A Generalization of Ky Fan's Inequality

A GENERALIZATION OF KY FAN'S INEQUALITY

TSZ HO CHAN AND PENG GAO

Abstract. Let $P_{n,r}(\mathbf{x})$ be the generalized weighted means. Let $F(x)$ be a C^1 function, $y = y(x)$ an implicit decreasing function defined by $f(x, y) = 0$ and $0 < m < M \leq m', n \geq 2, x_i \in [m, M], y_i \in [m', M']$. Then for $-1 \leq r \leq 1$, if $f_x/f_y \leq 1$

$$|\frac{F(P_{n,1}(\mathbf{y})) - F(P_{n,r}(\mathbf{y}))}{F(P_{n,1}(\mathbf{x})) - F(P_{n,r}(\mathbf{x}))}| < \frac{Max_{m' \leq \xi \leq M'} |F'(\xi)|}{Min_{m \leq \eta \leq M} |F'(\eta)|} \cdot \frac{M}{m'}$$

A similar result exists for $f_x/f_y \geq 1$. By specifying $f(x, y)$ and $F(x)$, we get various generalizations of Ky Fan's inequality. We also present some results on the comparison of $P^{\alpha}_{n,s}(\mathbf{y}) - P^{\alpha}_{n,r}(\mathbf{y})$ and $P^{\alpha}_{n,s}(\mathbf{x}) - P^{\alpha}_{n,r}(\mathbf{x})$ for $s \geq r, \alpha \in R$.

1. Introduction

Let $P_{n,r}(\mathbf{x})$ be the generalized weighted means: $P_{n,r}(\mathbf{x}) = (\sum_{i=1}^{n} \omega_i x_i^r)^{\frac{1}{r}}$, where $w_i, 1 \leq i \leq n$ are positive real numbers with $\sum_{i=1}^{n} w_i = 1$ and $\mathbf{x} = (x_1, x_2, \cdots, x_n)$. Here we denote $P_{n,0}(\mathbf{x})$ as $\lim_{r \to 0^+} P_{n,r}(\mathbf{x})$. Let $f(x, y)$ be a real function, we write $f(\mathbf{x}, \mathbf{y}) = 0$ for $\mathbf{y} = (y_1, y_2, \cdots, y_n)$ such that $f(x_i, y_i) = 0, 1 \leq i \leq n$.

In this paper, we always assume $x_1 \leq x_2 \leq \cdots \leq x_n$ and denote $x_1 = m, x_n = M$, $y_1 = M', y_n = m'$. We also write $A_n = P_{n,1}(\mathbf{x}), G_n = P_{n,0}(\mathbf{x}), H_n = P_{n,-1}(\mathbf{x}), A'_n = P_{n,1}(\mathbf{y})$, $G'_n = P_{n,0}(\mathbf{y}), H'_n = P_{n,-1}(\mathbf{y})$.

The following inequality, originally due to Ky Fan, was first published in the monograph *Inequalities* by Beckenbach and Bellman [6, p.5]:

Theorem I. *For $f(x, y) = x + y - 1$, $x_i \in [0, 1/2]$,*

$$\frac{A'_n}{G'_n} \leq \frac{A_n}{G_n} \tag{1.1}$$

with equality holding if and only if $x_1 = \cdots = x_n$.

Ky Fan's inequality has evoked the interest of several mathematicians and many papers appeared providing new proofs, generalizations and sharpenings of (1.1). We refer the reader to the survey article[3] and the references therein.

Under the same condition of theorem I, the following additive analogue of (1.1) was proved by H. Alzer[1]:

Theorem II.

$$A'_n - G'_n \leq A_n - G_n \tag{1.2}$$

with equality holding if and only if $x_1 = \cdots = x_n$.

Refinements of (1.2), (1.1) were obtained by H.Alzer([4], [5]) in the following two theorems, respectively:

Date: March 20, 2001.

1991 *Mathematics Subject Classification.* Primary 26D15, 26D20.

Key words and phrases. Ky Fan's inequality, generalized weighted means.

