A Survey of Recent Reverses for the Generalised Triangle Inequality in Normed Spaces

A SURVEY OF RECENT REVERSES FOR THE GENERALISED TRIANGLE INEQUALITY IN NORMED SPACES

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Abstract. Recent reverses of the generalised triangle inequality in normed linear spaces that complement the classical results of Diaz and Metcalf are surveyed.

1. Introduction

In [2], Diaz and Metcalf established the following reverse of the generalised triangle inequality in real or complex normed linear spaces.

If $F : X \to \mathbb{K}$, $\mathbb{K} = \mathbb{R}, \mathbb{C}$ is a linear functional of a unit norm defined on the normed linear space X endowed with the norm $\|\cdot\|$ and the vectors $x_1, \dots, x_n$ satisfy the condition

$$0 \le r \le \operatorname{Re} F(x_i), \qquad i \in \{1, \dots, n\}; \tag{1.1}$$

then

$$r \sum_{i=1}^{n} \|x_i\| \le \left\| \sum_{i=1}^{n} x_i \right\|, \tag{1.2}$$

where equality holds if and only if both

$$F\left(\sum_{i=1}^{n} x_i \right) = r \sum_{i=1}^{n} \|x_i\| \tag{1.3}$$

and

$$F\left(\sum_{i=1}^{n} x_i \right) = \left\| \sum_{i=1}^{n} x_i \right\|. \tag{1.4}$$

If $X = H$, $(H; \langle \cdot, \cdot \rangle)$ is an inner product space and $F(x) = \langle x, e \rangle$, $\|e\| = 1$, then the condition (1.1) may be replaced with the simpler

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assumption

$$0 \leq r \|x_i\| \leq \operatorname{Re}\langle x_i, e\rangle, \qquad i = 1, \dots, n, \tag{1.5}$$

which implies the reverse of the generalised triangle inequality (1.2). In this case the equality holds in (1.2) if and only if [2]

$$\sum_{i=1}^{n} x_i = r\left(\sum_{i=1}^{n} \|x_i\|\right) e. \tag{1.6}$$

Let $F_1, \dots, F_m$ be linear functionals on X, each of unit norm. As in [2], let consider the real number c defined by

$$c = \sup_{x \neq 0}\left[\frac{\sum_{k=1}^{m} |F_k(x)|^2}{\|x\|^2}\right];$$

it then follows that $1 \leq c \leq m$. Suppose the vectors $x_1, \dots, x_k$ whenever $x_i \neq 0$, satisfy

$$0 \leq r_k \|x_i\| \leq \operatorname{Re} F_k(x_i), \qquad i = 1, \dots, n, \; k = 1, \dots, m. \tag{1.7}$$

Then one has the following reverse of the generalised triangle inequality [2]

$$\left(\frac{\sum_{k=1}^{m} r_k^2}{c}\right)^{1/2} \sum_{i=1}^{n} \|x_i\| \leq \left\|\sum_{i=1}^{n} x_i\right\|, \tag{1.8}$$

where equality holds if and only if both

$$F_k\left(\sum_{i=1}^{n} x_i\right) = r_k \sum_{i=1}^{n} \|x_i\|, \qquad k = 1, \dots, m \tag{1.9}$$

and

$$\sum_{k=1}^{m}\left[F_k\left(\sum_{i=1}^{n} x_i\right)\right]^2 = c\left\|\sum_{i=1}^{n} x_i\right\|^2. \tag{1.10}$$

If $X = H$, an inner product space, then, for $F_k(x) = \langle x, e_k\rangle$, where $\{e_k\}_{k=\overline{1,n}}$ is an orthonormal family in H, i.e., $\langle e_i, e_j\rangle = \delta_{ij}$, $i, j \in \{1, \dots, k\}$, δ_{ij} is Kronecker delta, the condition (1.7) may be replaced by

$$0 \leq r_k \|x_i\| \leq \operatorname{Re}\langle x_i, e\rangle, \qquad i = 1, \dots, n, \; k = 1, \dots, m; \tag{1.11}$$

implying the following reverse of the generalised triangle inequality

$$\left(\sum_{k=1}^{m} r_k^2\right)^{\frac{1}{2}} \sum_{i=1}^{n} \|x_i\| \leq \left\|\sum_{i=1}^{n} x_i\right\|, \tag{1.12}$$

where the equality holds if and only if

$$\sum_{i=1}^{n} x_i = \left(\sum_{i=1}^{n} \|x_i\|\right) \sum_{k=1}^{m} r_k e_k. \tag{1.13}$$

The aim of this paper is to survey recent reverses of the triangle inequality obtained by the authors in [6] and [7]. Their versions in inner product spaces are analysed and applications for complex numbers are given as well.

For various inequalities related to the triangle inequality, see Chapter XVII of the book [10] and the references therein.

2. Some Inequalities of Diaz-Metcalf Type for m Functionals

2.1. **The Case of Normed Spaces.** The following result may be stated [6].

Theorem 1. *Let* $(X, \|\cdot\|)$ *be a normed linear space over the real or complex number field* $\mathbb{K}$ *and* $F_k : X \to \mathbb{K}$, $k \in \{1, \dots, m\}$ *continuous linear functionals on* X. *If* $x_i \in X \backslash \{0\}$, $i \in \{1, \dots, n\}$ *are such that there exists the constant* $r_k \geq 0$, $k \in \{1, \dots, m\}$ *with* $\sum_{k=1}^{m} r_k > 0$ *and*

$$\operatorname{Re} F_k (x_i) \geq r_k \|x_i\| \tag{2.1}$$

for each $i \in \{1, \dots, n\}$ *and* $k \in \{1, \dots, m\}$, *then*

$$\sum_{i=1}^{n} \|x_i\| \leq \frac{\left\|\sum_{k=1}^{m} F_k\right\|}{\sum_{k=1}^{m} r_k} \left\|\sum_{i=1}^{n} x_i\right\|. \tag{2.2}$$

The case of equality holds in (2.2) if both

$$\left(\sum_{k=1}^{m} F_k\right)\left(\sum_{i=1}^{n} x_i\right) = \left(\sum_{k=1}^{m} r_k\right) \sum_{i=1}^{n} \|x_i\| \tag{2.3}$$

and

$$\left(\sum_{k=1}^{m} F_k\right)\left(\sum_{i=1}^{n} x_i\right) = \left\|\sum_{k=1}^{m} F_k\right\| \left\|\sum_{i=1}^{n} x_i\right\|. \tag{2.4}$$

Proof. Utilising the hypothesis (2.1) and the properties of the modulus, we have

$$I := \left| \left(\sum_{k=1}^{m} F_k \right) \left(\sum_{i=1}^{n} x_i \right) \right| \geq \left| \operatorname{Re} \left[\left(\sum_{k=1}^{m} F_k \right) \left(\sum_{i=1}^{n} x_i \right) \right] \right| \geq \sum_{k=1}^{m} \operatorname{Re} F_k \left(\sum_{i=1}^{n} x_i \right) = \sum_{k=1}^{m} \sum_{i=1}^{n} \operatorname{Re} F_k (x_i) \geq \left(\sum_{k=1}^{m} r_k \right) \sum_{i=1}^{n} \|x_i\| . \tag{2.5}$$

On the other hand, by the continuity property of F_k, $k \in \{1, \ldots, m\}$ we obviously have

$$I = \left| \left(\sum_{k=1}^{m} F_k \right) \left(\sum_{i=1}^{n} x_i \right) \right| \leq \left\| \sum_{k=1}^{m} F_k \right\| \left\| \sum_{i=1}^{n} x_i \right\| . \tag{2.6}$$

Making use of (2.5) and (2.6), we deduce the desired inequality (2.2).

Now, if (2.3) and (2.4) are valid, then, obviously, the case of equality holds true in the inequality (2.2).

Conversely, if the case of equality holds in (2.2), then it must hold in all the inequalities used to prove (2.2). Therefore we have

$$\operatorname{Re} F_k (x_i) = r_k \|x_i\| \tag{2.7}$$

for each $i \in \{1, \ldots, n\}$, $k \in \{1, \ldots, m\}$;

$$\sum_{k=1}^{m} \operatorname{Im} F_k \left(\sum_{i=1}^{n} x_i \right) = 0 \tag{2.8}$$

and

$$\sum_{k=1}^{m} \operatorname{Re} F_k \left(\sum_{i=1}^{n} x_i \right) = \left\| \sum_{k=1}^{m} F_k \right\| \left\| \sum_{i=1}^{n} x_i \right\| . \tag{2.9}$$

Note that, from (2.7), by summation over i and k, we get

$$\operatorname{Re} \left[\left(\sum_{k=1}^{m} F_k \right) \left(\sum_{i=1}^{n} x_i \right) \right] = \left(\sum_{k=1}^{m} r_k \right) \sum_{i=1}^{n} \|x_i\| . \tag{2.10}$$

Since (2.8) and (2.10) imply (2.3), while (2.9) and (2.10) imply (2.4) hence the theorem is proved. ∎

Remark 1. *If the norms* $\|F_k\|$, $k \in \{1, \dots, m\}$ *are easier to find, then, from (2.2), one may get the (coarser) inequality that might be more useful in practice:*

$$\sum_{i=1}^{n} \|x_i\| \leq \frac{\sum_{k=1}^{m} \|F_k\|}{\sum_{k=1}^{m} r_k} \left\| \sum_{i=1}^{n} x_i \right\|. \tag{2.11}$$

2.2. **The Case of Inner Product Spaces.** The case of inner product spaces, in which we may provide a simpler condition for equality, is of interest in applications [6].

