A Taylor Like Formula for Mappings of Two Variables Defined on a Rectangle in the Plane

A TAYLOR LIKE FORMULA FOR MAPPINGS OF TWO VARIABLES DEFINED ON A RECTANGLE IN THE PLANE

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ABSTRACT. Using the Taylor Formula and a Grüss type inequality, with the integral remainder, we point out a Taylor's like formula for mappings of two variables defined on a interval and point out some upper bounds for the remainder.

1. INTRODUCTION

The following theorem is well known in the literature as Taylor's theorem with the integral remainder.

Theorem 1. *Let $I \subset \mathbb{R}$ be a closed interval, let $a \in I$ and let n be a positive integer. If $f : I \to \mathbb{R}$ is such that $f^{(n)}$ is absolutely continuous, then for each $x \in I$*

$$f(x) = T_n(f; a, x) + R_n(f; a, x), \tag{1.1}$$

where $T_n(f; a, x)$ is Taylor's polynomial, i.e.,

$$T_n(f; a, x) := \sum_{k=0}^{n} \frac{(x-a)^k}{k!} f^{(k)}(a) \tag{1.2}$$

(note that $f^{(0)} = f$ and $0! = 1$), and the remainder is given by

$$R_n(f; a, x) := \frac{1}{n!} \int_a^x (x-t)^n f^{(n+1)}(t)\, dt. \tag{1.3}$$

A simple proof of this theorem can be done by mathematical induction using the integration by parts formula.

The following corollary concerning the estimation of the remainder is useful when we want to approximate concrete functions by the Taylor's expansions.

Corollary 1.1. *With the above assumptions, we have the estimation:*

$$|R(f; a, x)| \le \frac{(x-a)^n}{n!} \int_a^x \left| f^{(n+1)}(t) \right| dt \tag{1.4}$$

or

$$|R(f; a, x)| \le \frac{1}{n!} \cdot \frac{(x-a)^{n+\frac{1}{q}}}{(nq+1)^{\frac{1}{q}}} \left(\int_a^x \left| f^{(n+1)}(t) \right|^p dt \right)^{\frac{1}{p}} \tag{1.5}$$

where $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, or the estimation

$$|R(f; a, x)| \le \frac{(x-a)^{n+1}}{(n+1)!} \max_{t \in (a,x)} \left| f^{(n+1)}(t) \right| \tag{1.6}$$

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for all $x \geq a$, $a \in I$.

Let f be in the class $C^{(q)}(\mathbb{R}^n)$ and $x,\ x_0 \in D \subset \mathbb{R}^n$ so that the line segment joining X and x_0 is contained in D. Let $h = x - x_0$. Then we have the Taylor's formula with the Lagrange type remainder:

$$\begin{aligned} f(x) &= f(x_0) + \sum_{i=1}^{n} f_i(x_0)\, h^i + \frac{1}{2!}\sum_{i,j=1}^{n} f_{ij}(x_0)\, h^i h^j + \dots \\ &\quad + \frac{1}{(q-1)!}\sum_{i_1,\dots,i_q=1}^{n} f_{i_1,\dots,i_q}(x_0)\, h^{i_1}\dots h^{i_{q-1}} + R_q(x), \end{aligned}$$

where $h^i = x^i - x_0^i$, $s \in (0,1)$ and

$$R_q(x) = \frac{1}{q!}\sum_{i_1,\dots,i_q=1}^{n} f_{i_1,\dots,i_q}(x_0 + sh)\, h^{i_1}\dots h^{i_q}.$$

2. The Results

The following theorem holds (see [1] or [2, p. 138] for a particular proof).

Theorem 2. *Let I, J be two closed intervals and $f : I \times J \to \mathbb{R}$ be a mapping so that the following partial derivatives $\frac{\partial^{i+m+1} f(a,\cdot)}{\partial x^i \partial y^{m+1}}$ $(i = 0, ..., n)$, $\frac{\partial^{j+n+1} f(\cdot,b)}{\partial x^{n+1} \partial y^j}$ $(j = 0, ..., m)$ and $\frac{\partial^{n+m+2} f(\cdot,\cdot)}{\partial x^{n+1} \partial y^{m+1}}$ exist on the intervals J, I and $I \times J$ respectively, where $a \in I$ and $b \in J$ are given. Let $x \in I$ and $y \in J$ and assume that $\frac{\partial^{i+m+1} f(a,\cdot)}{\partial x^i \partial y^{m+1}}$ are continuous on $[b, y]$ (for $i = 0, ..., n$), $\frac{\partial^{j+n+1} f(\cdot,b)}{\partial x^{n+1} \partial y^j}$ are continuous on $[a, x]$ (for $j = 0, ..., m$) and $\frac{\partial^{n+m+2} f(\cdot,\cdot)}{\partial x^{n+1} \partial y^{m+1}}$ is continuous on $[a, x] \times [b, y]$.*
Then we have the inequality:

