VICTORIA UNIVERSITY

MELBOURNE AUSTRALIA

Differences Between Means with Bounds from a Riemann-Stieltjes Integral

This is the Published version of the following publication

The publisher's official version can be found at

Note that access to this version may require subscription.

DIFFERENCES BETWEEN MEANS WITH BOUNDS FROM A RIEMANN-STIELTJES INTEGRAL

P. CERONE AND S.S. DRAGOMIR

ABSTRACT. An identity for the difference between two integral means is obtained in terms of a Riemann-Stieltjes integral. This enables bounds to be procured when the integrand is of bounded variation, Lipschitzian and monotonic. If f is absolutely continuous, bounds are also obtained for $f' \in L_p[a, b]$, $1 \le p < \infty$, the usual Lebesgue norms. This supplements earlier results involving $f' \in L_\infty[a, b]$.

1. INTRODUCTION

The following theorem was proved in Barnett et al. [2].

Theorem 1. *Let* $f : [a, b] \to \mathbb{R}$ *be an absolutely continuous mapping with the property that* $f' \in L_\infty[a, b]$, *i.e.,*

$$\|f'\|_\infty := ess \sup_{t \in [a,b]} |f'(t)| < \infty.$$

Then for $a \le c < d \le b$, *we have the inequality*

$$(1.1) \qquad \left| \frac{1}{b-a} \int_a^b f(t)\, dt - \frac{1}{d-c} \int_c^d f(u)\, du \right|$$

$$\le \left\{ \frac{1}{4} + \left[\frac{\frac{a+b}{2} - \frac{c+d}{2}}{(b-a)-(d-c)} \right]^2 \right\} [(b-a)-(d-c)] \|f'\|_\infty$$

$$\le \frac{1}{2} [(b-a)-(d-c)] \|f'\|_\infty.$$

The constant $\frac{1}{4}$ *is best possible in the first inequality and* $\frac{1}{2}$ *is best in the second one.*

They utilised the identity

$$(1.2) \qquad \frac{1}{b-a} \int_a^b f(t)\, dt - \frac{1}{d-c} \int_c^d f(u)\, du = \int_a^b K_{c,d}(s)\, f'(s)\, ds,$$

where

$$(1.3) \qquad K_{c,d}(s) := \begin{cases} \dfrac{a-s}{b-a} & \text{if} \quad s \in [a,c]; \\[2mm] \dfrac{s-c}{d-c} + \dfrac{a-s}{b-a} & \text{if} \quad s \in (c,d); \\[2mm] \dfrac{b-s}{b-a} & \text{if} \quad s \in [d,b]. \end{cases}$$

It was demonstrated that the Ostrowski inequality [12], represented by the following theorem, could be recaptured by using some limiting procedure:

Theorem 2. *Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b) and assume that $|f'(x)| \leq M$ for all $x \in (a,b)$. Then we have the inequality*

$$(1.4) \qquad \left| f(x) - \frac{1}{b-a} \int_a^b f(t)\,dt \right| \leq \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] (b-a)\,M$$

for all $x \in [a,b]$. The constant $\frac{1}{4}$ is the best possible.

For some generalisations and related results, see the book [11, p. 468 - 484], the papers [1] – [12] and the website http://rgmia.vu.edu.au/ where many papers devoted to this inequality can be accessed on line.

It is the aim of the current article to obtain bounds on

$$(1.5) \qquad D(f;a,c,d,b) := \mathcal{M}(f;a,b) - \mathcal{M}(f;c,d), \quad a \leq c < d \leq b,$$

where

$$\mathcal{M}(f;a,b) := \frac{1}{b-a} \int_a^b f(t)\,dt,$$

in terms of the Lebesgue norms $\|f'\|_p$, $1 \leq p < \infty$ with $f' \in L_p[a,b]$ implying $\left(\int_a^b |f'(t)|^p\,dt \right)^{\frac{1}{p}} < \infty$, $1 \leq p < \infty$.

Further bounds on $|D(f;a,c,d,b)|$ will be obtained under less restrictive assumptions than absolute continuity on f. Bounds are obtained for f Hölder continuous in Section 2 while in Section 3 bounds are obtained for f of bounded variation, Lipschitzian and monotonic.

2. Results for $f' \in L_p[a,b]$, $1 \leq p < \infty$

The following theorem holds (see also Cerone and Dragomir [3]).