Theorem 2. *Let* $(H; \langle \cdot, \cdot \rangle)$ *be an inner product space over the real or complex number field* $\mathbb{K}$, $e_k,\ x_i \in H \setminus \{0\}$, $k \in \{1, \dots, m\}$, $i \in \{1, \dots, n\}$. *If* $r_k \geq 0$, $k \in \{1, \dots, m\}$ *with* $\sum_{k=1}^{m} r_k > 0$ *satisfy*

$$\operatorname{Re} \langle x_i, e_k \rangle \geq r_k \|x_i\| \tag{2.12}$$

for each $i \in \{1, \dots, n\}$ *and* $k \in \{1, \dots, m\}$, *then*

$$\sum_{i=1}^{n} \|x_i\| \leq \frac{\left\| \sum_{k=1}^{m} e_k \right\|}{\sum_{k=1}^{m} r_k} \left\| \sum_{i=1}^{n} x_i \right\|. \tag{2.13}$$

The case of equality holds in (2.13) if and only if

$$\sum_{i=1}^{n} x_i = \frac{\sum_{k=1}^{m} r_k}{\left\| \sum_{k=1}^{m} e_k \right\|^2} \left(\sum_{i=1}^{n} \|x_i\| \right) \sum_{k=1}^{m} e_k. \tag{2.14}$$

Proof. By the properties of inner product and by (2.12), we have

$$\begin{aligned} &\left| \left\langle \sum_{i=1}^{n} x_i, \sum_{k=1}^{m} e_k \right\rangle \right| \\ &\geq \left| \sum_{k=1}^{m} \operatorname{Re} \left\langle \sum_{i=1}^{n} x_i, e_k \right\rangle \right| \geq \sum_{k=1}^{m} \operatorname{Re} \left\langle \sum_{i=1}^{n} x_i, e_k \right\rangle \\ &= \sum_{k=1}^{m} \sum_{i=1}^{n} \operatorname{Re} \langle x_i, e_k \rangle \geq \left(\sum_{k=1}^{m} r_k \right) \sum_{i=1}^{n} \|x_i\| > 0. \end{aligned} \tag{2.15}$$

Observe also that, by (2.15), $\sum_{k=1}^{m} e_k \neq 0$.

On utlising Schwarz's inequality in the inner product space $(H; \langle \cdot, \cdot \rangle)$ for $\sum_{i=1}^{n} x_i$, $\sum_{k=1}^{m} e_k$, we have

$$\left\| \sum_{i=1}^{n} x_i \right\| \left\| \sum_{k=1}^{m} e_k \right\| \geq \left| \left\langle \sum_{i=1}^{n} x_i, \sum_{k=1}^{m} e_k \right\rangle \right|. \tag{2.16}$$

Making use of (2.15) and (2.16), we can conclude that (2.13) holds.

Now, if (2.14) holds true, then, by taking the norm, we have

$$\left\|\sum_{i=1}^{n} x_i\right\| = \frac{\left(\sum_{k=1}^{m} r_k\right)\sum_{i=1}^{n}\|x_i\|}{\left\|\sum_{k=1}^{m} e_k\right\|^2}\left\|\sum_{k=1}^{m} e_k\right\| = \frac{\left(\sum_{k=1}^{m} r_k\right)}{\left\|\sum_{k=1}^{m} e_k\right\|}\left\|\sum_{i=1}^{n} x_i\right\|,$$

i.e., the case of equality holds in (2.13).

Conversely, if the case of equality holds in (2.13), then it must hold in all the inequalities used to prove (2.13). Therefore, we have

$$\operatorname{Re}\langle x_i, e_k\rangle = r_k\|x_i\| \tag{2.17}$$

for each $i \in \{1, \dots, n\}$ and $k \in \{1, \dots, m\}$,

$$\left\|\sum_{i=1}^{n} x_i\right\|\left\|\sum_{k=1}^{m} e_k\right\| = \left|\left\langle\sum_{i=1}^{n} x_i, \sum_{k=1}^{m} e_k\right\rangle\right| \tag{2.18}$$

and

$$\operatorname{Im}\left\langle\sum_{i=1}^{n} x_i, \sum_{k=1}^{m} e_k\right\rangle = 0. \tag{2.19}$$

From (2.17), on summing over i and k, we get

$$\operatorname{Re}\left\langle\sum_{i=1}^{n} x_i, \sum_{k=1}^{m} e_k\right\rangle = \left(\sum_{k=1}^{m} r_k\right)\sum_{i=1}^{n}\|x_i\|. \tag{2.20}$$

By (2.19) and (2.20), we have

$$\left\langle\sum_{i=1}^{n} x_i, \sum_{k=1}^{m} e_k\right\rangle = \left(\sum_{k=1}^{m} r_k\right)\sum_{i=1}^{n}\|x_i\|. \tag{2.21}$$

On the other hand, by the use of the following identity in inner product spaces

$$\left\|u - \frac{\langle u, v\rangle v}{\|v\|^2}\right\|^2 = \frac{\|u\|^2\|v\|^2 - |\langle u, v\rangle|^2}{\|v\|^2}, \quad v \neq 0,$$

the relation (2.18) holds if and only if

$$\sum_{i=1}^{n} x_i = \frac{\left\langle\sum_{i=1}^{n} x_i, \sum_{k=1}^{m} e_k\right\rangle}{\left\|\sum_{k=1}^{m} e_k\right\|^2}\sum_{k=1}^{m} e_k. \tag{2.22}$$

Finally, on utilising (2.21) and (2.22), we deduce that the condition (2.14) is necessary for the equality case in (2.13). ∎

Before we give a corollary of the above theorem, we need to state the following lemma that has been basically obtained in [3]. For the sake of completeness, we provide a short proof here as well.

Lemma 1. *Let* $(H; \langle \cdot, \cdot \rangle)$ *be an inner product space over the real or complex number field* $\mathbb{K}$ *and* $x, a \in H$, $r > 0$ *such that:*

$$\|x - a\| \leq r < \|a\| . \tag{2.23}$$

Then we have the inequality

$$\|x\| \left(\|a\|^2 - r^2\right)^{\frac{1}{2}} \leq \operatorname{Re} \langle x, a \rangle \tag{2.24}$$

or, equivalently

$$\|x\|^2 \|a\|^2 - [\operatorname{Re} \langle x, a \rangle]^2 \leq r^2 \|x\|^2 . \tag{2.25}$$

The case of equality holds in (2.24) (or in (2.25)) if and only if

$$\|x - a\| = r \quad \text{and} \quad \|x\|^2 + r^2 = \|a\|^2 . \tag{2.26}$$

Proof. From the first part of (2.23), we have

$$\|x\|^2 + \|a\|^2 - r^2 \leq 2 \operatorname{Re} \langle x, a \rangle . \tag{2.27}$$

By the second part of (2.23) we have $\left(\|a\|^2 - r^2\right)^{\frac{1}{2}} > 0$, therefore, by (2.27), we may state that

$$0 < \frac{\|x\|^2}{\left(\|a\|^2 - r^2\right)^{\frac{1}{2}}} + \left(\|a\|^2 - r^2\right)^{\frac{1}{2}} \leq \frac{2 \operatorname{Re} \langle x, a \rangle}{\left(\|a\|^2 - r^2\right)^{\frac{1}{2}}}. \tag{2.28}$$

Utilising the elementary inequality

$$\frac{1}{\alpha} q + \alpha p \geq 2\sqrt{pq}, \quad \alpha > 0,\ p > 0,\ q \geq 0;$$

with equality if and only if $\alpha = \sqrt{\frac{q}{p}}$, we may state (for $\alpha = \left(\|a\|^2 - r^2\right)^{\frac{1}{2}}$, $p = 1$, $q = \|x\|^2$) that

$$2 \|x\| \leq \frac{\|x\|^2}{\left(\|a\|^2 - r^2\right)^{\frac{1}{2}}} + \left(\|a\|^2 - r^2\right)^{\frac{1}{2}} . \tag{2.29}$$

The inequality (2.24) follows now by (2.28) and (2.29).