$$\begin{aligned} f(x,y) &= \sum_{i=0}^{n}\sum_{j=0}^{m} \frac{(x-a)^i}{i!}\cdot\frac{(y-b)^j}{j!}\frac{\partial^{i+j} f(a,b)}{\partial x^i \partial y^j} \\ &\quad + \frac{1}{m!}\sum_{i=0}^{n}\frac{(x-a)^i}{i!}\int_b^y (y-s)^m \frac{\partial^{i+m+1} f(a,s)}{\partial x^i \partial y^{m+1}} ds \\ &\quad + \frac{1}{n!}\sum_{j=0}^{m}\frac{(y-b)^j}{j!}\int_a^x (x-t)^n \frac{\partial^{j+n+1} f(t,b)}{\partial x^{n+1} \partial y^j} dt \\ &\quad + \frac{1}{n!m!}\int_a^x\int_b^y (x-t)^n (y-s)^m \frac{\partial^{n+m+2} f(t,s)}{\partial x^{n+1}\partial y^{m+1}} dsdt, \end{aligned} \tag{2.1}$$

where $n, m \in \mathbb{N}$.

Proof. For the sake of completeness we give here a short proof. Apply Taylor's formula (1.1) for the mapping $f(\cdot, y)$ to get

$$f(x,y) = \sum_{i=0}^{n}\frac{(x-a)^i}{i!}\cdot\frac{\partial^i f(a,y)}{\partial x^i} + \frac{1}{n!}\int_a^x (x-t)^n \frac{\partial^{n+1} f(t,y)}{\partial x^{n+1}} dt. \tag{2.2}$$

Also, by (1.1) applied for the partial derivatives $\frac{\partial^i f(a,\cdot)}{\partial x^i}$ $(i=0,...,n)$ we can state that

$$\begin{aligned}\frac{\partial^i f(a,y)}{\partial x^i} &= \sum_{j=0}^{m}\frac{(y-b)^j}{j!}\cdot\frac{\partial^{j+i} f(a,b)}{\partial x^i \partial y^j} \\ &\quad +\frac{1}{m!}\int_b^y (y-s)^m \frac{\partial^{i+m+1} f(a,s)}{\partial x^i \partial y^{m+1}}ds.\end{aligned}\tag{2.3}$$

Similarly, we have

$$\begin{aligned}\frac{\partial^{n+1} f(t,y)}{\partial x^{n+1}} &= \sum_{j=0}^{m}\frac{(y-b)^j}{j!}\cdot\frac{\partial^{j+n+1} f(a,b)}{\partial x^{n+1} \partial y^j} \\ &\quad +\frac{1}{m!}\int_b^y (y-s)^m \frac{\partial^{n+m+2} f(t,s)}{\partial x^i \partial y^{m+1}}ds.\end{aligned}\tag{2.4}$$

Using (2.3) and (2.4), the equation (2.2) becomes

$$\begin{aligned}& f(x,y) \\ =\;& \sum_{i=0}^{n}\frac{(x-a)^i}{i!}\left[\sum_{j=0}^{m}\frac{(y-b)^j}{j!}\cdot\frac{\partial^{j+i} f(a,b)}{\partial x^i \partial y^j}+\frac{1}{m!}\int_b^y (y-s)^m \frac{\partial^{i+m+1} f(a,s)}{\partial x^i \partial y^{m+1}}ds\right] \\ & +\frac{1}{n!}\int_a^x (x-t)^n \left[\sum_{j=0}^{m}\frac{(y-b)^j}{j!}\cdot\frac{\partial^{j+n+1} f(t,b)}{\partial x^{n+1} \partial y^j}\right. \\ & \left.+\frac{1}{m!}\int_b^y (y-s)^m \frac{\partial^{n+m+2} f(t,s)}{\partial x^{n+1} \partial y^{m+1}}ds\right]dt \\ =\;& \sum_{i=0}^{n}\sum_{j=0}^{m}\frac{(x-a)^i}{i!}\cdot\frac{(y-b)^j}{j!}\cdot\frac{\partial^{i+j} f(a,b)}{\partial x^i \partial y^j} \\ & +\frac{1}{m!}\sum_{i=0}^{n}\frac{(x-a)^i}{i!}\int_b^y (y-s)^m \frac{\partial^{i+m+1} f(a,s)}{\partial x^i \partial y^{m+1}}ds \\ & +\frac{1}{n!}\sum_{j=0}^{m}\frac{(y-b)^j}{j!}\int_a^x (x-t)^n \frac{\partial^{j+n+1} f(t,b)}{\partial x^{n+1} \partial y^j}dt \\ & +\frac{1}{n!m!}\int_a^x\int_b^y (x-t)^n (y-s)^m \frac{\partial^{n+m+2} f(t,s)}{\partial x^{n+1} \partial y^{m+1}}dsdt\end{aligned}$$

and the theorem is completely proved. ∎

Now using the above theorem, we can point out the following inequality.