Theorem III. *Let* $x_i \in (0, \frac{1}{2}]$ $(i = 1, 2, \cdots, n; n \geq 2)$ *and* $m < M$,

$$\frac{m}{1-m} < \frac{A'_n - G'_n}{A_n - G_n} < \frac{M}{1-M} \tag{1.3}$$

Theorem IV. *Let* $x_i \in [a, b]$ $(i = 1, 2, \cdots, n; 0 < a < b < 1)$,

$$(\frac{A_n}{G_n})^{(\frac{a}{1-a})^2} < \frac{A'_n}{G'_n} < (\frac{A_n}{G_n})^{(\frac{b}{1-b})^2} \tag{1.4}$$

Recently, A.M. Mercer obtained the following generalized Ky Fan's inequality[8]:

Theorem V. *For* $f(x, y) = x^p + y^p - 1, p \geq 1, n \geq 2, x_i \in [0, 2^{-(1/p)}]$,

$$P_{n,1}(\mathbf{x})P_{n,0}(\mathbf{y}) \geq P_{n,1}(\mathbf{y})P_{n,0}(\mathbf{x}) \tag{1.5}$$

with equality holding if and only if $x_1 = \cdots = x_n$.

In this paper, we will present the following theorem which will provide essentially a unified treatment of Theorem I - V and it also gives new extensions of Ky Fan's inequality:

Theorem 1.1. *Let* $F(x)$ *be a* C^1 *function,* $y = y(x)$ *an implicit decreasing function defined by* $f(x, y) = 0$ *and* $0 < m < M \leq m', n \geq 2$. *Then for* $-1 \leq r \leq 1$, *if* $f_x/f_y \leq 1$

$$|\frac{F(P_{n,1}(\mathbf{y})) - F(P_{n,r}(\mathbf{y}))}{F(P_{n,1}(\mathbf{x})) - F(P_{n,r}(\mathbf{x}))}| < \frac{Max_{m' \leq \xi \leq M'}|F'(\xi)|}{Min_{m \leq \eta \leq M}|F'(\eta)|} \cdot \frac{M}{m'} \tag{1.6}$$

If $f_x/f_y \geq 1$

$$\frac{Min_{m' \leq \xi \leq M'}|F'(\xi)|}{Max_{m \leq \eta \leq M}|F'(\eta)|} \cdot \frac{m}{M'} < |\frac{F(P_{n,1}(\mathbf{y})) - F(P_{n,r}(\mathbf{y}))}{F(P_{n,1}(\mathbf{x})) - F(P_{n,r}(\mathbf{x}))}| \tag{1.7}$$

provided the denominators on both sides are nonzero.

In section 3, applications to Ky Fan's inequality will be given by specifying the functions $f(x, y), F(x)$.

More generally, we can talk about the comparison of $P^\alpha_{n,s}(\mathbf{y}) - P^\alpha_{n,r}(\mathbf{y})$ and $P^\alpha_{n,s}(\mathbf{x}) - P^\alpha_{n,r}(\mathbf{x})$ for real α. The case of $A'^\alpha_n - G'^\alpha_n$ and $A^\alpha_n - G^\alpha_n$ was discussed in [5] and we will give some results related to the general case in section 4.

2. Proof of Theorem 1.1

Since the proofs of (1.6) and (1.7) are very similar, we only prove (1.6) for $r \neq 0$ here, the case $r = 0$ is also similar. We will consider the case $F(x) = x$ first, We define for $1 \leq i \leq n-1$ and $0 < x < x_{i+1}$:

$$\begin{aligned}
\mathbf{x_i} &= (x, \cdots, x, x_{i+1}, \cdots, x_n) \\
\mathbf{y_i} &= (y, \cdots, y, y_{i+1}, \cdots, y_n) \\
D(\mathbf{x}_i) &= x_n(P_{n,1}(\mathbf{x}_i) - P_{n,r}(\mathbf{x}_i)) - y_n(P_{n,1}(\mathbf{y}_i) - P_{n,r}(\mathbf{y}_i)) \\
g(\mathbf{x}_i) &= x_n P_{n,r}(\mathbf{x}_i)^{1-r} \cdot x^{r-1} + y_n P_{n,r}(\mathbf{y}_i)^{1-r} \cdot y^{r-1}
\end{aligned}$$

and $D_i(x) = D(\mathbf{x}_i), g_i(x) = g(\mathbf{x}_i)$.