From the above argument, it is clear that the equality holds in (2.24) if and only if it holds in (2.28) and (2.29). However, the equality holds in (2.28) if and only if $\|x - a\| = r$ and in (2.29) if and only if $\left(\|a\|^2 - r^2\right)^{\frac{1}{2}} = \|x\|$.

The proof is thus completed. ∎

We may now state the following corollary.

Corollary 1. *Let $(H;\langle\cdot,\cdot\rangle)$ be an inner product space over the real or complex number field $\mathbb{K}$, $e_k,\ x_i\in H\backslash\{0\},\ k\in\{1,\dots,m\},\ i\in\{1,\dots,n\}$. If $\rho_k\geq 0,\ k\in\{1,\dots,m\}$ with*

$$\|x_i-e_k\|\leq\rho_k<\|e_k\| \tag{2.30}$$

for each $i\in\{1,\dots,n\}$ and $k\in\{1,\dots,m\}$, then

$$\sum_{i=1}^{n}\|x_i\|\leq\frac{\left\|\sum_{k=1}^{m}e_k\right\|}{\sum_{k=1}^{m}\left(\|e_k\|^2-\rho_k^2\right)^{\frac{1}{2}}}\left\|\sum_{i=1}^{n}x_i\right\|. \tag{2.31}$$

The case of equality holds in (2.31) if and only if

$$\sum_{i=1}^{n}x_i=\frac{\sum_{k=1}^{m}\left(\|e_k\|^2-\rho_k^2\right)^{\frac{1}{2}}}{\left\|\sum_{k=1}^{m}e_k\right\|^2}\left(\sum_{i=1}^{n}\|x_i\|\right)\sum_{k=1}^{m}e_k.$$

Proof. Utilising Lemma 1, we have from (2.30) that

$$\|x_i\|\left(\|e_k\|^2-\rho_k^2\right)^{\frac{1}{2}}\leq\operatorname{Re}\langle x_i,e_k\rangle$$

for each $k\in\{1,\dots,m\}$ and $i\in\{1,\dots,n\}$.

Applying Theorem 2 for

$$r_k:=\left(\|e_k\|^2-\rho_k^2\right)^{\frac{1}{2}},\quad k\in\{1,\dots,m\},$$

we deduce the desired result. ∎

Remark 2. *If $\{e_k\}_{k\in\{1,\dots,m\}}$ are orthogonal, then (2.31) becomes*

$$\sum_{i=1}^{n}\|x_i\|\leq\frac{\left(\sum_{k=1}^{m}\|e_k\|^2\right)^{\frac{1}{2}}}{\sum_{k=1}^{m}\left(\|e_k\|^2-\rho_k^2\right)^{\frac{1}{2}}}\left\|\sum_{i=1}^{n}x_i\right\| \tag{2.32}$$

with equality if and only if

$$\sum_{i=1}^{n}x_i=\frac{\sum_{k=1}^{m}\left(\|e_k\|^2-\rho_k^2\right)^{\frac{1}{2}}}{\sum_{k=1}^{m}\|e_k\|^2}\left(\sum_{i=1}^{n}\|x_i\|\right)\sum_{k=1}^{m}e_k.$$

Moreover, if $\{e_k\}_{k\in\{1,\dots,m\}}$ is assumed to be orthonormal and

$$\|x_i-e_k\|\leq\rho_k\quad\text{for } k\in\{1,\dots,m\},\ i\in\{1,\dots,n\}$$

where $\rho_k\in[0,1)$ for $k\in\{1,\dots,m\}$, then

$$\sum_{i=1}^{n}\|x_i\|\leq\frac{\sqrt{m}}{\sum_{k=1}^{m}\left(1-\rho_k^2\right)^{\frac{1}{2}}}\left\|\sum_{i=1}^{n}x_i\right\| \tag{2.33}$$

with equality if and only if

$$\sum_{i=1}^{n} x_i = \frac{\sum_{k=1}^{m} (1-\rho_k^2)^{\frac{1}{2}}}{m} \left(\sum_{i=1}^{n} \|x_i\| \right) \sum_{k=1}^{m} e_k.$$

The following lemma may be stated as well [3].

Lemma 2. *Let $(H; \langle\cdot,\cdot\rangle)$ be an inner product space over the real or complex number field $\mathbb{K}$, $x, y \in H$ and $M \geq m > 0$. If*

$$\operatorname{Re}\langle My - x, x - my\rangle \geq 0 \tag{2.34}$$

or, equivalently,

$$\left\| x - \frac{m+M}{2} y \right\| \leq \frac{1}{2}(M-m)\|y\|, \tag{2.35}$$

then

$$\|x\|\,\|y\| \leq \frac{1}{2} \cdot \frac{M+m}{\sqrt{mM}} \operatorname{Re}\langle x, y\rangle. \tag{2.36}$$

The equality holds in (2.36) if and only if the case of equality holds in (2.34) and

$$\|x\| = \sqrt{mM}\,\|y\|. \tag{2.37}$$

Proof. Obviously,

$$\operatorname{Re}\langle My - x, x - my\rangle = (M+m)\operatorname{Re}\langle x, y\rangle - \|x\|^2 - mM\|y\|^2.$$

Then (2.34) is clearly equivalent to

$$\frac{\|x\|^2}{\sqrt{mM}} + \sqrt{mM}\,\|y\|^2 \leq \frac{M+m}{\sqrt{mM}} \operatorname{Re}\langle x, y\rangle. \tag{2.38}$$

Since, obviously,

$$2\|x\|\,\|y\| \leq \frac{\|x\|^2}{\sqrt{mM}} + \sqrt{mM}\,\|y\|^2, \tag{2.39}$$

with equality iff $\|x\| = \sqrt{mM}\,\|y\|$, hence (2.38) and (2.39) imply (2.36).

The case of equality is obvious and we omit the details. ∎

Finally, we may state the following corollary of Theorem 2.

Corollary 2. *Let $(H; \langle\cdot,\cdot\rangle)$ be an inner product space over the real or complex number field $\mathbb{K}$, e_k, $x_i \in H\backslash\{0\}$, $k \in \{1,\ldots,m\}$, $i \in \{1,\ldots,n\}$. If $M_k > \mu_k > 0$, $k \in \{1,\ldots,m\}$ are such that either*

$$\operatorname{Re}\langle M_k e_k - x_i, x_i - \mu_k e_k\rangle \geq 0 \tag{2.40}$$

or, equivalently,

$$\left\| x_i - \frac{M_k + \mu_k}{2} e_k \right\| \leq \frac{1}{2} (M_k - \mu_k) \|e_k\|$$

for each $k \in \{1, \dots, m\}$ *and* $i \in \{1, \dots, n\}$, *then*

$$\sum_{i=1}^{n} \|x_i\| \leq \frac{\left\| \sum_{k=1}^{m} e_k \right\|}{\sum_{k=1}^{m} \frac{2 \cdot \sqrt{\mu_k M_k}}{\mu_k + M_k} \|e_k\|} \left\| \sum_{i=1}^{n} x_i \right\|. \tag{2.41}$$

The case of equality holds in (2.41) if and only if

$$\sum_{i=1}^{n} x_i = \frac{\sum_{k=1}^{m} \frac{2 \cdot \sqrt{\mu_k M_k}}{\mu_k + M_k} \|e_k\|}{\left\| \sum_{k=1}^{m} e_k \right\|^2} \sum_{i=1}^{n} \|x_i\| \sum_{k=1}^{m} e_k.$$

Proof. Utilising Lemma 2, by (2.40) we deduce

$$\frac{2 \cdot \sqrt{\mu_k M_k}}{\mu_k + M_k} \|x_i\| \|e_k\| \leq \operatorname{Re} \langle x_i, e_k \rangle$$

for each $k \in \{1, \dots, m\}$ and $i \in \{1, \dots, n\}$.

Applying Theorem 2 for

$$r_k := \frac{2 \cdot \sqrt{\mu_k M_k}}{\mu_k + M_k} \|e_k\|, \quad k \in \{1, \dots, m\},$$

we deduce the desired result. ∎

3. Diaz-Metcal Inequality for Semi-Inner Products

In 1961, G. Lumer [9] introduced the following concept.

Definition 1. *Let* X *be a linear space over the real or complex number field* $\mathbb{K}$. *The mapping* $[\cdot, \cdot] : X \times X \to \mathbb{K}$ *is called a semi-inner product on* X, *if the following properties are satisfied (see also* [3, p. 17]*):*

(i) $[x + y, z] = [x, z] + [y, z]$ *for all* $x, y, z \in X$;
(ii) $[\lambda x, y] = \lambda [x, y]$ *for all* $x, y \in X$ *and* $\lambda \in \mathbb{K}$;
(iii) $[x, x] \geq 0$ *for all* $x \in X$ *and* $[x, x] = 0$ *implies* $x = 0$;
(iv) $|[x, y]|^2 \leq [x, x] [y, y]$ *for all* $x, y \in X$;
(v) $[x, \lambda y] = \bar{\lambda} [x, y]$ *for all* $x, y \in X$ *and* $\lambda \in \mathbb{K}$.