Theorem 3. *Assume that the mapping $f : I \times J \to \mathbb{R}$ fulfills the hypotheses from Theorem 2. Then we have the inequality*

$$\begin{aligned}
(2.5)\quad & \left| f(x,y) - \sum_{i=0}^{n}\sum_{j=0}^{m} \frac{(x-a)^i}{i!}\cdot\frac{(y-b)^j}{j!}\frac{\partial^{i+j}f(a,b)}{\partial x^i \partial y^j} \right. \\
& -\frac{1}{m!}\sum_{i=0}^{n}\frac{(x-a)^i}{i!}\int_b^y (y-s)^m \frac{\partial^{i+m+1}f(a,s)}{\partial x^i \partial y^{m+1}}ds \\
& \left. -\frac{1}{n!}\sum_{j=0}^{m}\frac{(y-b)^j}{j!}\int_a^x (x-t)^n \frac{\partial^{j+n+1}f(t,b)}{\partial x^{n+1} \partial y^{j}}dt \right| \\
\leq & \begin{cases}
\frac{1}{(n+1)!(m+1)!}(x-a)^{n+1}(y-b)^{m+1}\left\| \frac{\partial^{n+m+2}f}{\partial x^{n+1}\partial y^{m+1}} \right\|_{\infty;[a,x]\times[b,y]}; \\
if\, \frac{\partial^{n+m+2}f}{\partial x^{n+1}\partial y^{m+1}} \in L_\infty\Big[[a,x]\times[b,y]\Big] \\
\frac{1}{n!(nq+1)^{\frac{1}{q}}m!(mq+1)^{\frac{1}{q}}}(x-a)^{n+\frac{1}{q}}(y-b)^{m+\frac{1}{q}}\left\| \frac{\partial^{n+m+2}f}{\partial x^{n+1}\partial y^{m+1}} \right\|_{p;[a,x]\times[b,y]}; \\
if\, \frac{\partial^{n+m+2}f}{\partial x^{n+1}\partial y^{m+1}} \in L_p\Big[[a,x]\times[b,y]\Big] \qquad p>1,\, \frac{1}{p}+\frac{1}{q}=1 \\
\frac{1}{n!m!}(x-a)^{n}(y-b)^{m}\left\| \frac{\partial^{n+m+2}f}{\partial x^{n+1}\partial y^{m+1}} \right\|_{1;[a,x]\times[b,y]}, \\
if\, \frac{\partial^{n+m+2}f}{\partial x^{n+1}\partial y^{m+1}} \in L_1\left[[a,x]\times[b,y]\right]
\end{cases}
\end{aligned}$$

where $\|\cdot\|_{p,[a,x]\times[b,y]}$ is the usual p norm ($p \in [1,\infty]$) on the region $[a,x]\times[b,y]$.

Proof. Using the representation (2.1) and the property of modulus we have

$$\begin{aligned}
& \left| f(x,y) - \sum_{i=0}^{n}\sum_{j=0}^{m} \frac{(x-a)^i}{i!}\cdot\frac{(y-b)^j}{j!}\frac{\partial^{i+j}f(a,b)}{\partial x^i \partial y^j} \right. \\
& -\frac{1}{m!}\sum_{i=0}^{n}\frac{(x-a)^i}{i!}\int_b^y (y-s)^m \frac{\partial^{i+m+1}f(a,s)}{\partial x^i \partial y^{m+1}}ds \\
& \left. -\frac{1}{n!}\sum_{j=0}^{m}\frac{(y-b)^j}{j!}\int_a^x (x-t)^n \frac{\partial^{j+n+1}f(t,b)}{\partial x^{n+1} \partial y^{j}}dt \right| \\
\leq\ & \frac{1}{n!m!}\int_a^x\int_b^y |x-t|^n |y-s|^m \left| \frac{\partial^{n+m+2}f(t,s)}{\partial x^{n+1}\partial y^{m+1}} \right| dsdt =: M(x,y).
\end{aligned}$$