We need to show $D_1(x_1) > 0$ and differentiation yields

$$\begin{aligned}
\Omega_i^{-1} D'_i(x) &= x_n(1 - P_{n,r}(\mathbf{x}_i)^{1-r} \cdot x^{r-1}) + y_n \frac{f_x}{f_y}(1 - P_{n,r}(\mathbf{y}_i)^{1-r} \cdot y^{r-1}) \\
&\leq x_n(1 - P_{n,r}(\mathbf{x}_i)^{1-r} \cdot x^{r-1}) + y_n(1 - P_{n,r}(\mathbf{y}_i)^{1-r} \cdot y^{r-1}) \\
&= x_n + y_n - g_i(x)
\end{aligned}$$

where $\Omega_i = \sum_{j=1}^i \omega_i$.

Consider

$$\begin{aligned} g_i'(x) &= -(1-r)\sum_{j=i+1}^{n}\omega_j[(\frac{P_{n,r}(\mathbf{x}_i)}{x})^{1-2r}\cdot\frac{x_n x_j^r}{x^{r+1}}-\frac{f_x}{f_y}\cdot(\frac{P_{n,r}(\mathbf{y}_i)}{y})^{1-2r}\cdot\frac{y_n y_j^r}{y^{r+1}}] \\ &\leq -(1-r)\sum_{j=i+1}^{n}\omega_j[(\frac{P_{n,r}(\mathbf{x}_i)}{x})^{1-2r}\cdot\frac{x_n x_j^r}{x^{r+1}}-(\frac{P_{n,r}(\mathbf{y}_i)}{y})^{1-2r}\cdot\frac{y_n y_j^r}{y^{r+1}}]<0 \end{aligned}$$

The last inequality holds, since when $-1 \leq r \leq \frac{1}{2}, k = i+1, \cdots, n$, we have

$$(\frac{P_{n,r}(\mathbf{x}_i)}{x})^{1-2r} \geq (\frac{P_{n,r}(\mathbf{y}_i)}{y})^{1-2r}, \frac{x_k}{x} > \frac{y_k}{y}, \frac{x_n}{y_n}\cdot(\frac{x_j}{y_j})^r \geq (\frac{x_j}{y_j})^{1+r} > (\frac{x}{y})^{1+r}$$

when $\frac{1}{2} \leq r \leq 1$, we have

$$(\frac{P_{n,r}(\mathbf{x}_i)}{x})^{1-2r} \geq (\frac{x_n}{x})^{1-2r}, (\frac{P_{n,r}(\mathbf{y}_i)}{y})^{1-2r} \leq (\frac{y_n}{y})^{1-2r}$$

and $(\frac{x_n}{y_n})^{2-2r}\cdot(\frac{x_j}{y_j})^r \geq (\frac{x_j}{y_j})^{2-2r}\cdot(\frac{x_j}{y_j})^r = (\frac{x_j}{y_j})^{2-r} > (\frac{x}{y})^{2-r}$.

Thus $g_i(x) > g_i(x_{i+1}) = g_{i+1}(x_{i+1}) \geq g_{i+1}(x_{i+2}) \geq \cdots \geq g_{n-1}(x_{n-1}) \geq g_{n-1}(x_n) = x_n + y_n$, which implies $D_i'(x) < 0$ for all $x \in (0, x_{i+1})$, so

$$D_1(x_1) \geq D_1(x_2) = D_2(x_2) \geq D_2(x_3) \geq \cdots \geq D_{n-1}(x_{n-1}) \geq D_{n-1}(x_n) = 0 \tag{2.1}$$

Since D_i is strictly decreasing, we conclude from $m < M$ that $D_1(x_1) > 0$.

Next, for arbitrary F, by using the mean value theorem, (1.6) is equivalent to

$$\frac{F(P_{n,1}(\mathbf{y})) - F(P_{n,r}(\mathbf{y}))}{F(P_{n,1}(\mathbf{x})) - F(P_{n,r}(\mathbf{x}))} = \frac{F'(\xi)}{F'(\eta)}\cdot\frac{P_{n,1}(\mathbf{y}) - P_{n,r}(\mathbf{y})}{P_{n,1}(\mathbf{x}) - P_{n,r}(\mathbf{x})}$$

where $m' \leq \xi \leq M', m \leq \eta \leq M$. Taking absolute value and applying the result for $F(x) = x$, we get the desired inequality (1.6). This completes the proof. □

3. Consequences of Theorem 1.1

In this section, by choosing different functions $f(x,y), F(x)$, we will give several results of generalized Ky Fan's inequality of type (1.6). There are corresponding ones of type (1.7) and we leave the statements to the reader. To simplify expressions, we define:

$$\Delta_{s,r,\alpha} = \frac{P^{\alpha}_{n,s}(\mathbf{y}) - P^{\alpha}_{n,r}(\mathbf{y})}{P^{\alpha}_{n,s}(\mathbf{x}) - P^{\alpha}_{n,r}(\mathbf{x})} \tag{3.1}$$

with

$$\Delta_{s,r,0} = (\ln\frac{P_{n,s}(\mathbf{y})}{P_{n,r}(\mathbf{y})})/(\ln\frac{P_{n,s}(\mathbf{x})}{P_{n,r}(\mathbf{x})})$$

Also in order to include the case of equality for various inequalities in our discussion, we define $0/0 = 1$ from now on.