Theorem 3. *Let $f : [a,b] \to \mathbb{R}$ be an absolutely continuous mapping. Then for $a \leq c < d \leq b$ the inequalities*

$$(2.1) \qquad |D(f;a,c,d,b)|$$

$$\leq \begin{cases} \dfrac{b-a}{(q+1)^{\frac{1}{q}}} \left[1 + \left(\dfrac{\rho}{1-\rho} \right)^q \right]^{\frac{1}{q}} \left[\nu^{q+1} + \lambda^{q+1} \right]^{\frac{1}{q}} \|f'\|_p, \\ \qquad\qquad f' \in L_p[a,b], \;\; 1 \leq p < \infty, \; \frac{1}{p} + \frac{1}{q} = 1; \\[3mm] \left[1 - \rho + |\nu + \rho + \lambda| \right] \|f'\|_1, \;\; f' \in L_1[a,b], \end{cases}$$

where $(b-a)\nu = c-a$, $(b-a)\rho = d-c$, $(b-a)\lambda = b-d$.

Proof. From (1.2) and (1.5) we have on using Hölder's integral inequality that

$$(2.2) \quad |D(f; a, c, d, b)| \leq \left(\int_a^b |K_{c,d}(s)|^q \, dt \right)^{\frac{1}{q}} \|f'\|_p, \quad 1 \leq p < \infty, \quad \frac{1}{p} + \frac{1}{q} = 1.$$

Now,

$$(2.3) \quad \int_a^c |s - a|^q \, ds = \frac{(c-a)^{q+1}}{q+1} \quad \text{and} \quad \int_d^b |b - s|^q \, ds = \frac{(b-d)^{q+1}}{q+1}.$$

Further,

$$
\begin{aligned}
M \quad : \quad &= \int_c^d \left| \frac{s-c}{d-c} + \frac{a-s}{b-a} \right|^q ds \\
&= \frac{1}{(b-a)(d-c)} \int_c^d |[(b-a)-(d-c)]s - cb + ad|^q \, ds \\
&= \frac{b-a-(d-c)}{(b-a)(d-c)} \int_c^d |s - s_0|^q \, ds
\end{aligned}
$$

since $b - a > d - c$ and

$$(2.4) \quad s_0 = \frac{cb - ad}{(b-a)-(d-c)} \in [c, d].$$

Hence, as $c - s_0 < 0$ and $d - s_0 > 0$,

$$
\begin{aligned}
M \quad &= \quad \frac{b-a-(d-c)}{(b-a)(d-c)} \left[\int_c^{s_0} (s_0 - s)^q \, ds + \int_{s_0}^d (s - s_0)^q \, ds \right] \\
&= \quad \frac{b-a-(d-c)}{(b-a)(d-c)} \cdot \frac{(s_0 - c)^{q+1} + (d - s_0)^{q+1}}{q+1}.
\end{aligned}
$$

Further simplification may be accomplished since

$$s_0 - c = \frac{(c-a)(d-c)}{(b-a)(d-c)} \quad \text{and} \quad d - s_0 = \frac{(d-c)(b-d)}{(b-a)(d-c)},$$

giving

$$M = \frac{(d-c)^q}{(q+1)(b-a)[(b-a)-(d-c)]^q} \left[(c-a)^{q+1} + (b-d)^{q+1} \right].$$

Thus, combining the expression for M with (2.3) and using (2.2) gives (2.1) after some algebra.

Now, for the second inequality

$$|D(f; a, c, d, b)| \leq \sup_{s \in [a,b]} |K_{c,d}(s)| \int_a^b |f'(s)| \, ds.$$

From (1.3) it is easily seen that $K_{c,d}(s)$ is a piecewise linear and continuous function. It is negative on (a, s_0) and positive on (s_0, b). It reaches its extremities at c and d. Thus,

$$
\begin{aligned}
\sup_{s \in [a,b]} |K_{c,d}(s)| \quad &= \quad \max \left\{ \frac{c-a}{b-a}, \frac{b-d}{b-a} \right\} \\
&= \quad \frac{1}{b-a} \left\{ \frac{c-a+b-d}{2} + \left| \frac{(c-a)-(b-d)}{2} \right| \right\}.
\end{aligned}
$$

A simple rearrangement gives the result as stated. ∎

Remark 1. *If we take $q = 1$ in (2.1) then the first inequality in (1.1) is recaptured.*

3. Some Inequalities for Mappings of Hölder Type

If we drop the assumption of absolute continuity and allow f to be Hölder continuous, then the following result is valid.

