A Note on Bessel's Inequality

A NOTE ON BESSEL'S INEQUALITY

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Abstract. A monotonicity property of Bessel's inequality in inner product spaces is given.

1. Introduction

Let X be a linear space over the real or complex number field $\mathbb{K}$. A mapping $(\cdot,\cdot) : X \times X \to \mathbb{K}$ is said to be a *positive hermitian form* if the following conditions are satisfied:

(i) $(\alpha x + \beta y, z) = \alpha (x, z) + \beta (y, z)$ for all $x, y, z \in X$ and $\alpha, \beta \in \mathbb{K}$;
(ii) $(y, x) = \overline{(x, y)}$ for all $x, y \in X$;
(iii) $(x, x) \geq 0$ for all $x \in X$.

If $\|x\| := (x, x)^{\frac{1}{2}}$, $x \in X$ denotes the semi-norm associated to this form and $(e_i)_{i \in I}$ is an orthornormal family of vectors in X, i.e., $(e_i, e_j) = \delta_{ij}$ $(i, j \in I)$, then one has the following inequality [15]:

$$\|x\|^2 \geq \sum_{i \in I} |(x, e_i)|^2 \quad \text{for all } x \in X, \tag{1.1}$$

which is well known in the literature as Bessel's inequality.

Indeed, for every finite part H of I, one has:

$$\begin{aligned} 0 &\leq \left\| x - \sum_{i \in H} (x, e_i) e_i \right\|^2 = \left(x - \sum_{i \in H} (x, e_i) e_i, x - \sum_{j \in H} (x, e_j) e_j \right) \\ &= \|x\|^2 - \sum_{i \in H} |(x, e_i)|^2 - \sum_{j \in H} |(x, e_j)|^2 + \sum_{i,j \in H} (x, e_i)(e_j, x) \delta_{ij} \\ &= \|x\|^2 - \sum_{i \in H} |(x, e_i)|^2 , \end{aligned}$$

for all $x \in X$, which proves the assertion.

The main aim of this paper is to improve this result as follows.

2. Results

The following theorem holds.

Theorem 1. *Let X be a linear space and $(\cdot,\cdot)_2, (\cdot,\cdot)_1$ two hermitian forms on X such that $\|\cdot\|_2$ is greater than or equal to $\|\cdot\|_1$, i.e., $\|x\|_2 \geq \|x\|_1$ for all $x \in X$. Assume that $(e_i)_{i \in I}$ is an orthornormal family in $(X; (\cdot,\cdot)_2)$ and $(f_i)_{i \in J}$ is an*

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orthornormal family in $(X;(\cdot,\cdot)_1)$ *such that for any* $i\in I$ *there exists a finite* $K\subset J$ *so that*

$$e_i=\sum_{j\in K}\alpha_j f_j,\ \ \alpha_j\in\mathbb{K}\ (j\in K),\tag{F}$$

then one has the inequality:

$$\|x\|_2^2-\sum_{i\in I}|(x,e_i)_2|^2\geq\|x\|_1^2-\sum_{j\in J}\left|(x,f_j)_1\right|^2\geq 0,\tag{2.1}$$

for all $x\in X$.

In order to prove this fact, we require the following lemma.

Lemma 1. *Let* X *be a linear space endowed with a positive hermitian form* $(\cdot,\cdot)$ *and* $(g_k)_{k=\overline{1,n}}$ *be an orthornormal family in* $(X;(\cdot,\cdot))$. *Then*

$$\left\|x-\sum_{k=1}^{n}\lambda_k g_k\right\|^2\geq\|x\|^2-\sum_{k=1}^{n}|(x,g_k)|^2\geq 0,\tag{2.2}$$

for all $\lambda_k\in\mathbb{K}$ *and* $x\in X$ $(k=1,...,n)$.

Proof. We will prove this fact by induction over "n".

Suppose $n=1$. Then we must prove that

$$\|x-\lambda_1 g_1\|^2\geq\|x\|^2-|(x,g_1)|^2,\ x\in X,\ \lambda_1\in\mathbb{K}.$$

A simple computation shows that the above inequality is equivalent with

$$|\lambda_1|^2-2\operatorname{Re}(x,\lambda_1 g_1)+|(x,g_1)|^2\geq 0,\ x\in X,\ \lambda_1\in\mathbb{K}.$$

Since $\operatorname{Re}(x,\lambda_1 g_1)\leq|(x,\lambda_1 g_1)|$, one has

$$\begin{aligned}|\lambda_1|^2-2\operatorname{Re}(x,\lambda_1 g_1)+|(x,g_1)|^2&\geq|\lambda_1|^2-2|\lambda_1||(x,g_1)|+|(x,g_1)|^2\\&\geq(|\lambda_1|-|(x,g_1)|)^2\geq 0\end{aligned}$$

for all $\lambda_1\in\mathbb{K}$ and $x\in X$, which proves the statement.