It is well known that the mapping $X \ni x \longmapsto [x, x]^{\frac{1}{2}} \in \mathbb{R}$ is a norm on X and for any $y \in X$, the functional $X \ni x \overset{\varphi_y}{\longmapsto} [x, x]^{\frac{1}{2}} \in \mathbb{K}$ is a continuous linear functional on X endowed with the norm $\|\cdot\|$ generated by $[\cdot, \cdot]$. Moreover, one has $\|\varphi_y\| = \|y\|$ (see for instance [3, p. 17]).

Let $(X, \|\cdot\|)$ be a real or complex normed space. If $J : X \to_2 X^*$ is the *normalised duality mapping* defined on X, i.e., we recall that (see for instance [3, p. 1])

$$J(x) = \{\varphi \in X^* | \varphi(x) = \|\varphi\| \|x\|, \ \|\varphi\| = \|x\|\}, \quad x \in X,$$

then we may state the following representation result (see for instance [3, p. 18]):

Each semi-inner product $[\cdot, \cdot] : X \times X \to K$ that generates the norm $\|\cdot\|$ of the normed linear space $(X, \|\cdot\|)$ over the real or complex number field K, is of the form

$$[x, y] = \left\langle \tilde{J}(y), x \right\rangle \text{ for any } x, y \in X,$$

where $\tilde{J}$ is a selection of the normalised duality mapping and $\langle \varphi, x \rangle := \varphi(x)$ for $\varphi \in X^*$ and $x \in X$.

Utilising the concept of semi-inner products, we can state the following particular case of the Diaz-Metcalf inequality.

Corollary 3. *Let $(X, \|\cdot\|)$ be a normed linear space, $[\cdot, \cdot] : X \times X \to \mathbb{K}$ a semi-inner product generating the norm $\|\cdot\|$ and $e \in X$, $\|e\| = 1$. If $x_i \in X$, $i \in \{1, \dots, n\}$ and $r \geq 0$ such that*

$$r \|x_i\| \leq \operatorname{Re}[x_i, e] \quad \text{for each} \ i \in \{1, \dots, n\}, \tag{3.1}$$

then we have the inequality

$$r \sum_{i=1}^{n} \|x_i\| \leq \left\| \sum_{i=1}^{n} x_i \right\|. \tag{3.2}$$

The case of equality holds in (3.2) if and only if both

$$\left[\sum_{i=1}^{n} x_i, e \right] = r \sum_{i=1}^{n} \|x_i\| \tag{3.3}$$

and

$$\left[\sum_{i=1}^{n} x_i, e \right] = \left\| \sum_{i=1}^{n} x_i \right\|. \tag{3.4}$$

The proof is obvious from the Diaz-Metcalf theorem [2, Theorem 3] applied for the continuous linear functional $F_e(x) = [x, e]$, $x \in X$.

Before we provide a simpler necessary and sufficient condition of equality in (3.2), we need to recall the concept of strictly convex normed spaces and a classical characterisation of these spaces.

Definition 2. *A normed linear space $(X, \|\cdot\|)$ is said to be strictly convex if for every x, y from X with $x \neq y$ and $\|x\| = \|y\| = 1$, we have $\|\lambda x + (1 - \lambda) y\| < 1$ for all $\lambda \in (0, 1)$.*

The following characterisation of strictly convex spaces is useful in what follows (see [1], [8], or [3, p. 21]).

Theorem 3. *Let* $(X, \|\cdot\|)$ *be a normed linear space over* $\mathbb{K}$ *and* $[\cdot, \cdot]$ *a semi-inner product generating its norm. The following statements are equivalent:*

(i) $(X, \|\cdot\|)$ *is strictly convex;*
(ii) *For every* $x, y \in X$, $x, y \neq 0$ *with* $[x, y] = \|x\| \|y\|$, *there exists a* $\lambda > 0$ *such that* $x = \lambda y$.

The following result may be stated.

Corollary 4. *Let* $(X, \|\cdot\|)$ *be a strictly convex normed linear space,* $[\cdot, \cdot]$ *a semi-inner product generating the norm and* e, x_i $(i \in \{1, \dots, n\})$ *as in Corollary 3. Then the case of equality holds in (3.2) if and only if*

$$\sum_{i=1}^{n} x_i = r \left(\sum_{i=1}^{n} \|x_i\| \right) e. \tag{3.5}$$

Proof. If (3.5) holds true, then, obviously

$$\left\| \sum_{i=1}^{n} x_i \right\| = r \left(\sum_{i=1}^{n} \|x_i\| \right) \|e\| = r \sum_{i=1}^{n} \|x_i\|,$$

which is the equality case in (3.2).

Conversely, if the equality holds in (3.2), then by Corollary 3, we have that (3.3) and (3.4) hold true. Utilising Theorem 3, we conclude that there exists a $\mu > 0$ such that

$$\sum_{i=1}^{n} x_i = \mu e. \tag{3.6}$$

Inserting this in (3.3) we get

$$\mu \|e\|^2 = r \sum_{i=1}^{n} \|x_i\|$$

giving

$$\mu = r \sum_{i=1}^{n} \|x_i\|. \tag{3.7}$$

Finally, by (3.6) and (3.7) we deduce (3.5) and the corollary is proved. ∎

4. An Additive Reverse for the Triangle Inequality

4.1. **The Case of One Functional.** In the following we provide an alternative of the Diaz-Metcalf reverse of the generalised triangle inequality [7].

Theorem 4. *Let $(X, \|\cdot\|)$ be a normed linear space over the real or complex number field $\mathbb{K}$ and $F : X \to \mathbb{K}$ a linear functional with the property that $|F(x)| \leq \|x\|$ for any $x \in X$ (i.e., $\|F\| = 1$, we say that F is of unit norm). If $x_i \in X$, $k_i \geq 0$, $i \in \{1, \dots, n\}$ are such that*

$$(0 \leq)\, \|x_i\| - \operatorname{Re} F(x_i) \leq k_i \quad \text{for each} \quad i \in \{1, \dots, n\}, \tag{4.1}$$

then we have the inequality

$$(0 \leq) \sum_{i=1}^{n} \|x_i\| - \left\| \sum_{i=1}^{n} x_i \right\| \leq \sum_{i=1}^{n} k_i. \tag{4.2}$$

The equality holds in (4.2) if and only if both

$$F\left(\sum_{i=1}^{n} x_i\right) = \left\| \sum_{i=1}^{n} x_i \right\| \quad \text{and} \quad F\left(\sum_{i=1}^{n} x_i\right) = \sum_{i=1}^{n} \|x_i\| - \sum_{i=1}^{n} k_i. \tag{4.3}$$

Proof. If we sum in (4.1) over i from 1 to n, then we get

$$\sum_{i=1}^{n} \|x_i\| \leq \operatorname{Re}\left[F\left(\sum_{i=1}^{n} x_i\right)\right] + \sum_{i=1}^{n} k_i. \tag{4.4}$$

Taking into account that $|F(x)| \leq \|x\|$ for each $x \in X$, then we may state that

$$\begin{aligned} \operatorname{Re}\left[F\left(\sum_{i=1}^{n} x_i\right)\right] &\leq \left|\operatorname{Re} F\left(\sum_{i=1}^{n} x_i\right)\right| \\ &\leq \left|F\left(\sum_{i=1}^{n} x_i\right)\right| \leq \left\| \sum_{i=1}^{n} x_i \right\|. \end{aligned} \tag{4.5}$$

Now, making use of (4.4) and (4.5), we deduce (4.2).

Obviously, if (4.3) is valid, then the case of equality in (4.2) holds true.

Conversely, if the equality holds in (4.2), then it must hold in all the inequalities used to prove (4.2), therefore we have

$$\sum_{i=1}^{n} \|x_i\| = \operatorname{Re}\left[F\left(\sum_{i=1}^{n} x_i\right)\right] + \sum_{i=1}^{n} k_i$$

and

$$\operatorname{Re}\left[F\left(\sum_{i=1}^{n}x_i\right)\right]=\left|F\left(\sum_{i=1}^{n}x_i\right)\right|=\left\|\sum_{i=1}^{n}x_i\right\|,$$

which imply (4.3). ∎

The following corollary may be stated [7].