Theorem 4. *Assume that the mapping $f : [a, b] \to \mathbb{R}$ is of $r - H-$Hölder type, i.e.,*

$$(3.1) \qquad |f(t) - f(s)| \leq H |t - s|^r \quad for \ all \ \ t, s \in [a, b],$$

where $r \in (0, 1]$ and $H > 0$ are given.
     Then for $a < c < d < b$, we have the inequality

$$(3.2) \qquad \left| \frac{1}{b - a} \int_a^b f(t) \, dt - \frac{1}{d - c} \int_c^d f(t) \, dt \right| \leq \frac{(c - a)^{r+1} + (b - d)^{r+1}}{[(b - a) - (d - c)](r + 1)} \cdot H.$$

The inequality (2.2) is best in the sense that we cannot put in the right hand side a constant K less than 1.

Proof. Write

$$\frac{1}{b - a} \int_a^b f(t) \, dt = \int_0^1 f(ub + (1 - u) a) \, du$$

and similarly for the second term.
     Then

$$I \quad : \quad = \frac{1}{b - a} \int_a^b f(t) \, dt - \frac{1}{d - c} \int_c^d f(t) \, dt$$

$$= \int_0^1 [f(ub + (1 - u) a) - f(ud + (1 - u) c)] \, du.$$

Using the fact that f is of $r - H-$Hölder type, we have

$$(3.3) \qquad |I| \quad \leq \quad \int_0^1 |f(ub + (1 - u) a) - f(ud + (1 - u) c)| \, du$$

$$\leq \quad H \int_0^1 |ub + (1 - u) a - ud - (1 - u) c|^r \, du$$

$$= \quad H \int_0^1 |u[b - a - (d - c)] - (c - a)|^r \, du.$$

Now, as $b - a > d - c$, then $u_0 := \frac{c-a}{(b-a)-(d-c)} \in (0,1)$ and

$$\int_0^1 |u[b - a - (d - c)] - (c - a)|^r \, du$$

$$= \int_0^{u_0} [(c - a) - u[b - a - (d - c)]]^r \, du + \int_{u_0}^1 [u[b - a - (d - c)] - (c - a)]^r \, du$$

$$= -\frac{1}{b - a - (d - c)} \left. \frac{((c - a) - u[(b - a) - (d - c)])^{r+1}}{r + 1} \right|_0^{u_0}$$

$$+ \frac{1}{b - a - (d - c)} \left. \frac{(u[(b - a) - (d - c)] - (c - a))^{r+1}}{r + 1} \right|_{u_0}^1$$

$$= -\frac{1}{(b - a) - (d - c)} \frac{((c - a) - u_0[(b - a) - (d - c)])^{r+1}}{r + 1}$$

$$+ \frac{(c - a)^{r+1}}{(r + 1)(b - a - (d - c))} + \frac{1}{(b - a) - (d - c)} \frac{([b - a - (d - c)] - c + a)^{r+1}}{r + 1}$$

$$- \frac{(u_0[b - a - (d - c)] - (c - a))^{r+1}}{[b - a - (d - c)](r + 1)}$$

$$= \frac{(c - a)^{r+1} + (b - d)^{r+1}}{[(b - a) - (d - c)](r + 1)}.$$

Using (3.3) we deduce (3.2).

Assume that now, the inequality (2.2) holds with a constant $K > 0$, i.e.,

$$(3.4) \quad \left| \frac{1}{b - a} \int_a^b f(t) \, dt - \frac{1}{d - c} \int_c^d f(t) \, dt \right| \leq K \cdot \frac{(c - a)^{r+1} + (b - d)^{r+1}}{[(b - a) - (d - c)](r + 1)} H.$$

Choose $f_0 : [0, 1] \to \mathbb{R}$, $f_0(t) = t^r$, $r \in (0, 1]$. Then

$$|f_0(t) - f_0(s)| \leq |t - s|^r \quad \text{for all} \quad t, s \in [0, 1],$$

which shows that f_0 is of $r - H-$Hölder type with $H = 1$.

