,p(t)\,dt\right)^2 \left(\int_a^b p(t)\,dt\right)^{k-1}$$

$$+\frac{1}{k^2}\left[\int_a^b ... \int_a^b \left(f^2(t_1) + ... + f^2(t_k)\right) p(t_1)...p(t_{k+1})\,dt_1...dt_{k+1}\right.$$

$$\left.+ 2\sum_{1\le i<j\le k} \int_a^b ... \int_a^b f(t_i)\,f(t_j)\,p(t_1)...p(t_{k+1})\,dt_1...dt_{k+1}\right]$$

$$= \left(\int_a^b p(t)\,dt\right)^{k-1}\left[\int_a^b p(t)\,dt \int_a^b f^2(t)\,p(t)\,dt - 2\left(\int_a^b f(t)\,p(t)\,dt\right)^2\right.$$

$$\left.+\frac{1}{k}\left(k\int_a^b f^2(t)\,p(t)\,dt \int_a^b p(t)\,dt + \frac{2k(k-1)}{2}\left(\int_a^b f(t)\,p(t)\,dt\right)^2\right)\right]$$

$$= \frac{\left(\int_a^b p(t)\,dt\right)^{k-1}}{k}\left[k\int_a^b p(t)\,dt \int_a^b f^2(t)\,p(t)\,dt - 2k\left(\int_a^b f(t)\,p(t)\,dt\right)^2\right.$$

$$\left.+\int_a^b f^2(t)\,p(t)\,dt \int_a^b p(t)\,dt + \frac{(2k-2)}{2}\left(\int_a^b f(t)\,p(t)\,dt\right)^2\right]$$

$$= \frac{\left(\int_a^b p(t)\,dt\right)^{k-1}}{k}\left[(k+1)\int_a^b p(t)\,dt \int_a^b f^2(t)\,p(t)\,dt\right.$$

$$\left.-(k+1)\left(\int_a^b f(t)\,p(t)\,dt\right)^2\right]$$

$$= \frac{(k+1)}{k}\left(\int_a^b p(t)\,dt\right)^{k-1}\left[\int_a^b p(t)\,dt \int_a^b f^2(t)\,p(t)\,dt - \left(\int_a^b f(t)\,p(t)\,dt\right)^2\right]$$

Now, using the above inequality we deduce the desired result. □

The following corollary is interesting:

Corollary 4. *Let ϕ, f and p be as above and assume that $M := \sup_{x\in I}\left|\phi'_+(x)\right| < \infty$. Then we have:*

$$\lim_{n\to\infty}(\varphi_n - \varphi_{n+1})\,n^\alpha = 0 \text{ for all } 0 \le \alpha < 1.$$

Proof. By the above two theorems we have:

$$0 \le \varphi_n - \varphi_{n+1} \le \frac{M}{\sqrt{n(n+1)}}\left[\frac{\int_a^b f^2(t)\,p(t)\,dt}{\int_a^b p(t)\,dt} - \left(\frac{\int_a^b f(t)\,p(t)\,dt}{\int_a^b p(t)\,dt}\right)^2\right]^{\frac{1}{2}}$$

for all $n \ge 1$ which proves the statement. □

6. A Bound for the Difference $\varphi_k(u) - \varphi_k$

Now, for the mappings ϕ, p and f as above and for the positive weights $(u_i)_{i\in\mathbb{N}}$ we can introduce the following sequence:

$$\varphi_k(u) := \frac{1}{\left(\int_a^b p(t)\,dt\right)^k} \int_a^b \cdots \int_a^b \phi\left(\frac{u_1 f(t_1) + \ldots + u_k f(t_k)}{U_k}\right) p(t_1) \ldots p(t_k)\,dt_1 \ldots dt_k.$$

For $u_i = 1, i \in \mathbb{N}$ we have

$$\varphi_k(\mathbb{I}) = \varphi_k,$$

where $\mathbb{I} = (1, 1, \ldots, 1, \ldots)$ and, by Theorem 1 we have the inequalities:

$$\phi\left(\frac{\int_a^b f(t)\,p(t)\,dt}{\int_a^b p(t)\,dt}\right) \le \varphi_k \le \varphi_k(u) \le \frac{\int_a^b \phi(f(t))\,p(t)\,dt}{\int_a^b p(t)\,dt} \tag{6.1}$$

for all $u = (u_i), i \in \mathbb{N}$

It is natural to ask what happens to the difference $\varphi_k(u) - \varphi_k$.