Corollary 5. *Let $(X,\|\cdot\|)$ be a normed linear space, $[\cdot,\cdot]:X\times X\to\mathbb{K}$ a semi-inner product generating the norm $\|\cdot\|$ and $e\in X$, $\|e\|=1$. If $x_i\in X$, $k_i\geq 0$, $i\in\{1,\dots,n\}$ are such that*

$$(0\leq)\,\|x_i\|-\operatorname{Re}[x_i,e]\leq k_i\quad\textit{for each}\quad i\in\{1,\dots,n\},\tag{4.6}$$

then we have the inequality

$$(0\leq)\sum_{i=1}^{n}\|x_i\|-\left\|\sum_{i=1}^{n}x_i\right\|\leq\sum_{i=1}^{n}k_i.\tag{4.7}$$

The equality holds in (4.7) if and only if both

$$\left[\sum_{i=1}^{n}x_i,e\right]=\left\|\sum_{i=1}^{n}x_i\right\|\quad\textit{and}\quad\left[\sum_{i=1}^{n}x_i,e\right]=\sum_{i=1}^{n}\|x_i\|-\sum_{i=1}^{n}k_i.\tag{4.8}$$

Moreover, if $(X,\|\cdot\|)$ is strictly convex, then the case of equality holds in (4.7) if and only if

$$\sum_{i=1}^{n}\|x_i\|\geq\sum_{i=1}^{n}k_i\tag{4.9}$$

and

$$\sum_{i=1}^{n}x_i=\left(\sum_{i=1}^{n}\|x_i\|-\sum_{i=1}^{n}k_i\right)\cdot e.\tag{4.10}$$

Proof. The first part of the corollary is obvious by Theorem 4 applied for the continuous linear functional of unit norm F_e, $F_e(x)=[x,e]$, $x\in X$. The second part may be shown on utilising a similar argument to the one from the proof of Corollary 4. We omit the details. ∎

Remark 3. *If $X=H$, $(H;\langle\cdot,\cdot\rangle)$ is an inner product space, then from Corollary 5 we deduce the additive reverse inequality obtained in Theorem 7 of [5]. For further similar results in inner product spaces, see [4] and [5].*

4.2. **The Case of m Functionals.** The following result generalising Theorem 4 may be stated [7].

Theorem 5. *Let $(X, \|\cdot\|)$ be a normed linear space over the real or complex number field $\mathbb{K}$. If F_k, $k \in \{1, \dots, m\}$ are bounded linear functionals defined on X and $x_i \in X$, $M_{ik} \geq 0$ for $i \in \{1, \dots, n\}$, $k \in \{1, \dots, m\}$ such that*

$$\|x_i\| - \operatorname{Re} F_k(x_i) \leq M_{ik} \tag{4.11}$$

for each $i \in \{1, \dots, n\}$, $k \in \{1, \dots, m\}$, then we have the inequality

$$\sum_{i=1}^{n} \|x_i\| \leq \left\| \frac{1}{m} \sum_{k=1}^{m} F_k \right\| \left\| \sum_{i=1}^{n} x_i \right\| + \frac{1}{m} \sum_{k=1}^{m} \sum_{i=1}^{n} M_{ik}. \tag{4.12}$$

The case of equality holds in (4.12) if both

$$\frac{1}{m} \sum_{k=1}^{m} F_k \left(\sum_{i=1}^{n} x_i \right) = \left\| \frac{1}{m} \sum_{k=1}^{m} F_k \right\| \left\| \sum_{i=1}^{n} x_i \right\| \tag{4.13}$$

and

$$\frac{1}{m} \sum_{k=1}^{m} F_k \left(\sum_{i=1}^{n} x_i \right) = \sum_{i=1}^{n} \|x_i\| - \frac{1}{m} \sum_{k=1}^{m} \sum_{j=1}^{n} M_{ik}. \tag{4.14}$$

Proof. If we sum (4.11) over i from 1 to n, then we deduce

$$\sum_{i=1}^{n} \|x_i\| - \operatorname{Re} F_k \left(\sum_{i=1}^{n} x_i \right) \leq \sum_{i=1}^{n} M_{ik}$$

for each $k \in \{1, \dots, m\}$.

Summing these inequalities over k from 1 to m, we deduce

$$\sum_{i=1}^{n} \|x_i\| \leq \frac{1}{m} \sum_{k=1}^{m} \operatorname{Re} F_k \left(\sum_{i=1}^{n} x_i \right) + \frac{1}{m} \sum_{k=1}^{m} \sum_{i=1}^{n} M_{ik}. \tag{4.15}$$

Utilising the continuity property of the functionals F_k and the properties of the modulus, we have

$$\begin{aligned} \sum_{k=1}^{m} \operatorname{Re} F_k \left(\sum_{i=1}^{n} x_i \right) &\leq \left| \sum_{k=1}^{m} \operatorname{Re} F_k \left(\sum_{i=1}^{n} x_i \right) \right| \\ &\leq \left| \sum_{k=1}^{m} F_k \left(\sum_{i=1}^{n} x_i \right) \right| \leq \left\| \sum_{k=1}^{m} F_k \right\| \left\| \sum_{i=1}^{n} x_i \right\|. \end{aligned} \tag{4.16}$$

Now, by (4.15) and (4.16), we deduce (4.12).

Obviously, if (4.13) and (4.14) hold true, then the case of equality is valid in (4.12).

Conversely, if the case of equality holds in (4.12), then it must hold in all the inequalities used to prove (4.12). Therefore we have

$$\sum_{i=1}^{n}\|x_i\| = \frac{1}{m}\sum_{k=1}^{m}\operatorname{Re}F_k\left(\sum_{i=1}^{n}x_i\right) + \frac{1}{m}\sum_{k=1}^{m}\sum_{i=1}^{n}M_{ik},$$

$$\sum_{k=1}^{m}\operatorname{Re}F_k\left(\sum_{i=1}^{n}x_i\right) = \left\|\sum_{k=1}^{m}F_k\right\|\left\|\sum_{i=1}^{n}x_i\right\|$$

and

$$\sum_{k=1}^{m}\operatorname{Im}F_k\left(\sum_{i=1}^{n}x_i\right) = 0.$$

These imply that (4.13) and (4.14) hold true, and the theorem is completely proved. ∎

Remark 4. *If F_k, $k\in\{1,\dots,m\}$ are of unit norm, then, from (4.12), we deduce the inequality*

$$\sum_{i=1}^{n}\|x_i\| \le \left\|\sum_{i=1}^{n}x_i\right\| + \frac{1}{m}\sum_{k=1}^{m}\sum_{i=1}^{n}M_{ik}, \tag{4.17}$$

which is obviously coarser than (4.12), but perhaps more useful for applications.

4.3. **The Case of Inner Product Spaces.** The case of inner product spaces, in which we may provide a simpler condition of equality, is of interest in applications [7].

Theorem 6. *Let $(X,\|\cdot\|)$ be an inner product space over the real or complex number field $\mathbb{K}$, $e_k, x_i \in H\setminus\{0\}$, $k\in\{1,\dots,m\}$, $i\in\{1,\dots,n\}$. If $M_{ik}\ge 0$ for $i\in\{1,\dots,n\}$, $\{1,\dots,n\}$ such that*

$$\|x_i\| - \operatorname{Re}\langle x_i, e_k\rangle \le M_{ik} \tag{4.18}$$

for each $i\in\{1,\dots,n\}$, $k\in\{1,\dots,m\}$, then we have the inequality

$$\sum_{i=1}^{n}\|x_i\| \le \left\|\frac{1}{m}\sum_{k=1}^{m}e_k\right\|\left\|\sum_{i=1}^{n}x_i\right\| + \frac{1}{m}\sum_{k=1}^{m}\sum_{i=1}^{n}M_{ik}. \tag{4.19}$$

The case of equality holds in (4.19) if and only if

$$\sum_{i=1}^{n}\|x_i\| \ge \frac{1}{m}\sum_{k=1}^{m}\sum_{i=1}^{n}M_{ik} \tag{4.20}$$

and

$$\sum_{i=1}^{n}x_i = \frac{m\left(\sum_{i=1}^{n}\|x_i\| - \frac{1}{m}\sum_{k=1}^{m}\sum_{i=1}^{n}M_{ik}\right)}{\left\|\sum_{k=1}^{m}e_k\right\|^2}\sum_{k=1}^{m}e_k. \tag{4.21}$$

Proof. As in the proof of Theorem 5, we have

$$\sum_{i=1}^{n}\|x_i\| \le \operatorname{Re}\left\langle \frac{1}{m}\sum_{k=1}^{m}e_k, \sum_{i=1}^{n}x_i\right\rangle + \frac{1}{m}\sum_{k=1}^{m}\sum_{i=1}^{n}M_{ik}, \tag{4.22}$$

and $\sum_{k=1}^{m}e_k \neq 0$.

On utilising the Schwarz inequality in the inner product space $(H;\langle\cdot,\cdot\rangle)$ for $\sum_{i=1}^{n}x_i$, $\sum_{k=1}^{m}e_k$, we have

$$\begin{aligned}\left\|\sum_{i=1}^{n}x_i\right\|\left\|\sum_{k=1}^{m}e_k\right\| &\ge \left|\left\langle \sum_{i=1}^{n}x_i, \sum_{k=1}^{m}e_k\right\rangle\right| \\ &\ge \left|\operatorname{Re}\left\langle \sum_{i=1}^{n}x_i, \sum_{k=1}^{m}e_k\right\rangle\right| \\ &\ge \operatorname{Re}\left\langle \sum_{i=1}^{n}x_i, \sum_{k=1}^{m}e_k\right\rangle.\end{aligned} \tag{4.23}$$

By (4.22) and (4.23) we deduce (4.19).