Now, choose in (3.4) $a = 0$, $b = 1$, $c \in (0, 1)$, $d = c + \varepsilon$, ε small and such that $c + \varepsilon \in (0, 1)$. Then we get

$$\left| \frac{1}{r + 1} - \frac{(c + \varepsilon)^{r+1} - c^{r+1}}{\varepsilon (r + 1)} \right| \leq K \cdot \frac{c^{r+1} + (1 - c - \varepsilon)^{r+1}}{(1 - \varepsilon)(r + 1)},$$

which is clearly equivalent with

$$(3.5) \quad \left| 1 - \frac{(c + \varepsilon)^{r+1} - c^{r+1}}{\varepsilon} \right| \leq K \cdot \frac{c^{r+1} + (1 - c - \varepsilon)^{r+1}}{1 - \varepsilon}.$$

Now, if in (3.5) we let $\varepsilon \to 0+$, then we get

$$(3.6) \quad |1 - (r + 1)c^r| \leq K \left[c^{r+1} + (1 - c)^{r+1} \right] \quad \text{for all} \quad c \in (0, 1).$$

If in (3.6) we let $c \to 0+$, then we get $1 \leq K$, and the theorem is completely proved. ∎

4. Results for the Riemann-Stieltjes Integral

The results obtained to date for bounds for differences of integral means assume that f is differentiable. That is, f is absolutely continuous. This assumption may be relaxed somewhat and bounds on $D(f; a, c, d, b)$ may still be procured. The following lemma holds.

Lemma 1. Let $f : [a, b] \to \mathbb{R}$ be of bounded variation on $[a, b]$, then

$$(4.1) \qquad D(f; a, c, d, b) = \int_a^b K_{c,d}(s) \, df(s),$$

where $K_{c,d}$ is as given by (1.3) and $D(f; a, c, d, b)$ is as defined by (1.5).

Proof. The proof follows closely that used in obtaining (1.2). The integration by parts formula is used for Riemann-Stieltjes integrals to give

$$
\begin{aligned}
& (b - a)(d - c) \int_a^b K_{c,d}(s) \, df(s) \\
= \ & (d - c) \int_a^c (a - s) \, df(s) + \int_c^d [(b - a)(s - c) - (d - c)(s - a)] \, df(s) \\
& + (d - c) \int_d^b (b - s) \, df(s) \\
= \ & (d - c) \left\{ (a - s) f(s) \big]_a^c + \int_a^c f(s) \, ds \right\} + [(b - a)(s - c) - (d - c)(s - a)] f(s) \big]_c^d \\
& - [(b - a) - (d - c)] \int_c^d f(s) \, ds + (d - c) \left\{ (b - s) f(s) \big]_d^b + \int_d^b f(s) \, ds \right\} \\
= \ & (d - c) \left\{ (a - c) f(c) + \int_a^c f(s) \, ds \right\} + [(b - a)(d - c) - (d - c)(d - a)] f(d) \\
& + (d - c)(c - a) f(c) - [(b - a) - (d - c)] \int_c^d f(s) \, ds \\
& + (d - c) \left[\int_c^d f(s) \, ds - (b - d) f(d) \right] \\
= \ & (d - c) \int_c^d f(s) \, ds - (b - a) \int_c^d f(s) \, ds.
\end{aligned}
$$

Division by $(b - a)(d - c)$ produces (4.1) on noting the definition (1.5). ∎

The following well known lemmas involving Riemann-Stieltjes integrals are well known. They are stated here for clarity. (See Cerone and Dragomir [4] where they were applied to three point rules in numerical integration.)

Lemma 2. Let $g, v : [a, b] \to \mathbb{R}$ be such that g is continuous on $[a, b]$ and v is of bounded variation on $[a, b]$. Then the Riemann-Stieltjes integral $\int_a^b g(t) \, dv(t)$ exists and is such that

$$(4.2) \qquad \left| \int_a^b g(t) \, dv(t) \right| \leq \sup_{t \in [a,b]} |g(t)| \bigvee_a^b (v),$$

where $\bigvee_a^b (v)$ is the total variation of v on $[a, b]$.

Lemma 3. *Let $g, v : [a, b] \to \mathbb{R}$ be such that g is Riemann integrable on $[a, b]$ and v is L-Lipschitzian on $[a, b]$. Then*

$$(4.3) \qquad \left| \int_a^b g(t) \, dv(t) \right| \leq L \int_a^b |g(t)| \, dt$$

with v being $L-$Lipschitzian if it satisfies

$$|v(x) - v(y)| \leq L |x - y|$$

for all $x, y \in [a, b]$.