Theorem 8. *With the above assumptions for the mappings ϕ, p and f one has the inequality:*

$$\begin{aligned} 0 &\le \varphi_k(u) - \varphi_k \\ &\le \frac{1}{\left(\int_a^b p(t)\,dt\right)^k} \int_a^b \cdots \int_a^b \phi'_+\left(\frac{u_1 f(t_1) + \ldots + u_k f(t_k)}{U_k}\right) \\ &\quad \times \left(\frac{u_1 f(t_1) + \ldots + u_k f(t_k)}{U_k}\right) p(t_1) \ldots p(t_k)\,dt_1 \ldots dt_k \\ &\quad - \frac{1}{\left(\int_a^b p(t)\,dt\right)^k} \int_a^b \cdots \int_a^b \phi'_+\left(\frac{u_1 f(t_1) + \ldots + u_k f(t_k)}{U_k}\right) \\ &\quad \times \left(\frac{f(t_1) + \ldots + f(t_k)}{k}\right) p(t_1) \ldots p(t_k)\,dt_1 \ldots dt_k. \end{aligned} \tag{6.2}$$

for all $k > 0$.

Proof. By the convexity of ϕ we can write:

$$\begin{aligned} &\phi\left(\frac{f(t_1) + \ldots + f(t_k)}{k}\right) - \phi\left(\frac{u_1 f(t_1) + \ldots + u_k f(t_k)}{U_k}\right) \\ \ge\ &\phi'_+\left(\frac{u_1 f(t_1) + \ldots + u_k f(t_k)}{U_k}\right) \\ &\times \left(\frac{f(t_1) + \ldots + f(t_k)}{k} - \frac{u_1 f(t_1) + \ldots + u_k f(t_k)}{U_k}\right) \end{aligned}$$

for all $t_i \in [a, b]$.

Integrating the above inequality on $[a,b]^k$, we have

$$\begin{aligned}
& \frac{1}{\left(\int_a^b p(t)\,dt\right)^k}\int_a^b \cdots \int_a^b \phi\left(\frac{f(t_1)+\ldots+f(t_k)}{k}\right) p(t_1)\ldots p(t_k)\,dt_1\ldots dt_k \\
& -\frac{1}{\left(\int_a^b p(t)\,dt\right)^k}\int_a^b \cdots \int_a^b \phi\left(\frac{u_1 f(t_1)+\ldots+u_k f(t_k)}{U_k}\right) p(t_1)\ldots p(t_k)\,dt_1\ldots dt_k \\
\geq\ & \frac{1}{\left(\int_a^b p(t)\,dt\right)^k}\int_a^b \cdots \int_a^b \phi'_+\left(\frac{u_1 f(t_1)+\ldots+u_k f(t_k)}{U_k}\right) \\
& \times\left(\frac{f(t_1)+\ldots+f(t_k)}{k}\right) p(t_1)\ldots p(t_k)\,dt_1\ldots dt_k \\
& -\frac{1}{\left(\int_a^b p(t)\,dt\right)^k}\int_a^b \cdots \int_a^b \phi'_+\left(\frac{u_1 f(t_1)+\ldots+u_k f(t_k)}{U_k}\right) \\
& \times\left(\frac{u_1 f(t_1)+\ldots+u_k f(t_k)}{U_k}\right) p(t_1)\ldots p(t_k)\,dt_1\ldots dt_k
\end{aligned}$$

and the estimation (6.2) is proved. □

Theorem 9. *With the above assumptions for ϕ, p, f, and u, one has the estimation:*

$$\begin{aligned}
& \frac{1}{\left(\int_a^b p(t)\,dt\right)^k}\int_a^b \cdots \int_a^b \phi'_+\left(\frac{u_1 f(t_1)+\ldots+u_k f(t_k)}{U_k}\right) \\
& \times\left(\frac{u_1 f(t_1)+\ldots+u_k f(t_k)}{U_k}\right) p(t_1)\ldots p(t_k)\,dt_1\ldots dt_k \\
& -\frac{1}{\left(\int_a^b p(t)\,dt\right)^k}\int_a^b \cdots \int_a^b \phi'_+\left(\frac{u_1 f(t_1)+\ldots+u_k f(t_k)}{U_k}\right) \\
& \times\left(\frac{f(t_1)+\ldots+f(t_k)}{k}\right) p(t_1)\ldots p(t_k)\,dt_1\ldots dt_k \\
\leq\ & \sqrt{\frac{\sum_{i=1}^k (ku_i-U_k)^2}{k^2U_k^2}}\left[\frac{\int_a^b f^2(t)\,p(t)\,dt}{\int_a^b p(t)\,dt}-\left(\frac{\int_a^b f(t)\,p(t)\,dt}{\int_a^b p(t)\,dt}\right)^2\right]^{\frac{1}{2}} \\
& \times\left[\frac{1}{\left(\int_a^b p(t)\,dt\right)^k}\right. \\
& \left.\times\int_a^b \cdots \int_a^b \left[\phi'_+\left(\frac{u_1 f(t_1)+\ldots+u_k f(t_k)}{U_k}\right)\right]^2 p(t_1)\ldots p(t_k)\,dt_1\ldots dt_k\right]^{\frac{1}{2}}.
\end{aligned} \tag{6.3}$$