Now, assume that (2.2) is valid for "$(n-1)$". Then we have:

$$\begin{aligned}&\left\|x-\sum_{k=1}^{n}\lambda_k g_k\right\|^2\\&=\left\|(x-\lambda_n g_n)-\sum_{k=1}^{n-1}\lambda_k g_k\right\|\geq\|x-\lambda_n g_n\|^2-\sum_{k=1}^{n-1}|(x-\lambda_n g_n,g_k)|^2\\&=\|x-\lambda_n g_n\|^2-\sum_{k=1}^{n-1}|(x,g_k)|^2\geq\|x\|^2-|(x,g_n)|^2-\sum_{k=1}^{n-1}|(x,g_k)|^2\\&=\|x\|^2-\sum_{k=1}^{n}|(x,g_k)|^2,\end{aligned}$$

for all $\lambda_k\in\mathbb{K}$, $x\in X$ $(k=1,...,n)$, and the proof of the lemma is complete. ∎

Proof. (Theorem) Let H be a finite part of I. Since $\|\cdot\|_2$ is greater than $\|\cdot\|_1$, we have:

$$\begin{aligned}\|x\|_2^2 - \sum_{i\in H} |(x,e_i)_2|^2 &= \left\| x - \sum_{i\in H} (x,e_i)_2\, e_i \right\|_2^2 \\ &\geq \left\| x - \sum_{i\in H} (x,e_i)_2\, e_i \right\|_1^2, \quad x \in X.\end{aligned}$$

Since, by (F), we may state that for any $i \in H$ there exists a finite $K \subset J$ with

$$e_i = \sum_{j\in K} (e_i, f_j)_1\, f_j,$$

we have

$$\begin{aligned}\left\| x - \sum_{i\in H} (x,e_i)_2\, e_i \right\|_1^2 &= \left\| x - \sum_{i\in H} (x,e_i)_2 \sum_{j\in K} (e_i,f_j)_1\, f_j \right\|_1^2 \\ &= \left\| x - \sum_{j\in K} \left(\sum_{i\in H} (x,e_i)_2\, e_i, f_j \right)_1 f_j \right\|_1^2\end{aligned}$$

for all $x \in X$.

Applying the above lemma for $(\cdot,\cdot) = (\cdot,\cdot)_1$, $(g_k)_{k=\overline{1,n}} = (f_j)_{j\in K}$, we can conclude that

$$\left\| x - \sum_{j\in K} \lambda_j f_j \right\|_1^2 \geq \|x\|_1^2 - \sum_{j\in K} \left|(x,f_j)_1\right|^2, \quad x \in X,$$

where

$$\lambda_j = \left(\sum_{i\in H} (x,e_i)_2\, e_i, f_j \right)_1 \in \mathbb{K} \quad (j \in K).$$

Consequently, we have:

$$\|x\|_2^2 - \sum_{i\in H} |(x,e_i)_2|^2 \geq \|x\|_1^2 - \sum_{j\in K} \left|(x,f_j)_1\right|^2 \geq \|x\|_1^2 - \sum_{j\in J} \left|(x,f_j)_1\right|^2$$

for all $x \in X$ and H a finite part of I, from where results (2.1).

The proof is thus completed. ∎

Corollary 1. *Let $\|\cdot\|_1, \|\cdot\|_2 : X \to \mathbb{R}_+$ be as above. Then for all $x, y \in X$, we have the inequality:*

$$\|x\|_2^2 \|y\|_2^2 - |(x,y)_2|^2 \geq \|x\|_1^2 \|y\|_1^2 - |(x,y)_1|^2 \geq 0, \tag{2.3}$$

which is an improvement of the well known Cauchy-Scwartz inequality.

Proof. If $\|y\|_2 = 0$, then (2.3) holds with equality.

If $\|y\|_i \neq 0$, $(i = 1, 2)$, then for $\{e_1\} = \left\{ \frac{y}{\|y\|_2} \right\}$, $\{f_1\} = \left\{ \frac{y}{\|y\|_1} \right\}$, the above theorem yields that

$$\frac{\|x\|_2^2 \|y\|_2^2 - |(x,y)_2|^2}{\|y\|_2^2} \geq \frac{\|x\|_1^2 \|y\|_1^2 - |(x,y)_1|^2}{\|y\|_1^2}$$

and since $\|y\|_2 \geq \|y\|_1$, the inequality (2.3) is obtained. ∎

Remark 1. *For a different proof of (2.3), see also* [5].

Now, we will give some natural applications of the above theorem.

3. Applications

(1) Let $(X;(\cdot,\cdot))$ be an inner product space and $(e_i)_{i\in I}$ an orthornormal family in X. Assume that $A : X \to X$ is a linear operator such that $\|Ax\| \leq \|x\|$ for all $x \in X$ and $(Ae_i, Ae_j) = \delta_{ij}$ for all $i,j \in I$. Then one has the inequality

$$\|x\|^2 - \sum_{i\in I} |(x,e_i)|^2 \geq \|Ax\|^2 - \sum_{i\in I} |(Ax, Ae_i)|^2 \geq 0$$

for all $x \in X$.

The proof follows by the hermitian forms $(x,y)_2 = (x,y)$ and $(x,y)_1 = (Ax, Ay)$ for $x,y \in X$ and for the family $(f_i)_{i\in I} = (e_i)_{i\in I}$.

(2) If $A : X \to X$ is such that $\|Ax\| \geq \|x\|$ for all $x \in X$, then, with the previous assumptions, we also have

$$0 \leq \|x\|^2 - \sum_{i\in I} |(x,e_i)|^2 \leq \|Ax\|^2 - \sum_{i\in I} |(Ax, Ae_i)|^2 ,$$

for all $x \in X$.

(3) Suppose that $A : X \to X$ is a symmetric positive definite operator with $(Ax,x) \geq \|x\|^2$ for all $x \in X$. If $(e_i)_{i\in I}$ is an orthornormal family in X such that $(Ae_i, Ae_j) = \delta_{ij}$ for all $i,j \in I$, then one has the inequality

$$0 \leq \|x\|^2 - \sum_{i\in I} |(x,e_i)|^2 \leq (Ax,x) - \sum_{i\in I} |(Ax, e_i)|^2 ,$$

for all $x \in X$.

The proof follows from the above theorem for the choices $(x,y)_1 = (Ax,y)$ and $(x,y)_2 = (x,y)$, $x,y \in X$. We omit the details.

For other inequalities in inner product spaces, see the papers [1]-[14] and [7]-[6] where further references are given.

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