Taking the norm in (4.21) and using (4.20), we have

$$\left\|\sum_{i=1}^{n}x_i\right\| = \frac{m\left(\sum_{i=1}^{n}\|x_i\| - \frac{1}{m}\sum_{k=1}^{m}\sum_{i=1}^{n}M_{ik}\right)}{\left\|\sum_{k=1}^{m}e_k\right\|},$$

showing that the equality holds in (4.19).

Conversely, if the case of equality holds in (4.19), then it must hold in all the inequalities used to prove (4.19). Therefore we have

$$\|x_i\| = \operatorname{Re}\langle x_i, e_k\rangle + M_{ik} \tag{4.24}$$

for each $i \in \{1,\dots,n\}$, $k \in \{1,\dots,m\}$,

$$\left\|\sum_{i=1}^{n}x_i\right\|\left\|\sum_{k=1}^{m}e_k\right\| = \left|\left\langle \sum_{i=1}^{n}x_i, \sum_{k=1}^{m}e_k\right\rangle\right| \tag{4.25}$$

and

$$\operatorname{Im}\left\langle \sum_{i=1}^{n}x_i, \sum_{k=1}^{m}e_k\right\rangle = 0. \tag{4.26}$$

From (4.24), on summing over i and k, we get

$$\operatorname{Re}\left\langle \sum_{i=1}^{n}x_i, \sum_{k=1}^{m}e_k\right\rangle = m\sum_{i=1}^{n}\|x_i\| - \sum_{k=1}^{m}\sum_{i=1}^{n}M_{ik}. \tag{4.27}$$

On the other hand, by the use of the following identity in inner product spaces,

$$\left\| u - \frac{\langle u, v\rangle v}{\|v\|^2} \right\|^2 = \frac{\|u\|^2 \|v\|^2 - |\langle u, v\rangle|^2}{\|v\|^2}, \quad v \neq 0;$$

the relation (4.25) holds if and only if

$$\sum_{i=1}^{n} x_i = \frac{\langle \sum_{i=1}^{n} x_i, \sum_{k=1}^{m} e_k \rangle}{\|\sum_{k=1}^{m} e_k\|^2} \sum_{k=1}^{m} e_k,$$

giving, from (4.26) and (4.27), that

$$\sum_{i=1}^{n} x_i = \frac{m \sum_{i=1}^{n} \|x_i\| - \sum_{k=1}^{m} \sum_{i=1}^{n} M_{ik}}{\|\sum_{k=1}^{m} e_k\|^2} \sum_{k=1}^{m} e_k.$$

If the inequality holds in (4.19), then obviously (4.20) is valid, and the theorem is proved. ∎

Remark 5. *If in the above theorem the vectors* $\{e_k\}_{k=\overline{1,m}}$ *are assumed to be orthogonal, then (4.19) becomes:*

$$\sum_{i=1}^{n} \|x_i\| \leq \frac{1}{m} \left(\sum_{k=1}^{m} \|e_k\|^2 \right)^{\frac{1}{2}} \left\| \sum_{i=1}^{n} x_i \right\| + \frac{1}{m} \sum_{k=1}^{m} \sum_{i=1}^{n} M_{ik}. \tag{4.28}$$

Moreover, if $\{e_k\}_{k=\overline{1,m}}$ *is an orthonormal family, then (4.28) becomes*

$$\sum_{i=1}^{n} \|x_i\| \leq \frac{\sqrt{m}}{m} \left\| \sum_{i=1}^{n} x_i \right\| + \frac{1}{m} \sum_{k=1}^{m} \sum_{i=1}^{n} M_{ik}, \tag{4.29}$$

which has been obtained in [5].

Before we provide some natural consequences of Theorem 6, we need some preliminary results concerning reverses of Schwarz's inequality in inner product spaces (see for instance [4, p. 27]).

Lemma 3. *Let* $(X, \|\cdot\|)$ *be an inner product space over the real or complex number field* $\mathbb{K}$ *and* $x, a \in H$, $r > 0$. *If* $\|x - a\| \leq r$, *then we have the inequality*

$$\|x\| \|a\| - \operatorname{Re} \langle x, a\rangle \leq \frac{1}{2} r^2. \tag{4.30}$$

The case of equality holds in (4.30) if and only if

$$\|x - a\| = r \quad \text{and} \quad \|x\| = \|a\|. \tag{4.31}$$

Proof. The condition $\|x-a\| \leq r$ is clearly equivalent to

$$\|x\|^2 + \|a\|^2 \leq 2\operatorname{Re}\langle x, a\rangle + r^2. \tag{4.32}$$

Since

$$2\|x\|\|a\| \leq \|x\|^2 + \|a\|^2, \tag{4.33}$$

with equality if and only if $\|x\| = \|a\|$, hence by (4.32) and (4.33) we deduce (4.30).

The case of equality is obvious. ∎

Utilising the above lemma we may state the following corollary of Theorem 6 [7].

Corollary 6. *Let $(H;\langle\cdot,\cdot\rangle)$, e_k, x_i be as in Theorem 6. If $r_{ik} > 0$, $i \in \{1,\dots,n\}$, $k \in \{1,\dots,m\}$ such that*

$$\|x_i - e_k\| \leq r_{ik} \quad \text{for each } \ i \in \{1,\dots,n\} \ \text{ and } k \in \{1,\dots,m\}, \tag{4.34}$$

then we have the inequality

$$\sum_{i=1}^{n}\|x_i\| \leq \left\|\frac{1}{m}\sum_{k=1}^{m} e_k\right\|\left\|\sum_{i=1}^{n} x_i\right\| + \frac{1}{2m}\sum_{k=1}^{m}\sum_{i=1}^{n} r_{ik}^2. \tag{4.35}$$

The equality holds in (4.35) if and only if

$$\sum_{i=1}^{n}\|x_i\| \geq \frac{1}{2m}\sum_{k=1}^{m}\sum_{i=1}^{n} r_{ik}^2$$

and

$$\sum_{i=1}^{n} x_i = \frac{m\left(\sum_{i=1}^{n}\|x_i\| - \frac{1}{2m}\sum_{k=1}^{m}\sum_{i=1}^{n} r_{ik}^2\right)}{\left\|\sum_{k=1}^{m} e_k\right\|^2}\sum_{k=1}^{m} e_k.$$

The following lemma may provide another sufficient condition for (4.18) to hold (see also [4, p. 28]).

Lemma 4. *Let $(H;\langle\cdot,\cdot\rangle)$ be an inner product space over the real or complex number field $\mathbb{K}$ and $x, y \in H$, $M \geq m > 0$. If either*

$$\operatorname{Re}\langle My - x, x - my\rangle \geq 0 \tag{4.36}$$

or, equivalently,

$$\left\|x - \frac{m+M}{2}y\right\| \leq \frac{1}{2}(M-m)\|y\|, \tag{4.37}$$

holds, then

$$\|x\|\|y\| - \operatorname{Re}\langle x, y\rangle \leq \frac{1}{4}\cdot\frac{(M-m)^2}{m+M}\|y\|^2. \tag{4.38}$$

The case of equality holds in (4.38) if and only if the equality case is realised in (4.36) and

$$\|x\| = \frac{M+m}{2}\|y\|.$$

The proof is obvious by Lemma 3 for $a = \frac{M+m}{2}y$ and $r = \frac{1}{2}(M-m)\|y\|.$

Finally, the following corollary of Theorem 6 may be stated [7].

Corollary 7. *Assume that $(H, \langle\cdot,\cdot\rangle)$, e_k, x_i are as in Theorem 6. If $M_{ik} \geq m_{ik} > 0$ satisfy the condition*

$$\operatorname{Re}\langle M_k e_k - x_i, x_i - \mu_k e_k\rangle \geq 0$$

for each $i \in \{1,\dots,n\}$ and $k \in \{1,\dots,m\}$, then

$$\sum_{i=1}^{n}\|x_i\| \leq \left\|\frac{1}{m}\sum_{k=1}^{m}e_k\right\|\left\|\sum_{i=1}^{n}x_i\right\| + \frac{1}{4m}\sum_{k=1}^{m}\sum_{i=1}^{n}\frac{(M_{ik}-m_{ik})^2}{M_{ik}+m_{ik}}\|e_k\|^2.$$

5. Applications for Complex Numbers

Let $\mathbb{C}$ be the field of complex numbers. If $z = \operatorname{Re} z + i \operatorname{Im} z$, then by $|\cdot|_p : \mathbb{C} \to [0,\infty)$, $p \in [1,\infty]$ we define the $p-$*modulus* of z as

$$|z|_p := \begin{cases} \max\{|\operatorname{Re} z|, |\operatorname{Im} z|\} & \text{if } p = \infty, \\ \\ (|\operatorname{Re} z|^p + |\operatorname{Im} z|^p)^{\frac{1}{p}} & \text{if } p \in [1,\infty), \end{cases}$$

where $|a|$, $a \in \mathbb{R}$ is the usual modulus of the real number a.