Proof. The first membership of the above inequality is in fact:

$$\begin{aligned} I &= \frac{1}{\left(\int_a^b p(t)\,dt\right)^k}\int_a^b \cdots \int_a^b \phi'_+\left(\frac{u_1 f(t_1)+\dots+u_k f(t_k)}{U_k}\right) \\ &\quad\times\left(\frac{u_1 f(t_1)+\dots+u_k f(t_k)}{U_k}-\frac{f(t_1)+\dots+f(t_k)}{k}\right)p(t_1)\dots p(t_k)\,dt_1\dots dt_k. \end{aligned}$$

By the Schwartz-Buniakowsky-Cauchy integral inequality we have:

$$\begin{aligned} I &\le \\ &\left(\frac{1}{\left(\int_a^b p(t)\,dt\right)^k}\right. \\ &\left.\times\int_a^b \cdots \int_a^b \left[\phi'_+\left(\frac{u_1 f(t_1)+\dots+u_k f(t_k)}{U_k}\right)\right]^2 p(t_1)\dots p(t_k)\,dt_1\dots dt_k\right)^{\frac{1}{2}} \\ &\times\left(\frac{1}{\left(\int_a^b p(t)\,dt\right)^k}\int_a^b \cdots \int_a^b \left[\frac{u_1 f(t_1)+\dots+u_k f(t_k)}{U_k}\right.\right. \\ &\left.\left.-\frac{f(t_1)+\dots+f(t_k)}{k}\right]^2 p(t_1)\dots p(t_k)\,dt_1\dots dt_k\right)^{\frac{1}{2}}. \end{aligned}$$

We must compute the integral:

$$\begin{aligned} J_k &= \int_a^b \cdots \int_a^b \left[\frac{u_1 f(t_1)+\dots+u_k f(t_k)}{U_k}-\frac{f(t_1)+\dots+f(t_k)}{k}\right]^2 p(t_1)\dots p(t_k)\,dt_1\dots dt_k \\ &= \int_a^b \cdots \int_a^b \left[\frac{\sum_{i=1}^k (ku_i-U_k)f(t_i)}{U_k k}\right]^2 p(t_1)\dots p(t_k)\,dt_1\dots dt_k \\ &= \frac{1}{k^2U_k^2}\int_a^b \cdots \int_a^b \sum_{i,j=1}^k (ku_i-U_k)(ku_j-U_k)f(t_i)f(t_j)p(t_1)\dots p(t_k)\,dt_1\dots dt_k \\ &= \frac{1}{k^2U_k^2}\left[\int_a^b \cdots \int_a^b \sum_{i=1}^k (ku_i-U_k)^2 f^2(t_i)p(t_1)\dots p(t_k)\,dt_1\dots dt_k\right. \\ &\quad\left.+2\int_a^b \cdots \int_a^b \sum_{1\le i<j\le k}(ku_i-U_k)(ku_j-U_k)f(t_i)f(t_j)p(t_1)\dots p(t_k)\,dt_1\dots dt_k\right] \\ &= \frac{1}{k^2U_k^2}\left[\sum_{i=1}^k (ku_i-U_k)^2\left(\int_a^b p(t)\,dt\right)^{k-1}\int_a^b f^2(t)p(t)\,dt\right. \\ &\quad\left.+2\sum_{1\le i<j\le k}(ku_i-U_k)(ku_j-U_k)\left(\int_a^b f(t)p(t)\,dt\right)^2\left(\int_a^b p(t)\,dt\right)^{k-2}\right] \end{aligned}$$