As a generalization of theorem III, we have:

Corollary 3.1. *Let $f(x,y) = ax^p + by^p - 1, 0 < a \leq b, p \geq 1$, $0 < m < M \leq (a+b)^{-(1/p)}, n \geq 2$. For $\alpha \leq 1$, let $F(x) = \ln x, \alpha = 0$, or $F(x) = x^{\alpha}$, otherwise. Then for $-1 \leq r < 1$*

$$\Delta_{1,r,\alpha} < (\frac{M}{m'})^{2-\alpha} \tag{3.2}$$

Proof: This follows from $f_x/f_y \leq 1, Max_{m'\leq\xi\leq M'}|F'(\xi)|/Min_{m\leq\eta\leq M}|F'(\eta)| \leq (M/m')^{1-\alpha}$. □

We remark here in corollary 3.1, the case $\alpha = 0$ gives $\frac{P_{n,1}(\mathbf{y})}{P_{n,r}(\mathbf{y})} < (\frac{P_{n,1}(\mathbf{x})}{P_{n,r}(\mathbf{x})})^{(\frac{M}{m'})^2}$, which partially generalizes theorem IV. Also for the case $\alpha = 0$, by only assuming $x_i \in [0, (a+b)^{-(1/p)}]$, we get

$\frac{P_{n,1}(\mathbf{y})}{P_{n,r}(\mathbf{y})} \leq \frac{P_{n,1}(\mathbf{x})}{P_{n,r}(\mathbf{x})}$ for $-1 \leq r \leq 1$ with the equality holding if and only if $x_1 = \cdots = x_n$. This is a generalization of theorem V.

As a generalization of theorem II, we have:

Corollary 3.2. *Let $f(x,y) = ax^p + by^p - 1, 0 < a \leq b, p \geq 1, x_i \in [0,(a+b)^{-\frac{1}{p}}]$. For $\alpha \leq 1$, let $F(x) = \ln x, \alpha = 0$ and $F(x) = x^\alpha$, otherwise. Then for $-1 \leq r \leq p$*

$$0 \leq \Delta_{1,r,\alpha} \leq 1 \tag{3.3}$$

with equality holding if and only if $x_1 = \cdots = x_n$.

Proof: The first inequality is trivial and the second inequality for the case $-1 \leq r \leq 1$ follows from (3.2) by noticing $M/m' \leq 1$. For $1 \leq r \leq p$, we will prove the case $\alpha = 1$ and the general case follows from the using of the mean value theorem. We define for $1 \leq i \leq n-1$ and $x_{n-i} < x \leq M$

$$\begin{aligned} \mathbf{x_i} &= (x_1, \cdots, x_{n-i}, x, \cdots, x) \\ \mathbf{y_i} &= (y_1, \cdots, y_{n-i}, y, \cdots, y) \\ E(\mathbf{x}_i) &= P_{n,r}(\mathbf{x}_i) - A_n(\mathbf{x}_i) - P_{n,r}(\mathbf{y}_i) + A_n(\mathbf{y}_i) \end{aligned}$$

and $E_i(x) = E(\mathbf{x}_i)$.

We need to show $E_1(x_n) \geq 0$, notice first for $x_1 \leq \cdots \leq x_n$

$$P_{n,r}(\mathbf{x})^{1-r} \cdot x_n^{r-1} + P_{n,r}(\mathbf{y})^{1-r} \cdot y_n^{r-1} \geq 2(\frac{P_{n,r}(\mathbf{x})P_{n,r}(\mathbf{y})}{x_n y_n})^{\frac{1-r}{2}} \tag{3.4}$$

and

$$\frac{P_{n,r}(\mathbf{x})P_{n,r}(\mathbf{y})}{x_n y_n} = [\sum_{i=1}^{n} \omega_i^2 (\frac{x_i y_i}{x_n y_n})^r + \sum_{1 \leq i < j \leq n} \omega_i \omega_j ((\frac{x_i y_j}{x_n y_n})^r + (\frac{x_j y_i}{x_n y_n})^r)]^{\frac{1}{r}} \tag{3.5}$$

Since the function $x[\frac{1}{b}(1-ax^p)]^{\frac{1}{p}}$ is increasing for $0 \leq x^p \leq \frac{1}{2a}$, we have $(\frac{x_i y_i}{x_n y_n})^r \leq 1$ for all i.