It is easy to see that

$$\begin{aligned}
& M(x,y) \\
\leq\; & \left\| \frac{\partial^{n+m+2} f}{\partial x^{n+1} \partial y^{m+1}} \right\|_{\infty;[a,x]\times[b,y]} \frac{1}{n!m!} \int_a^x \int_b^y (x-t)^n (y-s)^m \, ds dt \\
=\; & \frac{1}{n!m!} \left[\frac{-(x-t)^{n+1}}{n+1} \Bigg|_a^x \right] \times \left[\frac{-(y-s)^{n+1}}{m+1} \Bigg|_b^y \right] \times \left\| \frac{\partial^{n+m+2} f}{\partial x^{n+1} \partial y^{m+1}} \right\|_{\infty;[a,x]\times[b,y]} \\
=\; & \frac{1}{(n+1)!\,(m+1)!} (x-a)^{n+1} (y-b)^{m+1} \left\| \frac{\partial^{n+m+2} f}{\partial x^{n+1} \partial y^{m+1}} \right\|_{\infty;[a,x]\times[b,y]}
\end{aligned}$$

and the first inequality in (2.5) is proved.

Using Hölder's inequality for double integrals, we have

$$\begin{aligned}
& M(x,y) \\
\leq\; & \frac{1}{n!m!} \left\| \frac{\partial^{n+m+2} f}{\partial x^{n+1} \partial y^{m+1}} \right\|_{p;[a,x]\times[b,y]} \left[\int_a^x \int_b^y (x-t)^{nq} (y-s)^{mq} \, ds dt \right]^{\frac{1}{q}} \\
=\; & \frac{1}{n!m!} \left\| \frac{\partial^{n+m+2} f}{\partial x^{n+1} \partial y^{m+1}} \right\|_{p;[a,x]\times[b,y]} \left[\frac{(x-a)^{nq+1} (y-b)^{mq+1}}{(nq+1)(mq+1)} \right]^{\frac{1}{q}} \\
=\; & \frac{1}{n!\,(nq+1)^{\frac{1}{q}}\, m!\,(mq+1)^{\frac{1}{q}}} (x-a)^{n+\frac{1}{q}} (y-b)^{m+\frac{1}{q}} \left\| \frac{\partial^{n+m+2} f}{\partial x^{n+1} \partial y^{m+1}} \right\|_{p;[a,x]\times[b,y]}
\end{aligned}$$

and the second inequality in (2.5) is proved.

Finally, we have

$$\begin{aligned}
M(x,y) \leq\; & \frac{1}{n!m!} \sup_{(t,s)\in[a,x]\times[b,y]} \left[(x-t)^n (y-s)^m \right] \left\| \frac{\partial^{n+m+2} f}{\partial x^{n+1} \partial y^{m+1}} \right\|_{1;[a,x]\times[b,y]} \\
=\; & \frac{1}{n!m!} (x-a)^n (y-b)^m \left\| \frac{\partial^{n+m+2} f}{\partial x^{n+1} \partial y^{m+1}} \right\|_{1;[a,x]\times[b,y]}
\end{aligned}$$

and the theorem is proved. ∎

The following approximation of the mapping $f(x,y)$ in terms of

$$\sum_{i=0}^{n} \sum_{j=0}^{m} \frac{(x-a)^i}{i!} \cdot \frac{(y-b)^j}{j!} \frac{\partial^{i+j} f(a,b)}{\partial x^i \partial y^j}$$

also holds.

Theorem 4. *Assume that the mapping $f : I \times J \to \mathbb{R}$ fulfills the hypothesis from Theorem 2. Then we have the inequality*

$$\left| f(x,y) - \sum_{i=0}^{n}\sum_{j=0}^{m} \frac{(x-a)^i}{i!} \cdot \frac{(y-b)^j}{j!} \frac{\partial^{i+j} f(a,b)}{\partial x^i \partial y^j} \right| \tag{2.6}$$

$$\leq \frac{1}{(m+1)!}(y-b)^{m+1} \sum_{i=0}^{n} \frac{(x-a)^i}{i!} \left\| \frac{\partial^{i+m+1} f(a,\cdot)}{\partial x^i \partial y^{m+1}} \right\|_{\infty,[b,y]}$$

$$+\frac{1}{(n+1)!}(x-a)^{n+1} \sum_{j=0}^{m} \frac{(y-b)^j}{j!} \left\| \frac{\partial^{j+n+1} f(\cdot,b)}{\partial x^{n+1} \partial y^{j}} \right\|_{\infty,[a,x]}$$

$$+\frac{1}{(n+1)!\,(m+1)!}(x-a)^{n+1}(y-b)^{m+1} \left\| \frac{\partial^{n+m+2} f}{\partial x^{n+1} \partial y^{m+1}} \right\|_{\infty;[a,x]\times[b,y]}.$$

The proof follows by using the formula (2.1) and the details will not be covered. Similar bounds in terms of the other norms may be pointed out, but we omit the details.