$$
\begin{aligned}
&= \frac{\left(\int_a^b p(t)\,dt\right)^{k-2}}{k^2U_k^2} \\
&\quad\times\left[\sum_{i=1}^{k}(ku_i-U_k)^2\left[\left(\int_a^b p(t)\,dt\right)\int_a^b f^2(t)\,p(t)\,dt-\left(\int_a^b f(t)\,p(t)\,dt\right)^2\right]\right. \\
&\quad+\left(\int_a^b f(t)\,p(t)\,dt\right)^2 \\
&\quad\times\left.\left[\sum_{i=1}^{k}(ku_i-U_k)^2+2\sum_{1\le i<j\le k}(ku_i-U_k)(ku_j-U_k)\right]\right] \\
&= \frac{\left(\int_a^b p(t)\,dt\right)^{k-2}}{k^2U_k^2} \\
&\quad\times\left\{\sum_{i=1}^{k}(ku_i-U_k)^2\left[\int_a^b p(t)\,dt\int_a^b f^2(t)\,p(t)\,dt-\left(\int_a^b f(t)\,p(t)\,dt\right)^2\right]\right. \\
&\quad+\left.\left(\int_a^b f(t)\,p(t)\,dt\right)^2\left(\sum_{i=1}^{k}(ku_i-U_k)\right)^2\right\}.
\end{aligned}
$$

As a simple calculation shows us that:

$$
\sum_{i=1}^{k}(ku_i-U_k)=kU_k-kU_k=0,
$$

we deduce that

$$
\begin{aligned}
J_k &= \frac{\left(\int_a^b p(t)\,dt\right)^{k-2}}{k^2U_k^2}\sum_{i=1}^{k}(ku_i-U_k)^2 \\
&\quad\times\left[\int_a^b p(t)\,dt\int_a^b f^2(t)\,p(t)\,dt-\left(\int_a^b f(t)\,p(t)\,dt\right)^2\right].
\end{aligned}
$$

Using the above inequality we deduce the desired estimation from (6.3). The theorem is thus proved. □

The following corollary is important for our purposes.

Corollary 5. *With the above assumptions and if* $M := \sup_{x\in I}\left|\phi'_+(x)\right| < \infty$ *then we have the estimation:*

$$
\begin{aligned}
0 &\le \varphi_k(u)-\varphi_k \\
&\le M\sqrt{\frac{\sum_{i=1}^{k}(ku_i-U_k)^2}{k^2U_k^2}}\left[\frac{\int_a^b f^2(t)\,p(t)\,dt}{\int_a^b p(t)\,dt}-\left(\frac{\int_a^b f(t)\,p(t)\,dt}{\int_a^b p(t)\,dt}\right)^2\right]^{\frac{1}{2}}
\end{aligned}
$$

for all $k\ge 1$ *and* $u>0$.

By the use of the above corollary we can complete our convergence result from Corollary 3:

Corollary 6. *With the above assumptions for ϕ, f, p and if $u_k > 0\,(k \in \mathbb{N})$ are such that:*

$$\lim_{k\to\infty} \frac{\sum_{i=1}^{k} (ku_i - U_k)^2}{k^2 U_k^2} = 0$$

then we also have:

$$\lim_{k\to\infty} \varphi_k(u) = \phi\left(\frac{\int_a^b f(t)\,p(t)\,dt}{\int_a^b p(t)\,dt}\right).$$

7. Some Applications for Hadamard's Inequalities

Let $\phi : I \to \mathbb{R}$ be a convex mapping on the interval of real numbers I. The following double inequality is well-known in the literature as Hadamard's inequality:

$$\phi\left(\frac{a+b}{2}\right) \le \int_a^b \phi(x)\,dx \le \frac{\phi(a)+\phi(b)}{2}. \tag{7.1}$$

In paper [4], S.S. Dragomir, J. Pečarić and J. Sándor proved the following interpolation of the first inequality in (7.1):

$$\begin{aligned} &\phi\left(\frac{a+b}{2}\right) \\ \le\; & \frac{1}{(b-a)^{k+1}} \int_a^b \cdots \int_a^b \phi\left(\frac{x_1+\dots+x_{k+1}}{k+1}\right) dx_1 \dots dx_{k+1} \\ \le\; & \frac{1}{(b-a)^{k}} \int_a^b \cdots \int_a^b \phi\left(\frac{x_1+\dots+x_{k}}{k}\right) dx_1 \dots dx_{k} \\ \le\; & \dots\dots \le \frac{1}{b-a}\int_a^b \phi(x)\,dx \end{aligned} \tag{7.2}$$

for all $k \ge 1$. Note that this inequality is also a particular case of (1.1).