Now for fixed $i \leq j$, define

$$h(x) = 2x_j^r y_j^r - y^r x_j^r - x^r y_j^r$$

then $h'(x_i) \leq r x_i^{p-1} y_i^{r-p} x_j^r - r x_i^{r-1} y_j^r \leq 0$ since $r \leq p$ and $\frac{x_i}{y_i} \leq 1 \leq \frac{y_j}{x_j}$. Thus $h(x_i) = 2x_j^r y_j^r - y_i^r x_j^r - x_i^r y_j^r \geq h(x_j) = 0$,which implies

$$(\frac{x_i y_j}{x_n y_n})^r + (\frac{x_j y_i}{x_n y_n})^r \leq (\frac{x_i y_j}{x_j y_j})^r + (\frac{x_j y_i}{x_j y_j})^r = (\frac{x_i}{x_j})^r + (\frac{y_i}{y_j})^r \leq 2 \tag{3.6}$$

Back to (3.5), we have:

$$\frac{P_{n,r}(\mathbf{x})P_{n,r}(\mathbf{y})}{x_n y_n} = ((\sum_{i=1}^{n} \omega_i \frac{x_i^r}{x_n^r})(\sum_{i=1}^{n} \omega_i \frac{y_i^r}{y_n^r}))^{\frac{1}{r}} \leq (\sum_{i=1}^{n} \omega_i^2 + \sum_{i<j} 2\omega_i\omega_j)^{\frac{1}{r}} = 1$$

In particular this gives(where $\Omega_i^{-1} = \sum_{k=n-i+1}^{n} \omega_k$)

$$\begin{aligned} \Omega_i^{-1} E_i'(x) &= P_{n,r}(\mathbf{x}_i)^{1-r} \cdot x^{r-1} - 1 + \frac{a}{b} \cdot \frac{x^{p-1}}{y^{p-1}} (P_{n,r}(\mathbf{y}_i)^{1-r} \cdot y^{r-1} - 1) \\ &\geq P_{n,r}(\mathbf{x}_i)^{1-r} \cdot x^{r-1} - 1 + P_{n,r}(\mathbf{y}_i)^{1-r} \cdot y^{r-1} - 1 \geq 0 \end{aligned} \tag{3.7}$$

Thus we deduce:

$$E_1(x_n) \geq E_1(x_{n-1}) = E_2(x_{n-1}) \geq \cdots \geq E_{n-1}(x_2) \geq E_{n-1}(x_1) = 0$$

A close look of the proof tells us the equality holds in (3.3) if and only if $x_1 = \cdots = x_n$ and the proof is completed. □

As a special case of the above corollary, we have $A_n' - H_n' \leq A_n - H_n$ for generalized weighted means, a proof of this for the special case $\omega_1 = \cdots = \omega_n$ was given in [2].

We remark here if $0 < b < a$, then in general $P_{n,1}(\mathbf{x}) - P_{n,r}(\mathbf{x})$ and $P_{n,1}(\mathbf{y}) - P_{n,r}(\mathbf{y})$ are not comparable. For example, if we let $a = 2, b = 1, n = 2, \omega_1 = \omega_2$, then when $x_1 = \frac{1}{3}, x_2 = \frac{1}{27}$, $A_2 - G_2 = A_2' - G_2'$; when $x_1 = \frac{1}{3}, x_2 = 0$, $A_2 - G_2 > A_2' - G_2'$ and when $x_1 = \frac{1}{3}, x_2 = \frac{1}{4}$, $A_2 - G_2 < A_2' - G_2'$.