3. A Grüss type inequality for double integrals

In this section we will produce, with complete proof, the following lemma representing a Grüss type inequality for double integrals.

Theorem 5. *We assume that*

$$|f(x,y) - f(u,v)| \leq M_1 |x-u|^{\alpha_1} + M_2 |y-v|^{\alpha_2}, \tag{3.1}$$

where

$$M_1, M_2 > 0, \alpha_1, \alpha_2 \in (0,1]$$

and

$$|g(x,y) - g(u,v)| \leq N_1 |x-u|^{\beta_1} + N_2 |y-v|^{\beta_2}, \tag{3.2}$$

where

$$N_1,\ N_2 > 0,\ \beta_1,\ \beta_2 \in (0,1]$$

then we have the following inequality:

$$\left| \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y)\, g(x,y)\, dydx \right. \tag{3.3}$$

$$\left. -\frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y)\, dydx \times \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d g(x,y)\, dydx \right|$$

$$\leq 4 \left[M_1 N_1 \frac{(b-a)^{\alpha_1+\beta_1}}{(\alpha_1+\beta_1+1)(\alpha_1+\beta_1+2)} + M_1 N_2 \frac{2(b-a)^{\alpha_1}(d-c)^{\beta_2}}{(\alpha_1+1)(\beta_2+1)(\alpha_1+2)(\beta_2+2)} \right.$$

$$\left. + M_2 N_1 \frac{2(b-a)^{\beta_1}(d-c)^{\alpha_2}}{(\alpha_2+1)(\alpha_2+2)(\beta_1+1)(\beta_1+2)} + M_2 N_2 \frac{(d-c)^{\alpha_2+\beta_2}}{(\alpha_2+\beta_2+1)(\alpha_2+\beta_2+2)} \right]$$

Proof. Multiplying (3.1) and (3.2), we get

$$\begin{aligned}
& \left|\left(f\left(x,y\right)-f\left(u,v\right)\right)\left(g\left(x,y\right)-g\left(u,v\right)\right)\right| \\
\leq\; & M_1N_1\left|x-u\right|^{\alpha_1+\beta_1}+M_1N_2\left|x-u\right|^{\alpha_1}\left|y-v\right|^{\beta_2} \\
& +M_2N_1\left|y-v\right|^{\alpha_2}\left|x-u\right|^{\beta_1}+M_2N_2\left|y-v\right|^{\alpha_2+\beta_2}.
\end{aligned}$$

Integrating on $[a,b]\times[c,d]^2$ over (x,y) and (u,v), we obtain

$$\begin{aligned}
(3.4)\quad & \int_a^b\int_c^d\int_a^b\int_c^d\left|\left(f\left(x,y\right)-f\left(u,v\right)\right)\left(g\left(x,y\right)-g\left(u,v\right)\right)\right|dydxdvdu \\
\leq\; & M_1N_1\int_a^b\int_c^d\int_a^b\int_c^d\left|x-u\right|^{\alpha_1+\beta_1}dydxdvdu \\
& +M_1N_2\int_a^b\int_c^d\int_a^b\int_c^d\left|x-u\right|^{\alpha_1}\left|y-v\right|^{\beta_2}dydxdvdu \\
& +M_2N_1\int_a^b\int_c^d\int_a^b\int_c^d\left|y-v\right|^{\alpha_2}\left|x-u\right|^{\beta_1}dydxdvdu \\
& +M_2N_2\int_a^b\int_c^d\int_a^b\int_c^d\left|y-v\right|^{\alpha_2+\beta_2}dydxdvdu \\
(3.5)\quad =\; & \left(I_1+I_2+I_3+I_4\right).
\end{aligned}$$

Applying Korkine's identities to the left side of (3.4) gives

$$\begin{aligned}
& \frac{1}{2}\int_a^b\int_c^d\int_a^b\int_c^d[\left(f\left(x,y\right)-f\left(u,v\right)\right)\left(g\left(x,y\right)-g\left(u,v\right)\right)]dydxdvdu \\
=\; & \frac{1}{2}\int_a^b\int_c^d\int_a^b\int_c^d[f\left(x,y\right)g\left(x,y\right)-f\left(x,y\right)g\left(u,v\right) \\
& -f\left(u,v\right)g\left(x,y\right)+f\left(u,v\right)g\left(u,v\right)]dydxdvdu \\
=\; & \frac{1}{2}\int_a^b\int_c^d\int_a^b\int_c^d f\left(x,y\right)g\left(x,y\right)dydxdvdu \\
& -\frac{1}{2}\int_a^b\int_c^d\int_a^b\int_c^d f\left(x,y\right)g\left(u,v\right)dydxdvdu \\
& -\frac{1}{2}\int_a^b\int_c^d\int_a^b\int_c^d f\left(u,v\right)g\left(x,y\right)dydxdvdu \\
& +\frac{1}{2}\int_a^b\int_c^d\int_a^b\int_c^d f\left(u,v\right)g\left(u,v\right)dydxdvdu
\end{aligned}$$