In paper [12], S.S. Dragomir pointed out the following weighted interpolation of Hadamard's first inequality:

$$\begin{aligned} \phi\left(\frac{a+b}{2}\right) &\le \frac{1}{(b-a)^{k}} \int_a^b \cdots \int_a^b \phi\left(\frac{q_1x_1+\dots+q_kx_k}{Q_k}\right) dx_1 \dots dx_{k} \\ &\le \frac{1}{b-a}\int_a^b \phi(x)\,dx \end{aligned} \tag{7.3}$$

where $q_i \ge 0\left(i = \overline{1,k}\right), Q_k := \sum_{i=1}^{k} q_i > 0$ and $k \ge 1$.
The last inequality was improved in the paper [19] by S.S. Dragomir and C. Buşe who proved among others that:

$$\begin{aligned} &\frac{1}{(b-a)^{k}} \int_a^b \cdots \int_a^b \phi\left(\frac{x_1+\dots+x_{k}}{k}\right) dx_1 \dots dx_{k} \\ \le\; & \frac{1}{(b-a)^{k}} \int_a^b \cdots \int_a^b \phi\left(\frac{q_1x_1+\dots+q_kx_k}{Q_k}\right) dx_1 \dots dx_{k} \end{aligned} \tag{7.4}$$

for all $q_i\left(i = \overline{1,k}\right)$ as above and $k \ge 1$.

Note that the above result follows now as a particular case of Theorem 1.

From Corollary 3 we can obtain the following result which was pointed out by another argument and in a more general case by C. Buşe, S.S. Dragomir and D. Barbu [20].

If $\phi : I \to \mathbb{R}$ is convex and $\sup_{x\in I} \left|\phi'_+(x)\right| = M < \infty$ and

$$\lim_{n\to\infty} \frac{q_1^2 + ... + q_n^2}{(q_1 + ... + q_n)^2} = 0 \, (q_i > 0, i \in \mathbb{N})$$

then we have the limit:

$$\begin{aligned} & \lim_{n\to\infty} \frac{1}{(b-a)^n} \int_a^b ... \int_a^b \phi\left(\frac{q_1x_1 + ... + q_nx_n}{Q_n}\right) dx_1...dx_n \\ = & \ \phi\left(\frac{a+b}{2}\right) \end{aligned}$$

Now, let us define the sequences:

$$h_n(q) := \frac{1}{(b-a)^n} \int_a^b ... \int_a^b \phi\left(\frac{q_1x_1 + ... + q_nx_n}{Q_n}\right) dx_1...dx_n$$

where $q = (q_i)_{i\in\mathbb{N}} \subset \mathbb{R}_+$ and

$$h_n := h_n(\mathbb{I}) \text{ where } \mathbb{I} = (1, 1, ..., 1, ...)$$

is a constant sequence.

Proposition 1. *Let $\phi : I \to \mathbb{R}$ be a convex mapping for which we have $M := \sup_{x\in I} \left|\phi'_+(x)\right| < \infty$. Then for all $a, b \in I$ with $a < b$ one has the converse inequality*

$$0 \le h_n - h_{n+1} \le \frac{M(b-a)}{2\sqrt{3}\sqrt{n(n+1)}}$$

for all $n \in \mathbb{N}$, $n \ge 1$.

Proof. As in the proof of Corollary 4 we have the inequality:

$$\begin{aligned} 0 & \le h_n - h_{n+1} \le \frac{M(b-a)}{\sqrt{n(n+1)}} \left[\frac{\int_a^b t^2 dt}{b-a} - \frac{\left(\int_a^b t dt\right)^2}{(b-a)^2}\right]^{\frac{1}{2}} \\ & = \frac{M(b-a)}{2\sqrt{3}\sqrt{n(n+1)}}. \end{aligned}$$

□

Remark 2. *The above inequality shows us that* $\lim_{n\to\infty} [n^\alpha (h_n - h_{n+1})] = 0$ *for* $\alpha \in [0, 1)$ *which is an improvement from the results of* [16].

The weighted case is embodied in the following proposition:

Proposition 2. *With the above assumptions one has the converse inequality*

$$0 \le h_n(q) - h_n \le \frac{M(b-a)}{2\sqrt{3}} \left[\frac{\sum_{i=1}^n (nq_i - Q_n)^2}{n^2 Q_n^2}\right]^{\frac{1}{2}}$$

for all $n \ge 1$ and $q_i > 0$ $\left(i = \overline{1, n}\right)$.