The classical case of Ky Fan's inequality corresponds to the choice of $f(x, y) = x + y - 1$ where $f_x/f_y = 1$. In this case both inequalities (1.6) and (1.7) hold and combinations of previous results yield:

Corollary 3.3. *For $f(x, y) = x + y - 1, 0 < m < M \leq \frac{1}{2}, n \geq 2$ then for $-1 \leq r \leq 1, \alpha \leq 1$*

$$(\frac{m}{1-m})^{2-\alpha} < \Delta_{1,r,\alpha} < (\frac{M}{1-M})^{2-\alpha} \tag{3.8}$$

4. The Comparison of $P^{\alpha}_{n,s}(\mathbf{y}) - P^{\alpha}_{n,r}(\mathbf{y})$ and $P^{\alpha}_{n,s}(\mathbf{x}) - P^{\alpha}_{n,r}(\mathbf{x})$

In this section, fixing $f(x, y) = x+y-1, x_i \in [0, 1/2]$, we give some results relating the comparison of $P^{\alpha}_{n,s}(\mathbf{y}) - P^{\alpha}_{n,r}(\mathbf{y})$ and $P^{\alpha}_{n,s}(\mathbf{x}) - P^{\alpha}_{n,r}(\mathbf{x})$, where $s > r, \alpha \in R$.

Our first result is the following lemma:

Lemma 4.1. *Given $s > r$, if for $\alpha_0 \in R$, we have $\Delta_{s,r,\alpha_0} \leq (\geq) 1$ with equality holding if and only if $x_1 = \cdots = x_n$, then for all $\alpha \leq (\geq) \alpha_0$, $\Delta_{s,r,\alpha} \leq (\geq) 1$ with equality holding if and only if $x_1 = \cdots = x_n$.*

Proof: Let $i = s, r, \mathbf{v} = \mathbf{x}, \mathbf{y}$, we can assume $P_{n,i}(\mathbf{v}) \neq 0$. If $\alpha_0 \neq 0$, write $P^{\alpha}_{n,s}(\mathbf{v}) - P^{\alpha}_{n,r}(\mathbf{v}) = (P^{\alpha_0}_{n,s}(\mathbf{v}))^{\alpha/\alpha_0} - (P^{\alpha_0}_{n,r}(\mathbf{v}))^{\alpha/\alpha_0} = \frac{\alpha}{\alpha_0}\xi^{\alpha-\alpha_0}(P^{\alpha_0}_{n,s}(\mathbf{v}) - P^{\alpha_0}_{n,r}(\mathbf{v}))$ with $P_{n,r}(\mathbf{v}) < \xi < P_{n,s}(\mathbf{v})$, where when $\alpha = 0$, we define $(P^{\alpha_0}_{n,i}(\mathbf{v}))^{0/\alpha_0} = \ln P^{\alpha_0}_{n,i}(\mathbf{v})$. By taking the quotient, we get the desired result. If $\alpha_0 = 0$, we write $P^{\alpha}_{n,i}(\mathbf{v}) = e^{\alpha \ln P_{n,i}(\mathbf{v})}$ and proceed similarly. □

For any $s \geq r$, the above lemma enables us to define a number $sup(\alpha)_{s,r}$ such that $\Delta_{s,r,\alpha} \leq 1$ holds for all $\alpha < sup(\alpha)_{s,r}$. A special case of this, $sup(\alpha)_{1,0} = 1$ was determined in [5].

The inequality $\Delta_{s,r,\alpha} \geq 1$ seems unusual but indeed it can happen, even for the case $r = 1$. Indeed we have the following theorem:

Theorem 4.1. *$\Delta_{s,1,\alpha} \geq 1$ for $\alpha \geq s \geq 2$; $\Delta_{s,1,\alpha} \leq 1$ for $1 < s \leq 2, \alpha \leq s$; $\Delta_{1,r,\alpha} \leq 1$ for $\alpha \leq r < 0$, in all cases the equality holds if and only if $x_1 = \cdots = x_n$.*

Proof: From lemma 4.1, it suffices to prove the theorem for $\alpha = s$ or r. In this case, for $x \in [0, \frac{1}{2}]$, consider the function $f(x) = x^t - (1-x)^t$ with $f''(x) = t(t-1)(x^{t-2} - (1-x)^{t-2})$. By considering the sign $f''(x)$ for various t and using corresponding Jensen's inequality: ($\sum \omega_i f(x_i) \geq (\leq) f(\sum \omega_i x_i)$, the above assertions follow. □

In theorem 4.1, by restricting $x_i \in [m, M], 0 < m < M \leq \frac{1}{2}$, we will get results similar to corollary 3.3 and we will leave the statements for the reader.