$$\begin{aligned}
&= (b-a)(d-c)\int_a^b\int_c^d f(x,y)\,g(x,y)\,dydx \\
&\quad -\int_a^b\int_c^d\int_a^b\int_c^d f(x,y)\,g(u,v)\,dydxdvdu \\
&= (b-a)(d-c)\int_a^b\int_c^d f(x,y)\,g(x,y)\,dydx \\
&\quad -\int_a^b\int_c^d f(x,y)\,dydx\int_a^b\int_c^d g(x,y)\,dydx.
\end{aligned}$$

For the right side of (3.4) the Lemma 1 proved in [4] will be used:

Lemma 1. *Let $a,b,c,d \in R$ with $a<b$ and $c<d$. Define*

$$C_\theta(a,b,c,d) := \int_a^b\int_c^d |x-y|^\theta dydx, \quad \theta \ge 0, \tag{3.6}$$

then

$$\begin{aligned}
&(\theta+1)(\theta+2)C_\theta(a,b,c,d) \\
&= |b-c|^{\theta+2} - |b-d|^{\theta+2} + |d-a|^{\theta+2} - |c-a|^{\theta+2}.
\end{aligned} \tag{3.7}$$

If $c=a$ and $d=a$, then from (3.7)

$$\begin{aligned}
D_\theta(a,b) = C_\theta(a,b,a,b) &= \int_a^b\int_a^b |x-y|^\theta dydx, \quad \theta \ge 0 \\
&= \frac{2(b-a)^{\theta+2}}{(\theta+1)(\theta+2)}.
\end{aligned} \tag{3.8}$$

Now, utilizing the result of Lemma 1 and returning to (3.5) we find that:

$$\begin{aligned}
I_1 &= \int_a^b\int_c^d\int_a^b\int_c^d |x-u|^{\alpha_1+\beta_1}\,dydxdvdu \\
&= (d-c)^2\int_a^b\int_a^b |x-u|^{\alpha_1+\beta_1}\,dxdu \\
&= (d-c)^2 D_{\alpha_1+\beta_1}(a,b)
\end{aligned} \tag{3.9}$$

and using (3.8) gives

$$I_1 = \frac{2\,(d-c)^2\,(b-a)^{\alpha_1+\beta_1+2}}{(\alpha_1+\beta_1+1)\,(\alpha_1+\beta_1+2)}.$$

Further, from (3.3) and using (3.7) gives

$$\begin{aligned}
I_2 &= \int_a^b\int_c^d\int_a^b\int_c^d |x-u|^{\beta_1}\,|y-v|^{\alpha_2}\,dydxdvdu \\
&= \int_a^b\int_a^b |x-u|^{\beta_1}\,dxdu\int_c^d\int_c^d |y-v|^{\alpha_2}\,dydv \\
&= D_{\beta_1}(a,b)D_{\alpha_2}(c,d)
\end{aligned} \tag{3.10}$$

and using (3.8) produces

$$I_2 = \frac{4(b-a)^{\beta_1+2}(d-c)^{\alpha_2+2}}{(\beta_1+1)(\beta_1+2)(\alpha_2+1)(\alpha_2+2)}. \tag{3.11}$$

Using a similar procedure we get for I_3 and I_4 as defined in (3.7),

$$I_3 = \frac{4(b-a)^{\alpha_1+2}(d-c)^{\beta_2+2}}{(\alpha_1+1)(\beta_2+1)(\alpha_1+2)(\beta_2+2)}, \tag{3.12}$$

and

$$I_4 = \frac{2(b-a)^2(d-c)^{\alpha_2+\beta_2+2}}{(\alpha_2+\beta_2+1)(\alpha_2+\beta_2+2)}. \tag{3.13}$$