For other results connected with Hadamard's inequality see the recent papers [3], [7], [9], [12], [13], [16], [18], [19] and [20] where further references are given.

8. A Concrete Example

Suppose that $\phi:\mathbb{R}\to\mathbb{R}$, $\phi(x)=\exp(x)=e^{x}$. As above, we have:

$$
\begin{aligned}
l_n(q) &:= \frac{1}{(b-a)^n}\int_a^b\cdots\int_a^b \exp\left(\frac{q_1x_1+\dots+q_nx_n}{Q_n}\right)dx_1\dots dx_n \\
&= \frac{1}{(b-a)^n}\int_a^b\cdots\int_a^b \exp\left(\frac{q_1x_1}{Q_n}\right)\dots\exp\left(\frac{q_nx_n}{Q_n}\right)dx_1\dots dx_n \\
&= \prod_{i=1}^{n}\left[\frac{\int_a^b \exp\left(\frac{q_ix}{Q_n}\right)dx}{(b-a)}\right] \\
&= \frac{Q_n^n}{\prod_{i=1}^n q_i}\prod_{i=1}^{n}\left(\frac{\exp\left(\frac{q_i}{Q_n}b\right)-\exp\left(\frac{q_i}{Q_n}a\right)}{b-a}\right)
\end{aligned}
\tag{8.1}
$$

for all $a<b$, $a,b\in\mathbb{R}$ and

$$
l_n(n)=l_n=n^n\left(\frac{\exp\left(\frac{b}{n}\right)-\exp\left(\frac{a}{n}\right)}{b-a}\right)^n
\tag{8.2}
$$

By the use of the above theorem we can state the following results.

(1) The sequence l_n given in (8.2) is monotonic nonincreasing and

$$
\exp\left(\frac{a+b}{2}\right)=\lim_{n\to\infty}l_n=\inf_{n\in\mathbb{N}^*}l_n\le\frac{\exp(b)-\exp(a)}{b-a},\ a<b;
$$

(2) One has the inequality:

$$
\begin{aligned}
& n^n\left(\frac{\exp\left(\frac{b}{n}\right)-\exp\left(\frac{a}{n}\right)}{b-a}\right)^n \\
\le\ & \frac{Q_n^n}{\prod_{i=1}^n q_i}\prod_{i=1}^{n}\left(\frac{\exp\left(\frac{q_i}{Q_n}b\right)-\exp\left(\frac{q_i}{Q_n}a\right)}{b-a}\right) \\
\le\ & \frac{\exp(b)-\exp(a)}{b-a},\ a<b;
\end{aligned}
$$

(3) If $\dfrac{q_1^2+\dots+q_n^2}{(q_1+\dots+q_n)^2}\to 0$ then $l_n(q)\to 0$ as $n\to\infty$.

(4) One has the converse inequality

$$
\begin{aligned}
0 &\le n^n\left(\frac{\exp\left(\frac{b}{n}\right)-\exp\left(\frac{a}{n}\right)}{b-a}\right)^n-(n+1)^{n+1}\left(\frac{\exp\left(\frac{b}{n+1}\right)-\exp\left(\frac{a}{n+1}\right)}{b-a}\right)^{n+1} \\
&\le \frac{b(b-a)}{2\sqrt{3}\sqrt{n(n+1)}}\text{ for }a<b;
\end{aligned}
$$

(5) One has the converse inequality:

$$
\begin{aligned}
0 \quad &\leq \quad \frac{Q_n^n}{\prod_{i=1}^n q_i} \prod_{i=1}^n \left(\frac{\exp\left(\frac{q_i}{Q_n} b\right) - \exp\left(\frac{q_i}{Q_n} a\right)}{b-a} \right) - n^n \left(\frac{\exp\left(\frac{b}{n}\right) - \exp\left(\frac{a}{n}\right)}{b-a} \right)^n \\
&\leq \quad \frac{b\,(b-a)}{2\sqrt{3}} \left[\frac{\sum_{i=1}^n (nq_i - Q_n)^2}{n^2 Q_n^2} \right]^{\frac{1}{2}}
\end{aligned}
$$

for all $q_i > 0, i \in \mathbb{N}$.

Similar inequalities can be obtained for some other concrete mappings such as: $\phi(x) = \ln x$, $x > 0$, $\phi(x) = -\cos x$, $x \in \left[0, \frac{\pi}{2}\right]$, $\phi(x) = \cosh x$, $x \in \mathbb{R}$, etc., but we omit the details.

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