For $p = 2$, we recapture the usual modulus of a complex number, i.e.,

$$|z|_2 = \sqrt{|\operatorname{Re} z|^2 + |\operatorname{Im} z|^2} = |z|, \quad z \in \mathbb{C}.$$

It is well known that $\left(\mathbb{C}, |\cdot|_p\right)$, $p \in [1,\infty]$ is a Banach space over the real number field $\mathbb{R}$.

Consider the Banach space $(\mathbb{C}, |\cdot|_1)$ and $F : \mathbb{C} \to \mathbb{C}$, $F(z) = az$ with $a \in \mathbb{C}$, $a \neq 0$. Obviously, F is linear on $\mathbb{C}$. For $z \neq 0$, we have

$$\frac{|F(z)|}{|z|_1} = \frac{|a|\,|z|}{|z|_1} = \frac{|a|\sqrt{|\operatorname{Re} z|^2 + |\operatorname{Im} z|^2}}{|\operatorname{Re} z| + |\operatorname{Im} z|} \leq |a|.$$

Since, for $z_0 = 1$, we have $|F(z_0)| = |a|$ and $|z_0|_1 = 1$, hence

$$\|F\|_1 := \sup_{z\neq 0}\frac{|F(z)|}{|z|_1} = |a|,$$

showing that F is a bounded linear functional on $(\mathbb{C}, |\cdot|_1)$ and $\|F\|_1 = |a|$.

We can apply Theorem 1 to state the following reverse of the generalised triangle inequality for complex numbers [6].

Proposition 1. *Let* $a_k, x_j \in \mathbb{C}$, $k \in \{1, \dots, m\}$ *and* $j \in \{1, \dots, n\}$. *If there exist the constants* $r_k \geq 0$, $k \in \{1, \dots, m\}$ *with* $\sum_{k=1}^m r_k > 0$ *and*

$$r_k [|\mathrm{Re}\, x_j| + |\mathrm{Im}\, x_j|] \leq \mathrm{Re}\, a_k \cdot \mathrm{Re}\, x_j - \mathrm{Im}\, a_k \cdot \mathrm{Im}\, x_j \tag{5.1}$$

for each $j \in \{1, \dots, n\}$ *and* $k \in \{1, \dots, m\}$, *then*

$$\sum_{j=1}^n [|\mathrm{Re}\, x_j| + |\mathrm{Im}\, x_j|] \leq \frac{|\sum_{k=1}^m a_k|}{\sum_{k=1}^m r_k} \left[\left| \sum_{j=1}^n \mathrm{Re}\, x_j \right| + \left| \sum_{j=1}^n \mathrm{Im}\, x_j \right| \right]. \tag{5.2}$$

The case of equality holds in (5.2) if both

$$\begin{aligned} &\mathrm{Re}\left(\sum_{k=1}^m a_k\right) \mathrm{Re}\left(\sum_{j=1}^n x_j\right) - \mathrm{Im}\left(\sum_{k=1}^m a_k\right) \mathrm{Im}\left(\sum_{j=1}^n x_j\right) \\ &= \left(\sum_{k=1}^m r_k\right) \sum_{j=1}^n [|\mathrm{Re}\, x_j| + |\mathrm{Im}\, x_j|] \\ &= \left|\sum_{k=1}^m a_k\right| \left[\left| \sum_{j=1}^n \mathrm{Re}\, x_j \right| + \left| \sum_{j=1}^n \mathrm{Im}\, x_j \right| \right]. \end{aligned}$$

The proof follows by Theorem 1 applied for the Banach space $(\mathbb{C}, |\cdot|_1)$ and $F_k(z) = a_k z$, $k \in \{1, \dots, m\}$ on taking into account that:

$$\left\| \sum_{k=1}^m F_k \right\|_1 = \left| \sum_{k=1}^m a_k \right|.$$

Now, consider the Banach space $(\mathbb{C}, |\cdot|_\infty)$. If $F(z) = dz$, then for $z \neq 0$ we have

$$\frac{|F(z)|}{|z|_\infty} = \frac{|d|\,|z|}{|z|_\infty} = \frac{|d| \sqrt{|\mathrm{Re}\, z|^2 + |\mathrm{Im}\, z|^2}}{\max\{|\mathrm{Re}\, z|, |\mathrm{Im}\, z|\}} \leq \sqrt{2}\,|d|.$$

Since, for $z_0 = 1 + i$, we have $|F(z_0)| = \sqrt{2}\,|d|$, $|z_0|_\infty = 1$, hence

$$\|F\|_\infty := \sup_{z \neq 0} \frac{|F(z)|}{|z|_\infty} = \sqrt{2}\,|d|,$$

showing that F is a bounded linear functional on $(\mathbb{C}, |\cdot|_\infty)$ and $\|F\|_\infty = \sqrt{2}\,|d|$.

If we apply Theorem 1, then we can state the following reverse of the generalised triangle inequality for complex numbers [6].

Proposition 2. *Let $a_k, x_j \in \mathbb{C}$, $k \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$. If there exist the constants $r_k \geq 0$, $k \in \{1, \ldots, m\}$ with $\sum_{k=1}^{m} r_k > 0$ and*

$$r_k \max\{|\operatorname{Re} x_j|, |\operatorname{Im} x_j|\} \leq \operatorname{Re} a_k \cdot \operatorname{Re} x_j - \operatorname{Im} a_k \cdot \operatorname{Im} x_j$$

for each $j \in \{1, \ldots, n\}$ and $k \in \{1, \ldots, m\}$, then

$$\sum_{j=1}^{n} \max\{|\operatorname{Re} x_j|, |\operatorname{Im} x_j|\} \leq \sqrt{2} \cdot \frac{|\sum_{k=1}^{m} a_k|}{\sum_{k=1}^{m} r_k} \max\left\{\left|\sum_{j=1}^{n} \operatorname{Re} x_j\right|, \left|\sum_{j=1}^{n} \operatorname{Im} x_j\right|\right\}. \tag{5.3}$$

The case of equality holds in (5.3) if both

$$\begin{aligned} &\operatorname{Re}\left(\sum_{k=1}^{m} a_k\right) \operatorname{Re}\left(\sum_{j=1}^{n} x_j\right) - \operatorname{Im}\left(\sum_{k=1}^{m} a_k\right) \operatorname{Im}\left(\sum_{j=1}^{n} x_j\right) \\ &= \left(\sum_{k=1}^{m} r_k\right) \sum_{j=1}^{n} \max\{|\operatorname{Re} x_j|, |\operatorname{Im} x_j|\} \\ &= \sqrt{2}\left|\sum_{k=1}^{m} a_k\right| \max\left\{\left|\sum_{j=1}^{n} \operatorname{Re} x_j\right|, \left|\sum_{j=1}^{n} \operatorname{Im} x_j\right|\right\}. \end{aligned}$$

Finally, consider the Banach space $\left(\mathbb{C}, |\cdot|_{2p}\right)$ with $p \geq 1$.

Let $F : \mathbb{C} \to \mathbb{C}$, $F(z) = cz$. By Hölder's inequality, we have

$$\frac{|F(z)|}{|z|_{2p}} = \frac{|c| \sqrt{|\operatorname{Re} z|^2 + |\operatorname{Im} z|^2}}{\left(|\operatorname{Re} z|^{2p} + |\operatorname{Im} z|^{2p}\right)^{\frac{1}{2p}}} \leq 2^{\frac{1}{2} - \frac{1}{2p}} |c|.$$

Since, for $z_0 = 1 + i$ we have $|F(z_0)| = 2^{\frac{1}{2}} |c|$, $|z_0| = 2^{\frac{1}{2p}}$ $(p \geq 1)$, hence

$$\|F\|_{2p} := \sup_{z \neq 0} \frac{|F(z)|}{|z|_{2p}} = 2^{\frac{1}{2} - \frac{1}{2p}} |c|,$$

showing that F is a bounded linear functional on $\left(\mathbb{C}, |\cdot|_{2p}\right)$, $p \geq 1$ and $\|F\|_{2p} = 2^{\frac{1}{2} - \frac{1}{2p}} |c|$.

If we apply Theorem 1, then we can state the following proposition [6].