Lemma 4. *Let $g, v \in [a, b] \to \mathbb{R}$ be such that g is Riemann integrable on $[a, b]$ and v is monotonic nondecreasing on $[a, b]$. Then*

$$(4.4) \qquad \left| \int_a^b g(t) \, dv(t) \right| \leq \int_a^b |g(t)| \, dv(t).$$

Theorem 5. *Let $f : [a, b] \to \mathbb{R}$ be of bounded variation on $[a, b]$. The following bounds hold*

$$(4.5) \qquad |D(f; a, c, d, b)|$$

$$\leq \begin{cases} \left[\dfrac{b - a - (d - c)}{2} + \left| \dfrac{c + d}{2} - \dfrac{a + b}{2} \right| \right] \dfrac{\bigvee_a^b (f)}{b - a}, & \\[3mm] \dfrac{(c - a)^2 + (b - d)^2}{2 \left[(b - a) - (d - c) \right]} L, & \text{for } f \; L - Lipschitzian; \\[3mm] \left(\dfrac{b - d}{b - a} \right) f(b) - \left(\dfrac{c - a}{b - a} \right) f(a) & \\[2mm] \quad + \left[\dfrac{c + d - (a + b)}{b - a} \right] f(s_0), & \text{for } f \text{ monotonic} \\[2mm] & \qquad \text{nondecreasing,} \end{cases}$$

where $S_0 = \frac{cb - ad}{(b-a) - (d-c)}$.

Proof. Using Lemma 2, we have from (4.2)

$$(4.6) \qquad \left| \int_a^b K_{c,d}(s) \, df(s) \right| \leq \sup_{s \in [a,b]} |K_{c,d}(s)| \bigvee_a^b (f).$$

Now, $K_{c,d}(a) = K_{c,d}(b) = K_{c,d}(s_0) = 0$. Further, $K_{c,d}(s)$ consists of straight line segments on $[a, c]$, $[c, d]$ and $[d, b]$. The extreme values occur at c and d. Thus,

$$\begin{aligned} \sup_{s \in [a,b]} |K_{c,d}(s)| &= \max \{ |K_{c,d}(c)|, |K_{c,d}(d)| \} = \max \left\{ \frac{c - a}{b - a}, \frac{b - d}{b - a} \right\} \\ &= \frac{1}{b - a} \left[\frac{c - a + b - d}{2} + \left| \frac{c - a - (b - d)}{2} \right| \right], \end{aligned}$$

which on rearrangement and using (4.6) gives the first inequality in (4.5).

Now, for the second inequality, we use Lemma 3 and so from (4.3)

$$(4.7) \qquad \left| \int_a^b K_{c,d}(s) \, df(s) \right| \leq L \int_a^b |K_{c,d}(s)| \, ds,$$

where

$$(4.8) \quad \int_a^b |K_{c,d}(s)| \, ds$$

$$= \quad \frac{1}{b-a} \int_a^c (s-a) \, ds + \frac{(b-a)-(d-c)}{(b-a)(d-c)} \int_c^d |s-s_0| \, ds + \frac{1}{b-a} \int_d^b (b-s) \, ds$$

with S_0 as given by (2.4).

Now,

$$\frac{1}{b-a} \int_a^c (s-a) \, ds \quad = \quad \frac{(c-a)^2}{2(b-a)},$$

$$\frac{1}{b-a} \int_d^b (b-s) \, ds \quad = \quad \frac{(b-d)^2}{2(b-a)}$$

and

$$\int_c^d |s-s_0| \, ds \quad = \quad \int_c^{s_0} (s_0-s) \, ds + \int_{s_0}^d (s-s_0) \, ds$$

$$= \quad \frac{(s_0-c)^2 + (d-s_0)^2}{2}.$$

That is, using (2.4), we have

$$\frac{(b-a)-(d-c)}{(b-a)(d-c)} \int_c^d |s-s_0| \, ds$$

$$= \quad \frac{(d-c)}{2(b-a)[(b-a)-(d-c)]} \left[(c-a)^2 + (b-d)^2 \right],$$

and so combining the above results and using (4.7) and (4.8) gives the second inequality.