Thus, using (3.5), (3.10), (3.11), (3.12) and(3.13) in (3.4), we get

$$\begin{aligned} & (b-a)(c-d)\int_a^b\int_c^d f(x,y)\,g(x,y)\,dydx - \int_a^b\int_c^d f(x,y)\,dydx\int_a^b\int_c^d g(x,y)\,dydx \\ \leq\ & 2\left[M_1N_1\frac{2(d-c)^2(b-a)^{\alpha_1+\beta_1+2}}{(\alpha_1+\beta_1+1)(\alpha_1+\beta_1+2)} + M_1N_2\frac{4(b-a)^{\alpha_1+2}(d-c)^{\beta_2+2}}{(\alpha_1+1)(\beta_2+1)(\alpha_1+2)(\beta_2+2)}\right. \\ & \left. +M_2N_1\frac{4(b-a)^{\beta_1+2}(d-c)^{\alpha_2+2}}{(\alpha_2+1)(\alpha_2+2)(\beta_1+1)(\beta_1+2)} + M_2N_2\frac{2(b-a)^2(d-c)^{\alpha_2+\beta_2+2}}{(\alpha_2+\beta_2+1)(\alpha_2+\beta_2+2)}\right] \end{aligned}$$

and dividing both sides by $(b-a)^2(d-c)^2$ completes the proof. ∎

Corollary 3.1. *(see also* [3, p. 305]*) When $\alpha_1 = \alpha_2 = 1$, $\beta_1 = \beta_2 = 1$, we have*

$$\begin{aligned} |f(x,y) - f(u,v)| &\leq L_1|x-u| + L_2|y-v|, \\ |g(x,y) - g(u,v)| &\leq K_1|x-u| + K_2|y-v|, \end{aligned}$$

and then (3.3) becomes

$$\begin{aligned} & \left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,g(x,y)\,dydx\right. \\ & \left.-\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx \times \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d g(x,y)\,dydx\right| \\ \leq\ & 4\left[L_1K_1\frac{(b-a)^2}{12} + L_1K_2\frac{(b-a)(d-c)}{18} + L_2K_1\frac{(b-a)(d-c)}{18} + L_2K_2\frac{(d-c)^2}{12}\right]. \end{aligned}$$

Corollary 3.2. *Let the conditions of Corollary 3.1 hold, then*

$$\begin{aligned} & \left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f^2(x,y)\,dx\,dy\right. \\ & \left.-\left[\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dx\,dy\right]^2\right| \\ \leq\ & 4\left[L_1^2\frac{(b-a)^2}{12} + L_1L_2\frac{(b-a)(d-c)}{9} + L_2^2\frac{(d-c)^2}{12}\right]. \end{aligned}$$

Proof. In (3.9) let $\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = 1$ and

$$f(\cdot,\cdot) = g(\cdot,\cdot).$$

∎

4. An Application for Taylor's Expansion

We may now state the following result.

Theorem 6. *With the conditions as in Theorem 4 and assuming that*

$$\left| \frac{\partial^{n+m+2} f(t,s)}{\partial x^{n+1} \partial y^{m+1}} - \frac{\partial^{n+m+2} f(u,v)}{\partial x^{n+1} \partial y^{m+1}} \right| \leq L_1 |x-u| + L_2 |y-v|,$$

we have the inequality

$$\begin{aligned} &\left| f(x,y) - \sum_{i=0}^{n} \sum_{j=0}^{m} \frac{(x-a)^i}{i!} \cdot \frac{(y-b)^j}{j!} \frac{\partial^{i+j} f(a,b)}{\partial x^i \partial y^j} \right. \\ &- \frac{1}{m!} \sum_{i=0}^{n} \frac{(x-a)^i}{i!} \int_b^y (y-s)^m \frac{\partial^{i+m+1} f(a,s)}{\partial x^i \partial y^{m+1}} ds \\ &- \frac{1}{n!} \sum_{j=0}^{m} \frac{(y-b)^j}{j!} \int_a^x (x-t)^n \frac{\partial^{j+n+1} f(t,b)}{\partial x^{n+1} \partial y^j} dt \\ &\left. - \frac{(x-a)^n (y-b)^n}{(n+1)!(m+1)!} \times \frac{\partial^{n+m} f(t,s)}{\partial x^n \partial y^m} dsdt \right| \\ \leq\; & \frac{(x-a)^{n+1} (y-b)^{m+1}}{3n!m!} G(n,m)\, p(a,x,b,y), \end{aligned} \tag{4.1}$$

where

$$G(n,m) = \left[\frac{1}{(2n+1)(2m+1)} - \frac{1}{(n+1)^2 (m+1)^2} \right]^{\frac{1}{2}}$$

and

$$p(a,x,b,y) = \left[3L_1^2 (x-a)^2 + 4L_1 L_2 (x-a)(y-b) + 3L_2^2 (y-b)^2 \right]^{\frac{1}{2}}.$$