Proposition 3. *Let $a_k, x_j \in \mathbb{C}$, $k \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$. If there exist the constants $r_k \geq 0$, $k \in \{1, \ldots, m\}$ with $\sum_{k=1}^{m} r_k > 0$ and*

$$r_k \left[|\operatorname{Re} x_j|^{2p} + |\operatorname{Im} x_j|^{2p}\right]^{\frac{1}{2p}} \leq \operatorname{Re} a_k \cdot \operatorname{Re} x_j - \operatorname{Im} a_k \cdot \operatorname{Im} x_j$$

for each $j \in \{1, \ldots, n\}$ *and* $k \in \{1, \ldots, m\}$, *then*

$$\sum_{j=1}^{n} \left[|\operatorname{Re} x_j|^{2p} + |\operatorname{Im} x_j|^{2p} \right]^{\frac{1}{2p}} \leq 2^{\frac{1}{2} - \frac{1}{2p}} \frac{\left| \sum_{k=1}^{m} a_k \right|}{\sum_{k=1}^{m} r_k} \left[\left| \sum_{j=1}^{n} \operatorname{Re} x_j \right|^{2p} + \left| \sum_{j=1}^{n} \operatorname{Im} x_j \right|^{2p} \right]^{\frac{1}{2p}}. \tag{5.4}$$

The case of equality holds in (5.4) if both:

$$\begin{aligned} &\operatorname{Re}\left(\sum_{k=1}^{m} a_k\right) \operatorname{Re}\left(\sum_{j=1}^{n} x_j\right) - \operatorname{Im}\left(\sum_{k=1}^{m} a_k\right) \operatorname{Im}\left(\sum_{j=1}^{n} x_j\right) \\ &= \left(\sum_{k=1}^{m} r_k\right) \sum_{j=1}^{n} \left[|\operatorname{Re} x_j|^{2p} + |\operatorname{Im} x_j|^{2p} \right]^{\frac{1}{2p}} \\ &= 2^{\frac{1}{2} - \frac{1}{2p}} \left| \sum_{k=1}^{m} a_k \right| \left[\left| \sum_{j=1}^{n} \operatorname{Re} x_j \right|^{2p} + \left| \sum_{j=1}^{n} \operatorname{Im} x_j \right|^{2p} \right]^{\frac{1}{2p}}. \end{aligned}$$

Remark 6. *If in the above proposition we choose* $p = 1$, *then we have the following reverse of the generalised triangle inequality for complex numbers*

$$\sum_{j=1}^{n} |x_j| \leq \frac{\left| \sum_{k=1}^{m} a_k \right|}{\sum_{k=1}^{m} r_k} \left| \sum_{j=1}^{n} x_j \right|$$

provided x_j, a_k, $j \in \{1, \ldots, n\}$, $k \in \{1, \ldots, m\}$ *satisfy the assumption*

$$r_k |x_j| \leq \operatorname{Re} a_k \cdot \operatorname{Re} x_j - \operatorname{Im} a_k \cdot \operatorname{Im} x_j$$

for each $j \in \{1, \ldots, n\}$, $k \in \{1, \ldots, m\}$. *Here* $|\cdot|$ *is the usual modulus of a complex number and* $r_k > 0$, $k \in \{1, \ldots, m\}$ *are given.*

We can apply Theorem 5 to state the following reverse of the generalised triangle inequality for complex numbers [7].

Proposition 4. *Let* a_k, $x_j \in \mathbb{C}$, $k \in \{1, \ldots, m\}$ *and* $j \in \{1, \ldots, n\}$. *If there exist the constants* $M_{jk} \geq 0$, $k \in \{1, \ldots, m\}$, $j \in \{1, \ldots, n\}$ *such that*

$$|\operatorname{Re} x_j| + |\operatorname{Im} x_j| \leq \operatorname{Re} a_k \cdot \operatorname{Re} x_j - \operatorname{Im} a_k \cdot \operatorname{Im} x_j + M_{jk} \tag{5.5}$$

for each $j \in \{1, \ldots, n\}$ *and* $k \in \{1, \ldots, m\}$, *then*

$$\begin{aligned} (5.6)\quad & \sum_{j=1}^{n} \left[\left| \operatorname{Re} x_j \right| + \left| \operatorname{Im} x_j \right| \right] \\ & \leq \frac{1}{m} \left| \sum_{k=1}^{m} a_k \right| \left[\left| \sum_{j=1}^{n} \operatorname{Re} x_j \right| + \left| \sum_{j=1}^{n} \operatorname{Im} x_j \right| \right] + \frac{1}{m} \sum_{k=1}^{m} \sum_{j=1}^{n} M_{jk}. \end{aligned}$$

The proof follows by Theorem 5 applied for the Banach space $(\mathbb{C}, |\cdot|_1)$ and $F_k(z) = a_k z$, $k \in \{1, \ldots, m\}$ on taking into account that:

$$\left\| \sum_{k=1}^{m} F_k \right\|_1 = \left| \sum_{k=1}^{m} a_k \right|.$$

If we apply Theorem 5 for the Banach space $(\mathbb{C}, |\cdot|_\infty)$, then we can state the following reverse of the generalised triangle inequality for complex numbers [7].

Proposition 5. *Let* $a_k, x_j \in \mathbb{C}$, $k \in \{1, \ldots, m\}$ *and* $j \in \{1, \ldots, n\}$. *If there exist the constants* $M_{jk} \geq 0$, $k \in \{1, \ldots, m\}$, $j \in \{1, \ldots, n\}$ *such that*

$$\max \left\{ \left| \operatorname{Re} x_j \right|, \left| \operatorname{Im} x_j \right| \right\} \leq \operatorname{Re} a_k \cdot \operatorname{Re} x_j - \operatorname{Im} a_k \cdot \operatorname{Im} x_j + M_{jk}$$

for each $j \in \{1, \ldots, n\}$ *and* $k \in \{1, \ldots, m\}$, *then*

$$\begin{aligned} (5.7)\quad & \sum_{j=1}^{n} \max \left\{ \left| \operatorname{Re} x_j \right|, \left| \operatorname{Im} x_j \right| \right\} \\ & \leq \frac{\sqrt{2}}{m} \left| \sum_{k=1}^{m} a_k \right| \max \left\{ \left| \sum_{j=1}^{n} \operatorname{Re} x_j \right|, \left| \sum_{j=1}^{n} \operatorname{Im} x_j \right| \right\} + \frac{1}{m} \sum_{k=1}^{m} \sum_{j=1}^{n} M_{jk}. \end{aligned}$$

Finally, consider the Banach space $\left(\mathbb{C}, |\cdot|_{2p}\right)$ with $p \geq 1$.

If we apply Theorem 5, then we can state the following proposition [7].

Proposition 6. *Let* a_k, x_j, M_{jk} *be as in Proposition 5. If*

$$\left[\left| \operatorname{Re} x_j \right|^{2p} + \left| \operatorname{Im} x_j \right|^{2p} \right]^{\frac{1}{2p}} \leq \operatorname{Re} a_k \cdot \operatorname{Re} x_j - \operatorname{Im} a_k \cdot \operatorname{Im} x_j + M_{jk}$$

for each $j \in \{1, \ldots, n\}$ *and* $k \in \{1, \ldots, m\}$, *then*

$$\sum_{j=1}^{n} \left[|\operatorname{Re} x_j|^{2p} + |\operatorname{Im} x_j|^{2p} \right]^{\frac{1}{2p}} \tag{5.8}$$

$$\leq \frac{2^{\frac{1}{2}-\frac{1}{2p}}}{m} \left| \sum_{k=1}^{m} a_k \right| \left[\left| \sum_{j=1}^{n} \operatorname{Re} x_j \right|^{2p} + \left| \sum_{j=1}^{n} \operatorname{Im} x_j \right|^{2p} \right]^{\frac{1}{2p}} + \frac{1}{m} \sum_{k=1}^{m} \sum_{j=1}^{n} M_{jk}.$$

where $p \geq 1$.

Remark 7. *If in the above proposition we choose* $p = 1$, *then we have the following reverse of the generalised triangle inequality for complex numbers*

$$\sum_{j=1}^{n} |x_j| \leq \left| \frac{1}{m} \sum_{k=1}^{m} a_k \right| \left| \sum_{j=1}^{n} x_j \right| + \frac{1}{m} \sum_{k=1}^{m} \sum_{j=1}^{n} M_{jk}$$

provided x_j, a_k, $j \in \{1, \ldots, n\}$, $k \in \{1, \ldots, m\}$ *satisfy the assumption*

$$|x_j| \leq \operatorname{Re} a_k \cdot \operatorname{Re} x_j - \operatorname{Im} a_k \cdot \operatorname{Im} x_j + M_{jk}$$

for each $j \in \{1, \ldots, n\}$, $k \in \{1, \ldots, m\}$. *Here* $|\cdot|$ *is the usual modulus of a complex number and* $M_{jk} > 0, j \in \{1, \ldots, n\}$, $k \in \{1, \ldots, m\}$ *are given.*

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