For the final inequality in (4.5) we use Lemma 4 giving from (4.4), for f monotonic nondecreasing,

$$(4.9) \qquad \left| \int_a^b K_{c,d}(s) \, df(s) \right| \le \int_a^b |K_{c,d}(s)| \, df(s).$$

Using the properties of $K_{c,d}(S)$ discussed earlier that $K_{c,d}(S) < 0$ for $S \in (c, s_0)$ and $K_{c,d}(S) > 0$ for $S \in (s_0, b)$ and zero at a, s_0 and b, then from (1.3)

$$(4.10) \qquad \int_a^b |K_{c,d}(s)| \, df(s)$$

$$= \quad \int_a^c \left(\frac{s-a}{b-a} \right) df(s) + \int_c^{s_0} \left(\frac{s-a}{b-a} + \frac{s-c}{d-c} \right) df(s)$$

$$+ \int_{s_0}^d \left(\frac{s-c}{d-c} + \frac{a-s}{b-a} \right) df(s) + \int_d^b \left(\frac{b-s}{b-a} \right) df(s).$$

Now, integration by parts of the Riemann-Stieltjes integrals on the right of (4.10) produces, after some simplification,

$$
\begin{aligned}
\int_a^b |K_{c,d}(s)|\, df(s) \;=\;& -\frac{1}{b-a}\int_a^c f(s)\,ds + \left(\frac{1}{d-c}-\frac{1}{b-a}\right)\int_c^{s_0} f(s)\,ds \\
& -\left(\frac{1}{d-c}-\frac{1}{b-a}\right)\int_{s_0}^d f(s)\,ds + \frac{1}{b-a}\int_d^b f(s)\,ds \\
\leq\;& -\left(\frac{c-a}{b-a}\right) f(a) + \left(\frac{1}{d-c}-\frac{1}{b-a}\right)(s_0-c) f(s_0) \\
& -\left(\frac{1}{d-c}-\frac{1}{b-a}\right)(d-s_0) f(s_0) + \left(\frac{b-d}{b-a}\right) f(b),
\end{aligned}
$$

where we have used the fact that f is monotonic nondecreasing to obtain the last inequality. Thus, from (3.5) we obtain the third inequality in (4.5) on grouping terms and simplifying. ∎

Remark 2. *From (4.10) we may use the fact that $\displaystyle\sup_{s\in[a,b]} |K_{c,d}(S)|$ occurs at c for $s\in[a,s_0]$ and at d for $s\in[s_0,b]$ to give for f monotonic nondecreasing:*

$$
\begin{aligned}
& \left|\int_a^b K_{c,d}(s)\, df(s)\right| \\
\leq\;& \left(\frac{c-a}{b-a}\right)(f(c)-f(a)) + \left(\frac{c-a}{b-a}\right)(f(s_0)-f(c)) \\
& +\left(1-\frac{d-a}{b-a}\right)(f(d)-f(s_0)) + \left(\frac{b-d}{b-a}\right)(f(b)-f(d)) \\
=\;& \left(\frac{c-a}{b-a}\right)(f(s_0)-f(a)) + \left(\frac{b-d}{b-a}\right)(f(b)-f(s_0)) \\
=\;& \left(\frac{b-d}{b-a}\right) f(b) - \left(\frac{c-a}{b-a}\right) f(a) + \frac{c+d-(a+b)}{b-a} f(s_0).
\end{aligned}
$$

Remark 3. *If we put $r=1$ and $H=L$ in (4.2), then we obtain the second inequality in (4.5). It should also be noted that if the parallelogram identity*

$$
2\left(x^2+y^2\right) = (x-y)^2 + (x+y)^2, \quad x,y\in\mathbb{R}
$$

is used then

$$
\frac{(c-a)^2+(b-d)^2}{2\left[(b-a)-(d-c)\right]} = \left[\frac{1}{4} + \left(\frac{\frac{a+b}{2}-\frac{c+d}{2}}{(b-a)-(d-c)}\right)^2\right]\left[(b-a)-(d-c)\right],
$$

with $\frac{1}{4}$ being the best constant attained when $\frac{a+b}{2}=\frac{c+d}{2}$.