We have omitted the case $0 < r < 1$ for theorem 4.1 since we have a stronger result as corollary 3.2. We point out an interesting phenomena here that when $s = 2$, $\Delta_{2,1,\alpha} \geq (\leq) 1$ for $\alpha \geq (\leq) 2$. We also remark here the proof of (1.1) follows by applying Jensen's inequality to the function $\ln x - \ln(1-x)$ for $x \in [0, \frac{1}{2}]$.

Notice $P_{n,s}(\mathbf{x}) - P_{n,r}(\mathbf{x}) \geq P_{n,s}(\mathbf{y}) - P_{n,r}(\mathbf{y})$ does not hold for arbitrary real numbers $s \geq r$, for otherwise we will have $P_{n,s}(\mathbf{x})/P_{n,r}(\mathbf{x}) \geq P_{n,s}(\mathbf{y})/P_{n,r}(\mathbf{y})$ which is not true in general according to a nice result by J. Chen and Z.Wang[7]:

Theorem VI. *For arbitrary $n, s > r, x_i \in (0, 1/2]$, $\Delta_{s,r,0} \leq 1$ holds if and only if $|r+s| \leq 3, 2^r/r \geq 2^s/s$ when $r > 0, r2^r \leq s2^s$ when $s < 0$.*

By using lemma 4.1 and the above theorem, we get the following theorem:

Theorem 4.2. *$sup(\alpha)_{s,r} \geq 0$ if $|r + s| \leq 3, 2^r/r \geq 2^s/s$ when $r > 0, r2^r \leq s2^s$ when $s < 0$. Moreover, $sup(\alpha)_{1,r} = 1$ for $-1 \leq r \leq 1$ and $sup(\alpha)_{s,1} = s$ for $1 < s \leq 2$.*

Proof: The first assertion follows from theorem VI and the definition for $sup(\alpha)_{s,r}$. From corollary 3.2, we know $sup(\alpha)_{1,r} \geq 1$ for $-1 \leq r \leq 1$ and when $\alpha > 1$, let $x_1 = 1/2, x_2 = \cdots = x_n = \epsilon$, and

$$f(\omega_1, \epsilon) = (P^{\alpha}_{n,1}(\mathbf{x}) - P^{\alpha}_{n,r}(\mathbf{x})) - (P^{\alpha}_{n,1}(\mathbf{y}) - P^{\alpha}_{n,r}(\mathbf{y}))$$

A simple calculation reveals that there exist positive real numbers δ and η such that we have $f(\omega_1, \epsilon) < 0,$ if $0 < \omega_1 < \delta$ and $0 < \epsilon < \eta$ and $f(\omega_1, \epsilon) > 0,$ if $1 - \delta < \omega_1 < 1$ and $0 < \epsilon < \eta$. Similar conclusion holds for $sup(\alpha)_{s,1}$ and this completes the proof. □

References

[1] H. Alzer, Ungleichungen für geometrische und arithmetische Mittelwerte, *Proc. Kon. Nederl. Akad. Wetensch.*, **91** (1988), 365-374.

[2] H. Alzer, An inequality for arithmetic and harmonic means, *Aequat. Math.*, **46** (1993), 257-263.

[3] H. Alzer, The inequality of Ky Fan and related results, *Acta Appl. Math.*, **38** (1995), 305-354.

[4] H. Alzer, On an additive analogue of Ky Fan's inequality, *Indag. Math.(N.S.)*, **8** (1997), 1-6.

[5] H. Alzer, Some inequalities for arithmetic and geometric means, *Proc. Roy. Soc. Edinburgh Sect. A*, **129** (1999), 221-228.

[6] E.F. Beckenbach and R. Bellman, "Inequalities," Springer-Verlag, Berlin, 1961.

[7] J.Chen and Z. Wang, Generalization of Ky Fan inequality, *Math. Balkanica(N.S.)* **5** (1991), 373-380.

[8] A.McD. Mercer, Bounds for A-G, A-H, G-H, and a family of inequalities of Ky Fan's type, using a general method, *J. Math. Anal. Appl.*, **243** (2000), 163-173.

Tsz Ho Chan and Peng Gao
2074 East Hall, 525 East University Avenue
Departmen of Mathematics, University of Michigan
Ann Arbor, MI 48109
email address: thchan@math.lsa.umich.edu, penggao@math.lsa.umich.edu