Proof. Writing (2.1) as

$$\begin{aligned} f(x,y) = & \sum_{i=0}^{n} \sum_{j=0}^{m} \frac{(x-a)^i}{i!} \cdot \frac{(y-b)^j}{j!} \cdot \frac{\partial^{i+j} f(a,b)}{\partial x^i \partial y^j} \\ & + \frac{1}{m!} \sum_{i=0}^{n} \frac{(x-a)^i}{i!} \int_b^y (y-s)^m \frac{\partial^{i+m+1} f(a,s)}{\partial x^i \partial y^{m+1}} ds \\ & + \frac{1}{n!} \sum_{j=0}^{m} \frac{(y-b)^j}{j!} \int_a^x (x-t)^n \frac{\partial^{j+n+1} f(t,b)}{\partial x^{n+1} \partial y^j} dt \\ & + R(f,a,x,b,y), \end{aligned} \tag{4.2}$$

where

$$R(f,a,x,b,y)=\frac{1}{n!m!}\int_a^x\int_b^y (x-t)^n (y-s)^m\cdot\frac{\partial^{n+m+2}f(t,s)}{\partial x^{n+1}\partial y^{m+1}}dsdt. \tag{4.3}$$

Now, let us consider that

$$h(t,s)=(x-t)^n(y-s)^m$$

and

$$g(t,s)=\frac{\partial^{n+m+2}f(t,s)}{\partial x^{n+1}\partial y^{m+1}}.$$

Furthermore, recalling Korkine's identity, we have

$$\begin{aligned} &R(f,a,x,b,y) \qquad (4.4)\\ &= \frac{1}{n!m!}\int_a^x\int_b^y h(t,s)\,g(t,s)\,dsdt\\ &= \frac{1}{n!m!\,(x-a)(y-b)}\int_a^x\int_b^y h(t,s)\,dsdt\int_a^x\int_b^y g(t,s)\,dsdt+R_1(f,a,x,b,y), \end{aligned}$$

where

$$\begin{aligned} &R_1(f,a,x,b,y) \qquad (4.5)\\ &= \frac{1}{2n!m!\,(x-a)(y-b)}\int_\Omega (h(t,s)-h(u,v))(g(t,s)-g(u,v))\,dtdsdudv, \end{aligned}$$

where

$$\Omega=\big[\,[a,x]\times[b,y]\big]^2. \tag{4.6}$$

In addition, applying the Cauchy-Schwartz inequality for (4.5), we get

$$\begin{aligned} &|R_1| \qquad (4.7)\\ &= \left|\frac{1}{2n!m!\,(x-a)(y-b)}\int_\Omega (h(t,s)-h(u,v))(g(t,s)-g(u,v))\,dtdsdudv\right|\\ &\le \frac{1}{2n!m!\,(x-a)(y-b)}\sqrt{\int_\Omega (h(t,s)-h(u,v))^2\,dtdsdudv}\\ &\quad\times\sqrt{\int_\Omega (g(t,s)-g(u,v))^2\,dtdsdudv}. \end{aligned}$$

By simple computation

$$\begin{aligned} &\int_\Omega (h(t,s)-h(u,v))^2\,dtdsdudv \qquad (4.8)\\ &= \int_\Omega ((x-t)^n(y-s)^m-(x-u)^n(y-v)^m)^2\,dtdsdudv\\ &= 2(x-a)^{2n+2}(y-b)^{2m+2}\left[\frac{1}{(2n+1)(2m+1)}-\frac{1}{(n+1)^2(m+1)^2}\right]. \end{aligned}$$

Now, we let

$$I = \int_{\Omega} (g(t,s) - g(u,v))^2 \, dtdsdudv = \int_{\Omega} \left(\frac{\partial^{n+m+2} f(t,s)}{\partial x^{n+1} \partial y^{m+1}} - \frac{\partial^{n+m+2} f(u,v)}{\partial x^{n+1} \partial y^{m+1}} \right)^2 dtdsdudv, \tag{4.9}$$

then

$$I = 2 \left[(x-a)(y-b) \int_a^x \int_b^y \left(\frac{\partial^{n+m+2} f(t,s)}{\partial x^{n+1} \partial y^{m+1}} \right)^2 dsdt - \left(\int_a^x \int_b^y \frac{\partial^{n+m+2} f(u,v)}{\partial x^{n+1} \partial y^{m+1}} dvdu \right)^2 \right]. \tag{4.10}$$

Applying Corollary 3.2, we have the following inequality

$$|I| \le 8 (x-a)^2 (y-b)^2 \times \left[\frac{L_1^2 (x-a)^2}{12} + L_1 L_2 \frac{(x-a)(y-b)}{9} + \frac{L_2^2 (y-b)^2}{12} \right]. \tag{4.11}$$

Utilising (4.8) and (4.11), and (4.4) and substituting in (4.2), the theorem is proved. ∎

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