Remark 4. *If we assume that there is a point $x\in(a,b)$ for which the functions is continuous, then we may recapture bounds for the Ostrowski functional*

$$
\theta(f)(x) := f(x) - \mathcal{M}(f).
$$

Indeed, if we assume that $c = x \in (a, b)$, $d = x + \varepsilon$ where $x + \varepsilon \in (a, b)$, then from (2.1) for example

$$
\begin{aligned}
&|D\left(f; a, x, x + \varepsilon, b\right)| \\
=\ & \left| \mathcal{M}\left(f; a, b\right) - \frac{1}{\varepsilon} \int_{x}^{x+\varepsilon} f\left(u\right) du \right| \\
\leq\ & \frac{b - a}{\left(q + 1\right)^{\frac{1}{q}}} \left[1 + \left(\frac{\varepsilon}{b - a - \varepsilon} \right)^{q} \right] \left[\left(\frac{x - a}{b - a} \right)^{q+1} + \left(\frac{b - x - \varepsilon}{b - a} \right)^{q+1} \right]^{\frac{1}{q}} \|f'\|_{p}.
\end{aligned}
$$

Now, taking the limit as $\varepsilon \to 0+$ gives

$$
\begin{aligned}
|\theta\left(f\right)\left(x\right)| \ &\leq\ \frac{b - a}{\left(q + 1\right)^{\frac{1}{q}}} \left[\left(\frac{x - a}{b - a} \right)^{q+1} + \left(\frac{b - x}{b - a} \right)^{q+1} \right]^{\frac{1}{q}} \|f'\|_{p}, \\
f' \ &\in\ L_{p}\left[a, b\right], \ 1 \leq p < \infty,
\end{aligned}
$$

recapturing a result of Dragomir and Wang [9].

References

[1] G.A. ANASTASSIOU, Ostrowski type inequalities, *Proc. of Amer. Math. Soc.,* **123**(12) (1999), 3775-3781.

[2] N.S. BARNETT, P. CERONE, S.S. DRAGOMIR and A.M. FINK, Comparing two integral means for absolutely continuous mappings whose derivatives are in $L_{\infty}\left[a, b\right]$ and applications, *Comp. and math. with Applics.* (accepted).

[3] P. CERONE and S.S. DRAGOMIR, On some inequalities arising from Montgomery's identity, *J. Comp. Anal. Applics.,* (accepted).

[4] P. CERONE and S.S. DRAGOMIR, Three point quadrature rules involving, at most, a first derivative, (submitted) (ON LINE: `http://rgmia.vu.edu.au/v2n4.html`).

[5] S.S. DRAGOMIR, On the Ostrowski's integral inequality for mappings with bounded variation and applications, *Math. Ineq. & Appl.,* **4**(1) (2001), 59-66. (ON LINE: `http://rgmia.vu.edu.au/v2n1.html`).

[6] S.S. DRAGOMIR, Ostrowski inequality for monotonous mappings and applications, *J. KSIAM,* **3**(1) (1999), 127-135.

[7] S.S. DRAGOMIR, The Ostrowski's integral inequality for Lipschitzian mappings and applications, *Comp. and Math. with Appl.,* **38** (1999), 33-37.

[8] S.S. DRAGOMIR, The Ostrowski integral inequality for mappings of bounded variation, *Bull. Austral. Math. Soc.,* **60** (1999), 495-508.

[9] S.S. DRAGOMIR and S. WANG, A new inequality of Ostrowski's type in $L_{p}-$norm and applications to some special means and to some numerical quadrature rules, *Indian Journal of Mathematics,* **40**(3) (1998), 299-304.

[10] A.M. FINK, Bounds on the derivative of a function from its averages, *Czech. Math. J.,* **42** (1992), 289-310.

[11] D.S. MITRINOVIĆ, J.E. PEČARIĆ and A.M. FINK, *Inequalities Involving Functions and Their Integrals and Derivatives,* Kluwer Academic Publishers, Dordrecht, 1994.

[12] A. OSTROWSKI, Uber die Absolutabweichung einer differentienbaren Funktionen von ihren Integralmittelwert, *Comment. Math. Hel,* **10** (1938), 226-227.

School of Communications and Informatics, Victoria University of Technology, PO Box 14428, Melbourne City, MC 8001,, Victoria, Australia.

E-mail address: `pc@matilda.vu.edu.au`

URL: `http://sci.vu.edu.au/staff/peterc.html`

E-mail address: `sever@matilda.vu.edu.au`

URL: `http://rgmia.vu.edu.au/SSDragomirWeb.html`